Maxwell Flux

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The Maxwell equations evolved in WARPXM are:

$$\frac{\partial \boldsymbol{E}}{\partial t} - (\omega_p \tau)^2 \left(\frac{\delta_p}{L}\right)^2 \nabla \times \boldsymbol{B} = - (\omega_p \tau)^2 \left(\frac{\delta_p}{L}\right) \sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha} \boldsymbol{u}_\alpha \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \tag{2}$$

Note that we write our conservations laws as:

$$\frac{\partial q_i}{\partial t} + \frac{\partial \mathcal{F}_{ij}}{\partial x_j} = S_i + \frac{\partial}{\partial x_j} \left(D_{ijkl} \frac{\partial q_l}{\partial x_k} \right) \tag{3}$$

In this form these laws become

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial x_j} \left[-\epsilon_{ijk} (\omega_p \tau)^2 \left(\frac{\delta_p}{L}\right)^2 B_k \right] = - (\omega_p \tau)^2 \left(\frac{\delta_p}{L}\right) \sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha} u_{\alpha_i} \tag{4}$$

and

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} \left[\epsilon_{ijk} E_k \right] = 0 \tag{5}$$

That is

$$\mathcal{F}_{ij_E} = - \epsilon_{ijk} (\omega_p \tau)^2 \left(\frac{\delta_p}{L}\right)^2 B_k \tag{6}$$

$$\mathcal{F}_{ij_B} = \epsilon_{ijk} E_k \tag{7}$$

Now we'll write this as a combined system:

$$\boldsymbol{q} = \begin{bmatrix} E_x & E_y & E_z & B_x & B_y & B_z \end{bmatrix}^T \tag{8}$$

So we can write this system as

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial \boldsymbol{G}}{\partial y} + \frac{\partial \boldsymbol{H}}{\partial z} \tag{9}$$

1 Flux-Splitting Approach

Consider the 1D system

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \tag{10}$$

if we have a hyperbolic problem we should have left and right-traveling waves, so we can split the fluxes into those:

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{F}^+}{\partial x} + \frac{\partial \boldsymbol{F}^-}{\partial x} = \boldsymbol{S} \tag{11}$$

At some boundary, the solution is affected by the net effect of these fluxes. To find these fluxes, we can decompose the fluxes using a flux Jacobian such that

$$\frac{\partial \boldsymbol{F}}{\partial x} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}} \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{x}} \equiv \overline{\overline{A}} \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{x}} \tag{12}$$

we can decompose $\overline{\overline{A}}$ by using the eigensystem $\overline{\overline{AX}} = \overline{\overline{X\Lambda}}$ where Λ is a diagonal matrix of eigenvalues of $\overline{\overline{A}}$ and $\overline{\overline{X}}$ is a matrix of the right eigenvectors or $\overline{\overline{A}}$. The eigenvalues should all be real (hyperbolic system). From this, $\overline{\overline{A}} = \overline{\overline{X\Lambda X}}^{-1}$. So we have the system

$$\frac{\partial \boldsymbol{q}}{\partial t} + \overline{\overline{X\Lambda X}}^{-1} \frac{\partial \boldsymbol{q}}{\partial x} = \boldsymbol{S} \tag{13}$$

which we can split

$$\frac{\partial \boldsymbol{q}}{\partial t} + \overline{\overline{X\Lambda}}^{+} \overline{\overline{X}}^{-1} \frac{\partial \boldsymbol{q}}{\partial x} + \overline{\overline{X\Lambda}}^{-} \overline{\overline{X}}^{-1} \frac{\partial \boldsymbol{q}}{\partial x} = \boldsymbol{S} \tag{14}$$

where $\overline{\overline{\Lambda}}^{+}$ contains positive diagonal entries (right-going waves) and $\overline{\overline{\Lambda}}^{-}$ contains negative diagonal entries (left-going waves). So we have can write

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial}{\partial x} \left(\underbrace{\overline{\overline{X\Lambda}}^{+} \overline{\overline{X}}^{-1} \boldsymbol{q}^-}_{\boldsymbol{F}^+} + \underbrace{\overline{\overline{X\Lambda}}^{-} \overline{\overline{X}}^{-1} \boldsymbol{q}^+}_{\boldsymbol{F}^-} \right) = \boldsymbol{S} \tag{15}$$

