Hall MHD Formulation

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March 28, 2024

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The Hall MHD Model used in WARPXM is given by Uri's model given in Ref. [1], given by Eqs. (30), (53), (54), (60), (61), and (9) (see Pg. 17). In the code, we reformulate the energy equation, Eq. (60) and Faraday Law, Eq. (9) as follows. First take the energy equation as formulated in Eq. (55),

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left((\varepsilon + p)\, \boldsymbol{v} + \frac{\gamma}{\gamma - 1} p_e \boldsymbol{w}_e \right) = (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{E}$$

Where by Eqs. (56) and (57)

$$\boldsymbol{w}_e = -\frac{\boldsymbol{J}}{n_e} = -\left(\frac{\delta_p}{L}\right) \frac{\nabla \times \boldsymbol{B}}{n_e}$$

Then using Eq. (54) for Ohm's Law, given by

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} - \boldsymbol{w}_e \times \boldsymbol{B} - \left(\frac{\delta_p}{L}\right) \frac{\nabla p_e}{n_e} \tag{1}$$

where the Hall terms are considered the blue terms, and calculating Eq. (59) without the additional identities, one achieves

$$\boldsymbol{E} \times \boldsymbol{B} = -\boldsymbol{v} \times \boldsymbol{B} \times \boldsymbol{B} - \boldsymbol{w}_e \times \boldsymbol{B} \times \boldsymbol{B} - \left(\frac{\delta_p}{L}\right) \frac{\nabla p_e}{n_e} \times \boldsymbol{B}$$

Substitution of this expression into Eq. (58)

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right) = -\nabla \times \boldsymbol{B} \cdot \boldsymbol{E} - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B})$$

and adding to Eq. (55) and using $e = \varepsilon + \frac{B^2}{2}$

$$\frac{\partial e}{\partial t} + \nabla \cdot \left((\varepsilon + p)\, \boldsymbol{v} + \frac{\gamma}{\gamma - 1} p_e \boldsymbol{w}_e \right) + \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = 0$$

or

$$\frac{\partial e}{\partial t}+\nabla\cdot\left(\left(\varepsilon+p\right)\boldsymbol{v}+\frac{\gamma}{\gamma-1}p_e\boldsymbol{w}_e\right)+\nabla\cdot\left(-\left(\boldsymbol{v}\times\boldsymbol{B}\right)\times\boldsymbol{B}-\left(\boldsymbol{w}_e\times\boldsymbol{B}\right)\times\boldsymbol{B}-\left(\frac{\delta_p}{L}\right)\frac{\nabla p_e}{n_e}\times\boldsymbol{B}\right)=0 \tag{2}$$

where again the blue terms are considered Hall terms and the rest are the ideal terms. One can arrive at the formulation given in Eq. (60) using vector identities on the vector triple product and as assuming $A_e\to 0$ as assumed in Ref. [1], $\varepsilon_e = p_e/\left(\gamma-1\right)$ as well as the assume total MHD energy definition, $e_e=\varepsilon_e+B^2/2$. This leads to $\frac{\gamma}{\gamma-1}p_e=\left(1+\frac{1}{\gamma-1}\right)p_e=p_e+\varepsilon_e$, and

$$\frac{\partial e}{\partial t}+\nabla\cdot\left(\left(\varepsilon+p+B^2\right)\boldsymbol{v}-\left(\boldsymbol{B}\cdot\boldsymbol{v}\right)\boldsymbol{B}\right)+\nabla\cdot\left(\frac{\gamma}{\gamma-1}p_e\boldsymbol{w}_e\right)+\nabla\cdot\left(B^2\boldsymbol{w}_e-\left(\boldsymbol{B}\cdot\boldsymbol{w}_e\right)\boldsymbol{B}-\left(\frac{\delta_p}{L}\right)\frac{\nabla p_e}{n_e}\times\boldsymbol{B}\right)=0$$

or

$$\frac{\partial e}{\partial t}+\nabla\cdot\left(\left(e+p+\frac{B^2}{2}\right)\boldsymbol{v}-\left(\boldsymbol{B}\cdot\boldsymbol{v}\right)\boldsymbol{B}\right)+\nabla\cdot\left(\left(e_e+p_e+\frac{B^2}{2}\right)\boldsymbol{w}_e-\left(\boldsymbol{B}\cdot\boldsymbol{w}_e\right)\boldsymbol{B}-\left(\frac{\delta_p}{L}\right)\frac{\nabla p_e}{n_e}\times\boldsymbol{B}\right)=0$$

Note that in Ref. [1], the sign of the $\left(\boldsymbol{B}\cdot\boldsymbol{v}\right)\boldsymbol{B}$ is incorrect.

The Warpxm formulation for the Hall terms is given in Eqs. (1) and (2). In the code "v_hall" refers to $\boldsymbol{w}_e$. There are three components then in Hall MHD that affect Eqs. (1) and (2):

1. drift_pressure_energy_term, $\frac{\gamma}{\gamma-1}p_e\boldsymbol{w}_e$ for Eq. (2)
2. (a) J_cross_B_energy_term, $-\left(\boldsymbol{w}_e\times\boldsymbol{B}\right)\times\boldsymbol{B}$ for Eq. (2)
   (b) J_cross_B_electric_field, $-\boldsymbol{w}_e\times\boldsymbol{B}$ for Eq. (1)
3. (a) grad_pe_energy_term, $-\left(\frac{\delta_p}{L}\right)\frac{\nabla p_e}{n_e}\times\boldsymbol{B}$ for Eq. (2)
   (b) diamagnetic_electric_field, $-\left(\frac{\delta_p}{L}\right)\frac{\nabla p_e}{n_e}$ for for Eq. (1)

The strength of each of these terms can be tuned manually, if desired.

References

[1] U. Shumlak. Extensible Normalization for Multi-Species Plasma Models. Reference Document, July 2016. Normalization scheme developed by Uri.