Description of collisional effects in WARPXM, including cylindrical source terms

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The model used in WARPXM for collisional effects is documented herein.

Sections 2 and 3 : Calculation of collision frequency is discussed. The frequencies given by Hinton and Braginskii versions for charged particles are shown. Collision frequency for neutral molecules is also addressed.

Section 4.1 : Interspecies collisional effects are presented, including interspecies thermal exchange and friction.

Section 5 : An implementation of unmagnetized intraspecies collisional effects is discussed. The implementation includes thermal conduction and viscosity effects.

Section 6 : A Braginskii-based implementation of intraspecies collisional effects is discussed. Diamagnetic fluxes in momentum and heat are included. In momentum, such fluxes are known as gyroviscous stress.

1 Collision frequency (neutral)

The collision frequency for a neutral gas is

$$\nu = \bar{c}\pi d^2 n, \tag{1}$$

where d is the molecule (or atom) diameter, and n is the number density. The RMS speed is

$$\bar{c} \equiv \left(\frac{8kT}{\pi m}\right)^{1/2}. \tag{2}$$

In WARPXM, a normalized form is used.

$$\nu = \left(\frac{8kT_0}{\pi m}\right)^{1/2} \pi d^2 n_0 \tilde{T}^{1/2} \tilde{n} \tag{3}$$

$$\tilde{\nu} = \nu\tau = \tau \left(\frac{8kT_0}{\pi m}\right)^{1/2} \pi d^2 n_0 \tilde{T}^{1/2} \tilde{n} \tag{4}$$

$$\rightarrow \tilde{\nu} = (\nu_0 \tau)\tilde{T}^{1/2}\tilde{n}, \tag{5}$$

The normalized characterstic collision frequency is

$$\nu_0\tau = \tau \left(\frac{8kT_0}{\pi m}\right)^{1/2} \pi d^2 n_0. \tag{6}$$

For hydrogenic neutral atoms, we can use $d_H \approx d_D \approx d_T \approx 0.24$ nm. For hydrogenic neutral atoms, we can use $d_{H2} \approx d_{D2} \approx d_{T2} \approx 0.27$ nm.

2 Collision frequency (Hinton)

In WARPXM, the following general collision frequency per Hinton [1] is used in calculations of transport coefficients.

$$\nu_{ab}\ (\text{cgs}) = \frac{4}{3\pi^{1/2}} \frac{n_b Z_b^2 4\pi Z_a^2 e^4 \ln\Lambda \left(1 + \frac{m_a}{m_b}\right)}{m_a^2 \left(2\frac{kT_a}{m_a} + 2\frac{kT_b}{m_b}\right)^{3/2}} \tag{7}$$

This expression assumes mass, density, charge, and energy in cgs units. To convert to SI units, a factor of $\approx 8.1 \times 10^{19}$ is required such that

$$\nu_{ab}\ (\text{SI}) = 6.09 \times 10^{19} \frac{n_b Z_b^2 4\pi Z_a^2 e^4 \ln\Lambda \left(1 + \frac{m_a}{m_b}\right)}{m_a^2 \left(2\frac{kT_a}{m_a} + 2\frac{kT_b}{m_b}\right)^{3/2}} \tag{8}$$

Now, following an earlier derivation of a normalized form by I. Datta,

$$\nu_{ab} = \frac{4\pi}{2^{3/2}} \times 6.09 \times 10^{19} \frac{n_0 e^4}{m_p^2 A_a^2 v_0^3} \frac{\tilde{n}_b Z_b^2 Z_a^2 \ln\Lambda \left(1 + \frac{A_a}{A_b}\right)}{\left(\frac{\tilde{T}_a}{A_a} + \frac{\tilde{T}_b}{A_b}\right)^{3/2}} \tag{9}$$

Introducing $\omega_p = (e^2 n_0/\epsilon_0 m_p)^{1/2}$,

$$\nu_{ab} = \frac{4\pi}{2^{3/2}} \times 6.09 \times 10^{19} \frac{(\omega_p\tau)^4 \epsilon_0^2}{A_a^2 \tau^4 v_0^3 n_0} \frac{\tilde{n}_b Z_b^2 Z_a^2 \ln\Lambda \left(1 + \frac{A_a}{A_b}\right)}{\left(\frac{\tilde{T}_a}{A_a} + \frac{\tilde{T}_b}{A_b}\right)^{3/2}}. \tag{10}$$

Substituting $v_0 = L/\tau$, and condensing the leading constants,

$$\nu_{ab} = 2.71 \times 10^{20} \frac{(\omega_p\tau)^4 \epsilon_0^2}{n_0 L^3 \tau} \frac{\tilde{n}_b Z_b^2 Z_a^2 \ln\Lambda \left(1 + \frac{A_a}{A_b}\right)}{A_a^2 \left(\frac{\tilde{T}_a}{A_a} + \frac{\tilde{T}_b}{A_b}\right)^{3/2}}. \tag{11}$$

Defining

$$\nu_p\tau \equiv 2.71 \times 10^{20} \frac{(\omega_p\tau)^4 \epsilon_0^2}{n_0 L^3} \tag{12}$$

we can now write an expression for $\tilde{\nu}_{ab}$, where $\tilde{\nu}_{ab}/\tau = \nu_{ab}$:

$$\tilde{\nu}_{ab} = \nu_p\tau \frac{\tilde{n}_b Z_b^2 Z_a^2 \ln\Lambda \left(1 + \frac{A_a}{A_b}\right)}{A_a^2 \left(\frac{\tilde{T}_a}{A_a} + \frac{\tilde{T}_b}{A_b}\right)^{3/2}}. \tag{13}$$

This expression is equivalent to the one derived by Miller and Datta, but it may be important to keep the ϵ_0 dependence in the definition of $\nu_p\tau$, which was absent in the Miller-Datta result (though Sean does show ν_p with the ϵ_0 dependence in his dissertation). If the speed of light is artificially reduced by enhancing ϵ_0, the change will be properly handled.

