Lattice gauge theory

- I follow my summer-school lectures quite closely (see website for link to PDF)
- All the texts on the web site have good introductions to this basic material.

* Discretizing gauge theory & studying it numerically (especially with fermions) is where the main research effort in the lattice community lies — primarily studying QCD.

This is because, unlike scalar field theories, QCD has a negative β-fcn $\Rightarrow$ even if the bare gauge coupling vanishes as $a \to 0$ the physical coupling remains large & long distance physics is non-perturbative.

Lattice methods are the only known way to obtain quantitative predictions in the low energy ($p \sim 1$ GeV) regime — spectrum of hadrons, scattering, decays, matrix elements, finite T properties, ...

* The methodology was invented by Ken Wilson

As for the scalar field theory, discretizing allows both the usual weak coupling perturbative expansion & also a strong coupling expansion which allows one to study the theory in the non-perturbative domain.

"Complete" control requires, however, numerical simulations.

Recall continuum (Euclidean) QCD

$$S = \int_x \left\{ \frac{1}{2} \mathrm{Tr}\left(F_{\mu\nu}(x) F_{\mu\nu}(x)\right) - \sum_q \bar{q}(x)(\not{D} + m_q)\, q(x) \right\}$$

assume in fundamental rep. of group.

$D_\mu \gamma_\mu$

quarks - Grassman variables in path integral

$$D_\mu = \partial_\mu - i g A_\mu \quad ; \quad A_\mu = \sum_a A_\mu^a T^a$$

coupling constant

matrix gauge field in Lie algebra

$a = 1$ $U(1)$

$a = 1-3$ $SU(2)$

$a = 1-8$ $SU(3)$

generators of gauge group. e.g. $\frac{\tau_1}{2}, \frac{\tau_2}{2}, \frac{\tau_3}{2}$ in $SU(2)$

Generator properties: $T^a = T^{a\dagger}$

$$[T^a, T^b] = i f^{abc} T^b$$

structure constants e.g. ε^{abc} in $SU(2)$

Can choose normalization: $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$

Field strength:
$$\begin{aligned} F_{\mu\nu} &= F_{\mu\nu}^a T^a \\ &= \frac{i}{g}[D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] \end{aligned}$$

To avoid issues w/ fermions on lattice, also consider scalar quarks $\phi(x)$ ← complex fields

$$S_\phi = \int_x \left\{ (D_\mu \phi)^\dagger (D_\mu \phi) + m_0^2 \phi^\dagger \phi + V(\phi^\dagger \phi) \right\}$$

also in fund. rep.

- A key point of the lattice construction is to maintain gauge invariance - ensures theory is unitary in pert-thy, & non-pert that transfer matrix is physical.

- So, recall gauge invariance in ctm:

$$q(x) \to V(x)\,q(x) \qquad V \in \text{gauge group.}$$

Let's assume SU(3) henceforth $\Rightarrow V^{\dagger} = V^{-1}$; $\det V = 1$

$$\phi(x) \to V(x)\,\phi(x)$$

$$\bar{q}(x) \to \bar{q}(x)\,V^{\dagger}(x) = \bar{q}(x)\,V^{-1}(x)$$

$$\phi^{\dagger}(x) \to \phi^{\dagger}(x)\,V^{\dagger}(x)$$

The transformation of A_μ

$$A_\mu(x) \to V(x)\,A_\mu(x)\,V^{-1}(x) + \frac{i}{g}\,V(x)\,\partial_\mu V^{-1}(x)$$

is such that covariant derivs are.... covariant

$$[D_\mu \phi](x) \to \partial_\mu(V\phi) - ig\left(VA_\mu V^{-1} + \frac{i}{g}\,V\partial_\mu V^{-1}\right)V\phi$$

Using $\partial_\mu(VV^{-1}) = 0 = V\partial_\mu V^{-1} + (\partial_\mu V)\,V^{-1}$ this can be written

$$= V(x)[D_\mu \phi](x)$$

One can write this as $D_\mu \to V D_\mu V^{-1}$

so that $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \to V\frac{i}{g}[D_\mu, D_\nu]\,V^{-1}$

$$= V(x)\,F_{\mu\nu}(x)\,V^{-1}(x)$$

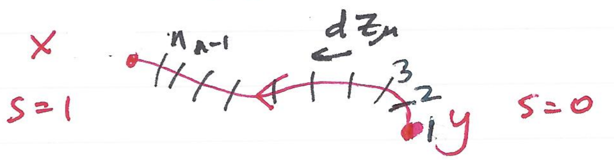
- It follows that S & S_ϕ are gauge invariant.