We can rewrite this

$$\begin{aligned}
\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial}{\partial x} \left(\overline{\overline{X\Lambda}}^{+} \overline{\overline{X}}^{-1} \boldsymbol{q}^- + \overline{\overline{X\Lambda}}^{-} \overline{\overline{X}}^{-1} \boldsymbol{q}^+ \right) &= \boldsymbol{S} \\
\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial}{\partial x} \left[\overline{\overline{X}} \left(\frac{\overline{\overline{\Lambda}} + \left|\overline{\overline{\Lambda}}\right|}{2} \right) \overline{\overline{X}}^{-1} \boldsymbol{q}^- + \overline{\overline{X}} \left(\frac{\overline{\overline{\Lambda}} - \left|\overline{\overline{\Lambda}}\right|}{2} \right) \overline{\overline{X}}^{-1} \boldsymbol{q}^+ \right] &= \boldsymbol{S} \\
\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial}{\partial x} \left[\frac{1}{2} \left(\overline{\overline{X\Lambda X}}^{-1} \boldsymbol{q}^- + \overline{\overline{X\Lambda X}}^{-1} \boldsymbol{q}^+ \right) + \frac{1}{2} \left(\overline{\overline{X}} \left|\overline{\overline{\Lambda}}\right| \overline{\overline{X}}^{-1} \right) \left(\boldsymbol{q}^- - \boldsymbol{q}^+ \right) \right] &= \boldsymbol{S} \\
\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial}{\partial x} \left[\frac{1}{2} \left(\boldsymbol{F}^- + \boldsymbol{F}^+ \right) + \frac{1}{2} \left(\overline{\overline{X}} \left|\overline{\overline{\Lambda}}\right| \overline{\overline{X}}^{-1} \right) \left(\boldsymbol{q}^- - \boldsymbol{q}^+ \right) \right] &= \boldsymbol{S} \qquad (16)
\end{aligned}$$

So our numerical flux is

$$\boldsymbol{F}^* = \frac{1}{2}\left(\boldsymbol{F}^- + \boldsymbol{F}^+\right) + \frac{1}{2}\left(\overline{\overline{X}}\left|\overline{\overline{\Lambda}}\right|\overline{\overline{X}}^{-1}\right)\left(\boldsymbol{q}^- - \boldsymbol{q}^+\right) \tag{17}$$

2 Application to Maxwell's Equations

For Maxwell's equations, we write $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ can be written using equations 6 and 7:

$$\boldsymbol{F} = \begin{pmatrix} -c^2\epsilon_{i1k}B_k \\ \epsilon_{i1k}E_k \end{pmatrix} = \begin{pmatrix} 0 \\ c^2B_z \\ -c^2B_y \\ 0 \\ -E_z \\ E_y \end{pmatrix} \tag{18}$$

$$\boldsymbol{G} = \begin{pmatrix} -c^2\epsilon_{i2k}B_k \\ \epsilon_{i2k}E_k \end{pmatrix} = \begin{pmatrix} -c^2B_z \\ 0 \\ c^2B_x \\ E_z \\ 0 \\ -E_x \end{pmatrix} \tag{19}$$

$$\boldsymbol{H} = \begin{pmatrix} -c^2\epsilon_{i3k}B_k \\ \epsilon_{i3k}E_k \end{pmatrix} = \begin{pmatrix} c^2B_y \\ -c^2B_x \\ 0 \\ -E_y \\ E_x \\ 0 \end{pmatrix} \tag{20}$$

with $\frac{c}{v_0} = (\omega_p\tau)\left(\frac{\delta_p}{L}\right)$. For ease of notation here assume $v_0 = 1$. So we can calculate the flux jacobian:

$$\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{q}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

Solving the eigenvalue problem, the matrix of eigenvalues is

$$\overline{\overline{\Lambda}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix} \tag{22}$$

and the matrix of column right eigenvectors is

$$\overline{\overline{X}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & c & 0 \\ 0 & 0 & 0 & c & 0 & -c \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \tag{23}$$

and its inverse

$$\overline{\overline{X}}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2c} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2c} & 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{24}$$

We now have our numerical flux for Maxwell's equations:

$$\boldsymbol{F}^* = \frac{1}{2}\left(\boldsymbol{F}^- + \boldsymbol{F}^+\right) + \frac{1}{2}\left(\overline{\overline{X}}\left|\overline{\overline{\Lambda}}\right|\overline{\overline{X}}^{-1}\right)\left(\boldsymbol{q}^- - \boldsymbol{q}^+\right)$$

