Artificial Dissipation

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1 Artificial Dissipation Based on Compressibility

Here we follow the idea of artificial dissipation as described by VonNeumnann [1,2]. The equations are added to the resistive MHD equations with the terms defining artificial dissipation in blue. Additional advection of the density diffusion term are given in red, though these are turned off by default as they appear to cause numerical instability.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{p} - \nabla \cdot (\rho \boldsymbol{v}_{\text{anom}}) = 0 \tag{1}$$

$$\frac{\partial \boldsymbol{p}}{\partial t} + \nabla \cdot (\boldsymbol{p}\boldsymbol{u} + p\mathcal{I}) - \nabla \cdot \overline{\overline{\Pi}}_{\text{anom}} + \nabla \cdot (\boldsymbol{p}\boldsymbol{v}_{\text{anom}}) = (\omega_c \tau)\, \boldsymbol{J} \times \boldsymbol{B} \tag{2}$$

$$\begin{aligned} \frac{\partial e_t}{\partial t} + \nabla \cdot [(e+p)\,\boldsymbol{u}] - \nabla \cdot \left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{u} + \boldsymbol{h}_{\text{anom}} \right) & \\ + \nabla \cdot \left[(e+p)\, \boldsymbol{v}_{\text{anom}} - \overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{v}_{\text{anom}} \right] &= -\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) \end{aligned} \tag{3}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \tag{4}$$

$$\boldsymbol{u} = \frac{\boldsymbol{p}}{\rho} \tag{5}$$

$$e_t = e + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2} = \frac{p}{\gamma - 1} + \frac{\boldsymbol{p} \cdot \boldsymbol{p}}{2\rho} + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2} \tag{6}$$

$$\boldsymbol{J} = \frac{\nabla \times \boldsymbol{B}}{(\omega_c \tau)} \tag{7}$$

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \frac{(\nu_p \tau)}{(\omega_c \tau)} \eta \boldsymbol{J} \tag{8}$$

$$\boldsymbol{v}_{\text{anom}} \equiv D_\rho \frac{\nabla \rho}{\rho} \tag{9}$$

$$\overline{\overline{\Pi}}_{\text{anom}} \equiv D_{\boldsymbol{p}} \rho \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T - \frac{2}{3} \nabla \cdot \boldsymbol{u} \right) \tag{10}$$

$$\boldsymbol{h}_{\text{anom}} \equiv D_h n \nabla T \tag{11}$$

$$D_\rho \equiv c_\rho^2 {\Delta x_{\text{eff}}}^2 \left| \nabla \cdot \boldsymbol{u} \right| \tag{12}$$

$$D_{\boldsymbol{p}} \equiv c_{\boldsymbol{p}}^2 {\Delta x_{\text{eff}}}^2 \left| \nabla \cdot \boldsymbol{u} \right| \tag{13}$$

$$D_h \equiv c_h^2 {\Delta x_{\text{eff}}}^2 \left| \nabla \cdot \boldsymbol{u} \right| \tag{14}$$

$$\nabla \boldsymbol{u} = \nabla \left(\frac{\boldsymbol{p}}{\rho} \right) = \frac{\nabla \boldsymbol{p} - \boldsymbol{u} \nabla \rho}{\rho} \tag{15}$$

$$\begin{aligned} p &= n_{\text{i}} T_{\text{i}} + n_{\text{e}} T_{\text{e}} \\ 0 &= Z_{\text{i}} n_{\text{i}} + Z_{\text{e}} n_{\text{e}} \\ Z_{\text{e}} &= -1 \\ T &= T_{\text{i}} = T_{\text{e}} \\ \rho &\approx \rho_{\text{i}} = A_{\text{i}} n_{\text{i}} \\ &\therefore \end{aligned}$$

$$T = \frac{p}{n_{\text{i}} (1 + Z_{\text{i}})} = \frac{p}{\rho} \frac{A_{\text{i}}}{1 + Z_{\text{i}}} \tag{16}$$

$$n \equiv \frac{\rho}{A_{\text{i}}} = n_{\text{i}} \tag{17}$$

$$\nabla T = \frac{A_{\text{i}}}{1 + Z_{\text{i}}} \left(\frac{\nabla p}{\rho} - \frac{p \nabla \rho}{\rho^2} \right) \tag{18}$$

