MHD Cylindrical Terms

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1 Ideal MHD Flux

Ideal mhd is written for $q = [\rho, \rho\boldsymbol{u}, e_t, \boldsymbol{B}]^T$, as

Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{1}$$

Momentum

$$\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \nabla \cdot \left[\rho \boldsymbol{u}\boldsymbol{u} + p\mathcal{I} - \left(\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\mathcal{I}\right)\right] = 0 \tag{2}$$

Energy

$$\frac{\partial e_t}{\partial t} + \nabla \cdot \left[\left(e_t + p + \frac{1}{2}B^2\right)\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{B})\,\boldsymbol{B}\right] = 0 \tag{3}$$

Magnetic Field

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{u}) = 0 \tag{4}$$

where $\boldsymbol{u} = (u, v, w)$ and $\boldsymbol{B} = (B_x, B_y, B_z)$, total energy e_t is related to pressure, p, by

$$e_t = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u} + \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right) + \frac{1}{2}\left(B_x^2 + B_y^2 + B_z^2\right) \tag{5}$$

Altogether this is written

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ e_t\\ B_x\\ B_y\\ B_z\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho uu-(B_xB_x-\frac{1}{2}B^2)+p\\ \rho uv-B_xB_y\\ \rho uw-B_xB_z\\ \left(e_t+p+\frac{1}{2}B^2\right)u-(\boldsymbol{B}\cdot\boldsymbol{u})B_x\\ 0\\ uB_y-B_xv\\ uB_z-B_xw\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho vu-B_yB_x\\ \rho vv-(B_yB_y-\frac{1}{2}B^2)+p\\ \rho vw-B_yB_z\\ \left(e_t+p+\frac{1}{2}B^2\right)v-(\boldsymbol{B}\cdot\boldsymbol{u})B_y\\ vB_x-B_yu\\ 0\\ vB_z-B_yw\end{pmatrix}$$
$$+\frac{\partial}{\partial z}\begin{pmatrix}\rho w\\ \rho wu-B_zB_x\\ \rho wv-B_zB_y\\ \rho ww-(B_zB_z-\frac{1}{2}B^2)+p\\ \left(e_t+p+\frac{1}{2}B^2\right)w-(\boldsymbol{B}\cdot\boldsymbol{u})B_z\\ wB_x-B_zu\\ wB_y-B_zv\\ 0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \tag{6}$$

2 Axisymmetric Cylindrical Form

We wish to express the equations in cylindrical form assuming axisymmetric geometry ($\frac{\partial}{\partial\theta}=0$).

2.1 Cylindrical Divergence

We will need cyldrical divergence of a vector and tensor, taken from the formulary:

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\cancel{\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}}+\frac{\partial A_z}{\partial z}=\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial r}+\frac{A_r}{r} \tag{7}$$

$$\left(\nabla\cdot\overline{\overline{T}}\right)_r=\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr})+\cancel{\frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta}}+\frac{\partial T_{zr}}{\partial z}-\frac{T_{\theta\theta}}{r}=\frac{\partial T_{zr}}{\partial z}+\frac{\partial T_{rr}}{\partial r}+\frac{T_{rr}-T_{\theta\theta}}{r} \tag{8a}$$

$$\left(\nabla\cdot\overline{\overline{T}}\right)_\theta=\frac{1}{r}\frac{\partial}{\partial r}(rT_{r\theta})+\cancel{\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}}+\frac{\partial T_{z\theta}}{\partial z}+\frac{T_{\theta r}}{r}=\frac{\partial T_{z\theta}}{\partial z}+\frac{\partial T_{r\theta}}{\partial r}+\frac{T_{r\theta}+T_{\theta r}}{r} \tag{8b}$$

$$\left(\nabla\cdot\overline{\overline{T}}\right)_z=\frac{1}{r}\frac{\partial}{\partial r}(rT_{rz})+\cancel{\frac{1}{r}\frac{\partial T_{\theta z}}{\partial\theta}}+\frac{\partial T_{zz}}{\partial z}=\frac{\partial T_{zz}}{\partial z}+\frac{\partial T_{rz}}{\partial r}+\frac{T_{rz}}{r} \tag{8c}$$

2.2 Cylindrical Divergence Applied to Ideal MHD

We go through each equation.

2.2.1 Continuity

Using Eq. (7)

$$\nabla\cdot(\rho\boldsymbol{u}) = \frac{\partial(\rho u_z)}{\partial z} + \frac{\partial(\rho u_r)}{\partial r} + \frac{(\rho u_r)}{r} \tag{9}$$

If we have on axis ($r = 0$) we use L'Hospital's rule for the source term

$$\lim_{r\to 0}\frac{(\rho u_r)}{r} = \frac{\frac{\partial}{\partial r}(\rho u_r)}{\frac{\partial}{\partial r}r} = \frac{\partial}{\partial r}(\rho u_r) \tag{10}$$

2.2.2 Momentum

For the radial component, Eq. (8a) yields

$$\begin{aligned}
&\nabla\cdot\left[\rho\boldsymbol{u}\boldsymbol{u} + p\mathcal{I} - \left(\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\mathcal{I}\right)\right]_r \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z)(\rho u_r)}{\rho} - B_zB_r\right] + \frac{\partial}{\partial r}\left[\frac{(\rho u_r)^2}{\rho} + p - \left(B_r^2 - \frac{1}{2}B^2\right)\right] \\
&+\frac{1}{r}\left[\left\{\frac{(\rho u_r)^2}{\rho} + p - \left(B_r^2 - \frac{1}{2}B^2\right)\right\} - \left\{\frac{(\rho u_\theta)^2}{\rho} + p - \left(B_\theta^2 - \frac{1}{2}B^2\right)\right\}\right] \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z)(\rho u_r)}{\rho} - B_zB_r\right] + \frac{\partial}{\partial r}\left[\frac{(\rho u_r)^2}{\rho} + p - \left(B_r^2 - \frac{1}{2}B^2\right)\right] + \frac{(\rho u_r)^2 - (\rho u_\theta)^2}{r\rho} - \frac{B_r^2 - B_\theta^2}{r}
\end{aligned} \tag{11}$$

If we have on axis ($r = 0$) we use L'Hospital's rule for the source term

$$\begin{aligned}
&\lim_{r\to 0}\frac{(\rho u_r)^2 - (\rho u_\theta)^2}{r\rho} - \frac{B_r^2 - B_\theta^2}{r} \\
=&\frac{\frac{\partial}{\partial r}(\rho u_r)^2 - \frac{\partial}{\partial r}(\rho u_\theta)^2}{\frac{\partial}{\partial r}(r\rho)} - \frac{\frac{\partial}{\partial r}B_r^2 - \frac{\partial}{\partial r}B_\theta^2}{\frac{\partial}{\partial r}r} \\
=&2\frac{(\rho u_r)\frac{\partial}{\partial r}(\rho u_r) - (\rho u_\theta)\frac{\partial}{\partial r}(\rho u_\theta)}{\rho + r\frac{\partial\rho}{\partial r}} - 2\left(B_r\frac{\partial}{\partial r}B_r - B_\theta\frac{\partial}{\partial r}B_\theta\right)
\end{aligned} \tag{12}$$

For the azimuthal component, Eq. (8b) yields

$$\begin{aligned}
&\nabla\cdot\left[\rho\boldsymbol{u}\boldsymbol{u} + p\mathcal{I} - \left(\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\mathcal{I}\right)\right]_\theta \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z)(\rho u_\theta)}{\rho} - B_zB_\theta\right] + \frac{\partial}{\partial r}\left[\frac{(\rho u_r)(\rho u_\theta)}{\rho} - B_rB_\theta\right] + \frac{1}{r}\left[\left\{\frac{(\rho u_r)(\rho u_\theta)}{\rho} - B_rB_\theta\right\} + \left\{\frac{(\rho u_\theta)(\rho u_r)}{\rho} - B_\theta B_r\right\}\right] \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z)(\rho u_\theta)}{\rho} - B_zB_\theta\right] + \frac{\partial}{\partial r}\left[\frac{(\rho u_r)(\rho u_\theta)}{\rho} - B_rB_\theta\right] + 2\left[\frac{(\rho u_r)(\rho u_\theta)}{r\rho} - \frac{B_rB_\theta}{r}\right]
\end{aligned} \tag{13}$$

