First-order predicate logic with identity: syntax and semantics

Syntax

Vocabulary

<table>
<thead>
<tr>
<th>Individual variables:</th>
<th>$u, v, w, x, y, z, u_1, ..., z_1, u_2, ...$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logical constants:</td>
<td>$\neg \land \lor \to \leftrightarrow ( ) \exists \forall =$</td>
</tr>
<tr>
<td>Variants:</td>
<td>$\sim &amp; \supset \equiv =$</td>
</tr>
</tbody>
</table>

Non-logical constants:

<table>
<thead>
<tr>
<th>Individual constants:</th>
<th>$a, b, c, d, ..., t, a_1, b_1, ... t_1, a_2, ...$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Predicate letters:</td>
<td>Capital letters with numerical superscripts and with or without numerical subscripts. The superscript indicates the degree of the predicate; the subscripts guarantee an infinite supply of predicate letters.</td>
</tr>
</tbody>
</table>

$$A^0, B^0, ..., Z^0, A_1^0, B_1^0, ..., Z_1^0, A_2^0, B_2^0, ..., Z_2^0, ...,$$

$$A^1, B^1, ..., Z^1, A_1^1, B_1^1, ..., Z_1^1, A_2^1, B_2^1, ..., Z_2^1, ...,$$

$$A^2, B^2, ..., Z^2, A_1^2, B_1^2, ..., Z_1^2, A_2^2, B_2^2, ..., Z_2^2, ...,$$

....

A predicate of degree $n$ is a predicate whose numerical superscript is $n$.

A sentential letter is a predicate of degree 0.

An individual symbol is either an individual variable or an individual constant.

Syntactic Rules

Atomic formulas: an atomic formula is either a sentential letter standing alone, or a predicate letter of degree $n$ followed by a string of $n$ individual symbols, or a string of the form $\alpha = \beta$, where $\alpha$ and $\beta$ are both individual symbols.

Formulas: A formula is either an atomic formula or else is built up out of atomic formulas by one or more of the following rules:
1. **Molecular formulas**: If \( \varphi \) and \( \psi \) are formulas, then:

\[
\neg \varphi \quad (\varphi \land \psi) \quad (\varphi \lor \psi) \quad (\varphi \to \psi) \quad (\varphi \leftrightarrow \psi)
\]

are all formulas.

2. **General formulas**: If \( \varphi \) is a formula and \( \alpha \) is a variable, then \( \forall \alpha \varphi \) and \( \exists \alpha \varphi \) are both formulas.

**Bound and free occurrences of variables**: an occurrence of a variable \( \alpha \) in a formula \( \varphi \) is **bound** if it is within an occurrence in \( \varphi \) of a formula of the form \( \forall \alpha \psi \) or of the form \( \exists \alpha \psi \). An occurrence that is not bound is **free**.

**Sentences**: a sentence is a formula in which no variable occurs free.

**Terminology**. Where \( \varphi \) and \( \psi \) are formulas and \( \alpha \) is a variable:

- \( \neg \varphi \) is the **negation** of \( \varphi \).
- \( (\varphi \land \psi) \) is a **conjunction**. \( \varphi \) and \( \psi \) are the **conjuncts**.
- \( (\varphi \lor \psi) \) is a **disjunction**. \( \varphi \) and \( \psi \) are the **disjuncts**.
- \( (\varphi \to \psi) \) is a **conditional**. \( \varphi \) is the **antecedent** and \( \psi \) is the **consequent**.
- \( (\varphi \leftrightarrow \psi) \) is a **biconditional**. \( \varphi \) and \( \psi \) are its components.

Where \( \beta \) and \( \delta \) are individual symbols, \( \beta = \delta \) is an **identity formula**. Where \( \beta \) and \( \delta \) are individual constants, \( \beta = \delta \) is an **identity sentence**.

An expression of the form \( \forall \alpha \) is a **universal quantifier**. \( \forall \alpha \varphi \) is the **universal generalization** of \( \varphi \) with respect to \( \alpha \).

An expression of the form \( \exists \alpha \) is an **existential quantifier**. \( \exists \alpha \varphi \) is the **existential generalization** of \( \varphi \) with respect to \( \alpha \).

For any formula \( \varphi \), variable \( \alpha \), and individual symbol \( \beta \), \( \varphi \alpha/\beta \) is the result of replacing all free occurrences of \( \alpha \) in \( \varphi \) with occurrences of \( \beta \).

**Variant terminology**:

What we are calling a formula is sometimes called a **well-formed formula**, or **wff** (pronounced ‘woof’). Also:

<table>
<thead>
<tr>
<th>We say:</th>
<th>sentence</th>
<th>formula</th>
<th>formula that is not a sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variant label:</td>
<td>closed sentence</td>
<td>sentence</td>
<td>open sentence</td>
</tr>
</tbody>
</table>

Semantics

Interpretations

An interpretation, $\mathcal{I}$, of the formal language we have just described consists of:

1. A non-empty domain, $\mathcal{D}$.
2. A mapping from constants of the language to elements (or other set-theoretic constructs out of elements) of $\mathcal{D}$.

