MHD BC

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1 Conducting wall boundary conditions

The question often comes up of what exactly the perfectly conducting ideal wall boundary conditions are and what their rationale is. Therefore we begin with a brief review of these conditions. Now, it can be found online in many places that, following from Maxwell's equations, one can show the following results regarding the normal and tangential components of the electric and magnetic fields $(\vec{E}, \vec{B})$ across any interface Σ from state 1 to state 2. Let $\hat{n}$ be the wall normal vector. Then,

- the normal component of $\vec{B}$ and tangential component of $\vec{E}$ are continuous:

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0, \tag{1}$$

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0, \tag{2}$$

- the tangential component of $\vec{B}$ and the normal component of $\vec{E}$ each jump by an amount equal to interfacial current and charge densities $\vec{\jmath}_\Sigma$ and ρ_Σ respectively,

$$\hat{n} \times (\vec{B}_1 - \vec{B}_2) = \mu_0 \vec{\jmath}_\Sigma, \tag{3}$$

$$\hat{n} \cdot (\vec{E}_1 - \vec{E}_2) = \varepsilon_0^{-1} \rho_\Sigma. \tag{4}$$

We now consider a computational domain bounded by a perfectly conducting material, which is known to satisfy the following conditions:

- The magnetic flux $\vec{B}$ maintains a constant value (flux pinning),
- The electric field vanishes in the material (ignoring such things as Josephson effect).

Provided that: i) the simulation time is much less than the resistive diffusion time of the material, and ii) the walls were not previously subject to sustained magnetic flux on resistive timescales prior to simulation initialization, then we can $\vec{B} = 0$ inside the material. From these conditions, we conclude the following boundary conditions from Eqs. 1 and 2, namely

$$\hat{n} \cdot \vec{B} = 0, \tag{5}$$

$$\vec{n} \times \vec{E} = 0. \tag{6}$$

Also, typically we take $\hat{n} \cdot \vec{v} = 0$ with $\vec{v}$ the fluid velocity for the zero-average flux of particles to/from a surface. As is widely well-known but not widely well-understood (particularly in current-carrying magnetoplasmas), the particle flux of ions to conducting electrodes can break this condition.

2 Conducting Wall for Resisistive MHD

For a conducting wall, the "tangential electric field and normal magnetic field must vanish",

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0 \tag{7a}$$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0 \tag{7b}$$

For resistive MHD, the, the electric field is given by the Generalized Ohm's Law with terms

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{j} \tag{8}$$

So Eq. (7a) becomes

$$\begin{aligned}\hat{\boldsymbol{n}} \times \boldsymbol{E} =& -\hat{\boldsymbol{n}} \times \boldsymbol{u} \times \boldsymbol{B} + \hat{\boldsymbol{n}} \times \eta \boldsymbol{j} \\ =& -\left[\cancel{(\hat{\boldsymbol{n}} \cdot \boldsymbol{B})}\boldsymbol{u} - \cancel{(\hat{\boldsymbol{n}} \cdot \boldsymbol{u})}\boldsymbol{B}\right] + \hat{\boldsymbol{n}} \times \eta \boldsymbol{j} \\ =& 0\end{aligned} \tag{9}$$

where the canceled terms come from Eq. (7b) and the stipulation that there can be no velocity through the wall. Note that $\hat{\boldsymbol{n}} \cdot \boldsymbol{u}$ is permits freeslip (flow-tangency) and noslip conditions. Equation (9) leads to the condition on $\boldsymbol{j}$:

$$\hat{\boldsymbol{n}} \times \boldsymbol{j} = \boldsymbol{0} \tag{10}$$

Notice that for MHD, $\boldsymbol{j}$ is given by

$$\boldsymbol{j} = \frac{\nabla \times \boldsymbol{B}}{(\omega_c \tau)} \tag{11}$$

Assuming Cartesian coordinates, Eq. (10) becomes

$$\hat{\boldsymbol{n}} \times \boldsymbol{j} = (n_y j_z - n_z j_y)\,\hat{\boldsymbol{x}} + (n_z j_x - n_x j_z)\,\hat{\boldsymbol{y}} + (n_x j_y - n_y j_x)\,\hat{\boldsymbol{z}} = \boldsymbol{0} \tag{12}$$

Equation (11) in Cartesian becomes

$$\boldsymbol{j} = \frac{1}{(\omega_c \tau)}\left[(\partial_y B_z - \partial_z B_y)\,\hat{\boldsymbol{x}} + (\partial_z B_x - \partial_x B_z)\,\hat{\boldsymbol{y}} + (\partial_x B_y - \partial_y B_x)\,\hat{\boldsymbol{z}}\right] \tag{13}$$