If we have on axis ($r = 0$) we use L'Hospital's rule for the source term

$$\begin{aligned}
&\lim_{r\to 0} 2\left[\frac{(\rho u_r)(\rho u_\theta)}{r\rho} - \frac{B_r B_\theta}{r}\right] \\
=&2\left[\frac{\frac{\partial}{\partial r}\{(\rho u_r)(\rho u_\theta)\}}{\frac{\partial}{\partial r}(r\rho)} - \frac{\frac{\partial}{\partial r}(B_r B_\theta)}{\frac{\partial}{\partial r} r}\right] \\
=&2\left[\frac{(\rho u_r)\frac{\partial}{\partial r}(\rho u_\theta) + (\rho u_\theta)\frac{\partial}{\partial r}(\rho u_r)}{\rho + r\frac{\partial \rho}{\partial r}} - B_r\frac{\partial}{\partial r}B_\theta - B_\theta\frac{\partial}{\partial r}B_r\right]
\end{aligned} \tag{14}$$

For the axial component, Eq. (8c) yields

$$\begin{aligned}
&\nabla\cdot\left[\rho\boldsymbol{u}\boldsymbol{u} + p\mathcal{I} - \left(\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\mathcal{I}\right)\right]_z \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z)^2}{\rho} + p - \left(B_z^2 - \frac{1}{2}B^2\right)\right] + \frac{\partial}{\partial r}\left[\frac{(\rho u_r)(\rho u_z)}{\rho} - B_r B_z\right] + \frac{1}{r}\left[\frac{(\rho u_r)(\rho u_z)}{\rho} - B_r B_z\right] \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z)^2}{\rho} + p - \left(B_z^2 - \frac{1}{2}B^2\right)\right] + \frac{\partial}{\partial r}\left[\frac{(\rho u_r)(\rho u_z)}{\rho} - B_r B_z\right] + \frac{(\rho u_r)(\rho u_z)}{r\rho} - \frac{B_r B_z}{r}
\end{aligned} \tag{15}$$

If we have on axis ($r = 0$) we use L'Hospital's rule for the source term

$$\begin{aligned}
&\lim_{r\to 0}\frac{(\rho u_r)(\rho u_z)}{r\rho} - \frac{B_r B_z}{r} \\
=&\frac{\frac{\partial}{\partial r}\{(\rho u_r)(\rho u_z)\}}{\frac{\partial}{\partial r}(r\rho)} - \frac{\frac{\partial}{\partial r}(B_r B_z)}{\frac{\partial}{\partial r} r} \\
=&\frac{(\rho u_r)\frac{\partial}{\partial r}(\rho u_z) + (\rho u_z)\frac{\partial}{\partial r}(\rho u_r)}{\rho + r\frac{\partial \rho}{\partial r}} - B_r\frac{\partial}{\partial r}B_z - B_z\frac{\partial}{\partial r}B_r
\end{aligned} \tag{16}$$

2.2.3 Energy

Using Eq. (7)

$$\begin{aligned}
&\nabla\cdot\left[\left(e_t + p + \frac{1}{2}B^2\right)\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{B})\,\boldsymbol{B}\right] \\
=&\frac{\partial}{\partial z}\left[\left(e_t + p + \frac{1}{2}B^2\right)u_z - (\boldsymbol{u}\cdot\boldsymbol{B})\,B_z\right] + \frac{\partial}{\partial r}\left[\left(e_t + p + \frac{1}{2}B^2\right)u_r - (\boldsymbol{u}\cdot\boldsymbol{B})\,B_r\right] \\
+&\frac{\left(e_t + p + \frac{1}{2}B^2\right)u_r}{r} - \frac{(\boldsymbol{u}\cdot\boldsymbol{B})\,B_r}{r} \\
=&\frac{\partial}{\partial z}\left[\frac{\left(e_t + p + \frac{1}{2}B^2\right)(\rho u_z)}{\rho} - \frac{[(\rho\boldsymbol{u})\cdot\boldsymbol{B}]\,B_z}{\rho}\right] + \frac{\partial}{\partial r}\left[\frac{\left(e_t + p + \frac{1}{2}B^2\right)(\rho u_r)}{\rho} - \frac{[(\rho\boldsymbol{u})\cdot\boldsymbol{B}]\,B_r}{\rho}\right] \\
+&\frac{\left(e_t + p + \frac{1}{2}B^2\right)(\rho u_r)}{r\rho} - \frac{[(\rho\boldsymbol{u})\cdot\boldsymbol{B}]\,B_r}{r\rho}
\end{aligned} \tag{17}$$

If we have on axis ($r = 0$) we use L'Hospital's rule for the source term

$$
\begin{aligned}
&\lim_{r\to 0} \frac{\left(e_t + p + \frac{1}{2}B^2\right)(\rho u_r)}{r\rho} - \frac{\left[(\rho \boldsymbol{u})\cdot \boldsymbol{B}\right] B_r}{r\rho} \\
=&\frac{\frac{\partial}{\partial r}\left[\left(e_t + p + \frac{1}{2}B^2\right)(\rho u_r)\right]}{\frac{\partial}{\partial r}(r\rho)} - \frac{\frac{\partial}{\partial r}\left[\left[(\rho \boldsymbol{u})\cdot \boldsymbol{B}\right] B_r\right]}{\frac{\partial}{\partial r}(r\rho)} \\
=&\frac{\left(e_t + p + \frac{1}{2}B^2\right)\frac{\partial}{\partial r}(\rho u_r) + (\rho u_r)\frac{\partial}{\partial r}\left(e_t + p + \frac{1}{2}B^2\right)}{\rho + r\frac{\partial \rho}{\partial r}} - \frac{\left[(\rho \boldsymbol{u})\cdot \boldsymbol{B}\right]\frac{\partial}{\partial r}B_r + B_r\frac{\partial}{\partial r}\left[(\rho \boldsymbol{u})\cdot \boldsymbol{B}\right]}{\rho + r\frac{\partial \rho}{\partial r}}
\end{aligned}
\tag{18}
$$

2.2.4 Faraday

For the radial component, Eq. (8a) yields

$$
\begin{aligned}
&\nabla \cdot (\boldsymbol{uB} - \boldsymbol{Bu})_r \\
=&\frac{\partial}{\partial z}(u_z B_r - B_z u_r) + \frac{\partial}{\partial r}(u_r B_r - B_r u_r) + \frac{1}{r}\left[(u_r B_r - B_r u_r) - (u_\theta B_\theta - B_\theta u_\theta)\right] \\
=&\frac{\partial}{\partial z}(u_z B_r - B_z u_r) \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z) B_r}{\rho} - \frac{B_z(\rho u_r)}{\rho}\right]
\end{aligned}
\tag{19}
$$

For the azimuthal component, Eq. (8b) yields

$$
\begin{aligned}
&\nabla \cdot (\boldsymbol{uB} - \boldsymbol{Bu})_\theta \\
=&\frac{\partial}{\partial z}(u_z B_\theta - B_z u_\theta) + \frac{\partial}{\partial r}(u_r B_\theta - B_r u_\theta) + \frac{1}{r}\left[(u_r B_\theta - B_r u_\theta) + (u_\theta B_r - B_\theta u_r)\right] \\
=&\frac{\partial}{\partial z}(u_z B_\theta - B_z u_\theta) + \frac{\partial}{\partial r}(u_r B_\theta - B_r u_\theta) \\
=&\frac{\partial}{\partial z}\left[\frac{(\rho u_z) B_\theta}{\rho} - \frac{B_z(\rho u_\theta)}{\rho}\right] + \frac{\partial}{\partial r}\left[\frac{(\rho u_r) B_\theta}{\rho} - \frac{B_r(\rho u_\theta)}{\rho}\right]
\end{aligned}
\tag{20}
$$

For the axial component, Eq. (8c) yields

$$
\begin{aligned}
&\nabla \cdot (\boldsymbol{uB} - \boldsymbol{Bu})_z \\
=&\frac{\partial}{\partial z}(u_z B_z - B_z u_z) + \frac{\partial}{\partial r}(u_r B_z - B_r u_z) + \frac{(u_r B_z - B_r u_z)}{r} \\
=&\frac{\partial}{\partial r}(u_r B_z - B_r u_z) + \frac{(u_r B_z - B_r u_z)}{r} \\
=&\frac{\partial}{\partial r}\left[\frac{(\rho u_r) B_z}{\rho} - \frac{B_r(\rho u_z)}{\rho}\right] + \frac{(\rho u_r) B_z}{r\rho} - \frac{B_r(\rho u_z)}{r\rho}
\end{aligned}
\tag{21}
$$