We will make extensive use of Wilson/Polyakov lines:

$$L(x,y) = \mathcal{P} \exp\left\{ ig \int_0^1 ds \, \frac{dz_\mu}{ds} A_\mu(z(s)) \right\} \in \text{Lie group}$$
e.g. $SU(3)$

$z(0) = y$ to $z(1) = x$

initial final



$\mathcal{P}$ means path order matrices A_μ (which don't, in general, commute), earlier to the right, later to the left.
– like T-ordering of $\hat{H}$ in QM

Can also write as the limit $ds \to 0$ of the product

$$\underbrace{\left(1 + ig \underbrace{ds \frac{dz_\mu}{ds}}_{dz_\mu^{(n)}} A_\mu\Big|_{s=1-\frac{1}{n}}\right) \cdots \left(1 + ig \underbrace{ds \frac{dz_\mu}{ds}}_{dz_\mu^{(1)}} A_\mu\Big|_{s=0}\right)}_{\in \text{ Lie group.}}$$

Now $1 + ig\, dz_\mu A_\mu(x') \xrightarrow{g.tr.}$

$$V(x')V^{-1}(x) + ig\, dz_\mu \left[V(x') A_\mu(x') V(x')^{-1} - \frac{i}{g} (\partial_\mu V(x')) V^{-1}(x') \right]$$

$$= V(x' + dz_\mu) \left(1 + ig\, dz_\mu A_\mu(x')\right) V^{-1}(x') + O(dz_\mu^2)$$

$$\Rightarrow L(x,y) \xrightarrow{g.tr.} V(x)\, L(x,y)\, V^{-1}(y)$$

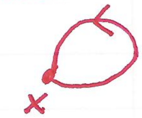
indep. of intermediate path

Thus $\phi^\dagger(x)\, L(x,y)\, \phi(y)$ is gauge invariant,

as is

$$\mathrm{tr}(L(x,x))$$

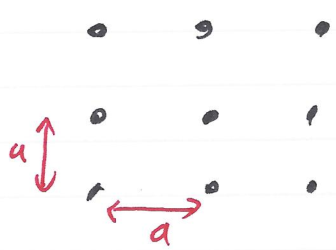
where intermediate path now forms a loop



so this is called a Wilson loop.

Wilson's discretization

Assume, for simplicity, an isotropic hypercubic lattice



Start w/ the complex scalar.

$$\int_x M^2 \phi^\dagger \phi \rightarrow \sum_n M_0^2\, \underbrace{\phi_n^\dagger \phi_n}_{\text{gauge inv. (see below)}}$$

as before

Similarly for $\int_x V(\phi^\dagger \phi)$

Kinetic term: without gauge fields

$$\int_x |\partial_\mu \phi|^2 \rightarrow \sum_{n,\mu} (\phi^\dagger_{n+\mu} - \phi^\dagger_n)(\phi_{n+\mu} - \phi_n)$$

drop "hats" $\hat{\mu} \rightarrow \mu$.

but not gauge invariant since $\phi_{n+\mu}$ & ϕ_n transform differently.

Ok, let's first be clear about gauge tran$^{\text{s}}$ on lattice:

$V(x) \to V_n$ (live on SITES)

"replaced by"

$\phi_n \xrightarrow[g.tr.]{} V_n \phi_n \quad ; \quad \phi_n^+ \longrightarrow \phi_n^+ V_n^+$

To give a gauge-invariant meaning to $\phi_{n+\mu} - \phi_n$ need to transport gauge transformation info. Wilson lines are the tool to do so:

In the continuum $\quad L(an, an+q\hat{\mu})\, \phi(an+q\hat{\mu})$

$$an \xleftarrow{L \sim U_{n,\mu}} an+q\hat{\mu}$$

$\xrightarrow{g.tr.} V(an)\,[\, L\, \phi \,]$

i.e. transforms the same way as $\phi(an)$ so can be compared.