So in terms of $\boldsymbol{B}$ gradients, Eq. (12) becomes

$$\begin{aligned}\hat{\boldsymbol{n}} \times \boldsymbol{j} =& \left[n_y\left(\partial_x B_y - \partial_y B_x\right) - n_z\left(\partial_z B_x - \partial_x B_z\right)\right]\hat{\boldsymbol{x}} + \left[n_z\left(\partial_y B_z - \partial_z B_y\right)\hat{\boldsymbol{x}} - n_x\left(\partial_x B_y - \partial_y B_x\right)\right]\hat{\boldsymbol{y}} \\ &+ \left[n_x\left(\partial_z B_x - \partial_x B_z\right) - n_y\left(\partial_y B_z - \partial_z B_y\right)\hat{\boldsymbol{x}}\right]\hat{\boldsymbol{z}} = \boldsymbol{0}\end{aligned} \tag{14}$$

If the geometry is axisymmetric cylindrical (z-r-θ), then Eq. (10) becomes

$$\hat{\boldsymbol{n}} \times \boldsymbol{j} = (n_r j_\theta - n_\theta j_r)\,\hat{\boldsymbol{z}} + (n_\theta j_z - n_z j_\theta)\,\hat{\boldsymbol{r}} + (n_z j_r - n_r j_z)\,\hat{\boldsymbol{\theta}} = \boldsymbol{0} \tag{15}$$

Equation (11) in axisymmetric Cylindrical ($\frac{\partial}{\partial\theta} = 0$) becomes

$$\boldsymbol{j} = \frac{1}{(\omega_c \tau)} \left[\left(\partial_r B_\theta + \frac{B_\theta}{r} \right) \hat{\boldsymbol{z}} + \left(-\partial_z B_\theta \right) \hat{\boldsymbol{r}} + \left(\partial_z B_r - \partial_r B_z \right) \hat{\boldsymbol{\theta}} \right] \tag{16}$$

So in terms of $\boldsymbol{B}$ gradients, Eq. (15) becomes

$$\begin{aligned} \hat{\boldsymbol{n}} \times \boldsymbol{j} = & \left[n_r \left(\partial_z B_r - \partial_r B_z \right) - n_\theta \left(-\partial_z B_\theta \right) \right] \hat{\boldsymbol{z}} + \left[n_\theta \left(\partial_r B_\theta + \frac{B_\theta}{r} \right) - n_z \left(\partial_z B_r - \partial_r B_z \right) \right] \hat{\boldsymbol{r}} \\ & + \left[n_z \left(-\partial_z B_\theta \right) - n_r \left(\partial_r B_\theta + \frac{B_\theta}{r} \right) \right] \hat{\boldsymbol{\theta}} = \boldsymbol{0} \end{aligned} \tag{17}$$

If in particular $\hat{\boldsymbol{n}} = n_r \hat{\boldsymbol{r}}$, we need $j_\theta = j_z = 0$, or $\partial_z B_r = \partial_r B_z$ and $\partial_r B_\theta = -\frac{B_\theta}{r}$.

3 Cartesian Procedure for Gradients

We want to define the ∇B such that Eq. (14), or equivalently Eq. (12), is satisfied. To do so, it is simplest to rotate into the frame of the boundary such that the local $\hat{\boldsymbol{x}}$ direction after rotation, denoted $\hat{\boldsymbol{x}}'$ is the normal to the boundary. In WARPXM this can be achieved using the $\overline{\overline{R}}$ matrix. The local normal is then

$$\hat{\boldsymbol{n}}' = \overline{\overline{R}} \hat{\boldsymbol{n}} = \begin{bmatrix} n_x & n_y & n_z \\ t_x & t_y & t_z \\ b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} n_z^2 + n_r^2 + n_z^2 \\ t_x n_x + t_y n_y + t_z n_z \\ b_x n_x + b_y n_y + b_z n_z \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{18}$$

We can rotate $\boldsymbol{j}|_{\text{in}}$ into this reference frame

$$\boldsymbol{j}'\big|_{\text{in}} = \overline{\overline{R}}\, \boldsymbol{j}|_{\text{in}} = \begin{bmatrix} n_x & n_y & n_z \\ t_x & t_y & t_z \\ b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix}\Bigg|_{\text{in}} = \begin{bmatrix} n_x j_x + n_y j_y + n_z j_z \\ t_x j_x + t_y j_y + t_z j_z \\ b_x j_x + b_y j_y + b_z j_z \end{bmatrix}\Bigg|_{\text{in}} = \begin{bmatrix} \hat{\boldsymbol{n}} \cdot \boldsymbol{j} \\ \hat{\boldsymbol{t}} \cdot \boldsymbol{j} \\ \hat{\boldsymbol{b}} \cdot \boldsymbol{j} \end{bmatrix}\Bigg|_{\text{in}} \equiv \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}. \tag{19}$$

Now notice

$$\hat{\boldsymbol{n}}' \times \boldsymbol{j}'\big|_{\text{in}} = \begin{bmatrix} 0 \\ -Z' \\ Y' \end{bmatrix}. \tag{20}$$