$$\overset{2}{\nabla} p = \nabla \left[(\gamma - 1) \left(e_t - \frac{\boldsymbol{p} \cdot \boldsymbol{p}}{2\rho} - \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2} \right) \right] \tag{19}$$

c_ρ, $c_{\boldsymbol{p}}$, and c_h are user-defined constants which are supposed to be near unity [2]. $\Delta x_{\text{eff}} = CFL\frac{\text{Element Volume}}{\max(\text{Element Areas})}$. The user is allowed to specify a minimum allowed Δx_{eff} by setting $\Delta x_{\text{eff}} = \max\left(\Delta x_{\text{cut}}, \Delta x_{\text{eff}}\right)$, which should effectively filter out high-k modes specified by the value of Δx_{cut}. The timestep used is $\Delta t = \frac{CFL_{\text{Diff}}\Delta x_{\text{eff}}^2}{\max(D_\rho, D_{\boldsymbol{p}}, D_h)}$, or if the density diffusion advection terms (red) are turned on $\Delta t = \min\left(\frac{CFL_{\text{Diff}}\Delta x_{\text{eff}}^2}{\max(D_\rho, D_{\boldsymbol{p}}, D_h)}, \frac{CFL\Delta x_{\text{eff}}}{|\boldsymbol{v}_{\text{anom}}| + |\boldsymbol{u}|}\right)$.

1.1 Cylindrical Source Terms

Cylindrical source terms can be calculated in an identical manner to that of other MHD terms. Since one runs into 2nd derivatives of $\boldsymbol{u}$ and T doing this at $r = 0$, Gaussian Quadrature nodes are required for these source terms.

1.1.1 Continuity

$$\nabla \cdot \left(\rho \boldsymbol{v}_{\text{anom}}\right) = \frac{\partial\left(\rho v_{\text{anom}_z}\right)}{\partial z} + \frac{\partial\left(\rho v_{\text{anom}_r}\right)}{\partial r} + \frac{\rho v_{\text{anom}_r}}{r} \tag{20}$$

1.1.2 Momentum

$$\nabla \boldsymbol{u} = \begin{pmatrix} (\nabla\boldsymbol{u})_{zz} & (\nabla\boldsymbol{u})_{zr} & (\nabla\boldsymbol{u})_{z\theta} \\ (\nabla\boldsymbol{u})_{rz} & (\nabla\boldsymbol{u})_{rr} & (\nabla\boldsymbol{u})_{r\theta} \\ (\nabla\boldsymbol{u})_{\theta z} & (\nabla\boldsymbol{u})_{\theta r} & (\nabla\boldsymbol{u})_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_z}{\partial z} & \frac{\partial u_z}{\partial r} & \cancel{\frac{1}{r}\frac{\partial u_z}{\partial \theta}} \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_r}{\partial r} & \cancel{\frac{1}{r}\frac{\partial u_r}{\partial \theta}} - \frac{u_\theta}{r} \\ \frac{\partial u_\theta}{\partial z} & \frac{\partial u_\theta}{\partial r} & \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{u_r}{r} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_z}{\partial z} & \frac{\partial u_z}{\partial r} & 0 \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_r}{\partial r} & -\frac{u_\theta}{r} \\ \frac{\partial u_\theta}{\partial z} & \frac{\partial u_\theta}{\partial r} & \frac{u_r}{r} \end{pmatrix} \tag{21}$$