So, on the lattice, introduce

$U_{n,\mu} \sim L(an, an+q\hat{\mu})$

↳ link variable, which we make transform like L

$$U_{n,\mu} \xrightarrow[g.tr.]{} V_n U_{n,\mu} V_{n+\mu}^{-1}$$

$U_{n,\mu} \in$ Lie group, e.g. $SU(3)$

Then $\displaystyle\sum_{n\mu} \left| U_{n,\mu}\phi_{n+\mu} - \phi_n \right|^2$ provides a gauge-invariant kinetic term for scalar.

$$= \sum_n \Big[8\,\phi_n^+\phi_n - \sum_{n,\mu} \underbrace{\left(\phi_n^+ U_{n,\mu}\, \phi_{n+\mu} + \phi_{n+\mu}^+ U_{n,\mu}^+ \phi_n \right)}_{\text{gauge-invariant hopping term}} \Big]$$

- Diagrammatically, hopping terms are

$$\phi_n^\dagger \xleftarrow{U_{n,\mu}} \phi_{n+\mu} \qquad \& \qquad \phi_n \xrightarrow{U_{n,\mu}^\dagger} \phi_{n+\mu}^\dagger$$

Note that $U_{n,\mu}^\dagger \xrightarrow[g.tr.]{} V_{n+\mu}\, U_{n,\mu}^\dagger\, V_n^{-1}$

"transports" $\phi_n \to n+\mu$ (or $\phi_{n+\mu}^\dagger$ to n)

This corresponds to the continuum result that

$$L(x,y)^\dagger = L(y,x)$$ (following the same path in reverse).

- So, we see that the natural lattice gauge variables are on LINKS (not sites) and live in the gauge GROUP (not the Lie algebra).

On the lattice there is no A_μ, only $U_{n,\mu}$.
However we can reintroduce A_μ to study slowly varying, classical-like fields

$$U_{n,\mu} \approx 1 - iga\, A_\mu(n+\hat{\mu}/2) + O(a^2)$$

(expanding out path-ordered exponential)

Then $$U_{n,\mu}\phi_{n+\mu} - \phi_n \approx a\left[\partial_\mu - ig A_\mu(n+\hat{\mu}/2)\right]\phi_n$$
$$\sim a^2 D_\mu \phi(an)$$

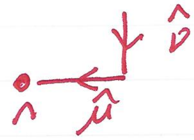
i.e. recover continuum form.

Now we get to the punch line : <u>lattice gauge action</u>

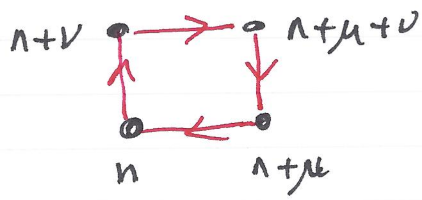
We want something built from $\mathcal{U}_{n,\mu}$ which, in classical cont.-limit goes over to $\mathrm{tr}\, F_{\mu\nu}^2$.

We recall $F_{\mu\nu} \propto [D_\mu, D_\nu]$

& that D_μ arises from shifting in μ'th dir^{n}.

$\Rightarrow$ want difference of  & 

Local object that has this difference built in is the "plaquette"



$$\mathrm{tr}\big(\underbrace{\mathcal{U}_{n,\mu}\,\mathcal{U}_{n+\mu,\nu}\,\mathcal{U}^{+}_{n+\nu,\mu}\,\mathcal{U}^{+}_{n,\nu}}_{\text{order fixed by gauge inv.}}\big) \equiv \mathrm{tr}(P_{\mu\nu,n})$$

makes gauge inv.

There are 12 such objects in 4 dim $(\mu \neq \nu)$, but

$$P_{\nu\mu,n} = P^{+}_{\mu\nu,n}$$

so only 6 are independent.

Note the similarity to GR – parallel transport around the plaquette gives something non trivial (which we will see is related to the curvature)

(−) sign since $dz_\nu = -a$

Once-in-a-lifetime exercise (HW)

Define A_ν by: $\quad \mathcal{U}_{n,\nu} \equiv \exp\left[-iag\,A_\nu\left(an + \frac{a\hat{\nu}}{2}\right)\right]$

treat as smooth continuum field

Expand $P_{\mu\nu}$ in powers of a

Find: $$P_{\mu\nu} = 1 - iga^2 F_{\mu\nu} - \frac{g^2}{2} a^4 F_{\mu\nu}^2 + ia^3 G_{\mu\nu} + ia^4 H_{\mu\nu} + O(a^5)$$

hermitian.