If we have on axis ($r = 0$) we use L'Hospital's rule for the source term

$$\begin{aligned}
&\lim_{r\to 0} \frac{(\rho u_r)\, B_z}{r\rho} - \frac{B_r\,(\rho u_z)}{r\rho} \\
=&\frac{B_z \frac{\partial}{\partial r}(\rho u_r) + (\rho u_r)\frac{\partial}{\partial r} B_z}{\rho + r\frac{\partial \rho}{\partial r}} - \frac{B_r \frac{\partial}{\partial r}(\rho u_z) + (\rho u_z)\frac{\partial}{\partial r} B_r}{\rho + r\frac{\partial \rho}{\partial r}}
\end{aligned} \tag{22}$$

2.2.5 $r = 0$ axis BC

According to Eric in "apps/5-moment/euler_cyl_source.cc": "But note that m=0 BC require at r=0 that radial derivatives of scalars and axial components of vectors are zero, and that radial and aziumuthal components of vectors are zero." That means Eq. (10) is unchanged:

$$\lim_{r\to 0} \frac{(\rho u_r)}{r} = \frac{\partial}{\partial r}(\rho u_r) \tag{23}$$

Equation (12) becomes

$$\lim_{r\to 0} \frac{(\rho u_r)^2 - (\rho u_\theta)^2}{r\rho} - \frac{B_r^2 - B_\theta^2}{r} = 0 \tag{24}$$

Equation (14) becomes

$$\lim_{r\to 0} 2\left[\frac{(\rho u_r)(\rho u_\theta)}{r\rho} - \frac{B_r B_\theta}{r}\right] = 0 \tag{25}$$

Equation (16) becomes

$$\lim_{r\to 0} \frac{(\rho u_r)(\rho u_z)}{r\rho} - \frac{B_r B_z}{r} = \frac{(\rho u_z)\frac{\partial}{\partial r}(\rho u_r)}{\rho} - B_z \frac{\partial}{\partial r} B_r \tag{26}$$

Equation (18) becomes

$$\lim_{r\to 0} \frac{\left(e_t + p + \frac{1}{2}B^2\right)(\rho u_r)}{r\rho} - \frac{[(\rho \boldsymbol{u}) \cdot \boldsymbol{B}]\, B_r}{r\rho} = \frac{\left(e_t + p + \frac{1}{2}B^2\right)\frac{\partial}{\partial r}(\rho u_r)}{\rho} - \frac{[(\rho \boldsymbol{u}) \cdot \boldsymbol{B}]\frac{\partial}{\partial r} B_r}{\rho} \tag{27}$$

Equation (22) becomes

$$\lim_{r\to 0} \frac{(\rho u_r)\, B_z}{r\rho} - \frac{B_r\,(\rho u_z)}{r\rho} = \frac{B_z \frac{\partial}{\partial r}(\rho u_r)}{\rho} - \frac{(\rho u_z)\frac{\partial}{\partial r} B_r}{\rho} \tag{28}$$

3 Effect of Ohm's Law

Ohm's Law gives a state equation for the electric field. This affectes the energy and magnetic field evolution equations. These are given by

$$\frac{\partial e_t}{\partial t} + \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = 0, \tag{29}$$

and

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0. \tag{30}$$

See Ref. [1] for details. It is instructive to derive cylindrical terms based on Eqs. (29) and (30), as this should generalize for resistive and Hall terms if they are added to Ohm's Law. Notice that Eq. (29) can be written out in Cartesian as

$$\frac{\partial e_t}{\partial t} + \frac{\partial}{\partial x}\left(E_y B_z - E_z B_y\right) + \frac{\partial}{\partial y}\left(E_z B_x - E_x B_z\right) + \frac{\partial}{\partial z}\left(E_x B_y - E_y B_x\right) = 0. \tag{31}$$

Equation (30) can also be written out in Cartesian as

$$\frac{\partial}{\partial t}\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} 0 \\ -E_z \\ +E_y \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} +E_z \\ 0 \\ -E_x \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} -E_y \\ +E_x \\ 0 \end{pmatrix} = 0 \tag{32}$$

The MHD equations that add terms to Ohm's Law such as resistive MHD can thus be written in Cartesian form as

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e_t \\ B_x \\ B_y \\ B_z \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ E_y B_z - E_z B_y \\ 0 \\ -E_z \\ +E_y \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ E_z B_x - E_x B_z \\ +E_z \\ 0 \\ -E_x \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ E_x B_y - E_y B_x \\ -E_y \\ +E_x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{33}$$

In a 2d cylindrical coordinate system (z-r-(theta)), this should translate to

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v_z \\ \rho v_r \\ \rho v_\theta \\ e_t \\ B_z \\ B_r \\ B_\theta \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ E_r B_\theta - E_\theta B_r \\ 0 \\ -E_\theta \\ +E_r \end{pmatrix} + \frac{\partial}{\partial r}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ E_\theta B_z - E_z B_\theta \\ +E_\theta \\ 0 \\ -E_z \end{pmatrix} = \text{Source Terms} \tag{34}$$

Now we work out the source terms

3.1 Energy equation

The energy equation, Eq. (29) is divergence of a vector, so Eq. (7) applies. That is

$$\begin{aligned} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) &= \frac{\partial}{\partial z}(\boldsymbol{E} \times \boldsymbol{B})_z + \frac{\partial}{\partial r}(\boldsymbol{E} \times \boldsymbol{B})_r + \frac{(\boldsymbol{E} \times \boldsymbol{B})_r}{r} \\ &= \frac{\partial}{\partial z}\left(E_r B_\theta - E_\theta B_r\right) + \frac{\partial}{\partial r}\left(E_\theta B_z - E_z B_\theta\right) + \frac{\left(E_\theta B_z - E_z B_\theta\right)}{r} \end{aligned} \tag{35}$$

3.2 Faraday equation

The Faraday equation, Eq. (30) is curl of a vector. Lets go through this in Cylindrical coordiinates

$$(\nabla \times \boldsymbol{A})_r = \cancel{\frac{1}{r}\frac{\partial A_z}{\partial \theta}} - \frac{\partial A_\theta}{\partial z} = -\frac{\partial A_\theta}{\partial z} \tag{36a}$$

$$(\nabla \times \boldsymbol{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \tag{36b}$$

$$(\nabla \times \boldsymbol{A})_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\theta\right) + \cancel{\frac{1}{r}\frac{\partial A_r}{\partial \theta}} = \frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r} \tag{36c}$$

Therefore the curl term in Eq. (30) becomes

$$(\nabla \times \boldsymbol{E})_r = -\frac{\partial E_\theta}{\partial z} \tag{37a}$$

$$(\nabla \times \boldsymbol{E})_\theta = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \tag{37b}$$

$$(\nabla \times \boldsymbol{E})_z = \frac{\partial E_\theta}{\partial r} + \frac{E_\theta}{r} \tag{37c}$$

4 Resistive MHD

Ohm's Law for resisitive MHD is

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + {\color{red}\left(\frac{\delta_p}{L}\right)\left(\nu_p \tau\right) \eta \boldsymbol{j},} \tag{38}$$

where the red represents the resistive term. This can be substituted into Eqs. (29) and (30). First notice though that for MHD, j can be written using the low-frequqncy version of Ampere's Law,

$$\boldsymbol{j} = \left(\frac{\delta_p}{L}\right) \nabla \times \boldsymbol{B}. \tag{39}$$

So in cylindrical, using Eqs. (36),

$$\boldsymbol{j} = \begin{pmatrix} j_r \\ j_\theta \\ j_z \end{pmatrix} = \left(\frac{\delta_p}{L}\right) \begin{pmatrix} (\nabla \times \boldsymbol{B})_r \\ (\nabla \times \boldsymbol{B})_\theta \\ (\nabla \times \boldsymbol{B})_z \end{pmatrix} = \left(\frac{\delta_p}{L}\right) \begin{pmatrix} -\frac{\partial B_\theta}{\partial z} \\ \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \\ \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \end{pmatrix}. \tag{40}$$

