CHAPTER 48

MASS, MOMENTUM, ENERGY

Once when I was with Einstein in order to read with him a work that contained many objections against his theory... he suddenly interrupted the discussion of the book, reached for a telegram that was lying on the windowsill, and handed it to me with the words, 'Here, this will perhaps interest you.' It was Eddington's cable with the results of measurements of the eclipse expedition. When I was giving expression to my joy that the results coincided with his calculations, he said quite unmoved, 'But I knew that the theory was correct'; and when I asked, what if there had been no confirmation of his prediction, he countered: 'Then I would have been sorry for the dear Lord – the theory *is* correct.

A Student of Einstein (1919)

48.1 INERTIA AND RELATIVITY

The law of inertia, one of the building blocks of classical mechanics, states that a body in motion tends to remain in motion with constant velocity. Newton incorporated this principle in his first law of motion. The second law focused on changes in motion. Calling force **F** the agent of change, we usually write it in the form

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt},$$

where the mass m is a measure of the difficulty in changing a body's velocity. The second law became the focal point of Newton's dynamical description of the world.

The law of inertia, in turn, is part of a deeper idea – the principle of relativity, which says that there is no such thing as absolute rest, so there is no reason for a body to come to rest. It keeps moving according to the law of inertia. The principle works well for mechanics, but if relativity is to be a general principle, it must work for all of physics. In particular, the principle of relativity must also describe the propagation of light. That leads to difficulties and contradictions unless we assume that every inertial observer will measure the same speed of light. Einstein started with the principle of relativity and the constancy of the speed of light as fundamental postulates and from them deduced the compelling consequences we have seen in Chapters 46 and 47. One of these is that nothing can move faster than the speed of light. Once we accept this fact, it's easy to see that Newton's second law $\mathbf{F} = m\mathbf{a}$ can't be valid at speeds approaching c.

To demonstrate this, apply a constant force that moves a body along a straight line and increases its speed. If the force is applied continuously, the speed of the body keeps increasing at a rate proportional to the force and therefore becomes arbitrarily large, eventually exceeding the speed of light, because there is no mechanism in Newton's second law to prevent it. But exceeding the speed of light contradicts one of the major deductions in Einstein's special theory of relativity.

There is a way to resolve this contradiction and at the same time preserve Newton's second law. First, we write Newton's second law more nearly as Newton actually stated it,

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}),$$

and realize that the mass m is not necessarily constant but may vary with speed. To understand why mass might vary with speed we turn to the principle of conservation of momentum.

48.2 MOMENTUM AND MASS

In classical mechanics the principle of conservation of momentum follows from Newton's second law of motion and it incorporates the law of inertia. Every body in motion possesses something that it can't get rid of, and that's what gives it its inertia. That something is Newtonian momentum, the product of mass and velocity. In Chapter 45 we showed that the law of conservation of Newtonian momentum is preserved under the Galilean transformation. That is, if we have two inertial frames S and S' related by the Galilean transformation, then in either frame the total Newtonian momentum of a system before collision is equal to that after collision. This was deduced from two facts: In either frame the total mass of the system is conserved, and velocities combine in a direct manner.

Because velocities do not combine in the same way under the Lorentz transformation, Newtonian momentum is not conserved in relativistic mechanics. This can be demonstrated by a simple example, illustrated in Fig. 48.1. An observer A on the ground (in frame S) sees two identical balls crash into each other with equal and opposite constant velocities u and $-u$. Each ball has mass m, and when they collide

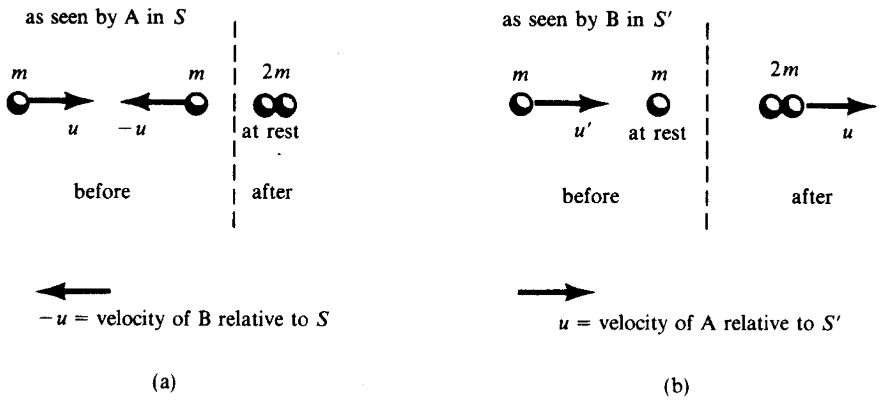


Figure 48.1 Newtonian momentum is conserved in (a) but not in (b).