where $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

$x = a\left(n + \frac{\hat{\mu}}{2} + \frac{\hat{\nu}}{2}\right)$

center of plaquette.

from $\mathrm{tr}\,\mathbb{1}$ for $SU(N_c)$

Thus $$\mathrm{Re}\,\mathrm{tr}\,P_{\mu\nu} = N_c - \frac{g^2}{2} a^4\, \mathrm{tr}\,F_{\mu\nu}^2 + O(a^5)$$

in fact $O(a^6)$ HW

$$S_{gauge} = \int_x \frac{1}{2}\,\underbrace{\mathrm{tr}\,F_{\mu\nu}^2}_{\substack{\text{implicit}\\ \text{sum over}\\ \mu\,\&\,\nu.}} \sim \sum_n \sum_{\mu\neq\nu} \frac{1}{g^2}\left(N_c - \mathrm{Re}\,\mathrm{tr}\,P_{\mu\nu}\right)$$

[$\mu = \nu$ gives no contrib.]

$$= \frac{2N_c}{g^2} \sum_{\square} \left(N_c^2 - \frac{\mathrm{Re}\,\mathrm{tr}\,\square}{N_c}\right)$$

can drop since constant

means sum over independent plaquettes i.e. $\mu < \nu$

$\square_\mu^\nu = P_{\mu\nu}$

Called, with malice, $\beta = \frac{2N_c}{g^2} \neq \frac{1}{kT}$

So, finally, our lattice action is.

$$S_{lat} = \sum_{\Box} -\beta \underbrace{\frac{\mathrm{Re}\,\mathrm{tr}\,\Pi}{N_c}}_{\substack{\text{normalized trace}\\ \text{max value is 1}}} + \sum_{n,\mu} |U_{n,\mu}\phi_{n+\mu} - \phi_n|^2 + \sum_n m_0^2 \phi_n^+ \phi_n + \sum_n V(\phi_n^+\phi_n)$$

Called "Wilson gauge action"

What about the partition fcn?

$$Z = \int \prod_{n,\mu} dU_{n,\mu} \prod_n \underbrace{d\phi_n\, d\phi_n^+}_{\text{equivalent to } d\,\mathrm{Re}\,\phi_n\, d\,\mathrm{Im}\,\phi_n} e^{-S_{lat}}$$

(this is gauge invariant).

What is this? Integration of gauge group.

Want this to be gauge invariant: $U_{n,\mu} \to V_n U_{n,\mu} V_{n+\mu}^{-1}$

Up to overall normalization, there is a unique measure on compact Lie groups invariant under such left- and right-"translations": <u>Haar measure</u>

Define normalization such that $\int dU = 1$

Can figure out many integrals using invariance

$$X = \int dU\, U = \int d(VU)\, V^{-1}(VU) = V^{-1}X \quad \text{for } \forall\, V \in \text{group}$$

$$\Rightarrow X = 0$$

More on this later.

[Also $\int dU\, f(U^+) = \int dU\, f(U)$ i.e. $d(U^+) = dU$.]

More on Haar measure (see Smit, Montvay & Münster)

Examples: $U(1)$: $\quad U = e^{i\phi} \quad \int dU = \frac{1}{2\pi}\int_0^{2\pi} d\phi$

$VU = e^{iv} e^{i\phi} = e^{i(\phi+v)}$ $\Rightarrow$ need translation invariance in ϕ

$SU(2)$: $\quad U = x^0 + \vec{x}\cdot\vec{\tau} \qquad \det U = x_\mu x_\mu = 1$

so $SU(2)$ manifold is S^3

"Translation" invariance $\Rightarrow$ measure invariant under arbitrary rotations of S^3.

unique result : $\quad dU = \frac{1}{\pi^2}\delta(x^2-1)\, d^4x$

Alternatively, parametrize group as

$$U = e^{i\theta\hat{n}\cdot\vec{\tau}/2} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\,\hat{n}\cdot\vec{\tau}$$

$\hat{n}$ = unit vector & $\quad 0 \le \theta < 2\pi$

Then $\quad dU = \frac{1}{4\pi^2}\, d\theta \left(\sin\frac{\theta}{2}\right)^2 \underbrace{d\Omega(\hat{n})}_{\text{uniform measure on } S^2}$

In general $\quad dU = \text{const.}\sqrt{\det g}\,\prod_k d\alpha_k$

where α_k are coords on group and

$$g_{k\ell} = \operatorname{tr}\left(\frac{\partial U}{\partial\alpha^k}\frac{\partial U^\dagger}{\partial\alpha^\ell}\right) \quad \text{is a metric.}$$

Easy to check that measure is invariant under arbitrary change of variables.