If we would like to keep a factor of $2^{1/2}$ in the quantity that $\nu_p\tau$ multiplies (as done by Miller et al. in the WARPXM code), this can be rewritten as

$$\tilde{\nu}_{ab} = (\nu_p\tau)^* \frac{2^{1/2}\tilde{n}_b Z_b^2 Z_a^2 \ln\Lambda \left(1 + \frac{A_a}{A_b}\right)}{A_a^2 \left(\frac{\tilde{T}_a}{A_a} + \frac{\tilde{T}_b}{A_b}\right)^{3/2}}, \tag{14}$$

where

$$(\nu_p\tau)^* \equiv 1.92 \times 10^{20} \frac{(\omega_p\tau)^4 \epsilon_0^2}{n_0 L^3}. \tag{15}$$

3 Collision frequency (Braginskii)

An alternative formulation of collision frequency for WARPXM follows Braginskii [3]. Braginskii gives formulas for τ_e and τ_i. Inverting those, and adjusting such that density is in SI units (m^{-3}), we have

$$\nu_e = 2.857 \times 10^{-12} \ln\Lambda \frac{Z^2 n_i}{T_e^{3/2}} \tag{16}$$

and

$$\nu_i = 3.333 \times 10^{-14} \ln\Lambda \left(\frac{2m_p}{m_i}\right)^{1/2} \frac{Z^4 n_i}{T_i^{3/2}}. \tag{17}$$

Note that twice the square root of the electron-proton mass ratio has been absorbed into the constant in front of ν_e. Restoring this constant gives

$$\nu_e = 3.333 \times 10^{-14} \ln\Lambda \left(\frac{2m_p}{m_e}\right)^{1/2} \frac{2^{1/2} Z^2 n_i}{T_e^{3/2}}. \tag{18}$$

Also notice that we have chosen to use ion density (n_i) to represent the plasma density — Braginskii assumes $n = n_e = Z n_i$. Normalizing the electron collision frequency gives

$$\nu_e = 4.714 \times 10^{-14} \frac{n_0}{T_0^{3/2}} \ln\Lambda \frac{2^{1/2} Z^2 \tilde{n}_i}{A_e^{1/2} \tilde{T}_e^{3/2}} \tag{19}$$

$$\rightarrow \tilde{\nu}_e = \nu_e \tau = (\nu_p\tau) \frac{\ln\Lambda}{10} \frac{2^{1/2} Z^2 \tilde{n}_i}{A_e^{1/2} \tilde{T}_e^{3/2}} \tag{20}$$

where $(\nu_p\tau) \equiv 4.714 \times 10^{-13} n_0 \tau / T_0^{3/2}$ has been used. Note that a factor of 10 has been absorbed into $\nu_p\tau$ such that the factor $\ln\Lambda/10$ (approximately 1) remains in the final expression, and the $\nu_p\tau$ approximately equals the normalized proton collision frequency.

Likewise, normalizing the ion collision frequency gives

$$\nu_i = 4.714 \times 10^{-14} \ln\Lambda \frac{n_0}{T_0^{3/2}} \frac{Z^4 \tilde{n}_i}{A_i^{1/2} \tilde{T}_i^{3/2}}. \tag{21}$$

$$\rightarrow \tilde{\nu}_i = \nu_i \tau = 4.714 \times 10^{-14} \ln\Lambda \frac{n_0 \tau}{T_0^{3/2}} \frac{Z^4 \tilde{n}_i}{A_i^{1/2} \tilde{T}_i^{3/2}}. \tag{22}$$

$$\rightarrow \tilde{\nu}_i = (\nu_p\tau) \frac{\ln\Lambda}{10} \frac{Z^4 \tilde{n}_i}{A_i^{1/2} \tilde{T}_i^{3/2}}. \tag{23}$$

4 Interspecies collisions

4.1 Interspecies collisions (Hinton / Miller)

To include interspecies collisional effects, $\mathbf{R}_{i,e}^{ie}$ and $Q_{i,e}^{ie}$ (following the nomenclature of [2]) must be computed. In WARPXM, a simple form for $\mathbf{R}_{i,e}^{ie}$ is used,

$$\mathbf{R}_e^{ie} = -\mathbf{R}_i^{ie} = -m_e n_e \nu_{ei}(\mathbf{v}_e - \mathbf{v}_i). \tag{24}$$

See also the discussion in work by Miller [4, 5] (Sect. 2.5.3). Normalized, this is (with tildes dropped)

$$\mathbf{R}_e^{ie} = -\mathbf{R}_i^{ie} = -A_e n_e \nu_{ei}(\mathbf{v}_e - \mathbf{v}_i). \tag{25}$$

The collisional exchange between two species that is coded in WARPXM, which follows Miller [4, 5], is

$$Q_\alpha = -\nu_{\alpha\beta} n_\alpha \frac{A_\alpha}{A_\alpha + A_\beta}(3T_{\alpha\beta} - A_\beta \mathbf{v}_{\alpha\beta}^2), \tag{26}$$

where $\mathbf{v}_{\alpha\beta} = \mathbf{v}_\alpha - \mathbf{v}_\beta$, and $T_{\alpha\beta} = T_\alpha - T_\beta$. This is essentially equation (81) of [5]. The first term is thermal exchange, and the second is frictional heating (leading to Ohmic heating). The second term is important if the relative speed is on the order of the thermal speed of species β. For Q_e, this corresponds to relative speed of order the ion thermal speed. For Q_i, this corresponds to relative speed of order the electron thermal speed. Note that in a total energy evolution, the source term associated with the 2^{nd} moment is $Q_\alpha + \mathbf{v}_\alpha \cdot \mathbf{R}_\alpha$. In the small-electron-mass limit, this agrees with Braginskii's result. Note also that the factor of 2/3 in equation (81) of [5] is not included above; in Miller's isotropic energy equation (24), there is a factor of 3/2 that cancels the 2/3.