Substitution of Eq. (40) into the resistive part of Eq. (38) and then into Eq. (34) using Eqs. (35) and (37) leads to

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v_z\\ \rho v_r\\ \rho v_\theta\\ e_t\\ B_z\\ B_r\\ B_\theta\end{pmatrix}+\frac{\partial}{\partial z}\left(\left(\frac{\delta_p}{L}\right)(\nu_p\tau)\,\eta\right)\begin{pmatrix}0\\0\\0\\0\\ j_rB_\theta-j_\theta B_r\\0\\-j_\theta\\+j_r\end{pmatrix}+\frac{\partial}{\partial r}\left(\left(\frac{\delta_p}{L}\right)(\nu_p\tau)\,\eta\right)\begin{pmatrix}0\\0\\0\\0\\ j_\theta B_z-j_zB_\theta\\+j_\theta\\0\\-j_z\end{pmatrix}$$
$$=\left(\frac{\delta_p}{L}\right)(\nu_p\tau)\,\eta\begin{pmatrix}0\\0\\0\\0\\-\frac{j_\theta B_z-j_zB_\theta}{r}\\-\frac{j_\theta}{r}\\0\\0\end{pmatrix}. \tag{41}$$

At $r=0$, we can find $\boldsymbol{j}$ as

$$\lim_{r\to 0}\boldsymbol{j}=\lim_{r\to 0}\left(\frac{\delta_p}{L}\right)\begin{pmatrix}-\frac{\partial B_\theta}{\partial z}\\ \frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}\\ \frac{\partial B_\theta}{\partial r}+\frac{B_\theta}{r}\end{pmatrix}=\left(\frac{\delta_p}{L}\right)\begin{pmatrix}-\left.\frac{\partial B_\theta}{\partial z}\right|_{r=0}\\ \left.\frac{\partial B_r}{\partial z}\right|_{r=0}-\left.\frac{\partial B_z}{\partial r}\right|_{r=0}\\ \left.\frac{\partial B_\theta}{\partial r}\right|_{r=0}+\frac{\frac{\partial}{\partial r}B_\theta|_{r=0}}{\frac{\partial}{\partial r}r|_{r=0}}\end{pmatrix}=\left(\frac{\delta_p}{L}\right)\begin{pmatrix}-\left.\frac{\partial B_\theta}{\partial z}\right|_{r=0}\\ \left.\frac{\partial B_r}{\partial z}\right|_{r=0}-\left.\frac{\partial B_z}{\partial r}\right|_{r=0}\\ \left.\frac{\partial B_\theta}{\partial r}\right|_{r=0}+\left.\frac{\partial B_\theta}{\partial r}\right|_{r=0}\end{pmatrix}$$
$$=\left(\frac{\delta_p}{L}\right)\begin{pmatrix}-\left.\frac{\partial B_\theta}{\partial z}\right|_{r=0}\\ \left.\frac{\partial B_r}{\partial z}\right|_{r=0}\\ 2\left.\frac{\partial B_\theta}{\partial r}\right|_{r=0}\end{pmatrix}, \tag{42}$$

where we use $\frac{\partial}{\partial r}\text{scalar} = \frac{\partial}{\partial r}\text{vector}_z = 0$ boundary condition at $r = 0$. Additionally, using $\text{vector}_r = \text{vector}_\theta = 0$ boundary condition at $r = 0$ the source term in Eq. (41) becomes

$$\begin{aligned}
\lim_{r\to 0}\left(\frac{\delta_p}{L}\right)(\nu_p\tau)\,\eta\begin{pmatrix}0\\0\\0\\0\\-\frac{j_\theta B_z - j_z B_\theta}{r}\\-\frac{j_\theta}{r}\\0\\0\end{pmatrix} &= \left(\frac{\delta_p}{L}\right)(\nu_p\tau)\,\eta\left.\begin{pmatrix}0\\0\\0\\0\\-\frac{\frac{\partial j_\theta}{\partial r}B_z+\frac{\partial B_z}{\partial r}j_\theta-\frac{\partial j_z}{\partial r}B_\theta-\frac{\partial B_\theta}{\partial r}j_z}{\frac{\partial}{\partial r}r}\\-\frac{\frac{\partial j_\theta}{\partial r}}{\frac{\partial}{\partial r}r}\\0\\0\end{pmatrix}\right|_{r=0}\\
&= \left(\frac{\delta_p}{L}\right)(\nu_p\tau)\,\eta\left.\begin{pmatrix}0\\0\\0\\0\\-\left(\frac{\partial j_\theta}{\partial r}B_z - \frac{\partial B_\theta}{\partial r}j_z\right)\\-\frac{\partial j_\theta}{\partial r}\\0\\0\end{pmatrix}\right|_{r=0}
\end{aligned} \tag{43}$$

Note the derivative on j_θ means a 2nd deriative on B_r for the source terms. It is probably easiest just to force calculation on Gaussian Quadrature nodes and avoid the issue.

5 Viscous MHD

We can also work with viscous terms. This affects the moment and energy equations, where the terms in question are

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \nabla\cdot\overline{\overline{\Pi}} = 0 \tag{44}$$

$$\frac{\partial e_t}{\partial t} + \nabla\cdot\left(\overline{\overline{\Pi}}\cdot\boldsymbol{u} + \boldsymbol{h}\right) = 0 \tag{45}$$

Equations (44) and (45) can be evaluated in cylindrical form using Eqs. (7) and (8). However, more work has to be done depending on the form of $\overline{\overline{\Pi}}$ and $\boldsymbol{h}$.

5.1 Unmagetized Form

In the unmagnetized case, the full pressure tensor $\overline{\overline{P}}$ is decomposed $\overline{\overline{P}} = p\overline{\overline{I}} + \overline{\overline{\Pi}}$ where $\overline{\overline{\Pi}}$ is given by the anisotropic Newtonian form seen for neutral fluids

$$\overline{\overline{\Pi}} = -\mu\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \frac{2}{3}(\nabla\cdot\boldsymbol{u})\overline{\overline{I}}\right] \tag{46}$$

The heat flux also has the Newtonian form

$$\boldsymbol{h} = -\,k\nabla T \tag{47}$$

First we need the gradient of a vector in cylindrical coordinates. Following the logic here and knowing that $\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}$ and $\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\boldsymbol{r}}$, see here, we can say if

$$\boldsymbol{u} = u_r\hat{\boldsymbol{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\boldsymbol{z}} \tag{48}$$

and if in Cylindrical coordinates

$$\nabla = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \tag{49}$$

We can then use

$$\frac{\partial \boldsymbol{u}}{\partial r} = \frac{\partial\left(u_r\hat{\boldsymbol{r}}\right)}{\partial r} + \frac{\partial\left(u_\theta\hat{\boldsymbol{\theta}}\right)}{\partial r} + \frac{\partial\left(u_z\hat{\boldsymbol{z}}\right)}{\partial r} = \frac{\partial u_r}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial u_\theta}{\partial r}\hat{\boldsymbol{\theta}} + \frac{\partial u_z}{\partial r}\hat{\boldsymbol{z}} \tag{50a}$$

$$\frac{\partial \boldsymbol{u}}{\partial \theta} = \frac{\partial\left(u_r\hat{\boldsymbol{r}}\right)}{\partial \theta} + \frac{\partial\left(u_\theta\hat{\boldsymbol{\theta}}\right)}{\partial \theta} + \frac{\partial\left(u_z\hat{\boldsymbol{z}}\right)}{\partial \theta} = \frac{\partial u_r}{\partial \theta}\hat{\boldsymbol{r}} + u_r\hat{\boldsymbol{\theta}} + \frac{\partial u_\theta}{\partial \theta}\hat{\boldsymbol{\theta}} - u_\theta\hat{\boldsymbol{r}} + \frac{\partial u_z}{\partial \theta}\hat{\boldsymbol{z}} \tag{50b}$$

$$\frac{\partial \boldsymbol{u}}{\partial z} = \frac{\partial\left(u_r\hat{\boldsymbol{r}}\right)}{\partial z} + \frac{\partial\left(u_\theta\hat{\boldsymbol{\theta}}\right)}{\partial z} + \frac{\partial\left(u_z\hat{\boldsymbol{z}}\right)}{\partial z} = \frac{\partial u_r}{\partial z}\hat{\boldsymbol{r}} + \frac{\partial u_\theta}{\partial z}\hat{\boldsymbol{\theta}} + \frac{\partial u_z}{\partial z}\hat{\boldsymbol{z}} \tag{50c}$$