and that $\nabla \cdot \boldsymbol{u}$ is given by

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}. \tag{22}$$

The $\overline{\overline{\Pi}}_{\text{anom}}$ tensor can thus be written

$$\begin{aligned} \overline{\overline{\Pi}}_{\text{anom}} &= D_{\boldsymbol{p}}\rho \begin{pmatrix} 2\left(\nabla\boldsymbol{u}\right)_{zz} - \frac{2}{3}(\nabla\cdot\boldsymbol{u}) & (\nabla\boldsymbol{u})_{zr} + (\nabla\boldsymbol{u})_{rz} & (\nabla\boldsymbol{u})_{z\theta} + (\nabla\boldsymbol{u})_{\theta z} \\ (\nabla\boldsymbol{u})_{rz} + (\nabla\boldsymbol{u})_{zr} & 2\left(\nabla\boldsymbol{u}\right)_{rr} - \frac{2}{3}(\nabla\cdot\boldsymbol{u}) & (\nabla\boldsymbol{u})_{r\theta} + (\nabla\boldsymbol{u})_{\theta r} \\ (\nabla\boldsymbol{u})_{\theta z} + (\nabla\boldsymbol{u})_{z\theta} & (\nabla\boldsymbol{u})_{\theta r} + (\nabla\boldsymbol{u})_{r\theta} & 2(\nabla\boldsymbol{u})_{\theta\theta} - \frac{2}{3}(\nabla\cdot\boldsymbol{u}) \end{pmatrix} \\ &= \begin{pmatrix} \Pi_{zz_{\text{anom}}} & \Pi_{zr} & \Pi_{z\theta_{\text{anom}}} \\ \Pi_{rz_{\text{anom}}} & \Pi_{rr_{\text{anom}}} & \Pi_{r\theta_{\text{anom}}} \\ \Pi_{\theta z_{\text{anom}}} & \Pi_{\theta r_{\text{anom}}} & \Pi_{\theta\theta_{\text{anom}}} \end{pmatrix} \end{aligned} \tag{23}$$

Then

$$\left(\nabla \cdot \overline{\overline{\Pi}}_{\text{anom}}\right)_r = \frac{\partial \Pi_{zr_{\text{anom}}}}{\partial z} + \frac{\partial \Pi_{rr_{\text{anom}}}}{\partial r} + \frac{\Pi_{rr_{\text{anom}}} - \Pi_{\theta\theta_{\text{anom}}}}{r} \tag{24a}$$

$$\left(\nabla \cdot \overline{\overline{\Pi}}_{\text{anom}}\right)_\theta = \frac{\partial \Pi_{z\theta_{\text{anom}}}}{\partial z} + \frac{\partial \Pi_{r\theta_{\text{anom}}}}{\partial r} + \frac{\Pi_{r\theta_{\text{anom}}} + \Pi_{\theta r_{\text{anom}}}}{r} \tag{24b}$$

$$\left(\nabla \cdot \overline{\overline{\Pi}}_{\text{anom}}\right)_z = \frac{\partial \Pi_{zz_{\text{anom}}}}{\partial z} + \frac{\partial \Pi_{rz_{\text{anom}}}}{\partial r} + \frac{\Pi_{rz_{\text{anom}}}}{r} \tag{24c}$$

1.1.3 Energy

$$\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{u} = \begin{pmatrix} \Pi_{zz_{\text{anom}}} & \Pi_{zr_{\text{anom}}} & \Pi_{z\theta_{\text{anom}}} \\ \Pi_{rz_{\text{anom}}} & \Pi_{rr_{\text{anom}}} & \Pi_{r\theta_{\text{anom}}} \\ \Pi_{\theta z_{\text{anom}}} & \Pi_{\theta r_{\text{anom}}} & \Pi_{\theta\theta_{\text{anom}}} \end{pmatrix} \cdot \begin{pmatrix} u_z \\ u_r \\ u_\theta \end{pmatrix} \tag{25}$$

So then

$$\begin{aligned} \nabla \cdot \left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{u}\right) =& \frac{\partial \left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{u}\right)_z}{\partial z} + \frac{\partial \left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{u}\right)_r}{\partial r} + \frac{\left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{u}\right)_r}{r} \\ =& \frac{\partial}{\partial z}\left(\Pi_{zz_{\text{anom}}} u_z + \Pi_{zr_{\text{anom}}} u_r + \Pi_{z\theta_{\text{anom}}} u_\theta\right) + \frac{\partial}{\partial r}\left(\Pi_{rz_{\text{anom}}} u_z + \Pi_{rr_{\text{anom}}} u_r + \Pi_{r\theta_{\text{anom}}} u_\theta\right) \\ &+ \frac{\Pi_{rz_{\text{anom}}} u_z + \Pi_{rr_{\text{anom}}} u_r + \Pi_{r\theta_{\text{anom}}} u_\theta}{r} \end{aligned} \tag{26}$$