4.2 Interspecies collisions (Braginskii)

Interspecies collisional effects include friction and thermal exchange ($\mathbf{R}_{i,e}^{ie}$ and $Q_{i,e}^{ie}$, following the nomenclature of [2]).

4.2.1 Braginskii friction

Here, describe a version of Braginskii's frictional force that includes the "correction" part that Braginskii labels "$\mathbf{R}^1$". We will allow for arbitrary magnetization.

Per Braginskii [3], the friction term is

$$\mathbf{R}_e^{ie} = -\mathbf{R}_i^{ie} = \mathbf{R}_u + \mathbf{R}_T. \tag{27}$$

Defining $\mathbf{u} = \mathbf{v}_e - \mathbf{v}_i$, Braginskii defines the two parts as

$$\mathbf{R}_u = -\alpha_{||}\mathbf{u}_{||} - \alpha_\perp \mathbf{u}_\perp + \alpha_\wedge \mathbf{b} \times \mathbf{u}, \tag{28}$$

and

$$\mathbf{R}_T = -\beta_{||}^{uT} \nabla_{||} T_e - \beta_\perp^{uT} \nabla_\perp T_e - \beta_\wedge^{uT} \mathbf{b} \times \nabla T_e, \tag{29}$$

where $\mathbf{b}$ is defined as $\mathbf{B}/|\mathbf{B}|$, and the various coefficients are defined by Braginskii in terms of tabulated constants and the variable $x = \omega_{ce}\tau_e$, where $\tau_e = 1/\nu_e$ as discussed in Section 3.

In axisymmetric Z-pinch modeling in the $r-z$ plane, $\mathbf{B} = B_\theta \hat{\theta}$, and $u_{||} = 0$. Also, parallel (azimuthal) gradients are zero by construction. The above expressions then reduce to

$$\mathbf{R}_u = -\alpha_\perp \mathbf{u}_\perp + \alpha_\wedge \mathbf{b} \times \mathbf{u}, \tag{30}$$

and

$$\mathbf{R}_T = -\beta_\perp^{uT} \nabla_\perp T_e - \beta_\wedge^{uT} \mathbf{b} \times \nabla T_e, \tag{31}$$

As discussed by Ji and Held [**?**], the $\alpha_\wedge$ and $\beta_\perp^{uT}$ coefficients have significant errors. Examining the values of the coefficients in our Z-pinch plasma, it seems that their counterparts $\alpha_\perp$ and $\beta_\wedge^{uT}$ are actually more significant. In our implementation, we keep $\alpha_\perp$, but neglect all the other terms. The neglected terms (especially $\beta_\wedge^{uT}$, an accurate version of which could be included easily per Braginskii) could possibly have some impact on instability growth rates, though we expect it to be minor.

To evaluate the effect of enhanced electron mass on ion-electron friction, let's consider the $\alpha_\perp \mathbf{u}_\perp$ term. In the strong magentization (large x) limit, $\alpha_\perp = m_e n_e \nu_e$, and so $\alpha_\perp \mathbf{u}_\perp = m_e n_e \nu_e \mathbf{u}_\perp$. Obviously, this has units of momentum per unit time. In a scenario with enhanced electron mass, if we would like the momentum exchange time to remain the same, we would like to use a physical ν_e (i.e., based on real electron mass). Thus, we would like to compute ν_e as

$$\rightarrow \tilde{\nu}_e = (\nu_p \tau) \ln\Lambda \frac{2^{1/2} Z^2 \tilde{n}_i}{A_{e,phys.}^{1/2} \tilde{T}_e^{3/2}}, \tag{32}$$

where the physical electron mass, $A_{e,phys.}$, is used.

4.2.2 Braginskii thermal exchange

Per Braginskii [3], the ion electron thermal exchange terms are

$$Q_i = Q_\Delta = \frac{3 m_e}{m_i} n_e \nu_e (T_e - T_i), \tag{33}$$

and

$$Q_e = -\mathbf{R}_e^{ie} \cdot \mathbf{u} - Q_\Delta, \tag{34}$$

To evaluate the effect of enhanced electron mass on thermal exchange, let's consider first the Q_Δ term. The exchange of thermal energy occurs on a time scale set by $\frac{m_e}{m_i}\nu_e$. Thus, we might like to keep this time scale fixed regardless of electron mass choice. That would require

$$\frac{m_e}{m_i}\nu_e \rightarrow \frac{A_{e,phys.}}{A_i}\tilde{\nu}_e, \tag{35}$$

where the arrow indicates conversion to normalized WARPXM notation. Note, however, that in the other term, $-\mathbf{R}_e^{ie} \cdot \mathbf{u}$, the friction force, $\mathbf{R}_e^{ie}$, should be computed as discussed above.

One should be careful when trying to get "physical" effects by using physical electron mass instead of the enhanced mass. In the specific cases discussed above, the approach described above should work fine. But, for example, $\mathbf{R}_T$ may require additional consideration.

Side note: we will not worry about accounting for physical vs. enhanced electron mass in viscosity and heat flux calculations (below). Electron perpendicular thermal conductivity is small

compared to the ion counterpart. Diamagnetic heat flux terms are independent of particle mass in the limit of strong magnetization.

