MHD Conducting Wall Boundary Conditions on Curved Boundary

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1 Adjustment of conducting wall BC for curved physical boundaries

In 3D MHD we get numerical instabilities due to the inexact resolution of the curved boundary. A possible solution in proposed in Ref. [1], where in the context of a neutral fluid, the normal velocity boundary condition,

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \tag{1}$$

where $\boldsymbol{v}$ is the velocity and $\boldsymbol{n}$ is the element normal. However, on curved boundaries, the approximation of a simplex element of the curvature of the actual physical boundary being modeled may not be so good. Really, we are trying to enforce

$$\boldsymbol{v} \cdot \boldsymbol{N} = 0 \tag{2}$$

where $\boldsymbol{N}$ is the actual normal of the physical boundary being modeled. Reference [1] provides a possible solution to apply Eq. (2) despite using the simplex elements. We try to adopt that formulation for the magnetic field boundary condition, thus similarly adjusting the conducting wall boundary condition using the element normal

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \tag{3}$$

to

$$\boldsymbol{B} \cdot \boldsymbol{N} = 0. \tag{4}$$

For the moment, assume at every node on a boundary, $\boldsymbol{n}$ and $\boldsymbol{N}$ are known. Then, applying Algorithm 2 in Ref. [1], which the author uses in shown results is determined in the following manner, where we consider $\boldsymbol{B}$ in instead of $\boldsymbol{u}$.

Assume

$$\boldsymbol{N} = N_1 \hat{\boldsymbol{x}} + N_2 \hat{\boldsymbol{y}}. \tag{5}$$

Then the physical surface tangent to the normal is given by

$$\begin{aligned}
\boldsymbol{N}\cdot\boldsymbol{T} =& 0\\
N_1T_1 + N_2T_2 =& 0\\
T_2 =& -\frac{N_1T_1}{N_2}\\
\therefore&\\
\boldsymbol{T} =& T_1\hat{\boldsymbol{x}} - \frac{N_1T_1}{N_2}\hat{\boldsymbol{y}}.
\end{aligned} \tag{6}$$

You can multiply through by N_2/T_1 to get a form with the same magnitude as $\boldsymbol{N}$:

$$\boldsymbol{T} = N_2\hat{\boldsymbol{x}} - N_1\hat{\boldsymbol{y}}. \tag{7}$$

Now going with Algorithm 2, and using our notation that "-" refers to the inside value and "+" refers to the outside value at an element boundary (in this case call "+" as "g" as it is the ghost value at the boundary), we can say the physical normal and tangential components of the magnetic field at the element boundary are

$$\boldsymbol{B}^-\cdot\boldsymbol{N} = B_x^-N_1 + B_y^-N_2 \tag{8}$$

and

$$\boldsymbol{B}^-\cdot\boldsymbol{T} = B_x^-N_2 - B_y^-N_1 \tag{9}$$

where $\boldsymbol{B}^- \equiv B_x^-\hat{\boldsymbol{x}} + B_y^-\hat{\boldsymbol{x}}$ is the inside magnetic field, where we are going go assume we are in the frame of the element normal $\boldsymbol{n}$, and $\boldsymbol{N}$ is described relative to that. The boundary condition proposed in Ref [1] is gien by

$$\boldsymbol{B}^g\cdot\boldsymbol{N} = -\boldsymbol{B}^-\cdot\boldsymbol{N} \tag{10}$$

and

$$\boldsymbol{B}^g\cdot\boldsymbol{T} = \boldsymbol{B}^-\cdot\boldsymbol{T}. \tag{11}$$

Substitution of Eqs. (8) and (9) into (10) and (11) using Eq. (7) yields a 2x2 system:

$$\begin{aligned}
B_x^gN_1 + B_y^gN_2 =& -B_x^-N_1 - B_y^-N_2\\
B_x^gN_2 - B_y^gN_1 =& B_x^-N_2 - B_y^-N_1\\
\begin{bmatrix} N_1 & N_2\\ N_2 & -N_1\end{bmatrix}\begin{bmatrix} B_x^g\\ B_y^g\end{bmatrix} =& \begin{bmatrix} -B_x^-N_1 - B_y^-N_2\\ B_x^-N_2 - B_y^-N_1\end{bmatrix}\\
\begin{bmatrix} B_x^g\\ B_y^g\end{bmatrix} =& \begin{bmatrix} N_1 & N_2\\ N_2 & -N_1\end{bmatrix}^{-1}\begin{bmatrix} -B_x^-N_1 - B_y^-N_2\\ B_x^-N_2 - B_y^-N_1\end{bmatrix}\\
\begin{bmatrix} B_x^g\\ B_y^g\end{bmatrix} =& \frac{1}{N_1^2+N_2^2}\begin{bmatrix} \left(N_2^2-N_1^2\right)B_x^- - 2N_1N_2B_y^-\\ \left(N_1^2-N_2^2\right)B_y^- - 2N_1N_2B_x^-\end{bmatrix}
\end{aligned} \tag{12}$$