For heat flux

$$\nabla \cdot \boldsymbol{h}_{\text{anom}} = \frac{\partial h_{z_{\text{anom}}}}{\partial z} + \frac{\partial h_{r_{\text{anom}}}}{\partial r} + \frac{h_{r_{\text{anom}}}}{r} = \frac{\partial}{\partial z}\left[D_h n \frac{\partial T}{\partial z}\right] + \frac{\partial}{\partial r}\left[D_h n \frac{\partial T}{\partial r}\right] + \frac{D_h n}{r}\left[\frac{\partial T}{\partial r}\right] \tag{27}$$

1.1.4 V_{anom} Terms

If V_{anom} is advected (red terms), then we also need

$$\nabla \cdot [\boldsymbol{p}\boldsymbol{v}_{\text{anom}}]_r = \frac{\partial}{\partial z}\left[p_z v_{r_{\text{anom}}}\right] + \frac{\partial}{\partial r}\left[p_r v_{r_{\text{anom}}}\right] + \frac{1}{r}\left[p_r v_{r_{\text{anom}}} - p_\theta v_{\theta_{\text{anom}}}\right] \tag{28}$$

$$\nabla \cdot [\boldsymbol{p}\boldsymbol{v}_{\text{anom}}]_\theta = \frac{\partial}{\partial z}\left[p_z v_{\theta_{\text{anom}}}\right] + \frac{\partial}{\partial r}\left[p_r v_{\theta_{\text{anom}}}\right] + \frac{1}{r}\left[p_r v_{\theta_{\text{anom}}} + p_\theta v_{r_{\text{anom}}}\right] \tag{29}$$

$$\nabla \cdot [\boldsymbol{p}\boldsymbol{v}_{\text{anom}}]_z = \frac{\partial}{\partial z}\left[p_z v_{z_{\text{anom}}}\right] + \frac{\partial}{\partial r}\left[p_r v_{z_{\text{anom}}}\right] + \frac{1}{r}\left[p_r v_{z_{\text{anom}}}\right] \tag{30}$$

$$\nabla \cdot \left[(e+p)\,\boldsymbol{v}_{\text{anom}}\right] = \frac{\partial}{\partial z}\left[(e+p)\,v_{z_{\text{anom}}}\right] + \frac{\partial}{\partial r}\left[(e+p)\,v_{r_{\text{anom}}}\right] + \frac{(e+p)\,v_{r_{\text{anom}}}}{r} \tag{31}$$

$$\begin{aligned} \nabla \cdot \left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{v}_{\text{anom}}\right) =& \frac{\partial \left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{v}_{\text{anom}}\right)_z}{\partial z} + \frac{\partial \left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{v}_{\text{anom}}\right)_r}{\partial r} + \frac{\left(\overline{\overline{\Pi}}_{\text{anom}} \cdot \boldsymbol{v}_{\text{anom}}\right)_r}{r} \\ =& \frac{\partial}{\partial z}\left(\Pi_{zz_{\text{anom}}} v_{z_{\text{anom}}} + \Pi_{zr_{\text{anom}}} v_{r_{\text{anom}}} + \Pi_{z\theta_{\text{anom}}} v_{\theta_{\text{anom}}}\right) \\ &+ \frac{\partial}{\partial r}\left(\Pi_{rz_{\text{anom}}} v_{z_{\text{anom}}} + \Pi_{rr_{\text{anom}}} v_{r_{\text{anom}}} + \Pi_{r\theta_{\text{anom}}} v_{\theta_{\text{anom}}}\right) \\ &+ \frac{\Pi_{rz_{\text{anom}}} v_{z_{\text{anom}}} + \Pi_{rr_{\text{anom}}} v_{r_{\text{anom}}} + \Pi_{r\theta_{\text{anom}}} v_{\theta_{\text{anom}}}}{r} \end{aligned} \tag{32}$$

References

[1] T. D. Rognlien and M. E. Rensink. Edge-plasma models and characteristics for magnetic fusion energy devices. *Fusion Engineering and Design*, 60(4):497–514, 7 2002.

[2] J. VonNeumann and R. D. Richtmyer. A Method for the Numerical Calculation of Hydrodynamic Shocks. *Journal of Applied Physics*, 21(3):232–237, 1950.