Note that the boundary condition in Eq. (10) is imposing

$$\hat{\boldsymbol{n}}' \times \boldsymbol{j}'\big|_{\text{wall}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{21}$$

Note that this is achieved if

$$\boldsymbol{j}'\big|_{\text{wall}} = \begin{bmatrix} \text{Anything} \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

Here I'm choosing to copy the normal component such that

$$\boldsymbol{j}'\big|_{\text{wall}} = \begin{bmatrix} X' \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

To achieve this we set $\boldsymbol{j}'\big|_{\text{out}}$ as

$$\boldsymbol{j}'\big|_{\text{out}} = \begin{bmatrix} X' \\ -Y' \\ -Z' \end{bmatrix}. \tag{24}$$

What remains is to antirotate $\boldsymbol{j}'\big|_{\text{out}}$ and set $\nabla \boldsymbol{B}$ accordingly. First, the antirotation is given by

$$\boldsymbol{j}\big|_{\text{out}} = \overline{\overline{R}}^{-1} \boldsymbol{j}'\big|_{\text{out}} = \begin{bmatrix} n_x & t_x & b_x \\ n_y & t_y & b_y \\ n_z & t_z & b_z \end{bmatrix} \begin{bmatrix} X' \\ -Y' \\ -Z' \end{bmatrix} = \begin{bmatrix} n_x X' - t_x Y' - b_x Z' \\ n_y X' - t_y Y' - b_y Z' \\ n_z X' - t_z Y' - b_z Z' \end{bmatrix} \equiv \begin{bmatrix} X_{\text{out}} \\ Y_{\text{out}} \\ Z_{\text{out}} \end{bmatrix} \tag{25}$$

Then using Eq. (13)

$$\begin{aligned} \frac{\left.\frac{\partial B_z}{\partial y}\right|_{\text{out}} - \left.\frac{\partial B_y}{\partial z}\right|_{\text{out}}}{\omega_c \tau} &= X_{\text{out}} \\ \left.\frac{\partial B_z}{\partial y}\right|_{\text{out}} &= (\omega_c \tau)\, X_{\text{out}} + \left.\frac{\partial B_y}{\partial z}\right|_{\text{out}} \end{aligned} \tag{26a}$$

$$\begin{aligned} \frac{\left.\frac{\partial B_x}{\partial z}\right|_{\text{out}} - \left.\frac{\partial B_z}{\partial x}\right|_{\text{out}}}{\omega_c \tau} &= Y_{\text{out}} \\ \left.\frac{\partial B_x}{\partial z}\right|_{\text{out}} &= (\omega_c \tau)\, Y_{\text{out}} + \left.\frac{\partial B_z}{\partial x}\right|_{\text{out}} \end{aligned} \tag{26b}$$

$$\begin{aligned} \frac{\left.\frac{\partial B_y}{\partial x}\right|_{\text{out}} - \left.\frac{\partial B_x}{\partial y}\right|_{\text{out}}}{\omega_c \tau} &= Z_{\text{out}} \\ \left.\frac{\partial B_y}{\partial x}\right|_{\text{out}} &= (\omega_c \tau)\, Z_{\text{out}} + \left.\frac{\partial B_x}{\partial y}\right|_{\text{out}} \end{aligned} \tag{26c}$$

To apply Eq. (25), we set arbitrarily set reverse copy to the derivatives

$$\left.\frac{\partial B_y}{\partial z}\right|_{\text{out}} = -\left.\frac{\partial B_y}{\partial z}\right|_{\text{in}} \tag{27a}$$

$$\left.\frac{\partial B_z}{\partial x}\right|_{\text{out}} = -\left.\frac{\partial B_z}{\partial x}\right|_{\text{in}} \tag{27b}$$

$$\left.\frac{\partial B_x}{\partial y}\right|_{\text{out}} = -\left.\frac{\partial B_x}{\partial y}\right|_{\text{in}} \tag{27c}$$

This should be ok because it is only the difference in the gradients that matter. The other B derivatives do not figure into the resistive mhd equations since it is only the $\boldsymbol{j}$ term that has an effect. Therefore it doesn't matter what I do. I just copy out $\frac{\partial B_x}{\partial x}$, $\frac{\partial B_y}{\partial y}$, and $\frac{\partial B_z}{\partial z}$.