Then we work out the gradients

$$\nabla \boldsymbol{u}_{rr} = \nabla_r \boldsymbol{u}\cdot\hat{\boldsymbol{r}} = \frac{\partial \boldsymbol{u}}{\partial r}\cdot\hat{\boldsymbol{r}} = \frac{\partial u_r}{\partial r} \tag{51a}$$

$$\nabla \boldsymbol{u}_{r\theta} = \nabla_\theta \boldsymbol{u}\cdot\hat{\boldsymbol{r}} = \frac{1}{r}\frac{\partial \boldsymbol{u}}{\partial \theta}\cdot\hat{\boldsymbol{r}} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \tag{51b}$$

$$\nabla \boldsymbol{u}_{rz} = \nabla_z \boldsymbol{u}\cdot\hat{\boldsymbol{r}} = \frac{\partial \boldsymbol{u}}{\partial z}\cdot\hat{\boldsymbol{r}} = \frac{\partial u_r}{\partial z} \tag{51c}$$

$$\nabla \boldsymbol{u}_{\theta r} = \nabla_r \boldsymbol{u}\cdot\hat{\boldsymbol{\theta}} = \frac{\partial \boldsymbol{u}}{\partial r}\cdot\hat{\boldsymbol{\theta}} = \frac{\partial u_\theta}{\partial r} \tag{51d}$$

$$\nabla \boldsymbol{u}_{\theta\theta} = \nabla_\theta \boldsymbol{u}\cdot\hat{\boldsymbol{\theta}} = \frac{1}{r}\frac{\partial \boldsymbol{u}}{\partial \theta}\cdot\hat{\boldsymbol{\theta}} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \tag{51e}$$

$$\nabla \boldsymbol{u}_{\theta z} = \nabla_z \boldsymbol{u} \cdot \hat{\boldsymbol{\theta}} = \frac{\partial \boldsymbol{u}}{\partial z} \cdot \hat{\boldsymbol{\theta}} = \frac{\partial u_\theta}{\partial z} \tag{51f}$$

$$\nabla \boldsymbol{u}_{zr} = \nabla_r \boldsymbol{u} \cdot \hat{\boldsymbol{z}} = \frac{\partial \boldsymbol{u}}{\partial r} \cdot \hat{\boldsymbol{z}} = \frac{\partial u_z}{\partial r} \tag{51g}$$

$$\nabla \boldsymbol{u}_{z\theta} = \nabla_\theta \boldsymbol{u} \cdot \hat{\boldsymbol{z}} = \frac{1}{r}\frac{\partial \boldsymbol{u}}{\partial \theta} \cdot \hat{\boldsymbol{z}} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} \tag{51h}$$

$$\nabla \boldsymbol{u}_{zz} = \nabla_z \boldsymbol{u} \cdot \hat{\boldsymbol{z}} = \frac{\partial \boldsymbol{u}}{\partial z} \cdot \hat{\boldsymbol{z}} = \frac{\partial u_z}{\partial z} \tag{51i}$$

or in matrix form

$$\nabla \boldsymbol{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{52}$$

If I understood covariant/contravariant stuff I probably could just write this down very easily...

So then the transpose is

$$(\nabla \boldsymbol{u})^T = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\theta}{\partial r} & \frac{\partial u_z}{\partial r} \\ \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\theta}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{53}$$

So then

$$\begin{aligned} \overline{\overline{\Pi}} =& -\mu \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \frac{2}{3} (\nabla \cdot \boldsymbol{u}) \overline{\overline{I}} \right] \\ =& -\mu \left[\begin{pmatrix} 2\frac{\partial u_r}{\partial r} - \frac{2}{3}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) & \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} & \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\ \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} & 2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - \frac{2}{3}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) & \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \\ \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} & 2\frac{\partial u_z}{\partial z} - \frac{2}{3}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) \end{pmatrix} \right] \end{aligned} \tag{54}$$

Now since we are going for axisymmetric formulation, cancel $\frac{\partial}{\partial \theta}$ terms:

$$\overline{\overline{\Pi}} = -\mu \left[\begin{pmatrix} 2\frac{\partial u_r}{\partial r} - \frac{2}{3}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) & -\frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} & \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\ \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} & 2\left(\frac{u_r}{r}\right) - \frac{2}{3}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} & \frac{\partial u_\theta}{\partial z} & 2\frac{\partial u_z}{\partial z} - \frac{2}{3}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) \end{pmatrix} \right] \tag{55}$$

Now we take the divergence of Eq. (55) according to Eq. (8) for the flux term in Eq. (44).

$$\begin{aligned}\left(\nabla\cdot\overline{\overline{\Pi}}\right)_r =&\frac{\partial \Pi_{zr}}{\partial z}+\frac{\partial \Pi_{rr}}{\partial r}+\frac{\Pi_{rr}-\Pi_{\theta\theta}}{r}\\
=&\frac{\partial}{\partial z}\left[-\mu\left\{\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right\}\right]+\frac{\partial}{\partial r}\left[-\mu\left\{2\frac{\partial u_r}{\partial r}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}\right]\\
&+\frac{1}{r}\left[-\mu\left\{2\frac{\partial u_r}{\partial r}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)-\left(2\left(\frac{u_r}{r}\right)-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right)\right\}\right]\\
=&\frac{\partial}{\partial z}\left[-\mu\left\{\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right\}\right]+\frac{\partial}{\partial r}\left[-\mu\left\{2\frac{\partial u_r}{\partial r}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}\right]\\
&-2\mu\left[\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right]\end{aligned}\tag{56a}$$

$$\begin{aligned}\left(\nabla\cdot\overline{\overline{\Pi}}\right)_\theta =&\frac{\partial \Pi_{z\theta}}{\partial z}+\frac{\partial \Pi_{r\theta}}{\partial r}+\frac{\Pi_{r\theta}+\Pi_{\theta r}}{r}\\
=&\frac{\partial}{\partial z}\left[-\mu\left\{\frac{\partial u_\theta}{\partial z}\right\}\right]+\frac{\partial}{\partial r}\left[-\mu\left\{-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right\}\right]+\frac{1}{r}\left[-\mu\left\{-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right\}\right]\\
=&\frac{\partial}{\partial z}\left[-\mu\frac{\partial u_\theta}{\partial z}\right]+\frac{\partial}{\partial r}\left[-\mu\left\{-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right\}\right]-2\mu\left[-\frac{u_\theta}{r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}\right]\end{aligned}\tag{56b}$$

$$\begin{aligned}\left(\nabla\cdot\overline{\overline{\Pi}}\right)_z =&\frac{\partial \Pi_{zz}}{\partial z}+\frac{\partial \Pi_{rz}}{\partial r}+\frac{\Pi_{rz}}{r}\\
=&\frac{\partial}{\partial z}\left[-\mu\left\{2\frac{\partial u_z}{\partial z}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}\right]+\frac{\partial}{\partial r}\left[-\mu\left\{\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right\}\right]-\frac{\mu}{r}\left[\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right]\end{aligned}\tag{56c}$$

For the energy equation, Eq. (45), one can also determine the flux terms using Eq. (7). First notice

$$\begin{aligned}\overline{\overline{\Pi}}\cdot\boldsymbol{u}=&\begin{pmatrix}\Pi_{rr}&\Pi_{r\theta}&\Pi_{rz}\\\Pi_{\theta r}&\Pi_{\theta\theta}&\Pi_{\theta z}\\\Pi_{zr}&\Pi_{z\theta}&\Pi_{zz}\end{pmatrix}\cdot\begin{pmatrix}u_r\\u_\theta\\u_z\end{pmatrix}=\begin{pmatrix}\Pi_{rr}u_r+\Pi_{r\theta}u_\theta+\Pi_{rz}u_z\\\Pi_{\theta r}u_r+\Pi_{\theta\theta}u_\theta+\Pi_{\theta z}u_z\\\Pi_{zr}u_r+\Pi_{z\theta}u_\theta+\Pi_{zz}u_z\end{pmatrix}\\
=&-\mu\begin{pmatrix}\left\{2\frac{\partial u_r}{\partial r}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}u_r+\left\{-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right\}u_\theta+\left\{\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right\}u_z\\\left\{\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right\}u_r+\left\{2\left(\frac{u_r}{r}\right)-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}u_\theta+\left\{\frac{\partial u_\theta}{\partial z}\right\}u_z\\\left\{\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right\}u_r+\left\{\frac{\partial u_\theta}{\partial z}\right\}u_\theta+\left\{2\frac{\partial u_z}{\partial z}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}u_z\end{pmatrix}\end{aligned}\tag{57}$$

