Philosophy 453 S. M. Cohen
Philosophy of Language

First-order predicate logic with identity: syntax and semantics

Syntax

Vocabulary

Individual variables: $u, v, w, x, y, z, u_1, \ldots, z_1, u_2, \ldots$.

Logical constants: $\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \quad (\quad) \quad \exists \quad \forall \quad =$

Variants: $\sim \quad \& \quad \supset \quad \equiv$

Non-logical constants:

Individual constants: $a, b, c, d, \ldots, t, a_1, b_1, \ldots t_1, a_2, \ldots$.

Predicate letters: Capital letters with numerical superscripts and with or without numerical subscripts. The *superscript* indicates the *degree* of the predicate; the subscripts guarantee an infinite supply of predicate letters.

$A^0, B^0, \ldots, Z^0, A^0_1, B^0_1, \ldots, Z^0_1, A^0_2, B^0_2, \ldots, Z^0_2, \ldots,$

$A^1, B^1, \ldots, Z^1, A^1_1, B^1_1, \ldots, Z^1_1, A^1_2, B^1_2, \ldots, Z^1_2, \ldots,$

$A^2, B^2, \ldots, Z^2, A^2_1, B^2_1, \ldots Z^2_1, A^2_2, B^2_2, \ldots Z^2_2, \ldots,$

$\ldots$.

A **predicate of degree *n*** is a predicate whose numerical superscript is *n*.

A **sentential letter** is a predicate of degree 0.

An **individual symbol** is either an individual variable or an individual constant.

Syntactic Rules

Atomic formulas: an atomic formula is either a sentential letter standing alone, or a predicate letter of degree *n* followed by a string of *n* individual symbols, or a string of the form $\alpha = \beta$, where α and β are both individual symbols.

Formulas: A formula is either an atomic formula or else is built up out of atomic formulas by one or more of the following rules:

1. **Molecular formulas**: If φ and ψ are formulas, then:

   $\neg\varphi \qquad (\varphi \wedge \psi) \qquad (\varphi \vee \psi) \qquad (\varphi \rightarrow \psi) \qquad (\varphi \leftrightarrow \psi)$

   are all formulas.

2. **General formulas**: If φ is a formula and α is a variable, then $\forall\alpha\ \varphi$ and $\exists\alpha\ \varphi$ are both formulas.

Bound and free occurrences of variables: an occurrence of a variable α in a formula φ is **bound** if it is within an occurrence in φ of a formula of the form $\forall\alpha\ \psi$ or of the form $\exists\alpha\ \psi$. An occurrence that is not bound is **free**.

Sentences: a sentence is a formula in which no variable occurs free.

Terminology. Where φ and ψ are formulas and α is a variable:

$\neg\varphi$ is the **negation** of φ.

$(\varphi \wedge \psi)$ is a **conjunction**. φ and ψ are the **conjuncts**.

$(\varphi \vee \psi)$ is a **disjunction**. φ and ψ are the **disjuncts**.

$(\varphi \rightarrow \psi)$ is a **conditional**. φ is the **antecedent** and ψ is the **consequent**.

$(\varphi \leftrightarrow \psi)$ is a **biconditional**. φ and ψ are its components.

Where β and δ are individual symbols, $\beta = \delta$ is an **identity formula**. Where β and δ are individual constants, $\beta = \delta$ is an **identity sentence**.

An expression of the form $\forall\alpha$ is a **universal quantifier**. $\forall\alpha\ \varphi$ is the **universal generalization** of φ with respect to α.

An expression of the form $\exists\alpha$ is an **existential quantifier**. $\exists\alpha\ \varphi$ is the **existential generalization** of φ with respect to α.

For any formula φ, variable α , and individual symbol β , $\boldsymbol{\varphi\ \alpha/\beta}$ is the result of replacing all free occurrences of α in φ with occurrences of β.

Variant terminology:

What we are calling a formula is sometimes called a *well-formed formula*, or *wff* (pronounced ‘woof’). Also:

We say:	sentence	formula	formula that is not a sentence
Variant label:	closed sentence	sentence	open sentence

Semantics

Interpretations

An **interpretation**, $\mathfrak{I}$, of the formal language we have just described consists of:

1. A non-empty **domain**, $\mathfrak{D}$.

2. A **mapping** from constants of the language to elements (or other set-theoretic constructs out of elements) of $\mathfrak{D}$.