In general, if one of these Braginskii transport terms is thought to be dominant, e.g., electron thermal conduction in the tokamak SOL, it may be particularly important to adjust the transport such that it is at a physical level, despite enhanced electron mass.

5 Intraspecies collisions (unmagnetized)

The full species pressure tensor, $\mathbb{P}$, is decomposed as $\mathbb{P} = p\mathbb{I} + \Pi$, where p is the isotropic scalar pressure, and Π is the stress tensor. The stress tensor is

$$\Pi = -\mu \left(\nabla \mathbf{v} + \nabla \mathbf{v}^{\intercal} - \mathbb{I} \frac{2}{3} \nabla \cdot \mathbf{v} \right) \tag{36}$$

The heat flux vector is $\mathbf{h} = -\kappa \nabla T$.

In cylindrical coordinates, when taking the divergence of these quantities, special modifications must be made. For the heat flux vector, the changes are relatively simple. In cylindrical coordinates (r, θ, z), letting $\partial/\partial\theta \to 0$,

$$\nabla \cdot \mathbf{h} = \frac{1}{r} \frac{\partial}{\partial r}(r h_r) + \frac{\partial h_z}{\partial z}. \tag{37}$$

To take advantage of the existing machinery in WARPXM for computing the divergence of fluxes in cartesian coordinates, we can rework the first term to yield a cartesian gradient plus an additional term,

$$\nabla \cdot \mathbf{h} = \frac{\partial h_r}{\partial r} + \frac{\partial h_z}{\partial z} + \frac{h_r}{r}. \tag{38}$$

By including the final term as a source term in WARPXM, the divergence in cylindrical coordinates is properly represented. Turning to the divergence of the stress tensor, there are coordinate-system-specific modifications associated with the gradient of vectors (e.g., $\nabla \mathbf{v}$) and with the divergence of a tensor. The gradient of a vector is

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_r}{r \partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{\partial v_\theta}{r \partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{\partial v_z}{r \partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}. \tag{39}$$

Letting $\partial/\partial\theta \to 0$ and adding $\nabla \mathbf{v}^{\intercal}$,

$$\nabla \mathbf{v} + \nabla \mathbf{v}^{\intercal} = \begin{bmatrix} 2\frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} & \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \\ \cdot & 2\frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \cdot & \cdot & 2\frac{\partial v_z}{\partial z} \end{bmatrix}. \tag{40}$$

Thus, the full stress tensor is

$$\Pi = -\mu \begin{bmatrix} 2\frac{\partial v_r}{\partial r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right) & \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} & \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \\ \cdot & 2\frac{v_r}{r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right) & \frac{\partial v_\theta}{\partial z} \\ \cdot & \cdot & 2\frac{\partial v_z}{\partial z} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right) \end{bmatrix}. \tag{41}$$

The divergence of this tensor is

$$\nabla \cdot \Pi = \begin{bmatrix} \frac{\partial \Pi_{rr}}{\partial r} + \frac{\partial \Pi_{zr}}{\partial z} - \frac{\Pi_{\theta\theta}}{r} + \frac{\Pi_{rr}}{r} \\ \frac{\partial \Pi_{r\theta}}{\partial r} + \frac{\partial \Pi_{z\theta}}{\partial z} + 2\frac{\Pi_{r\theta}}{r} \\ \frac{\partial \Pi_{rz}}{\partial r} + \frac{\partial \Pi_{zz}}{\partial z} + \frac{\Pi_{rz}}{r} \end{bmatrix}^{\mathsf{T}}. \tag{42}$$

Written out, the components of $\nabla \cdot \Pi$ are

$$(\nabla \cdot \Pi)_r = \frac{\partial}{\partial r}\left\{-\mu\left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right)\right]\right\} + \frac{\partial}{\partial z}\left\{-\mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\right\}$$
$$+2\mu\left(\frac{v_r}{r^2} - \frac{1}{r}\frac{\partial v_r}{\partial r}\right) \tag{43}$$

$$(\nabla \cdot \Pi)_\theta = \frac{\partial}{\partial r}\left\{-\mu\left[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right]\right\} + \frac{\partial}{\partial z}\left\{-\mu\left[\frac{\partial v_\theta}{\partial z}\right]\right\}$$
$$+2\mu\left(\frac{v_\theta}{r^2} - \frac{1}{r}\frac{\partial v_\theta}{\partial r}\right) \tag{44}$$

$$(\nabla \cdot \Pi)_z = \frac{\partial}{\partial r}\left\{-\mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\right\} + \frac{\partial}{\partial z}\left\{-\mu\left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right)\right]\right\}$$
$$-\mu\frac{1}{r}\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \tag{45}$$

The red terms in the radial fluxes of $(\nabla \cdot \Pi)_{r,\theta}$ and in the axial flux of $(\nabla \cdot \Pi)_z$ may be included in those fluxes; including them there is probably easier than handling the derivatives of μ required to move the terms into the sources. The remaining red terms are source terms.

Notes:

- From wiki page

  `https://en.wikipedia.org/wiki/Strain-rate_tensor#In_continuum_mechanics`

  "For a two-dimensional flow, the divergence of v has only two terms and quantifies the change in area rather than volume. The factor 1/3 in the expansion rate term should be replaced by 1/2 in that case."

- See online course notes by Pedlosky

  `https://www.whoi.edu/profile/jpedlosky`

  `http://www.whoi.edu/sites/12800-2014`

These notes provide an excellent discussion of the stress tensor. The "Course Material" link provides all notes in ".doc" format. See especially

Chapter 3: The Stress Tensor for a Fluid and the Navier Stokes Equations

- See also another wiki page

  `https://en.wikipedia.org/wiki/Volume_viscosity`

  which clearly presents the full-fledged incompressible Navier-Stokes equation.

