1. **Classical continuum limit of lattice gauge action.** The lattice action is built from the plaquette variable

\[ P_{\mu\nu;n} = U_{n,\mu}U_{n+\mu,\nu}U_{n+\nu,\mu}^\dagger U_{n,\nu}^\dagger. \]

Here the \( U_{n,\mu} \) are elements of the gauge group, which we assume to be \( SU(N_c) \). In this problem we write the \( U \)'s in terms of continuum \( A \) fields which we assume are smooth and slowly varying. Specifically, we define \( A_\mu \) by

\[ U_{n,\mu} \equiv \exp[-iagA_\mu(an + a\hat{\mu}/2)].\]

Note that we place the continuum \( A_\mu \) at the midpoint of the link. The problem is to insert this form into \( P_{\mu\nu;n} \) and show that

\[ \text{Re tr}(P_{\mu\nu;n}) = N_c - a^4g^2 \frac{1}{2} \text{tr}(F_{\mu\nu}^2(x)) + O(a^6), \]

where

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \]

and \( x = an + a\hat{\mu}/2 + a\hat{\nu}/2 \) is the midpoint of the plaquette. Note that part of the problem is to show that the first corrections are of \( O(a^6) \) and not \( O(a^5) \).

This sounds like a tedious exercise, and to some extent it is, but we can save a lot of work using general considerations. Here is one suggested path:

(a) Writing \( P_{\mu\nu;n} = e^J \), show that \( J \) begins at \( O(a^2) \) and determine the form of \( J \). You will need to use Baker-Campbell-Hausdorff (e.g. \( e^{A}e^{B} = e^{A+B+\frac{1}{2}[A,B]+...} \)) as well as Taylor expansions of \( A_\mu \) about the midpoint of the plaquette.

(b) Writing

\[ J = -ia^2gF_{\mu\nu} + ia^3G_{\mu\nu} + ia^4H_{\mu\nu} + O(a^5), \]

show that \( F_{\mu\nu}, G_{\mu\nu} \) and \( H_{\mu\nu} \) are hermitian.

(c) Use these results to determine \( \text{Re tr}(P_{\mu\nu;n}) \) including terms proportional to \( a^4 \).

(d) Show that \( \text{tr}(P_{\mu\nu;n}) \) is an even function of \( a \), so that the first corrections are proportional to \( a^6 \).
2. Reflection positivity of pure gauge theory. Demonstrate the site and link reflection positivity of the pure $SU(N_c)$ gauge theory with Wilson action

$$S = \sum_{n,\mu<\nu} -\beta \text{Re} \text{tr} P_{n,\mu\nu},$$

where the plaquette is

$$P_{n,\mu\nu} = U_{n,\mu} U_{n+\mu,\nu} U_{n+\nu,\mu}^\dagger U_{n,\nu}^\dagger.$$

This shows that a physical transfer matrix and Hilbert space can be constructed (though we won’t do so here).

We first fix the theory to temporal gauge, $U_{n,4} = 1$. As discussed in class, this leads to an unconstrained path integral over the remaining (spatial) links. We assume the lattice has infinite length in the time direction, with boundary conditions at infinity such that we can set all $U_{n,4}$ to unity. (This is in contrast to a lattice of finite time extent where one time-directed link cannot be fixed to unity.) We are also using the fact that the only observables of interest are gauge invariant, and so are unchanged by gauge transformations. (Recall that expectation values of gauge non-invariant quantities vanish.)

Time-reflection for links is defined in general by

$$\Theta U(n, m) = U(\theta n, \theta m)^*,$$

where $U(x, y)$ is the link from site $n$ to site $m$. The only non-zero such links are $U(n, n + \mu) \equiv U_{n,\mu}$ and $U(n, n - \mu) \equiv U_{n,-\mu,\mu}^\dagger$. Here we are only interested in spatial links, for which we have

$$\Theta U_{n,j} = U_{\theta n,j}^*.$$

The statement of reflection positivity is then

$$\int [DU_{\text{spatial}}] e^{-S(U_+)} \Theta F(U_+) \geq 0.$$

For site-reflection, $U_+$ is the set of spatial links on timeslices with $n_4 \geq 0$, while for link-reflection the condition is $n_4 \geq 1$.

(a) Show that the action is invariant under $\Theta$, i.e. $\Theta S(U) = S(U)$.