$$= \begin{pmatrix} \frac{1}{2}\left[\left(-c^2\epsilon_{i1k}B_k\right)^- + \left(-c^2\epsilon_{i1k}B_k\right)^+\right] \\ \frac{1}{2}\left[\left(\epsilon_{i1k}E_k\right)^- + \left(\epsilon_{i1k}E_k\right)^+\right] \end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & c & 0 \\ 0 & 0 & 0 & c & 0 & -c \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2c} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2c} & 0 & \frac{1}{2} & 0 \end{pmatrix}\left(\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix}^- - \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix}^+\right)$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ c^2\left(B_z^- + B_z^+\right) \\ -c^2\left(B_y^- + B_y^+\right) \\ 0 \\ -\left(E_z^- + E_z^+\right) \\ \left(E_y^- + E_y^+\right) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}\left(\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix}^- - \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix}^+\right)$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ c^2\left(B_z^- + B_z^+\right) \\ -c^2\left(B_y^- + B_y^+\right) \\ 0 \\ -\left(E_z^- + E_z^+\right) \\ \left(E_y^- + E_y^+\right) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ c\left(E_y^- - E_y^+\right) \\ c\left(E_z^- - E_z^+\right) \\ 0 \\ c\left(B_y^- - B_y^+\right) \\ c\left(B_z^- - B_z^+\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2}c^2\left(B_z^- + B_z^+\right) + \frac{1}{2}c\left(E_y^- - E_y^+\right) \\ -\frac{1}{2}c^2\left(B_y^- + B_y^+\right) + \frac{1}{2}c\left(E_z^- - E_z^+\right) \\ 0 \\ -\frac{1}{2}\left(E_z^- + E_z^+\right) + \frac{1}{2}c\left(B_y^- - B_y^+\right) \\ \frac{1}{2}\left(E_y^- + E_y^+\right) + \frac{1}{2}c\left(B_z^- - B_z^+\right) \end{pmatrix} \tag{25}$$

The assumption of 1D is effectively calculating $(\boldsymbol{F}\cdot\boldsymbol{n})^*$. In index form, this can be written

$$\left(F_{ij}n_j\right)^*_{(\lambda)\boldsymbol{E}} = \left[\frac{1}{2}\left(-\epsilon_{ijk}c^2\left(B_k^{(\lambda)} + B_k^{(\mu)}\right)\right)n_j + \frac{cn_j}{2}\epsilon_{ijk}\epsilon_{klm}\left(E_l^{(\lambda)} - E_l^{(\mu)}\right)n_m\right], \tag{26}$$

$$\left(F_{ij}n_j\right)^*_{(\lambda)\boldsymbol{B}} = \left[\frac{1}{2}\left(\epsilon_{ijk}\left(E_k^{(\lambda)} + E_k^{(\mu)}\right)\right)n_j + \frac{cn_j}{2}\epsilon_{ijk}\epsilon_{klm}\left(B_l^{(\lambda)} - B_l^{(\mu)}\right)n_m\right], \tag{27}$$

where $c = (\omega_p\tau)\left(\frac{\delta_p}{L}\right)$ is the speed of light. $\left(F_{ij}n_j\right)^*_{(\lambda)\boldsymbol{E}}$ refers to the numerical flux on the evolution of $\boldsymbol{E}$ (Ampère's Law) where the analytical flux is $F_{ij} = -\epsilon_{ijk}c^2B_k$. $\left(F_{ij}n_j\right)^*_{(\lambda)\boldsymbol{B}}$ refers to the numerical flux on the evolution of $\boldsymbol{B}$ (Faraday's Law) where the analytical flux is $F_{ij} = \epsilon_{ijk}E_k$. λ

refers to "inside" element node and μ refers to "outside" element node. In WARPXM notation we signify the inside element as λ and outside element as γ and the wall node associated with each with another letter index, which becomes k. So:

$$\tilde{\mathcal{F}}^{\lambda\gamma}_{iq_{\boldsymbol{E}}} \equiv \left(\hat{\mathcal{F}}_{ij}n_j\right)^{\lambda\gamma}_{q_{\boldsymbol{E}}} = \left[\frac{1}{2}\left(-\epsilon_{ijk}c^2\left(B^{\lambda}_{kq}+B^{\gamma}_{kq}\right)\right)n_j + \frac{cn_j}{2}\epsilon_{ijk}\epsilon_{klm}\left(E^{\lambda}_{lq}-E^{\gamma}_{lq}\right)n_m\right],$$
$$\tilde{\mathcal{F}}^{\lambda\gamma}_{iq_{\boldsymbol{B}}} \equiv \left(\hat{\mathcal{F}}_{ij}n_j\right)^{\lambda\gamma}_{q_{\boldsymbol{B}}} = \left[\frac{1}{2}\left(\epsilon_{ijk}\left(E^{\lambda}_{kq}+E^{\gamma}_{kq}\right)\right)n_j + \frac{cn_j}{2}\epsilon_{ijk}\epsilon_{klm}\left(B^{\lambda}_{lq}-B^{\gamma}_{lq}\right)n_m\right]$$