4 Axisymmetric Cylindrical Geometry General Case

Equation (17) dictates some of the magnetic field gradients in cylindrical geometry. The axisymmetric formulation simulations, in addition to having $\frac{\partial}{\partial\theta} = 0$, are also in the $r - z$ plane, meaning, $n_\theta = 0$, reducing Eq. (17) to

$$\hat{\boldsymbol{n}} \times \boldsymbol{j} = \left[n_r\left(\partial_z B_r - \partial_r B_z\right)\right]\hat{\boldsymbol{z}} + \left[-n_z\left(\partial_z B_r - \partial_r B_z\right)\right]\hat{\boldsymbol{r}} + \left[n_z\left(-\partial_z B_\theta\right) - n_r\left(\partial_r B_\theta + \frac{B_\theta}{r}\right)\right]\hat{\boldsymbol{\theta}} = \boldsymbol{0} \tag{28}$$

Now we rotate $\boldsymbol{j}$ in Eq. (16) and then apply Eq. (17), which allows us to determine some of the gradients. Note that

$$\hat{\boldsymbol{n}}' = \overline{\overline{R}}\hat{\boldsymbol{n}} = \begin{bmatrix} n_z & n_r & 0 \\ t_z & t_r & 0 \\ 0 & 0 & b_\theta \end{bmatrix}\begin{bmatrix} n_z \\ n_r \\ 0 \end{bmatrix} = \begin{bmatrix} n_z^2 + n_r^2 \\ t_z n_z + t_r n_r \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

$$\left.\boldsymbol{j}'\right|_{\text{in}} = \overline{\overline{R}}\left.\boldsymbol{j}\right|_{\text{in}} = \begin{bmatrix} n_z & n_r & 0 \\ t_z & t_r & 0 \\ 0 & 0 & b_\theta \end{bmatrix}\left.\begin{bmatrix} j_z \\ j_r \\ j_\theta \end{bmatrix}\right|_{\text{in}} = \left.\begin{bmatrix} n_z j_z + n_r j_r \\ t_z j_z + t_r j_r \\ b_\theta j_\theta \end{bmatrix}\right|_{\text{in}} \equiv \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \tag{30}$$

So we have

$$\hat{\boldsymbol{n}}' \times \left.\boldsymbol{j}'\right|_{\text{in}} = \begin{bmatrix} 0 \\ -b_\theta j_\theta \\ t_z j_z + t_r j_r \end{bmatrix} = \begin{bmatrix} 0 \\ -Z' \\ Y' \end{bmatrix} \tag{31}$$

Basicaly in the rotated frame, we set $Y' = Z' = 0$, which corresponds to setting a reverse copy

$$\left.\boldsymbol{j}'\right|_{\text{out}} = \begin{bmatrix} X' \\ -Y' \\ -Z' \end{bmatrix} \tag{32}$$

so that

$$\left.\boldsymbol{j}'\right|_{\text{wall}} = \begin{bmatrix} X' \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

Again we antirotate $\boldsymbol{j}|_{\text{out}}$

$$\boldsymbol{j}|_{\text{out}} = \overline{\overline{R}}^{-1}\, \boldsymbol{j}'|_{\text{out}} = \begin{bmatrix} n_z & t_z & 0 \\ n_r & t_r & 0 \\ 0 & 0 & b_\theta \end{bmatrix} \begin{bmatrix} X' \\ -Y' \\ -Z' \end{bmatrix} = \begin{bmatrix} n_z X' - t_z Y' \\ n_r X' - t_r Y' \\ -b_\theta Z' \end{bmatrix} \equiv \begin{bmatrix} X_{\text{out}} \\ Y_{\text{out}} \\ Z_{\text{out}} \end{bmatrix} \tag{34}$$

Then we use Eq. (16),

$$\begin{aligned} \frac{\left.\frac{\partial B_\theta}{\partial r}\right|_{\text{out}} + \frac{B_\theta|_{\text{out}}}{r|_{\text{wall}}}}{\omega_c \tau} &= X_{\text{out}} \\ \left.\frac{\partial B_\theta}{\partial r}\right|_{\text{out}} &= (\omega_c \tau)\, X_{\text{out}} - \frac{B_\theta|_{\text{out}}}{r|_{\text{wall}}} \end{aligned} \tag{35a}$$

$$\begin{aligned} \frac{-\left.\frac{\partial B_\theta}{\partial z}\right|_{\text{out}}}{\omega_c \tau} &= Y_{\text{out}} \\ \left.\frac{\partial B_\theta}{\partial z}\right|_{\text{out}} &= -\,(\omega_c \tau)\, Y_{\text{out}} \end{aligned} \tag{35b}$$

$$\begin{aligned} \frac{\left.\frac{\partial B_r}{\partial z}\right|_{\text{out}} - \left.\frac{\partial B_z}{\partial r}\right|_{\text{out}}}{\omega_c \tau} &= Z_{\text{out}} \\ \left.\frac{\partial B_r}{\partial z}\right|_{\text{out}} &= (\omega_c \tau)\, Z_{\text{out}} + \left.\frac{\partial B_z}{\partial r}\right|_{\text{out}} \end{aligned} \tag{35c}$$

Since in Eq. (35c), only the difference the gradients matter, I arbitrarily set a reverse copy

$$\left.\frac{\partial B_z}{\partial r}\right|_{\text{out}} = -\left.\frac{\partial B_z}{\partial r}\right|_{\text{in}} \tag{36}$$