6 Intraspecies collisions (Braginskii magnetized)

6.1 Viscosity

Braginskii [3] defines a rate-of-strain tensor,

$$\mathbb{W} = \nabla\mathbf{v} + \nabla\mathbf{v}^\intercal - \frac{2}{3}\mathbb{I}\nabla\cdot\mathbf{v}. \tag{46}$$

Following the discussion in Braginskii's Sect. 4, the stress tensor is then expressed in terms of $\mathbb{W}$ as

$$\Pi = -\eta_0\mathbb{W}_0 - \eta_1\mathbb{W}_1 - \eta_2\mathbb{W}_2 + \eta_3\mathbb{W}_3 + \eta_4\mathbb{W}_4. \tag{47}$$

The sum of the three tensors $\mathbb{W}_0$, $\mathbb{W}_1$, $\mathbb{W}_2$, gives $\mathbb{W}$. The $\mathbb{W}_3$ and $\mathbb{W}_4$ tensors are constructed from components of $\mathbb{W}$. Here, we map Braginskii's coordinate system with z-aligned magnetic field (x, y, z) to a coordinate system with θ-aligned magnetic field (r, θ, z). Then the tensors are

$$\mathbb{W}_0 = \begin{bmatrix} \frac{1}{2}\left(\mathbb{W}_{zz} + \mathbb{W}_{rr}\right) & 0 & 0 \\ 0 & \mathbb{W}_{\theta\theta} & 0 \\ 0 & 0 & \frac{1}{2}\left(\mathbb{W}_{zz} + \mathbb{W}_{rr}\right) \end{bmatrix}, \tag{48}$$

$$\mathbb{W}_1 = \begin{bmatrix} \frac{1}{2}\left(\mathbb{W}_{rr} - \mathbb{W}_{zz}\right) & 0 & \mathbb{W}_{rz} \\ 0 & 0 & 0 \\ \mathbb{W}_{zr} & 0 & \frac{1}{2}\left(\mathbb{W}_{zz} - \mathbb{W}_{rr}\right) \end{bmatrix}, \tag{49}$$

$$\mathbb{W}_2 = \begin{bmatrix} 0 & \mathbb{W}_{r\theta} & 0 \\ \mathbb{W}_{\theta r} & 0 & \mathbb{W}_{\theta z} \\ 0 & \mathbb{W}_{z\theta} & 0 \end{bmatrix}, \tag{50}$$

$$\mathbb{W}_3 = \begin{bmatrix} \mathbb{W}_{zr} & 0 & \frac{1}{2}\left(\mathbb{W}_{zz} - \mathbb{W}_{rr}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(\mathbb{W}_{zz} - \mathbb{W}_{rr}\right) & 0 & -\mathbb{W}_{zr} \end{bmatrix}, \tag{51}$$

$$\mathbb{W}_4 = \begin{bmatrix} 0 & \mathbb{W}_{z\theta} & 0 \\ \mathbb{W}_{\theta z} & 0 & -\mathbb{W}_{\theta r} \\ 0 & -\mathbb{W}_{r\theta} & 0 \end{bmatrix}. \tag{52}$$

Note that, as discussed by Haines [6], a straight magnetic field is assumed. Implicit in this assumption is that the gyroradius is smaller than the radius of curvature of the field (and because ion stress usually dominates over electrons stress, usually it is the ion gyroradius that is relevant). This is, of course, not the case near $r = 0$ in the Z pinch.

From these pieces, the entire Π tensor may be constructed. The components correspond to those shown by Braginskii in his Eq. (2.21).

$$\Pi_{\theta\theta} = -\eta_0 \mathbb{W}_{\theta\theta} \tag{53}$$

$$\Pi_{zz} = -\eta_0 \frac{1}{2}\left(\mathbb{W}_{zz} + \mathbb{W}_{rr}\right) - \eta_1 \frac{1}{2}\left(\mathbb{W}_{zz} - \mathbb{W}_{rr}\right) - \eta_3 \mathbb{W}_{zr} \tag{54}$$

$$\Pi_{rr} = -\eta_0 \frac{1}{2}\left(\mathbb{W}_{zz} + \mathbb{W}_{rr}\right) - \eta_1 \frac{1}{2}\left(\mathbb{W}_{rr} - \mathbb{W}_{zz}\right) + \eta_3 \mathbb{W}_{zr} \tag{55}$$

$$\Pi_{zr} = \Pi_{rz} = -\eta_1 \mathbb{W}_{zr} + \eta_3 \frac{1}{2}\left(\mathbb{W}_{zz} - \mathbb{W}_{rr}\right) \tag{56}$$

$$\Pi_{z\theta} = \Pi_{\theta z} = -\eta_2 \mathbb{W}_{z\theta} - \eta_4 \mathbb{W}_{r\theta} \tag{57}$$

$$\Pi_{r\theta} = \Pi_{\theta r} = -\eta_2 \mathbb{W}_{r\theta} + \eta_4 \mathbb{W}_{z\theta} \tag{58}$$

As discussed by Braginskii, the coefficient η_0 is associated with unmagnetized viscosity, η_1 and η_2 with perpendicular viscosity, and η_3 and η_4 with gyroviscosity. Modifications to the Braginskii formulation of these coefficients to make them more numerically tractable is discussed in Section 1. The $\mathbb{W}$ tensor is represented using $\nabla \mathbf{v}$ and $\nabla \cdot \mathbf{v}$ as outlined in Sect. 5, accounting for the cylindrical coordinate system. Explicitly, we have

$$\mathbb{W} = \begin{bmatrix} 2\frac{\partial v_r}{\partial r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right) & \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} & \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \\ . & 2\frac{v_r}{r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right) & \frac{\partial v_\theta}{\partial z} \\ . & . & 2\frac{\partial v_z}{\partial z} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right) \end{bmatrix}. \tag{59}$$

where we have assumed $\partial/\partial\theta \to 0$.