We then just swap $j \leftrightarrow m$ and $k \leftrightarrow q$ to get

$$\tilde{\mathcal{F}}^{\lambda\gamma}_{ik_{\boldsymbol{E}}} \equiv \left(\hat{\mathcal{F}}_{im}n_m\right)^{\lambda\gamma}_{k_{\boldsymbol{E}}} = \left[\frac{1}{2}\left(-\epsilon_{imq}c^2\left(B^{\lambda}_{qk}+B^{\gamma}_{qk}\right)\right)n_m + \frac{cn_m}{2}\epsilon_{imq}\epsilon_{qlj}\left(E^{\lambda}_{lk}-E^{\gamma}_{lk}\right)n_j\right], \tag{28}$$
$$\tilde{\mathcal{F}}^{\lambda\gamma}_{ik_{\boldsymbol{B}}} \equiv \left(\hat{\mathcal{F}}_{im}n_m\right)^{\lambda\gamma}_{k_{\boldsymbol{B}}} = \left[\frac{1}{2}\left(\epsilon_{imq}\left(E^{\lambda}_{qk}+E^{\gamma}_{qk}\right)\right)n_m + \frac{cn_m}{2}\epsilon_{imq}\epsilon_{qlj}\left(B^{\lambda}_{lk}-B^{\gamma}_{lk}\right)n_j\right] \tag{29}$$

Now note the second term:

$$\begin{aligned}
&\epsilon_{imq}\epsilon_{qlj}n_m q_{lk} n_j \\
=&\epsilon_{imq}\epsilon_{ljq}n_m q_{lk} n_j \\
=&\left(\delta_{il}\delta_{mj}-\delta_{ij}\delta_{ml}\right)n_m q_{lk} n_j \\
=&n_m q_{ik} n_m - n_m q_{mk} n_i
\end{aligned} \tag{30}$$

if $n = n_x$ then (i is component):

$$= \begin{pmatrix} q_{xk} - q_{xk} = 0 \\ q_{yk} - 0 = q_{yk} \\ q_{zk} - 0 = q_{zk} \end{pmatrix} \tag{31}$$

Also the first term

$$\epsilon_{imq}q_{qk}n_m = \begin{pmatrix} 0 \\ -q_{zk} \\ q_{yk} \end{pmatrix} \tag{32}$$

This matches equation 25.

3 Application to Perfectly Hyperbolic Maxwell's Equations

The perfectly hyperbolic Maxwell Equations [2] with our normalization is

$$\frac{\partial \boldsymbol{E}}{\partial t} - (\omega_p\tau)^2\left(\frac{\delta_p}{L}\right)^2 \nabla\times\boldsymbol{B} + \chi(\omega_p\tau)^2\left(\frac{\delta_p}{L}\right)^2\nabla\Phi = -(\omega_p\tau)^2\left(\frac{\delta_p}{L}\right)\sum_\alpha \frac{Z_\alpha\rho_\alpha}{A_\alpha}\boldsymbol{u}_\alpha \tag{33}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} + \gamma \nabla \Psi = 0 \tag{34}$$

$$\frac{\partial \Phi}{\partial t} + \chi \nabla \cdot \boldsymbol{E} = \chi \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right) \rho_c \tag{35}$$

$$\frac{\partial \Psi}{\partial t} + \gamma \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 \nabla \cdot \boldsymbol{B} = 0 \tag{36}$$

So this means we have

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial x_j} \left[-\epsilon_{ijk} \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 B_k + \chi \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 \Phi \delta_{ij} \right] = - \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right) \sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha} u_{\alpha_i} \tag{37}$$

and

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} \left[\epsilon_{ijk} E_k + \gamma \Psi \delta_{ij} \right] = 0 \tag{38}$$

$$\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x_j} \left[\chi E_j \right] = \chi \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right) \rho_c \tag{39}$$

$$\frac{\partial \Psi}{\partial t} + \frac{\partial}{\partial x_j} \left[\gamma \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 B_j \right] = 0 \tag{40}$$

That is

$$\mathcal{F}_{i=\boldsymbol{E}j} = -\epsilon_{ijk} \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 B_k + \chi \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 \Phi \delta_{ij} \tag{41}$$

$$\mathcal{F}_{i=\boldsymbol{B}j} = \epsilon_{ijk} E_k + \gamma \Psi \delta_{ij} \tag{42}$$

$$\mathcal{F}_{i=\Phi j} = \chi E_j \tag{43}$$

$$\mathcal{F}_{i=\Psi j} = \gamma \left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 B_j \tag{44}$$