The other B derivatives do not figure into the resistive mhd equations since it is only the $\boldsymbol{j}$ term that has an effect. Therefore it doesn't matter what I do. By the axisymmetric assumption I just reverse copy $\frac{\partial B_z}{\partial \theta}$, $\frac{\partial B_r}{\partial \theta}$, and $\frac{\partial B_\theta}{\partial \theta}$. I also just copy out $\frac{\partial B_z}{\partial z}$ and $\frac{\partial B_r}{\partial r}$. Note that Eq. (35a) becomes problematic at $r|_{\text{wall}} = 0$. For that we can use L'Hospital's rule

$$\begin{aligned} \left.\frac{\partial B_\theta}{\partial r}\right|_{\text{out}}^{(r=0)} &= (\omega_c \tau)\, X_{\text{out}} - \lim_{r \to 0} \frac{B_\theta|_{\text{out}}}{r|_{\text{wall}}} \\ &= (\omega_c \tau)\, X_{\text{out}} - \frac{\left(\frac{\partial}{\partial r}\, B_\theta|_{\text{out}}\right)^{r=0}}{\left(\frac{\partial}{\partial r}\, r|_{\text{wall}}\right)^{r=0}} \\ &= (\omega_c \tau)\, X_{\text{out}} - \left.\frac{\partial B_\theta}{\partial r}\right|_{\text{out}}^{(r=0)} \\ \left.\frac{\partial B_\theta}{\partial r}\right|_{\text{out}}^{(r=0)} &= \frac{(\omega_c \tau)\, X_{\text{out}}}{2} \end{aligned} \tag{37}$$

5 Conducting wall for dissipative Hall MHD

In this section we apply the perfectly conducting wall boundary conditions $\hat{n} \times \vec{E} = 0$ and $\hat{n} \cdot \hat{b} = 0$ (with $\hat{b} \equiv \vec{B}/B$) to the generalized Ohm's law for the Hall MHD system,

$$\vec{E} + \vec{v} \times \vec{B} = \frac{1}{ne}(\vec{\jmath} \times \vec{B} - \nabla p_e) + \eta \vec{\jmath}. \tag{38}$$

From before we have $\hat{n} \times (\vec{E} + \vec{v} \times \vec{B}) = 0$ at the perfectly conducting fluxless ($\hat{n} \cdot \hat{b} = 0$) surface. We define the Hall coefficient as $\chi_H \equiv B/ne\eta$ which for the Spitzer model reduces to $\chi_H = \omega_{ce}/\nu_e$. Multiplying by ne/B we obtain the nonlinear boundary condition as the system of equations

$$\hat{n} \cdot \hat{b} = 0, \tag{39}$$

$$\hat{n} \times (\vec{\jmath} \times \hat{b} - \nabla p_e/B + \chi_H^{-1} \vec{\jmath}) = 0. \tag{40}$$

We note the limit $\chi_H \to \infty$ (either collisionless or strongly magnetized) and $p_e = 0$ as reducing to the force-free condition $\hat{n} \times (\vec{\jmath} \times \hat{b}) = 0$ indicating either the two-dimensional $\hat{n} \cdot \vec{\jmath} = \hat{n} \cdot \hat{b} = 0$ solution or the three-dimensional Beltrami/Taylor states $\vec{\jmath} = \alpha \vec{B}$ as the boundary condition. Regardless, for general values of χ_H we can apply Eq. 39 to the triple cross product in Eq. 40 to find

$$\hat{n} \times (\vec{\jmath} \times \hat{b}) = \vec{\jmath}(\hat{n} \cdot \hat{b}) - \hat{b}(\hat{n} \cdot \vec{\jmath}) = -\hat{b}(\hat{n} \cdot \vec{\jmath}). \tag{41}$$

Therefore we write Eq. 40 in the form

$$\chi_H^{-1}(\hat{n} \times \vec{\jmath}) = \hat{b}(\hat{n} \cdot \vec{\jmath}) + \hat{n} \times \nabla p_e/B. \tag{42}$$

We will both "dot" and "cross" this equation by $\hat{b}$. First "crossing", and applying Eq. 39, we find

$$\chi_H^{-1} \hat{b} \times (\hat{n} \times \vec{\jmath}) = \hat{b} \times (\hat{n} \times \nabla p_e/B) \tag{43}$$

$$\implies \hat{b} \cdot (\chi_H^{-1} \vec{\jmath} - \nabla p_e/B) = 0. \tag{44}$$

Equation 44 indicates a potential for sophisticated three-dimensional solutions depending on the distribution of pressure isosurfaces. However, for two-dimensional dynamics with an out-of-plane magnetic field, it is consistent to apply the two-dimensional solution $\hat{b} \cdot \vec{\jmath} = \hat{b} \cdot \nabla p_e = 0$, the two of which add to the constraint $\hat{n} \cdot \hat{b} = 0$. Now "dotting" Eq. (42) by $\hat{b}$, we obtain the equation

$$\chi_H^{-1} \vec{\jmath} \cdot (\hat{b} \times \hat{n}) = \hat{n} \cdot (\vec{\jmath} - \vec{\jmath}_{de}) \tag{45}$$

where the electron diamagnetic current is defined as $\vec{\jmath}_{de} \equiv -\nabla p_e \times \hat{b}/B$. We now make a closure assumption for the electron pressure p_e's relation to the total plasma pressure p, namely that electron and ion pressures p_e and p_i are in a constant ratio

$$\frac{p_e}{p_i} \equiv \theta \implies \frac{p_e}{p} = \frac{\theta}{1+\theta}. \tag{46}$$