So then

$$\begin{aligned}
\nabla\cdot\left(\overline{\overline{\overline{\Pi}}}\cdot\boldsymbol{u}\right) =& \frac{\partial\left(\overline{\overline{\overline{\Pi}}}\cdot\boldsymbol{u}\right)_z}{\partial z}+\frac{\partial\left(\overline{\overline{\overline{\Pi}}}\cdot\boldsymbol{u}\right)_r}{\partial r}+\frac{\left(\overline{\overline{\overline{\Pi}}}\cdot\boldsymbol{u}\right)_r}{r}\\
=&\frac{\partial}{\partial z}\left[-\mu\left\{\left\{\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right\}u_r+\left\{\frac{\partial u_\theta}{\partial z}\right\}u_\theta+\left\{2\frac{\partial u_z}{\partial z}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}u_z\right\}\right]\\
&+\frac{\partial}{\partial r}\left[\left\{-\mu\left\{2\frac{\partial u_r}{\partial r}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}u_r+\left\{-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right\}u_\theta+\left\{\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right\}u_z\right\}\right]\\
&-\frac{\mu}{r}\left[\left\{2\frac{\partial u_r}{\partial r}-\frac{2}{3}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)\right\}u_r+\left\{-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right\}u_\theta+\left\{\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right\}u_z\right]
\end{aligned} \tag{58}$$

For the heat flux, notice the gradient definition of scalar for axisymmtric cylindrical coordinates

$$(\nabla f)_r = \frac{\partial f}{\partial r} \tag{59a}$$

$$(\nabla f)_\theta = \cancel{\frac{1}{r}}\frac{\cancel{\partial f}}{\partial\theta} = 0 \tag{59b}$$

$$(\nabla f)_z = \frac{\partial f}{\partial z} \tag{59c}$$

Then,

$$\nabla\cdot\boldsymbol{h} = \frac{\partial h_z}{\partial z}+\frac{\partial h_r}{\partial r}+\frac{h_r}{r} = \frac{\partial}{\partial z}\left[-k\frac{\partial T}{\partial z}\right]+\frac{\partial}{\partial r}\left[-k\frac{\partial T}{\partial r}\right]-\frac{k}{r}\left[\frac{\partial T}{\partial r}\right] \tag{60}$$

5.1.1 How to implement

The most direct way to realize the flux terms in Eqs. (56) and (58) is to realize the axisymmetric $\nabla\boldsymbol{u}$ in Eq. (52) become (rearranging coordinate basis order to z-r-θ),

$$\nabla\boldsymbol{u} = \begin{pmatrix} (\nabla\boldsymbol{u})_{zz} & (\nabla\boldsymbol{u})_{zr} & (\nabla\boldsymbol{u})_{z\theta}\\ (\nabla\boldsymbol{u})_{rz} & (\nabla\boldsymbol{u})_{rr} & (\nabla\boldsymbol{u})_{r\theta}\\ (\nabla\boldsymbol{u})_{\theta z} & (\nabla\boldsymbol{u})_{\theta r} & (\nabla\boldsymbol{u})_{\theta\theta}\end{pmatrix} = \begin{pmatrix} \frac{\partial u_z}{\partial z} & \frac{\partial u_z}{\partial r} & \cancel{\frac{1}{r}\frac{\partial u_z}{\partial\theta}}\\ \frac{\partial u_r}{\partial z} & \frac{\partial u_r}{\partial r} & \cancel{\frac{1}{r}\frac{\partial u_r}{\partial\theta}}-\frac{u_\theta}{r}\\ \frac{\partial u_\theta}{\partial z} & \frac{\partial u_\theta}{\partial r} & \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}}+\frac{u_r}{r}\end{pmatrix} = \begin{pmatrix} \frac{\partial u_z}{\partial z} & \frac{\partial u_z}{\partial r} & 0\\ \frac{\partial u_r}{\partial z} & \frac{\partial u_r}{\partial r} & -\frac{u_\theta}{r}\\ \frac{\partial u_\theta}{\partial z} & \frac{\partial u_\theta}{\partial r} & \frac{u_r}{r}\end{pmatrix} \tag{61}$$

and that $\nabla\cdot\boldsymbol{u}$ is given by

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}. \tag{62}$$

Not that in the $r\to 0$ limits:

$$\lim_{r\to 0}\frac{u_\theta}{r} = \frac{\frac{\partial}{\partial r}u_\theta}{\frac{\partial}{\partial r}r} = \frac{\partial u_\theta}{\partial r} \tag{63a}$$

$$\lim_{r\to 0}\frac{u_r}{r} = \frac{\frac{\partial}{\partial r}u_r}{\frac{\partial}{\partial r}r} = \frac{\partial u_r}{\partial r} \tag{63b}$$

The Π tensor can thus be written

$$\overline{\overline{\Pi}} = -\mu\begin{pmatrix} 2\left(\nabla\boldsymbol{u}\right)_{zz} - \frac{2}{3}(\nabla\cdot\boldsymbol{u}) & \left(\nabla\boldsymbol{u}\right)_{zr} + \left(\nabla\boldsymbol{u}\right)_{rz} & \left(\nabla\boldsymbol{u}\right)_{z\theta} + \left(\nabla\boldsymbol{u}\right)_{\theta z} \\ \left(\nabla\boldsymbol{u}\right)_{rz} + \left(\nabla\boldsymbol{u}\right)_{zr} & 2\left(\nabla\boldsymbol{u}\right)_{rr} - \frac{2}{3}(\nabla\cdot\boldsymbol{u}) & \left(\nabla\boldsymbol{u}\right)_{r\theta} + \left(\nabla\boldsymbol{u}\right)_{\theta r} \\ \left(\nabla\boldsymbol{u}\right)_{\theta z} + \left(\nabla\boldsymbol{u}\right)_{z\theta} & \left(\nabla\boldsymbol{u}\right)_{\theta r} + \left(\nabla\boldsymbol{u}\right)_{r\theta} & 2(\nabla\boldsymbol{u})_{\theta\theta} - \frac{2}{3}(\nabla\cdot\boldsymbol{u}) \end{pmatrix} = \begin{pmatrix} \Pi_{zz} & \Pi_{zr} & \Pi_{z\theta} \\ \Pi_{rz} & \Pi_{rr} & \Pi_{r\theta} \\ \Pi_{\theta z} & \Pi_{\theta r} & \Pi_{\theta\theta} \end{pmatrix} \tag{64}$$

Then

$$\left(\nabla\cdot\overline{\overline{\Pi}}\right)_r = \frac{\partial\Pi_{zr}}{\partial z} + \frac{\partial\Pi_{rr}}{\partial r} + \frac{\Pi_{rr} - \Pi_{\theta\theta}}{r} \tag{65a}$$

$$\left(\nabla\cdot\overline{\overline{\Pi}}\right)_\theta = \frac{\partial\Pi_{z\theta}}{\partial z} + \frac{\partial\Pi_{r\theta}}{\partial r} + \frac{\Pi_{r\theta} + \Pi_{\theta r}}{r} \tag{65b}$$

$$\left(\nabla\cdot\overline{\overline{\Pi}}\right)_z = \frac{\partial\Pi_{zz}}{\partial z} + \frac{\partial\Pi_{rz}}{\partial r} + \frac{\Pi_{rz}}{r} \tag{65c}$$

Also

$$\overline{\overline{\Pi}}\cdot\boldsymbol{u} = \begin{pmatrix} \Pi_{zz} & \Pi_{zr} & \Pi_{z\theta} \\ \Pi_{rz} & \Pi_{rr} & \Pi_{r\theta} \\ \Pi_{\theta z} & \Pi_{\theta r} & \Pi_{\theta\theta} \end{pmatrix}\cdot\begin{pmatrix} u_z \\ u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \Pi_{zz}u_z + \Pi_{zr}u_r + \Pi_{z\theta}u_\theta \\ \Pi_{rz}u_z + \Pi_{rr}u_r + \Pi_{r\theta}u_\theta \\ \Pi_{\theta z}u_z + \Pi_{\theta r}u_r + \Pi_{\theta\theta}u_\theta \end{pmatrix} \tag{66}$$