So for these perfectly hyperbolic Maxwell's equations,

$$\boldsymbol{q} = \begin{bmatrix} E_x & E_y & E_z & B_x & B_y & B_z & \Phi & \Psi \end{bmatrix}^T \tag{45}$$

so we write $\boldsymbol{F}$

$$\boldsymbol{F} = \begin{pmatrix} -c^2\epsilon_{i1k}B_k + \chi c^2\Phi\delta_{i1} \\ \epsilon_{i1k}E_k + \gamma\Psi\delta_{i1} \\ \chi E_1 \\ \gamma c^2 B_1 \end{pmatrix} = \begin{pmatrix} \chi c^2\Phi \\ c^2 B_z \\ -c^2 B_y \\ \gamma\Psi \\ -E_z \\ E_y \\ \chi E_x \\ \gamma c^2 B_x \end{pmatrix} \tag{46}$$

$$\boldsymbol{G} = \begin{pmatrix} -c^2\epsilon_{i2k}B_k + \chi c^2\Phi\delta_{i2} \\ \epsilon_{i2k}E_k + \gamma\Psi\delta_{i2} \\ \chi E_2 \\ \gamma c^2 B_2 \end{pmatrix} = \begin{pmatrix} -c^2 B_z \\ \chi c^2\Phi \\ c^2 B_x \\ E_z \\ \gamma\Psi \\ -E_x \\ \chi E_y \\ \gamma c^2 B_y \end{pmatrix} \tag{47}$$

$$\boldsymbol{H} = \begin{pmatrix} -c^2\epsilon_{i3k}B_k + \chi c^2\Phi\delta_{i23} \\ \epsilon_{i3k}E_k + \gamma\Psi\delta_{i3} \\ \chi E_3 \\ \gamma c^2 B_3 \end{pmatrix} = \begin{pmatrix} c^2 B_y \\ -c^2 B_x \\ \chi c^2\Phi \\ -E_y \\ E_x \\ \gamma\Psi \\ \chi E_z \\ \gamma c^2 B_z \end{pmatrix} \tag{48}$$

with $\frac{c}{v_0} = (\omega_p\tau)\left(\frac{\delta_p}{L}\right)$. For ease of notation here assume $v_0 = 1$. So we can calculate the flux jacobian:

$$\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{q}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \chi c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma c^2 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{49}$$

Solving the eigenvalue problem, the matrix of eigenvalues is

$$\overline{\overline{\Lambda}} = \begin{pmatrix} -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c\chi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\chi \end{pmatrix} \tag{50}$$

also

$$\left|\overline{\overline{\Lambda}}\right| = \begin{pmatrix} c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c\chi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\chi \end{pmatrix} \tag{51}$$

and the matrix of column right eigenvectors is

$$\overline{\overline{X}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -c & c \\ -c & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{c} & \frac{1}{c} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \tag{52}$$

and its inverse

$$\overline{\overline{X}}^{-1} = \begin{pmatrix} 0 & -\frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2c} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2c} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{c}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2c} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2c} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{53}$$

We now have our numerical flux for Maxwell's equations:

$$
\begin{aligned}
\boldsymbol{F}^* =& \frac{1}{2}\left(\boldsymbol{F}^- + \boldsymbol{F}^+\right) + \frac{1}{2}\left(\overline{\overline{X}}\left|\overline{\overline{\Lambda}}\right|\overline{\overline{X}}^{-1}\right)\left(\boldsymbol{q}^- - \boldsymbol{q}^+\right) \\
=& \begin{pmatrix} \frac{1}{2}\left[\left(-c^2\epsilon_{i1k}B_k + \chi c^2\Phi\delta_{i1}\right)^- + \left(-c^2\epsilon_{i1k}B_k + \chi c^2\Phi\delta_{i1}\right)^+\right] \\ \frac{1}{2}\left[\left(\epsilon_{i1k}E_k + \gamma\Psi\delta_{i1}\right)^- + \left(\epsilon_{i1k}E_k + \gamma\Psi\delta_{i1}\right)^+\right] \\ \frac{1}{2}\left[\left(\chi E_1\right)^- + \left(\chi E_1\right)^+\right] \\ \frac{1}{2}\left[\left(\gamma c^2 B_1\right)^- + \left(\gamma c^2 B_1\right)^+\right] \end{pmatrix} \\
&+\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -c & c \\ -c & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{c} & \frac{1}{c} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}
\begin{pmatrix} c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c\chi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\chi \end{pmatrix}
\begin{pmatrix} 0 & -\frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2c} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2c} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2c} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{c}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2c} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2c} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \\
&\times\left(\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \\ \Phi \\ \Psi \end{pmatrix}^- - \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \\ \Phi \\ \Psi \end{pmatrix}^+\right) \\
=& \frac{1}{2}\begin{pmatrix} \chi c^2\left(\Phi^- + \Phi^+\right) \\ c^2\left(B_z^- + B_z^+\right) \\ -c^2\left(B_y^- + B_y^+\right) \\ \gamma\left(\Psi^- + \Psi^+\right) \\ -\left(E_z^- + E_z^+\right) \\ \left(E_y^- + E_y^+\right) \\ \chi\left(E_x^- + E_x^+\right) \\ \gamma c^2\left(B_x^- + B_x^+\right) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} c\chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c\chi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\gamma \end{pmatrix}\left(\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \\ \Phi \\ \Psi \end{pmatrix}^- - \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \\ \Phi \\ \Psi \end{pmatrix}^+\right) \\
=& \frac{1}{2}\begin{pmatrix} \chi c^2\left(\Phi^- + \Phi^+\right) \\ c^2\left(B_z^- + B_z^+\right) \\ -c^2\left(B_y^- + B_y^+\right) \\ \gamma\left(\Psi^- + \Psi^+\right) \\ -\left(E_z^- + E_z^+\right) \\ \left(E_y^- + E_y^+\right) \\ \chi\left(E_x^- + E_x^+\right) \\ \gamma c^2\left(B_x^- + B_x^+\right) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} c\chi\left(E_x^- - E_x^+\right) \\ c\left(E_y^- - E_y^+\right) \\ c\left(E_z^- - E_z^+\right) \\ c\gamma\left(B_x^- - B_x^+\right) \\ c\left(B_y^- - B_y^+\right) \\ c\left(B_z^- - B_z^+\right) \\ c\chi\left(\Phi^- - \Phi^+\right) \\ c\gamma\left(\Psi^- - \Psi^+\right) \end{pmatrix} \\
=& \begin{pmatrix} \frac{1}{2}\chi c^2\left(\Phi^- + \Phi^+\right) + \frac{1}{2}\chi c\left(E_x^- - E_x^+\right) \\ \frac{1}{2}c^2\left(B_z^- + B_z^+\right) + \frac{1}{2}c\left(E_y^- - E_y^+\right) \\ -\frac{1}{2}c^2\left(B_y^- + B_y^+\right) + \frac{1}{2}c\left(E_z^- - E_z^+\right) \\ \frac{1}{2}\gamma\left(\Psi^- + \Psi^+\right) + \frac{1}{2}\gamma c\left(B_x^- - B_x^+\right) \\ -\frac{1}{2}\left(E_z^- + E_z^+\right) + \frac{1}{2}c\left(B_y^- - B_y^+\right) \\ \frac{1}{2}\left(E_y^- + E_y^+\right) + \frac{1}{2}c\left(B_z^- - B_z^+\right) \\ \frac{1}{2}\chi\left(E_x^- + E_x^+\right) + \frac{1}{2}\chi c\left(\Phi^- - \Phi^+\right) \\ \frac{1}{2}\gamma c^2\left(B_x^- + B_x^+\right) + \frac{1}{2}\gamma c\left(\Psi^- - \Psi^+\right) \end{pmatrix} \qquad (54)
\end{aligned}
$$

4 Application to Parabolically Cleaned Maxwell's Equations

Parabolically claned Maxwell's equations are designed to locally remove divergence errors using a diffusion operator. The following version is derived following the method as described in Ref. [1].

First take the divergence of Ampere's Law, Eq. (1),

$$\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right)-\left(\omega_p\tau\right)^2\left(\frac{\delta_p}{L}\right)^2\cancel{\nabla\cdot\nabla\times\boldsymbol{B}}=-\left(\omega_p\tau\right)^2\left(\frac{\delta_p}{L}\right)\sum_\alpha\frac{Z_\alpha\rho_\alpha}{A_\alpha}\nabla\cdot\boldsymbol{u}_\alpha. \tag{55}$$