Examining the components of Π closely, we can recognize some terms that may be ignored in our axisymmetric simulations. Again, derivatives with respect to θ may be dropped. Also, v_θ is zero in our simulations, so derivatives of v_θ will be zero. Applying these assumptions, we can see that the terms involving η_2 and η_4 (i.e., $\Pi_{z\theta}$, $\Pi_{\theta z}$, $\Pi_{r\theta}$, and $\Pi_{\theta r}$) vanish. Here we have used $\mathbb{W}_{\theta r} = \mathbb{W}_{r\theta} = \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} = 0$ and $\mathbb{W}_{\theta z} = \mathbb{W}_{z\theta} = \frac{\partial v_\theta}{\partial z} = 0$. Now substituting the appropriate components of $\mathbb{W}$,

$$\Pi_{\theta\theta} = -\eta_0 \left(2\frac{v_r}{r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right)\right) \tag{60}$$

$$\Pi_{zz} = -\eta_0 \left(\frac{1}{3}\frac{\partial v_r}{\partial r} + \frac{1}{3}\frac{\partial v_z}{\partial z} - \frac{2}{3}\frac{v_r}{r}\right) - \eta_1 \left(\frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r}\right) - \eta_3 \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \tag{61}$$

$$\Pi_{rr} = -\eta_0 \left(\frac{1}{3}\frac{\partial v_r}{\partial r} + \frac{1}{3}\frac{\partial v_z}{\partial z} - \frac{2}{3}\frac{v_r}{r}\right) + \eta_1 \left(\frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r}\right) + \eta_3 \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \tag{62}$$

$$\Pi_{zr} = \Pi_{rz} = -\eta_1 \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) + \eta_3 \left(\frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r} \right) \tag{63}$$

$$\Pi_{z\theta} = \Pi_{\theta z} = 0 \tag{64}$$

$$\Pi_{r\theta} = \Pi_{\theta r} = 0 \tag{65}$$

Here, it is worth noting that the unmagnetized viscosity coefficient, η_0, appears despite an absence of derivatives in the direction of the magnetic field ($\partial/\partial\theta \to 0$). The related stress components are, as discussed by Braginskii, due to an increase in pressure when the plasma is compressed rather than a transport of momentum *per se*.

The divergence of Π, in cylindrical coordinates, is

$$\nabla \cdot \Pi = \begin{bmatrix} \frac{\partial \Pi_{rr}}{\partial r} + \frac{\partial \Pi_{zr}}{\partial z} - \frac{\Pi_{\theta\theta}}{r} + \frac{\Pi_{rr}}{r} \\ 0 \\ \frac{\partial \Pi_{rz}}{\partial r} + \frac{\partial \Pi_{zz}}{\partial z} + \frac{\Pi_{rz}}{r} \end{bmatrix}^\intercal . \tag{66}$$

This is implemented in WARPXM in a straightforward way by using Gaussian quadrature to avoid division by r at $r = 0$, and the associated multiple applications of L'Hopital's rule that would be required, which in turn would require second derivatives of our primary variables.

The magnetized viscosity coefficients, η_1 and η_3 are accurate to within 7% for all x [**?**], and are implemented according to the prescription of Braginskii.

6.1.1 Another look at the components of Π

Above, components of Π have been written out. Let's write it again, but leave the $\nabla \cdot \mathbf{v}$ term intact.

$$\Pi_{\theta\theta} = -\eta_0 \left(2\frac{v_r}{r} - \frac{2}{3}\nabla \cdot \mathbf{v} \right) \tag{67}$$

$$\Pi_{zz} = -\eta_0 \left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} - \frac{2}{3}\nabla \cdot \mathbf{v} \right) - \eta_1 \left(\frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r} \right) - \eta_3 \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \tag{68}$$

$$\Pi_{rr} = -\eta_0 \left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} - \frac{2}{3}\nabla \cdot \mathbf{v} \right) + \eta_1 \left(\frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r} \right) + \eta_3 \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \tag{69}$$

$$\Pi_{zr} = \Pi_{rz} = -\eta_1 \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) + \eta_3 \left(\frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r} \right) \tag{70}$$

$$\Pi_{z\theta} = \Pi_{\theta z} = 0 \tag{71}$$

$$\Pi_{r\theta} = \Pi_{\theta r} = 0 \tag{72}$$

Written this way, it is more obvious that if $\eta_1 = \eta_0 = \mu$, the tensor is identical to the Π used for intraspecies collisions, except for the presence of the η_3 contributions.

6.1.2 Modifications to viscosity coefficients (take 1)

Formulae for the relevant Braginskii's dimensionless viscosity coefficients are as follows.

$$\hat{\eta}_0 = g_i^0$$

$$\hat{\eta}_1 = (g_i^{1'} x_{i2} + g_i^{0'})/\Delta_2$$
$$\hat{\eta}_3 = x_{i2}(g_i^{1''} x_{i2} + g_i^{0''})/\Delta_2, \tag{73}$$

where the "g" coefficients are given by Braginskii, the magnetization is $x_{i2} \equiv 2\omega_{ci}/\nu_i$, and Δ_2 is

$$\Delta_2 = x_{i2}^4 + 13.8 x_{i2}^2 + 11.6. \tag{74}$$

Momentum diffusivities (i.e., kinetmatic viscosities) are found as $\eta_n^d = \hat{\eta}_n T/(\nu_i m_i)$.