(Terminology: 'Element' means the same as 'member'. The converse of 'mapping' is 'assignment': we **map** a constant of the language **onto** an object in the domain; we **assign** that object **to** the constant that is mapped onto it. A mapping is a **function** in the mathematical sense; that is, a mapping gives each constant a **unique** value.)

1. Each individual constant is mapped onto exactly one element of $\mathfrak{D}$.

2. Each predicate of degree *n* is mapped onto exactly one set of ordered *n*-tuples of elements of $\mathfrak{D}$.

Explanation: each predicate of degree 2 is mapped onto a set of ordered pairs of elements of $\mathfrak{D}$, each predicate of degree 3 is mapped onto a set of ordered triples of elements of $\mathfrak{D}$, etc. We consider an ordered 1-tuple of elements of $\mathfrak{D}$ to be simply an element of $\mathfrak{D}$. Thus, each predicate of degree 1 is mapped onto a set of elements of $\mathfrak{D}$. Finally, we arbitrarily define 'set of ordered 0-tuples of elements of $\mathfrak{D}$' to be one or the other of the two **truth-values**, **T** or **F**. Thus, each sentential letter (predicate of degree 0) is mapped onto one of these two truth-values.

Truth under an interpretation

We will define 'φ is true under $\mathfrak{I}$', where φ is a sentence and $\mathfrak{I}$ is an interpretation. To deal with general sentences, we will need one additional definition — of the notion of the "β–variant" of an interpretation:

> Where $\mathfrak{I}$ and $\mathfrak{I}'$ are interpretations and β is an individual constant, $\mathfrak{I}$ is a β–variant of $\mathfrak{I}'$ iff $\mathfrak{I}$ and $\mathfrak{I}'$ differ at most in what they assign to β.

Our definition of truth is **recursive**: it will state the conditions under which the simplest sentences are true, and then state how the truth-values of more complex sentences depend upon those of simpler ones.

1. If φ is a sentential letter, then φ is true under $\mathfrak{I}$ iff $\mathfrak{I}$ assigns **T** to φ.

2. If φ is an identity sentence, β = δ, then φ is true under $\mathfrak{I}$ iff $\mathfrak{I}$ assigns the same object to both β and δ.

3. If φ is atomic and not a sentential letter and not an identity sentence, then φ contains a predicate of degree n (for $n \geq 1$). Then φ is true under $\Im$ iff the ordered n-tuple of objects that $\Im$ assigns to the individual constants of φ (taken in the order in which their corresponding constants occur in φ) is an element of the set of ordered n-tuples that $\Im$ assigns to the predicate occurring in φ.

4. If $\varphi = \neg\psi$, then φ is true under $\Im$ iff ψ is not true under $\Im$.

5. If $\varphi = (\psi \vee \chi)$, then φ is true under $\Im$ iff either ψ is true under $\Im$ or χ is true under $\Im$, or both.

6. If $\varphi = (\psi \wedge \chi)$, then φ is true under $\Im$ iff both ψ is true under $\Im$ and χ is true under $\Im$.

7. If $\varphi = (\psi \rightarrow \chi)$, then φ is true under $\Im$ iff either ψ is not true under $\Im$ or χ is true under $\Im$, or both.

8. If $\varphi = (\psi \leftrightarrow \chi)$, then φ is true under $\Im$ iff either ψ and χ are both true under $\Im$ or ψ and χ are both not true under $\Im$.

   Let β be the (alphabetically) first individual constant not occurring in φ. Then:

9. If $\varphi = \forall\alpha\ \psi$, then φ is true under $\Im$ iff $\psi\ \alpha/\beta$ is true under every β–variant of $\Im$.

10. If $\varphi = \exists\alpha\ \psi$, then φ is true under $\Im$ iff $\psi\ \alpha/\beta$ is true under at least one β–variant of $\Im$.

φ is false under $\Im$ iff φ is not true under $\Im$.

Logical Truth

A sentence φ is a **logical truth** iff φ is true under every interpretation.

A sentence φ is **logically false** iff φ is false under every interpretation.

A set of sentences Γ is **consistent** iff there is at least one interpretation under which every member of Γ is true.

A sentence φ is a **logical consequence** of a set of sentences Γ iff there is no interpretation under which all the members of Γ are true and φ is false.

A pair of sentences are **logically equivalent** iff there is no interpretation under which they differ in truth-value.