Since the net diamagnetic current is $\vec{\jmath}_d \equiv -\nabla p \times \hat{b}/B$, we modify Eq. 45 to read

$$\chi_H^{-1} \vec{\jmath} \cdot (\hat{b} \times \hat{n}) = \hat{n} \cdot \left(\vec{\jmath} - \frac{\theta}{1+\theta} \vec{\jmath}_d\right). \tag{47}$$

The left-hand side of Eq. 47 is the signed-magnitude of the wall-tangential current $\vec{j}_\perp \equiv j_\perp \hat{b} \times \hat{n}$. For example, in Cartesian coordinates (x, y, z) with $\hat{n} = \hat{x}$ and $\hat{b} = \hat{z}$, the wall-tangential current flows in the $\hat{y}$ direction provided the right-hand side of Eq. 47 is positive. We let $\sigma \equiv \sin^{-1}(\hat{b} \times \hat{n})$ be the sign of the direction of $\hat{n} \times \hat{b}$, and conclude that

$$j_\perp = \sigma \chi_H \hat{n} \cdot \left(\vec{j} - \frac{\theta}{1+\theta} \vec{j}_d \right) \tag{48}$$

is our desired boundary condition.

5.1 Boundary condition with WARPXM normalization

The relevant normalized equations of the Hall-MHD system are the following:

$$\nabla \times \vec{B} = (\omega_c \tau) \vec{j} \tag{49}$$

$$\vec{E} + \vec{v} \times \vec{B} = \frac{1}{n_e} (\vec{j} \times \vec{B} - \frac{1}{\omega_c \tau} \nabla p_e) + \frac{\nu_p \tau}{\omega_c \tau} \eta \vec{j} \tag{50}$$

As before, apply the boundary condition Eq. (6) to Eq. (50) and multiply through by (n_e/B). Defining the normalized Hall parameter

$$\chi_H = \frac{B}{n_\mathrm{e} \eta}, \tag{51}$$

while $\omega_c \tau / \nu_p \tau$ represents the reference Hall parameter, we can obtain

$$\hat{n} \times (\vec{j} \times \hat{b} - (\omega_c \tau)^{-1} \nabla p_e / B + \frac{\nu_p \tau}{\omega_c \tau} \chi_H^{-1} \vec{j}) = 0 \tag{52}$$

Equation (52) is the normalized analog of Eq. (40). From this, we conclude that the physical Hall parameter from the normalized B, n_e, and η is

$$\chi_{H,\mathrm{phys}} = \frac{B}{n_e \eta} \frac{(\omega_c \tau)}{(\nu_p \tau)}. \tag{53}$$

Now, we can conclude from this result that everything of the boundary condition is unchanged aside from rescaling the pressure gradient and the Hall parameter as so. Equation (42) becomes

$$\frac{(\nu_p \tau)}{(\omega_c \tau)} \chi_H^{-1} (\hat{n} \times \vec{j}) = \hat{b} (\hat{n} \cdot \vec{j}) + \frac{1}{(\omega_c \tau)} \hat{n} \times \nabla p_e / B. \tag{54}$$

Equation (44) becomes

$$\hat{b} \cdot \left[\frac{(\nu_p \tau)}{(\omega_c \tau)} \chi_H^{-1} \vec{j} - \frac{1}{(\omega_c \tau)} \frac{\nabla p_e}{B} \right] = 0. \tag{55}$$

Equation (45) becomes

$$\frac{(\nu_p \tau)}{(\omega_c \tau)} \chi_H^{-1} \vec{j} \cdot (\hat{b} \times \hat{n}) = \hat{n} \cdot (\vec{j} - \vec{j}_{de}) \tag{56}$$

for $\vec{\jmath}_{de} \equiv -\frac{1}{(\omega_c\tau)}\nabla p_e \times \hat{b}/B$. Assuming $\vec{\jmath}_d \equiv -\frac{1}{\omega_c\tau}\nabla p \times \hat{b}/B$, Eq. (47) becomes

$$\frac{(\nu_p\tau)}{(\omega_c\tau)}\chi_H^{-1}\vec{\jmath}\cdot(\hat{b}\times\hat{n}) = \hat{n}\cdot(\vec{\jmath} - \frac{\theta}{1+\theta}\vec{\jmath}_d). \tag{57}$$