Note the conservation of charge

$$\frac{\partial\rho_c}{\partial t}=-\nabla\cdot\boldsymbol{j}=-\nabla\cdot\sum_\alpha\frac{Z_\alpha\rho_\alpha}{A_\alpha}\cdot\boldsymbol{u}_\alpha=-\sum_\alpha\frac{Z_\alpha\rho_\alpha}{A_\alpha}\nabla\cdot\boldsymbol{u}_\alpha, \tag{56}$$

where $\rho_c=\sum_\alpha\rho_\alpha\frac{Z_\alpha}{A_\alpha}$. Substitution of Eq. (56) into Eq. (55) yields

$$\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right)=\left(\omega_p\tau\right)^2\left(\frac{\delta_p}{L}\right)\frac{\partial\rho_c}{\partial t}. \tag{57}$$

The next step is to formulate a form of Gauss law to clean electric field divergence. Notice the normalized Poisson equation

$$-\frac{1}{\left(\omega_p\tau\right)^2}\left(\frac{L}{\delta_p}\right)\nabla^2\phi=\rho_c \tag{58}$$

and electric field definition

$$\boldsymbol{E}=-\nabla\phi. \tag{59}$$

Guass Law is then

$$\frac{1}{\left(\omega_p\tau\right)^2}\left(\frac{L}{\delta_p}\right)\nabla\cdot\boldsymbol{E}=\rho_c. \tag{60}$$

Let the parabolically-cleaned Gauss law indicate the discrepancy

$$\frac{1}{\left(\omega_p\tau\right)^2}\left(\frac{L}{\delta_p}\right)\nabla\cdot\boldsymbol{E}-\rho_c\equiv\Psi_E. \tag{61}$$

Equation (61) allows Eq. (57) to be written

$$\frac{1}{\left(\omega_p\tau\right)^2}\left(\frac{L}{\delta_p}\right)\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right)-\frac{\partial\rho_c}{\partial t}=\frac{\partial\Psi_E}{\partial t}=0. \tag{62}$$

Equation (62) shows the divergence error should be constant, at least analytically, though this may not be the case numerically. Equation (62) can then be modified into a diffusion equation as

$$\frac{\partial \Psi_E}{\partial t} \equiv \nabla \cdot \left(\chi_E \nabla \Psi_E\right). \tag{63}$$

Equation (63) can be worked from backward to achieve a modified Ampere's Law

$$\begin{aligned}
\frac{1}{\left(\omega_p \tau\right)^2}\left(\frac{L}{\delta_p}\right)\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right) - \frac{\partial \rho_c}{\partial t} =& \nabla\cdot\left(\chi_E \nabla \Psi_E\right)\\
\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right) - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\frac{\partial \rho_c}{\partial t} =& \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\nabla\cdot\left(\chi_E \nabla \Psi_E\right)\\
\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right) - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\frac{\partial \rho_c}{\partial t} - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)^2\nabla\cdot\nabla\times\boldsymbol{B} =& \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\nabla\cdot\left(\chi_E \nabla \Psi_E\right)\\
\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right) + \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha}\nabla\cdot\boldsymbol{u}_\alpha - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)^2\nabla\cdot\nabla\times\boldsymbol{B} =& \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\nabla\cdot\left(\chi_E \nabla \Psi_E\right)\\
\frac{\partial \boldsymbol{E}}{\partial t} + \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha}\boldsymbol{u}_\alpha - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)^2\nabla\times\boldsymbol{B} =& \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\chi_E \nabla \Psi_E\\
\frac{\partial \boldsymbol{E}}{\partial t} + \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha}\boldsymbol{u}_\alpha - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)^2\nabla\times\boldsymbol{B} =& \chi_E \nabla\left[\nabla\cdot\boldsymbol{E} - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\rho_c\right]
\end{aligned}$$

$$\boxed{\frac{\partial \boldsymbol{E}}{\partial t} - \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)^2\nabla\times\boldsymbol{B} - \chi_E\nabla\nabla\cdot\boldsymbol{E} + \left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\chi_E\nabla\rho_c = -\left(\omega_p \tau\right)^2\left(\frac{\delta_p}{L}\right)\sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha}\boldsymbol{u}_\alpha}. \tag{64}$$

The same procedure can be applied to Faraday's Law. Taking the divergence of Eq. (2) yields

$$\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{B}\right) + \cancel{\nabla\cdot\nabla\times\boldsymbol{E}} = 0. \tag{65}$$

Note the magnetic Gauss law

$$\nabla\cdot\boldsymbol{B} = 0. \tag{66}$$

Similarly to Eq. (61), let the parabolically-cleaned magnetic Gauss law indicate the discrepancy

$$\nabla\cdot\boldsymbol{B} \equiv \Psi_B. \tag{67}$$

Equation (65) can thus be written

$$\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{B}\right) = \frac{\partial \Psi_B}{\partial t} = 0. \tag{68}$$