So then

$$\begin{aligned} \nabla\cdot\left(\overline{\overline{\Pi}}\cdot\boldsymbol{u}\right) &= \frac{\partial\left(\overline{\overline{\Pi}}\cdot\boldsymbol{u}\right)_z}{\partial z} + \frac{\partial\left(\overline{\overline{\Pi}}\cdot\boldsymbol{u}\right)_r}{\partial r} + \frac{\left(\overline{\overline{\Pi}}\cdot\boldsymbol{u}\right)_r}{r} \\ &= \frac{\partial}{\partial z}\left(\Pi_{zz}u_z + \Pi_{zr}u_r + \Pi_{z\theta}u_\theta\right) + \frac{\partial}{\partial r}\left(\Pi_{rz}u_z + \Pi_{rr}u_r + \Pi_{r\theta}u_\theta\right) + \frac{\Pi_{rz}u_z + \Pi_{rr}u_r + \Pi_{r\theta}u_\theta}{r} \end{aligned} \tag{67}$$

Finally, we need to find the temperature gradients for Eq. (60). We know from Eq. (59), the gradient of a scalar in the axisymmetric cylindrical coordinates should produce no extra source terms. Then we calculate the ∇T using

$$\begin{aligned} p &= nT \\ T &= \frac{p}{n} \\ \nabla T &= \left(\frac{p}{n}\right) = \frac{(\nabla p)\,n - p\,(\nabla n)}{n^2} = \frac{(\nabla p)\,\frac{\rho}{A} - p\left(\nabla\frac{\rho}{A}\right)}{\left(\frac{\rho}{A}\right)^2} = A\frac{(\nabla p)\,\rho - p\,(\nabla\rho)}{\rho^2} \end{aligned} \tag{68}$$

We also first need ∇p (see Ref. [1] or work it out):

$$
\begin{aligned}
\nabla p =& \nabla \left[(\gamma - 1) \left(e_t - \frac{1}{2} \frac{(\rho \boldsymbol{u})^2}{\rho} - \frac{1}{2} \boldsymbol{B}^2 \right) \right] \\
=& (\gamma - 1) \left[\nabla e_t - \frac{(\rho u) \nabla (\rho u) + (\rho v) \nabla (\rho v) + (\rho w) \nabla (\rho w)}{\rho} + \frac{1}{2} \left(\frac{(\nabla \rho) \left((\rho u)^2 + (\rho v)^2 + (\rho w)^2 \right)}{\rho^2} \right) \right. \\
& \left. - (B_x \nabla B_x + B_y \nabla B_y + B_z \nabla B_z) \right]
\end{aligned} \tag{69}
$$

Since all these gradients are for scalars, there should be no additional terms required for a Cylindrical calculation versus Cartesian.

5.1.2 Units

Specifically lets look at k. Equation (47) unitswise is

$$
\begin{aligned}
\left[\frac{W}{m^2} \right] &= \left[\frac{W}{mK} \right] \frac{1}{m} [K] \\
\left[\frac{M}{T^3} \right] &= \left[\frac{ML}{T^3 \theta} \right] \frac{1}{L} [\theta] \\
&= \left[\frac{ML}{T^3 \theta \frac{\frac{ML^2}{T^2}}{\theta}} \right] \frac{1}{L} \left[\theta \frac{\frac{ML^2}{T^2}}{\theta} \right] \\
&= \underbrace{\left[\frac{1}{LT} \right]}_{k} \underbrace{\frac{1}{L}}_{\nabla} \underbrace{\left[\frac{ML^2}{T^2} \right]}_{T}
\end{aligned} \tag{70}
$$

If we directly dimensionalize the energy equation for viscous terms

$$
\begin{aligned}
\frac{e_0}{\tau} \frac{\partial e_t}{\partial t} + \frac{1}{L} \nabla \cdot (p_0 v_0 \Pi \cdot \boldsymbol{u} + h_0 \boldsymbol{h}) &= 0 \\
\frac{\partial e_t}{\partial t} + \frac{\tau}{e_0} \frac{1}{L} \nabla \cdot (p_0 v_0 \Pi \cdot \boldsymbol{u} + h_0 \boldsymbol{h}) &= 0 \\
\frac{\partial e_t}{\partial t} + \nabla \cdot \left(\frac{p_0 v_0}{e_0 v_0} \Pi \cdot \boldsymbol{u} + \frac{h_0}{e_0 v_0} \boldsymbol{h} \right) &= 0 \\
\frac{\partial e_t}{\partial t} + \nabla \cdot \left(\frac{m_0 n_0 v_0^2 v_0}{m_0 n_0 v_0^2 v_0} \Pi \cdot \boldsymbol{u} + \frac{h_0}{m_0 n_0 v_0^2 v_0} \boldsymbol{h} \right) &= 0 \\
\frac{\partial e_t}{\partial t} + \nabla \cdot \left(\Pi \cdot \boldsymbol{u} + \frac{h_0}{m_0 n_0 v_0^3} \boldsymbol{h} \right) &= 0
\end{aligned} \tag{71}
$$

If we then set

$$
h_0 = m_0 n_0 v_0^3 = \left[\frac{M}{L^3} \frac{L^3}{T^3} \right] = \left[\frac{M}{T^3} \right] \tag{72}
$$

which agrees with the dimensions in Eq. (70). Then we really do have $[k] \equiv \left[\frac{1}{LT}\right]$. If we input κ as a diffusivity ($[\kappa] = \frac{L^2}{T}$)

$$k = \kappa n = \left[\frac{L^2}{T}\right]\left[\frac{1}{L^3}\right] = \left[\frac{1}{LT}\right] \tag{73}$$

6 Hall MHD

Ohm's Law for Hall MHD is

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \left(\frac{\delta_p}{L}\right)(\nu_p \tau)\eta \boldsymbol{j} + \frac{1}{n_\mathrm{e}}\left(\boldsymbol{j} \times \boldsymbol{B} - \left(\frac{\delta_p}{L}\right)\nabla p_e\right), \tag{74}$$

where the red represents the Hall terms. This can be substituted into Eqs. (29) and (30). First, notice that component-wise, the Hall part of Eq. (74) can be written

$$\begin{aligned}\boldsymbol{E}_\mathrm{Hall} =& \frac{1}{n_\mathrm{e}}\left[(j_r B_\theta - j_\theta B_r) - \left(\frac{\delta_p}{L}\right)(\nabla p_e)_z\right]\hat{\boldsymbol{z}} \\ &+ \frac{1}{n_\mathrm{e}}\left[(j_\theta B_z - j_z B_\theta) - \left(\frac{\delta_p}{L}\right)(\nabla p_e)_r\right]\hat{\boldsymbol{r}} \\ &+ \frac{1}{n_\mathrm{e}}\left[(j_z B_r - j_r B_z) - \left(\frac{\delta_p}{L}\right)(\nabla p_e)_\theta\right]\hat{\boldsymbol{\theta}}\end{aligned} \tag{75}$$

The components of $\boldsymbol{j}$ are found using Eq. (40). Equation (75) can be then substituted into Eq. (34) using Eqs. (35) and (37) leads to

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho v_z \\ \rho v_r \\ \rho v_\theta \\ e_t \\ B_z \\ B_r \\ B_\theta\end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix}0 \\ 0 \\ 0 \\ 0 \\ E_{\mathrm{Hall},r} B_\theta - E_{\mathrm{Hall},\theta} B_r \\ 0 \\ -E_{\mathrm{Hall},\theta} \\ +E_{\mathrm{Hall},r}\end{pmatrix} + \frac{\partial}{\partial r}\begin{pmatrix}0 \\ 0 \\ 0 \\ 0 \\ E_{\mathrm{Hall},\theta} B_z - E_{\mathrm{Hall},z} B_\theta \\ +E_{\mathrm{Hall},\theta} \\ 0 \\ -E_{\mathrm{Hall},z}\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0 \\ 0 \\ -\frac{E_{\mathrm{Hall},\theta} B_z - E_{\mathrm{Hall},z} B_\theta}{r} \\ -\frac{E_{\mathrm{Hall},\theta}}{r} \\ 0 \\ 0\end{pmatrix}. \tag{76}$$

As with resistive MHD, we can avoid $1/r$ issues by just using Gaussian Quadrature nodes.