Equation (48) becomes

$$j_\perp = \frac{(\omega_c\tau)}{(\nu_p\tau)}\sigma\chi_H\hat{n}\cdot\left(\vec{\jmath} - \frac{\theta}{1+\theta}\vec{\jmath}_d\right). \tag{58}$$

5.2 Changes to the pressure gradient boundary condition

In hydrodynamic modeling, it's often the case that a wall is subject to both pressure gradient and body forces from an equation of motion such as

$$\rho\frac{d\vec{v}}{dt} = -\nabla p + \vec{f}. \tag{59}$$

Given that $\hat{n}\cdot\vec{v} = 0$ on an ideal surface, Eq. 59 leads to the boundary condition on ∇p as

$$\hat{n}\cdot\nabla p = \hat{n}\cdot\vec{f}. \tag{60}$$

For example, in the subject of flows in gravity the pressure gradient is balanced by $\rho\hat{n}\cdot\vec{g}$. Similarly, in the MHD problem we balance the pressure gradient by the Laplace body force $(\omega_c\tau)\,\vec{\jmath}\times\vec{B}$ as

$$\hat{n}\cdot\nabla p = (\omega_c\tau)\,\hat{n}\cdot(\vec{\jmath}\times\vec{B}) \tag{61}$$

Using vector identities, Eq. (61) can also be written

$$\hat{n}\cdot\nabla p = (\omega_c\tau)\,\vec{B}\cdot(\hat{n}\times\vec{\jmath}) \tag{62}$$

In the RMHD model we take $\hat{n}\cdot\nabla p = 0$ (*i.e.* reverse-copy pressure gradient) because $\hat{n}\times\vec{\jmath} = 0$.

5.3 Implementation Details

We wish to spell out the wall boundary condition for $\nabla p = (\nabla p)_x\,\hat{\boldsymbol{x}} + (\nabla p)_y\,\hat{\boldsymbol{y}} + (\nabla p)_z\,\hat{\boldsymbol{z}}$, and $\boldsymbol{j} = j_x\hat{\boldsymbol{x}} + j_y\hat{\boldsymbol{y}} + j_z\hat{\boldsymbol{z}}$ on a conducting wall boundary. We assume we are in the rotated frame where the normal is in the "x" direction, $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}$. Note that due to Eq. (5), $\boldsymbol{B} = B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}$ (and consequently $\boldsymbol{b} = b_y\hat{\boldsymbol{y}} + b_z\hat{\boldsymbol{z}}$) in this frame. We'll follow a procedure for $\boldsymbol{j}$ in this wall frame as in Eq. (23) (dropping the $'$ for convenience). Looking at Eq. (6) for the Hall MHD case, we find this equation becomes Eq. (54). Substitution of Eq. (46) yields

$$\frac{(\nu_p\tau)}{(\omega_c\tau)}\chi_H^{-1}(\hat{n}\times\vec{\jmath}) = \hat{b}(\hat{n}\cdot\vec{\jmath}) + \frac{1}{(\omega_c\tau)\,B}\frac{\theta}{1+\theta}\hat{n}\times\nabla p. \tag{63}$$

Writing this out, one gets

$$\begin{aligned}\frac{(\nu_p\tau)}{(\omega_c\tau)}\chi_H^{-1}\left[\hat{\boldsymbol{x}}\times\left(j_x\hat{\boldsymbol{x}}+j_y\hat{\boldsymbol{y}}+j_z\hat{\boldsymbol{z}}\right)\right]&=\left(b_y\hat{\boldsymbol{y}}+b_z\hat{\boldsymbol{z}}\right)j_x+\frac{1}{(\omega_c\tau)\,B}\frac{\theta}{1+\theta}\hat{\boldsymbol{x}}\times\Big((\nabla p)_x\,\hat{\boldsymbol{x}}+(\nabla p)_y\,\hat{\boldsymbol{y}}+(\nabla p)_z\,\hat{\boldsymbol{z}}\Big)\\ \frac{(\nu_p\tau)}{(\omega_c\tau)}\chi_H^{-1}\left[j_y\hat{\boldsymbol{z}}-j_z\hat{\boldsymbol{y}}\right]&=\left(b_y\hat{\boldsymbol{y}}+b_z\hat{\boldsymbol{z}}\right)j_x+\frac{1}{(\omega_c\tau)\,B}\frac{\theta}{1+\theta}\Big((\nabla p)_y\,\hat{\boldsymbol{z}}-(\nabla p)_z\,\hat{\boldsymbol{y}}\Big)\\ j_y\hat{\boldsymbol{z}}-j_z\hat{\boldsymbol{y}}&=\frac{(\omega_c\tau)}{(\nu_p\tau)}\chi_H\left(b_y\hat{\boldsymbol{y}}+b_z\hat{\boldsymbol{z}}\right)j_x+\frac{\chi_H}{(\nu_p\tau)\,B}\frac{\theta}{1+\theta}\Big((\nabla p)_y\,\hat{\boldsymbol{z}}-(\nabla p)_z\,\hat{\boldsymbol{y}}\Big)\end{aligned} \tag{64}$$