As shown in Eq. (62) the divergence error should be constant, at least analytically, though this may not be the case numerically. Equation (68) can then be modified into a diffusion equation as

$$\frac{\partial \Psi_B}{\partial t} \equiv \nabla \cdot \left(\chi_B \nabla \Psi_B\right). \tag{69}$$

As before, Eq. (69) can be worked from backward to achieve a modified Faraday's Law

$$\begin{aligned} \frac{\partial}{\partial t}\left(\nabla \cdot \boldsymbol{B}\right) =& \nabla \cdot \left(\chi_B \nabla \Psi_B\right) \\ \frac{\partial}{\partial t}\left(\nabla \cdot \boldsymbol{B}\right) + \nabla \cdot \nabla \times \boldsymbol{E} =& \nabla \cdot \left(\chi_B \nabla \nabla \cdot \boldsymbol{B}\right) \\ & \boxed{\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = \chi_B \nabla \nabla \cdot \boldsymbol{B}}. \end{aligned} \tag{70}$$

Equations (64) and (70) are the PCMaxwell equations, with the red terms indicating the additional terms that the model adds to Maxwell's equations. χ_E and χ_B are constant diffusivities. In index notation, these equations can be written

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial x_j}\left[-\epsilon_{ijk}\left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)^2 B_k + \chi_E \left\{\left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)\rho_c - \frac{\partial E_k}{\partial x_k}\right\}\delta_{ij}\right] = -\left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right) \sum_\alpha \frac{Z_\alpha \rho_\alpha}{A_\alpha} u_{\alpha_i}, \tag{71}$$

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j}\left[\epsilon_{ijk} E_k - \chi_B \frac{\partial B_k}{\partial x_k}\delta_{ij}\right] = 0. \tag{72}$$

For PCMaxwell's equations, we write $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ can be written using equations 71 and 72:

$$\boldsymbol{F} = \begin{pmatrix} -c^2 \epsilon_{i1k} B_k + \chi_E \left\{\left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)\rho_c - \frac{\partial E_k}{\partial x_k}\right\}\delta_{i1} \\ \epsilon_{i1k} E_k - \chi_B \frac{\partial B_k}{\partial x_k}\delta_{i1} \end{pmatrix} = \begin{pmatrix} \chi_E \left\{\left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)\rho_c - \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\right\} \\ c^2 B_z \\ -c^2 B_y \\ -\chi_B \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) \\ -E_z \\ E_y \end{pmatrix} \tag{73}$$

$$\boldsymbol{G} = \begin{pmatrix} -c^2 \epsilon_{i2k} B_k + \chi_E \left\{\left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)\rho_c - \frac{\partial E_k}{\partial x_k}\right\}\delta_{i2} \\ \epsilon_{i2k} E_k - \chi_B \frac{\partial B_k}{\partial x_k}\delta_{i2} \end{pmatrix} = \begin{pmatrix} -c^2 B_z \\ \chi_E \left\{\left(\omega_p \tau\right)^2 \left(\frac{\delta_p}{L}\right)\rho_c - \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\right\} \\ c^2 B_x \\ E_z \\ -\chi_B \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) \\ -E_x \end{pmatrix} \tag{74}$$

$$\boldsymbol{H} = \begin{pmatrix} -c^2\epsilon_{i3k}B_k + \chi_E\left\{(\omega_p\tau)^2\left(\frac{\delta_p}{L}\right)\rho_c - \frac{\partial E_k}{\partial x_k}\right\}\delta_{i3} \\ \epsilon_{i3k}E_k - \chi_B\frac{\partial B_k}{\partial x_k}\delta_{i3} \end{pmatrix} = \begin{pmatrix} c^2 B_y \\ -c^2 B_x \\ \chi_E\left\{(\omega_p\tau)^2\left(\frac{\delta_p}{L}\right)\rho_c - \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\right\} \\ -E_y \\ E_x \\ -\chi_B\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) \end{pmatrix} \tag{75}$$

So just use LDG or IP on field gradient terms and an average for ρ_c terms?

References

[1] Sean Miller. *Modeling collisional processes in plasmas using discontinuous numerical methods*. PhD thesis, University of Washington, Seattle, WA, 2016.

[2] C.-D. Munz, P. Ommes, and R. Schneider. A three-dimensional finite-volume solver for the maxwell equations with divergence cleaning on unstructured meshes. *Computer Physics Communications*, 1-2(130):83–117, 7 2000.