6.1 Hyperresistivity

For Hall MHD, we might add hyperresistivity to Ohm's Law. That looks like in normalized form:

$$\boldsymbol{E}_\mathrm{hyper} = -\nu \nabla^2 \boldsymbol{j} = -\nu \nabla \cdot \nabla \boldsymbol{j}. \tag{77}$$

To incorporate hyperresistivity correctly in the Cylindrical formulation, we need to correctly calculate $\boldsymbol{j}$, $\nabla\boldsymbol{j}$, and $\boldsymbol{E}_{\text{hyper}}$. We already know how to calculate $\boldsymbol{j}$ using the gradient variable adjuster for $\nabla\boldsymbol{B}$ and Eq. (40). $\nabla\boldsymbol{j}$ can be calculated by using the gradient variable adjuster again, and then correcting in the same manner as for $\nabla\boldsymbol{u}$ as shown in Eqs. (61) and (63). Th last step is to evaluate Eq. (77) with appropriate Cylindrical corrections.

As an aside, note that the gradient variable adjuster as a DG calculator derived in the following manner, where $\boldsymbol{\sigma}$ is the auxiliary variable for the gradient:

$$\begin{aligned}
\boldsymbol{\sigma} \equiv& \nabla q \\
\int\limits_{\Omega} \boldsymbol{\sigma}\boldsymbol{\phi} dV =& \int\limits_{\Omega} \nabla q \boldsymbol{\phi} dV \\
=& \int\limits_{\partial\Omega} q\boldsymbol{\phi}\hat{\boldsymbol{n}} dS - \int\limits_{\Omega} q \nabla\boldsymbol{\phi} dV \\
\hat{\boldsymbol{\sigma}} \int\limits_{\Omega} \boldsymbol{\psi}\boldsymbol{\phi} dV =& \hat{q}^* \int\limits_{\partial\Omega} \boldsymbol{\psi}\boldsymbol{\phi}\hat{\boldsymbol{n}} dS - \hat{q} \int\limits_{\Omega} \boldsymbol{\psi}\nabla\boldsymbol{\phi} dV
\end{aligned} \tag{78}$$

where $\boldsymbol{\phi}$ is the test function, $\boldsymbol{\psi}$ is the expansion basis such that for some variable, $u = \sum_i \hat{u}_i \psi_i$, and $\hat{q}^*$ is an element surface value of $\hat{q}$ calculated using some formulation such as central flux, LDG, or interior penalty. We use the Galerkin choice of setting $\boldsymbol{\psi} = \boldsymbol{\phi}$. Equation (78) can be then solved for $\hat{\boldsymbol{\sigma}}$.

Now to evaluate Eq. (77), we could take another gradient of $\boldsymbol{j}$ and calculate the divergence of $\nabla\boldsymbol{j}$ from $\nabla\nabla\boldsymbol{j}$. However, $\nabla\nabla\boldsymbol{j}$ would be 27 elements, which reduces to 3 when calculating $\boldsymbol{E}_{\text{hyper}}$, which would be inefficient, not to mention we'd have to work out the cylindrical correction. The better way is to notice that we can rewrite Eq. (77) as

$$\boldsymbol{E}_{\text{hyper}} + \nabla \cdot \overline{\overline{F}} = 0, \tag{79}$$

where

$$\overline{\overline{F}} \equiv + \nu \nabla \boldsymbol{j}. \tag{80}$$

Note that the flux term is moved to the LHS, hence the sign change. Note that since we are in Cylindrical coordinates, we can incorporate source terms that come from $\nabla \cdot \overline{\overline{F}}$ by writing Eq. (79) as

$$\boldsymbol{E}_{\text{hyper}} + \nabla \cdot \overline{\overline{F}} = \boldsymbol{S}, \tag{81}$$

Note that this is exactly the conservation law form that we derive the DG method from, except in instead, we are solving for $\boldsymbol{E}_{\text{hyper}}$ as opposed to $\frac{\partial \boldsymbol{q}}{\partial t}$. This becomes

$$
\begin{aligned}
\boldsymbol{E}_{\text{hyper}} + \nabla \cdot \overline{\overline{F}} &= \boldsymbol{S} \\
\int_{\Omega} \boldsymbol{E}_{\text{hyper}} \boldsymbol{\phi} dV + \int_{\Omega} \nabla \cdot \overline{\overline{F}} \boldsymbol{\phi} dV &= \int_{\Omega} \boldsymbol{S} \boldsymbol{\phi} dV \qquad (82) \\
\int_{\Omega} \boldsymbol{E}_{\text{hyper}} \boldsymbol{\phi} dV + \int_{\partial\Omega} \overline{\overline{F}} \boldsymbol{\phi} \cdot \hat{\boldsymbol{n}} dS - \int_{\Omega} \overline{\overline{F}} \cdot \nabla \boldsymbol{\phi} dV &= \int_{\Omega} \boldsymbol{S} \boldsymbol{\phi} dV \\
\hat{\boldsymbol{E}}_{\text{hyper}} \int_{\Omega} \boldsymbol{\psi} \boldsymbol{\phi} dV + \hat{\overline{\overline{F}}}^{*} \cdot \int_{\partial\Omega} \boldsymbol{\psi} \boldsymbol{\phi} \hat{\boldsymbol{n}} dS - \hat{\overline{\overline{F}}} \cdot \int_{\Omega} \boldsymbol{\psi} \nabla \boldsymbol{\phi} dV &= \int_{\Omega} \boldsymbol{S} \boldsymbol{\phi} dV \qquad (83)
\end{aligned}
$$

This is identical to the DG calculation. For this, a variable adjuster named

`wxm::dfem::variable_adjuster::auxiliary_variables::DivergenceIntegralByParts`

is used. For this variable adjuster, the app,

`wxm::apps::hyperresistivity::OhmsLaw`

defines $\overline{\overline{F}}$ in Eq. (80) and takes care of the cylindrical corrections to $\nabla \boldsymbol{j}$ as mentioned already using Eqs. (61) and (63). The app,

`wxm::apps::hyperresistivity::OhmsLawCylSource`

defines $\boldsymbol{S}$, the Cylindrical source terms arising from the calculation of $\nabla \cdot \overline{\overline{F}}$, using Eqs. (8). Finally, $\boldsymbol{E}_{\text{hyper}}$ is incorporated into MHD (energy equation and Faraday Law) as written in Sec. 3. The apps

`wxm::apps::hallmhd::hyperresistivityElectricFieldEffect`

and

`wxm::apps::hallmhd::hyperresistivityElectricFieldEffectCylSource`

do this and are put into the dg solver.

Note: Previously, we had a variable adjuster named

`wxm::dfem::variable_adjuster::auxiliary_variables::FullDivergenceVolumeIntegral`

which attempted to calculate Eq. (82), bypassing the integration by parts, as described in Appx. C of Ref. [1]. However, this was not found to be stable when stress testing it on a pure hyperdiffusion problem. It is akin to just using the derivative matrix to determine the gradient variable, as shown in the beginning of Ch. 7 of Ref. [2], which lacks convergence and stability at high resolution. Therefore, that variable adjuster has been removed. However,

`wxm::dfem::variable_adjuster::auxiliary_variables::DivergenceIntegralByParts`

needs to be used with boundary conditions set on all necessary variables, as required by the problem. This means, $\boldsymbol{B}$, $\nabla \boldsymbol{B}$, $\boldsymbol{j}$ (even though its calculated from $\nabla \boldsymbol{B}$, it would only be calculated on interior nodes without another boundary condition), $\nabla \boldsymbol{j}$, and $\boldsymbol{E}_{\text{hyper}}$, all need to have boundary conditions set.

References

[1] Iman Datta. *A Domain-Hybridized Plasma Model Using Discontinuous Galerkin Finite Elements*. PhD thesis, University of Washington, Seattle, WA, 2021.

[2] Jan S. Hesthaven and Tim Warburton. *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*. Springer Science+Business Media, LLC, New York, USA, 2008.