Componentwise, this is

$$j_z=-\frac{(\omega_c\tau)}{(\nu_p\tau)}\chi_H b_y j_x+\frac{\chi_H}{(\nu_p\tau)\,B}\frac{\theta}{1+\theta}\,(\nabla p)_z \tag{65a}$$

$$j_y=\frac{(\omega_c\tau)}{(\nu_p\tau)}\chi_H b_z j_x+\frac{\chi_H}{(\nu_p\tau)\,B}\frac{\theta}{1+\theta}\,(\nabla p)_y \tag{65b}$$

Note, as before, there is no condition on j_x so that is copied out just like in the resistive MHD case. We continue to reverse copy $(\nabla p)_y$ and $(\nabla p)_z$, allowing for the prescription of j_z and j_y in Eq. (65).

For now, we still apply $(\nabla p)_x = 0$ (set wall value to 0, or equivalently, reverse copying the inside value to the outside value). We are currently not applying Eq. (62), which would lead to $(\nabla p)_x = (\omega_c\tau)\,(B_z j_y - B_y j_z)$, as this is causing numerical instability and may not apply to our conservative scheme.

It might also be simpler substitute Eq. (51) to Eq. (65) to obtain

$$j_z=-\frac{(\omega_c\tau)}{(\nu_p\tau)}\frac{B_y}{n_\mathrm{e}\eta}j_x+\frac{1}{n_\mathrm{e}\eta\,(\nu_p\tau)}\frac{\theta}{(\theta+1)}\,(\nabla p)_z \tag{66a}$$

$$j_y=\frac{(\omega_c\tau)}{(\nu_p\tau)}\frac{B_z}{n_\mathrm{e}\eta}j_x+\frac{1}{n_\mathrm{e}\eta\,(\nu_p\tau)}\frac{\theta}{(\theta+1)}\,(\nabla p)_y \tag{66b}$$

5.4 Limits of the boundary condition with varying Hall parameter

In this section we recover the boundary condition for resistive MHD and note the limiting behavior in the Hall-dominated regime. Letting $\chi_H \to 0$ and $\chi_H \to \infty$ we discover the following limits of the MHD boundary conditions at a perfectly conducting surface,

$$\chi_H \to \infty, \quad j_n = j_{de,n}, \quad j_\perp \text{ is unconstrained,} \tag{67}$$

$$\chi_H \to 0, \quad j_\perp = 0, \quad j_n \text{ is unconstrained.} \tag{68}$$

The latter condition is our resistive MHD boundary condition where the normal component of current is unconstrained, and there is no wall-tangential current. On the other hand, in the Hall-dominated regime the wall-normal current is constrained and limits to the wall-normal electron diamagnetic current. In other words, a tangential electron pressure gradient remains as the only physics capable of allowing current to be drawn from the wall due to the electrical “demand” of inductive flows within the domain. In fact, in the cold electron limit $\theta = 0$ the condition remaining is that $j_n = 0$ and currents are prevented from leaving the wall entirely, leading to electron vortex build-up near the boundary or lines of current “grazing” the boundary.

5.5 Prediction of wall-tangential flow from large Hall parameters

In the typical situation of non-zero electron pressure (in our closure model, $\theta \neq 0$), normal currents may be supplied by the electron diamagnetic current in the limit $\chi_H \to \infty$. However, if $p_i \neq 0$ then this cannot be maintained in static equilibrium, as the net force $\vec{\jmath} \times \vec{B} - \nabla p$ is unbalanced. Equilibrium can only be maintained by a flow force,

$$\hat{n} \times \left((\vec{v} \times \nabla)\vec{v} + \frac{\nabla p_i}{\rho} \right) = 0. \tag{69}$$

In two-dimensional flow there will be no vortex force due to swirling flow tangential to the boundary,

$$\hat{n} \times (\vec{v} \times \vec{\omega}) = \vec{v}(\hat{n} \cdot \vec{\omega}) - \vec{\omega}(\hat{n} \cdot \vec{v}) = 0 \tag{70}$$

and thus the condition would simplify to a potential flow on the boundary,

$$\hat{n} \times \left(\nabla(v^2/2) + \frac{\nabla p_i}{\rho} \right) = 0. \tag{71}$$

In any of the three typical flow cases: i) incompressible, ii) isothermal, iii) isentropic, along the boundary, Eq. 71 reduces to a wall-normal ion isoenergetic condition of the form $\hat{n} \times \nabla p_{ti} = 0$ where p_{ti} is the relevant ion Bernoulli quantity (*e.g.* the ion total pressure or specific enthalpy). Thus we can expect a wall-tangential flow to develop on the free-slip surface where the tangential flow velocity varies along with the ion pressure gradient, meaning that plasma streamlines are quite likely to either peel-away from or be directed towards the wall in regions with wall-normal current.