(b) Show site-reflection positivity.
(c) Show link-reflection positivity. This is more tricky and we need to
know something about the “character-expansion” on group mani-
folds. Let \( r \) be a label for inequivalent unitary representations of
\( SU(N_c) \). (For example, in \( SU(2) \) this label is \( \ell = 0, 1/2, 1, 3/2, \ldots \))
The representation matrices are \( D^{(r)}(U) \), and satisfy the group alge-
bra. The characters are their traces:
\[
\chi_r(U) \equiv \text{tr} D^{(r)}(U),
\]
and so are a mapping from the group to complex numbers. Characters are used often for finite groups, but are also very useful for
continuous groups. This is in particular because they form a com-
plete basis for square-integrable functions on the group. Thus any
such function \( f(U) \) can be decomposed as
\[
f(U) = \sum_r f_r \chi_r(U), \quad f_r = \int dU \chi_r^*(U)f(U),
\]
where the integral uses the normalized Haar measure.
The final piece of information you’ll need is that if \( f(U) = \exp(c \text{Re tr} U) \),
with \( c \) a positive real constant, then the expansion coefficients \( f_r \) are
all positive.
Good luck!

3. String tension in \( U(1) \) gauge theory in strong coupling expansion using
characters. In class, we calculated the string tension in the strong coupling
expansion, and found, for the \( U(1) \) theory,
\[
\sigma = \ln(2/\beta), \quad (\beta = 1/g^2).
\]
In this problem we go beyond this leading order result in two ways. First,
we replace a simple expansion in powers of \( \beta \) with the character expansion.
This is simple for \( U(1) \), which is why we stick to this group. This allows us
to some up an infinite series of terms in the \( \beta \) expansion—those associated
with the plaquette action appearing repeatedly on the same site. This
changes the argument of the logarithm from \( 2/\beta \) to a function of \( \beta \) which
you will determine.
The second extension is to go beyond the minimal tiling to obtain the
first correction, which turns out to be proportional for small \( \beta \) to \( \beta^4 \), and
which you should calculate.
Here is the key result for doing the character expansion for \( U(1) \):
\[
e^{\beta \text{Re} U} = \sum_{r=-\infty}^{\infty} I_r(\beta)U^r,
\]
with the modified Bessel functions satisfying
\[ I_{-r}(\beta) = I_r(\beta) = (\beta/2)^r \frac{1}{|r|!} \left[ 1 + O(\beta^2) \right]. \]

Here we are using the results that the irreps of \( U(1) \) are labeled by integers with the representative of \( U \) being \( U^r \), and that the trace is irrelevant since all irreps are 1-dimensional. Thus \( \chi_r(U) = U^r \).

Recall that the \( U(1) \) action is
\[ S = -\sum_{n,\mu<\nu} \beta \text{Re} P_{n,\mu\nu}, \]
where \( P_{n,\mu\nu} \) is the usual plaquette. Also, the string tension is defined by
\[ \ln[\langle W(L,T) \rangle] = -LT\sigma + c_1(L+T) + c_2 + \ldots \]
for large \( L \) and \( T \), where \( W(L,T) \) is the rectangular Wilson loop.

4. **Tensor-scalar glueball mass splitting in \( U(1) \) gauge theory at strong coupling.** As discussed in class, we can determine the mass of glueballs from two-point correlators:
\[ \sum_{\vec{n}} \langle \mathcal{O}_{\vec{n},\mu_4} \mathcal{O}_0 \rangle_{\text{conn}} = ce^{-mn_4} + \text{higher mass exponentials}. \]

We also saw how, at leading order in the strong coupling expansion, the scalar and tensor glueballs were degenerate.

Here we go to higher order in the \( U(1) \) gauge theory, both by using the character expansion (as explained in the previous problem) and by keeping higher order “diagrams” beyond the leading-order “tube” configuration discussed in class.

The operator which couples to the scalar glueball is
\[ \mathcal{O}^{(s)}_n = \frac{1}{\sqrt{3}} \text{Re}(P_{n,12} + P_{n,23} + P_{n,13}), \]
while that which couples to the tensor glueball is
\[ \mathcal{O}^{(t)}_n = \frac{1}{\sqrt{2}} \text{Re}(P_{n,12} - P_{n,13}). \]

There are four other tensor glueball operators, which live in two irreps of the lattice cubic group, but we will focus on this one here.

(a) Calculate the common mass \( m_s = m_t \), at leading order in the strong-coupling expansion using the character methods described in the previous problem.

(b) Calculate \( m_t - m_s \) to leading non-trivial order.