Another useful example of Haar measure, needed for pert. thy.

For a (compact) Lie group, write $\mathcal{U} = e^{\sum_a i \alpha_a t_a}$

real parameters $a = 1 - (N_c^2 - 1)$ for $SU(N_c)$

hermitian generator matrices

$$g_{ab} = \mathrm{tr}\left(\frac{\partial \mathcal{U}}{\partial \alpha_a} \frac{\partial \mathcal{U}^+}{\partial \alpha_b} \right)$$

For $\alpha_a \ll 1$, $\forall a$ ($\mathcal{U}$ close to $\mathbb{1}$)

$$\frac{\partial \mathcal{U}}{\partial \alpha_a} \approx i t_a + \mathcal{O}(\alpha)$$

$\Rightarrow \quad g_{ab} \approx -\mathrm{tr}\, t_a t_b = -\frac{1}{2} \delta_{ab}$ (conventional normalization)

Thus $\det g \approx$ constant $\left[= \left(-\frac{1}{2}\right)^{N_c^2-1} \right]$

and $d\mathcal{U} \approx \text{const.} \prod_a d\alpha_a$

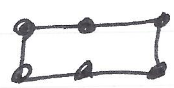
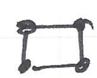
Bottom line:

- Haar measure can be constructed as needed.
- If lattice volume Ω is finite, then Z is completely defined
- No need to fix a gauge to calculate:

  $\int \prod_{n,\mu} dU_{n,\mu}$ contains all gauge transforms as a subset (for which S_{lat} invariant)

  but the "gauge volume" is finite.

  Very different from continuum P.T. where have to fix the gauge otherwise gluon propagator vanishes in "gauge directions"

- Definition valid for all g – non-perturbative
- Could have used  instead of 

  – contains $F_{\mu\nu}^2$ in classical expansion

  This is the usual freedom of lattice actions, which can be exploited to reduce a^2 errors (indeed, to eliminate them)

- Construction holds also for discrete groups e.g. Z_2 (first studied by Wegner)

  (although in this case classical field A_μ makes no sense)

Can one spontaneously break a (local) gauge symmetry? NO! Elitzur's theorem.

$$\langle f(\mathcal{U}, \phi, \phi^{+}) \rangle = 0 \quad \text{if } f \text{ is gauge non-invariant}$$

e.g. $\langle \mathcal{U}_{n,\mu} \rangle = \langle \phi_n \rangle = \langle \phi_n^{+} \rangle = 0$

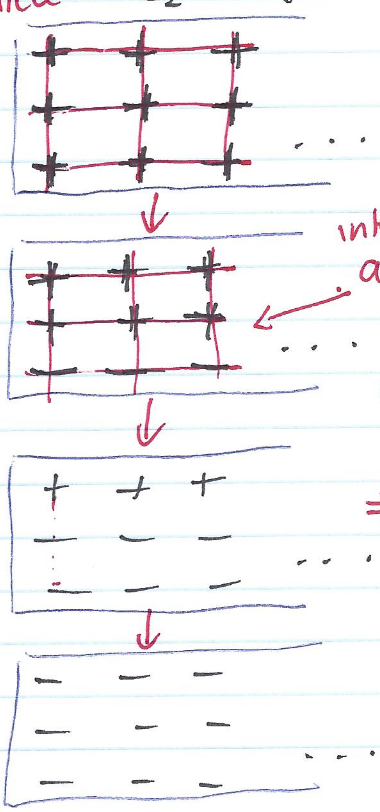
where $\langle tr(P_{\mu\nu,n}) \rangle \neq 0 \neq \langle \phi_n^{+} \phi_n \rangle$

gauge inv.

Wait, you say! In finite volume, if have a global symmetry, e.g. Z_2 in non-gauge ϕ^4 theory, then expectation value $\langle \phi_n \rangle = 0$. Needed to think about action= "energy" cost of transforming between vacua, or add source, to understand SSB.