The Braginskii coefficients are modified such that their rapid r-dependent variation may be more easily resolved numerically. Specifically, three changes are made: 1) the magnetization, x_{i2} used to determine the viscosities is modified and 2) the viscosities resulting from (1) are multiplied by an additional factor and 3) η_0 and η_1 are multplied by a uniform reducing factor f_{01} and 4) all viscosities are limited to a maximum diffusivity D_{lim}.

A smooth transition function is used to restrict modifications to the region $0 < r < f_L r_{Li}$. We use the function

$$\mathbb{S} = \frac{1}{2}\left[1 - \cos\left(\frac{\pi r}{f_L r_{Li}}\right)\right] \quad ; \quad r < f_L r_{Li},$$
$$\mathbb{S} = 1; \quad r \geq f_L r_{Li}.$$

For η_1 and η_3, a modified x_{i2} is used,

$$x_{i2,\eta 1} = f_{x,\eta 1} x_{i2},$$

$$x_{i2,\eta 3} = f_{x,\eta 3} x_{i2},$$

with

$$f_{x,\eta 1} = f_{x,\eta 1}^0 + (1 - f_{x,\eta 1}^0)\mathbb{S}$$
$$f_{x,\eta 3} = f_{x,\eta 3}^0 + (1 - f_{x,\eta 3}^0)\mathbb{S}.$$

Notice that the expressions vary sinsoidally from some constant at $r = 0$ to 1 at $r = f_L r_{Li}$. With the constants ($f_{x,\eta 1}^0$ and $f_{x,\eta 3}^0$) between 0 and 1, the resulting modified magnetization $x_{i2,\eta 1}$ has the effect of generating viscosity that is *less* well magnetized than it normally would be.

An additional multiplier, $f_{\eta 1}$, is applied to η_1,

$$f_{\eta 1} = 1 + (1/f_{01} - 1)\mathbb{S}$$

Because η_1 is also multiplied by the factor f_{01}, the effect of the $f_{\eta 1}$ multiplier is that for $r \geq f_L r_{Li}$, η_1 is the unchanged.

An additional multiplier, $f_{\eta 3}$, is applied to η_3,

$$f_{\eta 3} = f_{\eta 3}^0 + (1 - f_{\eta 3}^0)\mathbb{S}$$

This multiplier is small (≈ 0.1) at $r = 0$ and is 1 for $r \geq f_L r_{Li}$.

With these modifications, $\eta_{1,3}$ change more gradually as $r \to 0$, as shown in Fig. 1. The free parameters in these modifications — $f_{01}, f_L, f_{x,\eta 1}^0, f_{x,\eta 3}^0, f_{\eta 3}^0$, and D_{lim} — are shown in order in the figure as "tuning params".

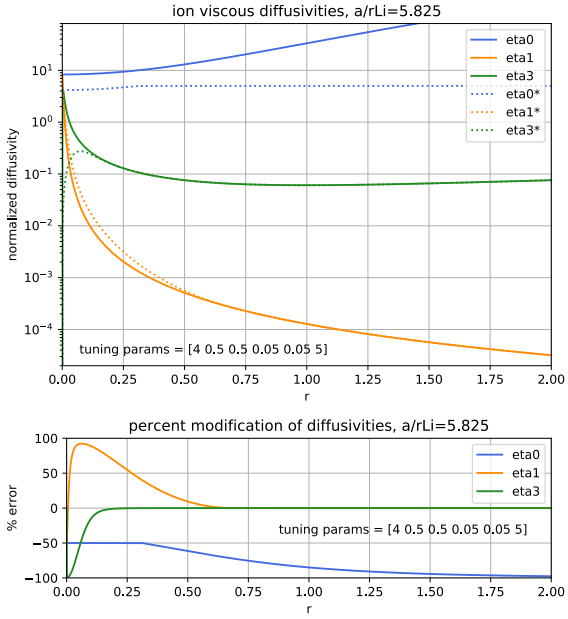

Figure 1: Modified Braginskii viscous diffusivities.

6.1.3 Modifications to viscosity coefficients (take 2)

The parallel (unmagnetized) diffusion of momentum and heat captured by the Braginskii model can give unphysically high fluxes that can significantly exceed the thermal fluxes. Using full kinetic modeling for guidance, kinetic corrections to fluid closures have been developed [**?**]. Although the corrections are by nature *ad hoc*, and will not allow fluid models to reproduce true kinetic transport effects, they offer a clear improvement in certain situations. For unmagnetized transport, we adopt a common form for the correction factor,

$$f_{corr}^{||} = [1 + c_{||}\lambda/\ell]^{-1}, \tag{75}$$

where λ is the mean free path, ℓ is the gradient scale length, and $c_{||}$ is a tunable constant that is of order unity. For $\lambda \gg \ell$, $f_{corr} \to 0$. Larger $c_{||}$ would give a stronger correction, i.e., reduction of the transport coefficient. In this work, we exclusively use $c_{||} = 1$. The corrected unmagnetized viscosity would then be $\eta_0^* = f_{corr}^{||}\eta_0$. The value of ℓ is a characteristic length scale of features of interest for a particular problem.

Where the Larmor orbit size exceeds the length scale of interest, and near $r = 0$, the Braginskii cross-field transport model is not valid. In such regions, the η_0 effect in the perpendicular plane is also inapplicable. To correct the model in regions of large Larmor radius, we use a smooth function,

$$\mathbb{S} = \frac{1}{2}\left[1 - \cos\left(\frac{\pi s}{2c_\times r_L}\right)\right] \quad \text{where } r_L > \frac{s}{2c_\times}, \text{ and} \tag{76}$$

$$\mathbb{S} = 1 \quad \text{elsewhere}, \tag{77}$$

where s is a controlling length scale. For $r_L = c_\times s$, $\mathbb{S} = 1/2$, while for $r_L = c_\times s/2$, $\mathbb{S} = 1$. Using

a larger tunable constant, $c_\times$, makes the kinetic correction more aggressive. Except for electron transport, as noted below, we use $c_\times = 1$.