1.1 Determination of $\boldsymbol{N}$

Reference [1] describes this in the general case which I'm not certain on. But in the circular arc case with origin at 0, you can simply say

$$\theta = \text{arctan2}(y/x) \tag{13}$$

and

$$\boldsymbol{N} = \hat{\boldsymbol{r}} = \cos\theta\hat{\boldsymbol{x}} + \sin\theta\hat{\boldsymbol{y}}. \tag{14}$$

1.2 Modification for case of initial $\boldsymbol{B}\cdot\boldsymbol{n}$

Technically, conducting walls have a boundary condition of

$$\frac{\partial\boldsymbol{B}}{\partial t}\cdot\boldsymbol{n} = 0. \tag{15}$$

That means we have initial a finite $\boldsymbol{B}\cdot\boldsymbol{n}$ initially, we have a modified version of Eqs. (3) and (4) to be instead

$$\boldsymbol{B}\cdot\boldsymbol{n} = \boldsymbol{B}^{\text{ic}}\cdot\boldsymbol{n}, \tag{16}$$

which is modified to

$$\boldsymbol{B}\cdot\boldsymbol{N} = \boldsymbol{B}^{\text{ic}}\cdot\boldsymbol{N}. \tag{17}$$

There, in this case, assuming the boundary condition is given by $\boldsymbol{B}^{\text{ic}}\cdot\boldsymbol{N} = \frac{1}{2}\left(\boldsymbol{B}^{-}\cdot\boldsymbol{N} + \boldsymbol{B}^{g}\cdot\boldsymbol{N}\right)$, the boundary condition the normal component of magnetic field becomes a modified version of Eqs (10)

$$\boldsymbol{B}^{g}\cdot\boldsymbol{N} = 2\boldsymbol{B}^{\text{ic}}\cdot\boldsymbol{N} - \boldsymbol{B}^{-}\cdot\boldsymbol{N}. \tag{18}$$

Combining Eq. (18) with Eq. (11) and following the same derivation as in Sec. 1, substituting Eqs. (8) and (9), and applying Eq. (7), yields

$$\begin{aligned}
B_x^g N_1 + B_y^g N_2 =& 2B_x^{\text{ic}} N_1 + 2B_y^{\text{ic}} N_2 - B_x^- N_1 - B_y^- N_2 \\
B_x^g N_2 - B_y^g N_1 =& B_x^- N_2 - B_y^- N_1 \\
\begin{bmatrix} N_1 & N_2 \\ N_2 & -N_1 \end{bmatrix}\begin{bmatrix} B_x^g \\ B_y^g \end{bmatrix} =& \begin{bmatrix} \left(2B_x^{\text{ic}} - B_x^-\right) N_1 + \left(2B_y^{\text{ic}} - B_y^-\right) N_2 \\ B_x^- N_2 - B_y^- N_1 \end{bmatrix} \\
\begin{bmatrix} B_x^g \\ B_y^g \end{bmatrix} =& \begin{bmatrix} N_1 & N_2 \\ N_2 & -N_1 \end{bmatrix}^{-1}\begin{bmatrix} \left(2B_x^{\text{ic}} - B_x^-\right) N_1 + \left(2B_y^{\text{ic}} - B_y^-\right) N_2 \\ B_x^- N_2 - B_y^- N_1 \end{bmatrix} \\
\begin{bmatrix} B_x^g \\ B_y^g \end{bmatrix} =& \begin{bmatrix} N_1 & N_2 \\ N_2 & -N_1 \end{bmatrix}^{-1}\begin{bmatrix} \left(2B_x^{\text{ic}} - B_x^-\right) & \left(2B_y^{\text{ic}} - B_y^-\right) \\ -B_y^- & B_x^- \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \\
\begin{bmatrix} B_x^g \\ B_y^g \end{bmatrix} =& \frac{1}{N_1^2 + N_2^2}\begin{bmatrix} \left(N_2^2 - N_1^2\right) B_x^- + 2N_1^2 B_x^{\text{ic}} + 2N_1 N_2 \left(B_y^{\text{ic}} - B_y^-\right) \\ \left(N_1^2 - N_2^2\right) B_y^- + 2N_2^2 B_y^{\text{ic}} + 2N_1 N_2 \left(B_x^{\text{ic}} - B_x^-\right) \end{bmatrix}
\end{aligned} \tag{19}$$

2 A simpler implementation

You can avoid having to use Eqs (12) or Eq. (19), but just rotating from the global frame of the element surface into the frame of the arc normal, $\boldsymbol{N}$ given in Eq. (14), and applying Eq. (10) (or Eq. (18)) and Eq. (11) directly in that frame, and rotating back into global frame. Warpxm has the infrastructure in the function `wxm::dfem::constructTangentAndBinormalFromNormal(N, T, B)` where $\boldsymbol{N}$ is the normal calculated in Eq. (14), $\boldsymbol{T}$ is the tangent vector, and $\boldsymbol{B}$ is the binormal vector.

References

[1] Lilia Krivodonova and Marsha Berger. High-order accurate implementation of solid wall boundary conditions in curved geometries. *Journal of Computational Physics*, 211:482–512, 2006.