So let's compare action cost in Z_2 field & gauge theory (i.e. $\phi_n \to \pm 1$ or $\mathcal{U}_{n,\mu} \to \pm 1$)

L^4 lattice $\quad Z_2$ FT (= Ising)



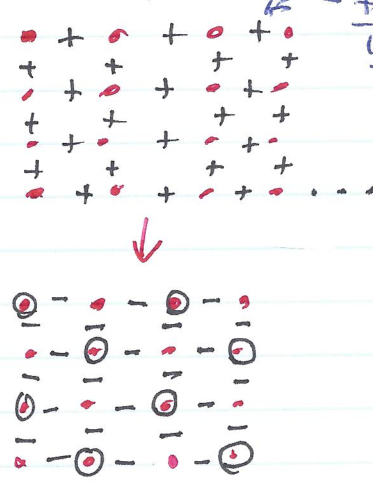
interface action $\propto L^3$

& need L steps.

$\Rightarrow$ doesn't happen when $L \to \infty$

Z_2 gauge th.

+ on links



Can "flip" links but NO action cost by doing gauge transform at sites ⊙

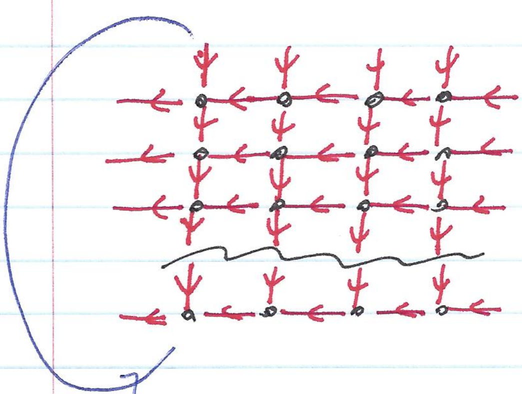
$\Rightarrow$ No barrier.

$SU(N_c)$ gauge theory with the Wilson (single-plaquette) action is reflection positive, and it is not too hard to construct the transfer matrix.

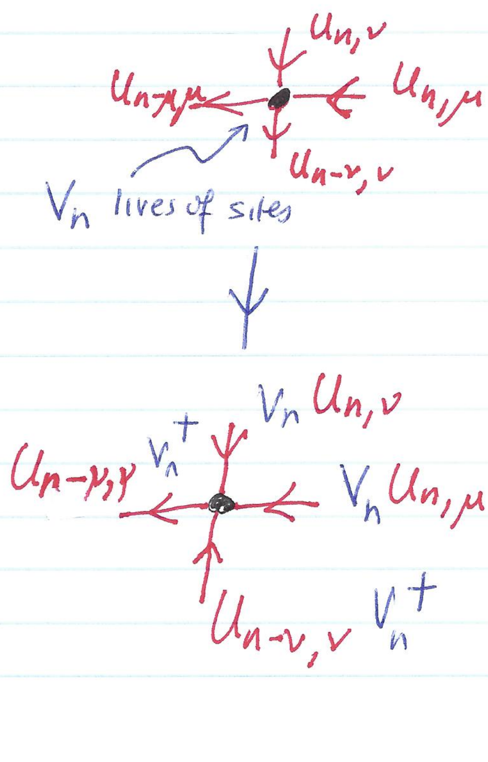
I won't go through this in detail (see Montvay & Münster for some discussion) but I want to note that the construction involves gauge fixing to "temporal gauge".

This is the lattice version of $A_{\mu=4} = 0$, which becomes $U_{n,\mu=4} = 1$.

Although we don't need to gauge fix to define Z, we can do so. It is worth understanding something about it. We do so in the context of a lattice w/ N_t time links & PBC.



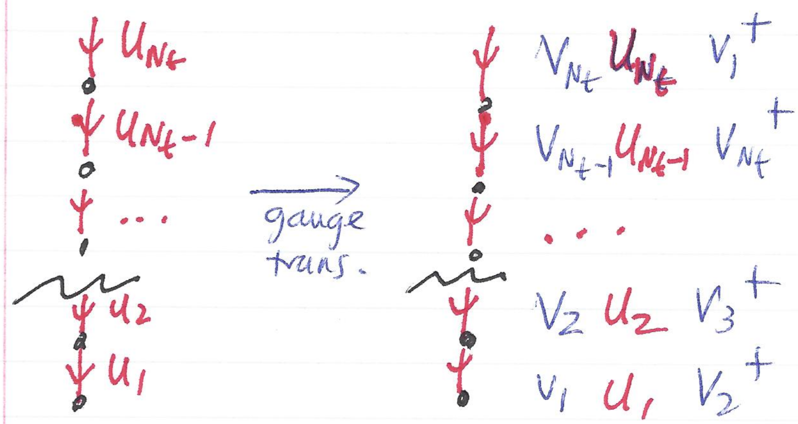
Recall what a gauge transformation does:



In 4-d changes 8 links.