For the purpose of correcting the gyroviscosity and diamagnetic heat flux terms, we use $s = \ell$. The correction simply $f^\wedge_{corr} = \mathbb{S}|_{s=\ell}$, such that $\eta^*_3 = f^\wedge_{corr}\eta_3$ and $\kappa^{\alpha *}_\wedge = f^\wedge_{corr}\kappa^\alpha_\wedge$. One might imagine using a correction approach similar to the one described above for unmangetized transport, $f^\wedge_{corr} = [1 + c^\times_{corr} r_L/\ell]^{-1}$, where r_L is the relevant Larmor radius. We find that having linear dependence on r_L allows significant transport even where the Larmor radius is larger than the feature size. We find that the form of Eq. (77) provides a stronger reduction for large Larmor radius, and seems more physically plausible.

Without any kinetic correction, perpendicular transport of momentum and heat rises to the unmagnetized level at $r = 0$ where Larmor radii become infinite. Physically, particles near $r = 0$ that move perpendicular to the z axis encounter significant magnetic field not far from $r = 0$. Such particles will be contained by the magnetic field after traveling a distance characterized by a critical radius, r_{crit}, corresponding to the radius at which r_L coincides with r. In the FuZE-like Bennett equilibrium, $r_{crit} \approx 0.3$ for ions. We employ a scheme in which perpendicular viscosity and heat flux coefficients transition, over a region determined by r_{crit}, from their uncorrected values to a corrected on-axis value. The on-axis value can be chosen using the usual notion of diffusive transport. Looking at η_1, for example, in regions that are well magnetized, Braginskii finds $\eta_1 = 0.3\tau_i p_i/x_i^2$, where $x_i = \omega_{ci}\tau_i$. Using standard definitions of ω_{ci} and r_L, it is easily shown that Braginskii's expression corresponds to a diffusivity $\eta_1/(m_i n_i) = 0.15 r_L^2/\tau_i$. Using this same expression, but substituting r_{Li}^{crit}, we have $\eta_1^{r=0} = 0.15 m_i n_i (r_L^{crit})^2/\tau_i$. To correct perpendicular transport coefficients, the function of Eq. (77) with $s = r_{crit}$ is employed, i.e., $f^\perp_{corr} = \mathbb{S}|_{s=r_{crit}}$. So, $\eta_1^* = \mathbb{S}|_{s=r_{crit}}\eta_1$, and $\kappa^{\alpha *}_\perp = \mathbb{S}|_{s=r_{crit}}\kappa^\alpha_\perp$.

- for eta0, do we then use a double-corrected value, $eta0** = \mathbb{S}|_{s=r_{crit}} eta0*$?

- NB this means that we eschew the Braginskii general magnetization scheme in favor of this transition in terms of S and rLcrit. The Braginskii scheme is really not meant for situations in which the magnetization x varies enormously over distances comparable to the Larmor radius. Braginskii's approach is more aimed at situations with small x due to large collisionality such that particles do not complete a Larmor orbit before colliding.

The value of r_L^{crit} is based on the following logic.

Continue... address * eta0 reducing to eta1 value * enhancement of correction for electrons.

- Determining a meaningful value of ℓ ahead of time is not possible, in general. A typical approach for determining length scales is to compare variable magnitude to its gradient; for temperature gradient, for example, $\ell_T = T/\nabla T$. For nonlinear simulations, this approach could be useful. For linear modeling, gradients are presumably tiny, giving large ℓ_T despite small feature sizes. ...

- Note that the approach above may be considered a starting point for exploration of modeling Braginskii transport in the Z pinch. Improvements and further exploration are certainly possible in future work.

6.2 Heat flux

Braginskii gives results for electron and ion heat fluxes ($\mathbf{q}_e$ and $\mathbf{q}_i$). Electron heat flux is composed of two parts,

$$\mathbf{q}_e = \mathbf{q}^e_u + \mathbf{q}^e_T, \tag{78}$$

where

$$\mathbf{q}_e^T = \beta_{||}^{Tu} u_{||} + \beta_{\perp}^{Tu} \mathbf{u}_{\perp} + \beta_{\wedge}^{Tu} \mathbf{b} \times \mathbf{u}. \tag{79}$$

The β coefficients in this expression are related to those in the $\mathbf{R}_T$ (see above) by

$$\beta_{||}^{Tu} = \beta_{||}^{uT} T_e \quad \beta_{\perp}^{Tu} = \beta_{\perp}^{uT} T_e \quad \beta_{\wedge}^{Tu} = \beta_{\wedge}^{uT} T_e. \tag{80}$$

By the same reasoning as used for $\mathbf{R}_T$, the term reltaed to $\beta_{||}^{Tu}$ is dropped. As discussed in connection with $\mathbf{R}_T$, the $\beta_{\perp}^{Tu}$ coefficient is inaccurate, and probably has minimal impact on plasma behavior, and the $\beta_{\wedge}^{Tu}$ term could be included, but probably has minor effects and is neglected here.

The $\mathbf{q}_u^T$ term is represented as

$$\mathbf{q}_u^T = -\kappa_{||}^e \nabla_{||} T_e - \kappa_{\perp} \nabla_{\perp} T_e - \kappa_{\wedge} \mathbf{b} \times \nabla_{\wedge} T_e. \tag{81}$$

The $\kappa_{\perp}$ and $\kappa_{\wedge}$ coefficients are accurate, for $Z = 1$, to within 29% for all x [**?**], and both are implemented in WARPXM following the prescription of Braginskii.

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