(Terminology: ‘Element’ means the same as ‘member’. The converse of ‘mapping’ is ‘assignment’: we map a constant of the language onto an object in the domain; we assign that object to the constant that is mapped onto it. A mapping is a function in the mathematical sense; that is, a mapping gives each constant a unique value.)

1. Each individual constant is mapped onto exactly one element of $\mathcal{D}$.
2. Each predicate of degree $n$ is mapped onto exactly one set of ordered $n$-tuples of elements of $\mathcal{D}$.

Explanation: each predicate of degree 2 is mapped onto a set of ordered pairs of elements of $\mathcal{D}$; each predicate of degree 3 is mapped onto a set of ordered triples of elements of $\mathcal{D}$, etc. We consider an ordered 1-tuple of elements of $\mathcal{D}$ to be simply an element of $\mathcal{D}$. Thus, each predicate of degree 1 is mapped onto a set of elements of $\mathcal{D}$. Finally, we arbitrarily define ‘set of ordered 0-tuples of elements of $\mathcal{D}$’ to be one or the other of the two truth-values, $T$ or $F$. Thus, each sentential letter (predicate of degree 0) is mapped onto one of these two truth-values.

Truth under an interpretation

We will define ‘$\varphi$ is true under $\mathcal{I}$’, where $\varphi$ is a sentence and $\mathcal{I}$ is an interpretation. To deal with general sentences, we will need one additional definition — of the notion of the “$\beta$–variant” of an interpretation:

Where $\mathcal{I}$ and $\mathcal{I}'$ are interpretations and $\beta$ is an individual constant, $\mathcal{I}$ is a $\beta$–variant of $\mathcal{I}'$ iff $\mathcal{I}$ and $\mathcal{I}'$ differ at most in what they assign to $\beta$.

Our definition of truth is recursive: it will state the conditions under which the simplest sentences are true, and then state how the truth-values of more complex sentences depend upon those of simpler ones.

1. If $\varphi$ is a sentential letter, then $\varphi$ is true under $\mathcal{I}$ iff $\mathcal{I}$ assigns $T$ to $\varphi$.
2. If $\varphi$ is an identity sentence, $\beta = \delta$, then $\varphi$ is true under $\mathcal{I}$ iff $\mathcal{I}$ assigns the same object to both $\beta$ and $\delta$. 
3. If $\varphi$ is atomic and not a sentential letter and not an identity sentence, then $\varphi$ contains a predicate of degree $n$ (for $n \geq 1$). Then $\varphi$ is true under $\mathcal{I}$ iff the ordered $n$-tuple of objects that $\mathcal{I}$ assigns to the individual constants of $\varphi$ (taken in the order in which their corresponding constants occur in $\varphi$) is an element of the set of ordered $n$-tuples that $\mathcal{I}$ assigns to the predicate occurring in $\varphi$.

4. If $\varphi = \neg \psi$, then $\varphi$ is true under $\mathcal{I}$ iff $\psi$ is not true under $\mathcal{I}$.

5. If $\varphi = (\psi \lor \chi)$, then $\varphi$ is true under $\mathcal{I}$ iff either $\psi$ is true under $\mathcal{I}$ or $\chi$ is true under $\mathcal{I}$, or both.

6. If $\varphi = (\psi \land \chi)$, then $\varphi$ is true under $\mathcal{I}$ iff both $\psi$ is true under $\mathcal{I}$ and $\chi$ is true under $\mathcal{I}$.

7. If $\varphi = (\psi \rightarrow \chi)$, then $\varphi$ is true under $\mathcal{I}$ iff either $\psi$ is not true under $\mathcal{I}$ or $\chi$ is true under $\mathcal{I}$, or both.

8. If $\varphi = (\psi \leftrightarrow \chi)$, then $\varphi$ is true under $\mathcal{I}$ iff either $\psi$ and $\chi$ are both true under $\mathcal{I}$ or $\psi$ and $\chi$ are both not true under $\mathcal{I}$.

Let $\beta$ be the (alphabetically) first individual constant not occurring in $\varphi$. Then:

9. If $\varphi = \forall \alpha \psi$, then $\varphi$ is true under $\mathcal{I}$ iff $\psi \alpha/\beta$ is true under every $\beta$–variant of $\mathcal{I}$.

10. If $\varphi = \exists \alpha \psi$, then $\varphi$ is true under $\mathcal{I}$ iff $\psi \alpha/\beta$ is true under at least one $\beta$–variant of $\mathcal{I}$.

$\varphi$ is false under $\mathcal{I}$ iff $\varphi$ is not true under $\mathcal{I}$.

**Logical Truth**

A sentence $\varphi$ is a **logical truth** iff $\varphi$ is true under every interpretation.

A sentence $\varphi$ is **logically false** iff $\varphi$ is false under every interpretation.

A set of sentences $\Gamma$ is **consistent** iff there is at least one interpretation under which every member of $\Gamma$ is true.

A sentence $\varphi$ is a **logical consequence** of a set of sentences $\Gamma$ iff there is no interpretation under which all the members of $\Gamma$ are true and $\varphi$ is false.

A pair of sentences are **logically equivalent** iff there is no interpretation under which they differ in truth-value.