We are interested in setting time-directed links $\to \mathbb{1}$, and don't care what happens to the spatial links.

Focus on one time-chain. Naively have N_t V's so can set all N_t U's $\Rightarrow \mathbb{1}$.



Not correct!

$\mathrm{tr}(U_1 U_2 \ldots U_{N_t})$ = Polyakov line is gauge invariant & $\neq N_c$ in general.

$\Rightarrow$ one link is "left over"

e.g.
$$V_1 = U_{N_t-1}^+ U_{N_t-2}^+ \cdots U_1^+$$
$$V_2 = U_{N_t-1}^+ \cdots U_2^+$$
$$\cdots$$
$$V_{N_t-2} = U_{N_t-1}^+ U_{N_t-2}^+$$
$$V_{N_t-1} = U_{N_t-1}^+$$
$$V_{N_t} = \mathbb{1}$$

$\Rightarrow U_1' = \ldots = U_{N_t-1}' = \mathbb{1}$

& $U_{N_t}' = U_{N_t} U_1 \ldots U_{N_t-2} U_{N_t-1}$

$\Rightarrow \mathrm{tr}\, U_{N_t}'$ = Polyakov line

- Can put the "left over" link on any timeslice
- Leaves the freedom of time indep. gauge transforms, but with arbitrary spatial dependence

$\Rightarrow$ can set all (but one) links in one spatial direction on one time slice to unity.

etc. etc. Gives a "maximal tree," w/ no closed loops

* What is the Jacobian for gauge fixing?

Unity!

$$Z_{gauge} = \underbrace{\int [D\mathcal{U}]_{\substack{\text{unfixed}\\ \text{links}}}}_{\text{integrate over unfixed links just as before}} e^{-S_{gauge}}$$

action evaluated with gauge-fixed links fixed.

In fact, Z_{gauge} is independent of the choices of the values of the gauge-fixed links.

* Follows from invariance of Haar measure, $D\mathcal{U} = D(V\mathcal{U}W^+)$, and gauge invariance of action.

* Let's see this if we gauge fix just 1 link

Call this $\mathcal{U}$; then $Z = \int d\mathcal{U} \; I(\mathcal{U})$

integrate $\mathcal{U}$ last

result of doing all other integrals

$$I(\mathcal{U}) = \int [D\mathcal{U}_{other}] \, e^{-S_{gauge}(\mathcal{U}_{other}, \mathcal{U})}$$

$$= I(V\mathcal{U}W^+)$$

since this is a part of a gauge transform, under which S_{gauge} & $[D\mathcal{U}_{other}]$ are invariant

$\Rightarrow I(\mathcal{U})$ indep. of $\mathcal{U}$

$\Rightarrow Z = I \int d\mathcal{U} = I$

* Can extend link by link until fully gauge fixed

* At that point cannot "absorb" a gauge transform on $\mathcal{U}$'s without introducing new variables

- Temporal gauge is useful for determining the transfer matrix & showing reflection positivity, but it clearly breaks rot^n invariance

- In practice, when gauge fixing is used on the lattice, it is most often to

  Landau Gauge : $\partial_\mu A_\mu = 0$

  Smooth fields, equivalent to minimizing $\int_x tr(A_\mu A_\mu)$

  Lattice version is to maximize.

$$\mathrm{Re} \sum_{n,\mu} tr(\mathcal{U}_{n,\mu})$$

  - makes $\mathcal{U}_{n,\mu}$ globally as close to $\mathbb{1}$ as possible.

  In perturbation theory this picks out a unique A_μ for small fields, but for large fields (even in the ctm) there are multiple solutions: "Gribov copies"

  On the lattice, the number of copies grows enormously – there are many local maxima.

- I stress, however, that gauge fixing is not needed in general, so we don't care much about Gribov copies