they stick together and form a body of mass $2m$ at rest, as shown in Fig. 48.1a. Before collision, one ball has Newtonian momentum mu, the other $-mu$, so the total Newtonian momentum is zero. This is equal to the total Newtonian momentum after collision, $(2m)0 = 0$, so Newtonian momentum is conserved in frame S.

A second observer B in an airplane (frame S') flies over the same experiment with constant velocity $-u$ relative to the ground, as shown in Fig. 48.1a. Before the collision he sees one of the masses moving with velocity 0, as shown in Fig. 48.1b, and the other moving with velocity u' given by the relativistic composition law

$$u' = \frac{2u}{1 + u^2/c^2}. \tag{48.1}$$

According to B, the total Newtonian momentum before collision is mu' while the total Newtonian momentum after collision is $2mu$, and these are not equal because $u' \neq 2u$. In other words, Newtonian momentum is not conserved in relativistic mechanics. Therefore, if we want to have a law of conservation of momentum in relativistic mechanics that is consistent with the Lorentz transformation, we must adopt a different definition of momentum.

The solution of the problem turns out to be very simple, once we incorporate an idea proposed by Einstein. He pointed out that it's a matter of choice to redefine momentum or mass. We can regard the mass of a moving body in a given frame not as constant, but as a quantity that varies with its speed u. When $u = 0$ the body has Newtonian inertial mass m_0, which we refer to as the *rest mass*. But when the body has speed u it has relativistic mass $m(u)$, a function of u with $m(0) = m_0$. We will show presently that this function is given by the formula

$$m(u) = \gamma(u)m_0, \tag{48.2}$$

where

$$\gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}}. \tag{48.3}$$

Despite similarity in appearance, this is not the dilation factor γ that occurs in the Lorentz transformation, because u is not the speed of one reference frame relative to another but rather the speed of the particle in a given frame.

Having defined relativistic mass by Eq. (48.2), we now define the relativistic momentum of the particle in frame S to be the product of the relativistic mass and its velocity $\mathbf{u}$,

$$\mathbf{p} = m(u)\mathbf{u}. \tag{48.4}$$

Note that relativistic momentum is a vector quantity completely analogous to Newtonian momentum. The only difference is that the mass is no longer constant but varies with speed.

In Newtonian mechanics, mass is a conserved quantity, in the same sense that electric charge is conserved: The mass of each body is always the same. But if we regard the mass of a body to be a function of its speed, we've obviously given up that simple aspect of mass conservation. We need to examine mass conservation in terms of variable mass. We'll assume that mass continues to be conserved in the same sense that momentum is conserved: The total relativistic mass of a collection of bodies doesn't change so long as there's no outside force acting on them. Later we will see the deeper significance of this assumption.

To see why the function $m(u)$ is chosen as indicated in Eq. (48.2), we refer once again to the inelastic collision of Fig. 48.1 and recalculate the total relativistic momentum before and after the collision in each of the two frames. In this discussion we assume that $m(u)$ is some unspecified function of u with $m(0) = m_0$, the rest mass. Then we show that if relativistic mass and momentum are conserved in both frames S and S' we must necessarily take for $m(u)$ the function defined by Eq. (48.2).

In frame S, shown in Fig. 48.2a, the two masses are traveling with the same speed because they have equal and opposite velocities, so they have the same relativistic mass $m(u)$. Before collision the sum of the relativistic momenta is $m(u)u - m(u)u = 0$, and after collision there is one body of mass M_0 at rest so its relativistic momentum is also zero. In other words, relativistic momentum is conserved in frame S. (We do not use vector notation in this discussion because the motion is along a line.)

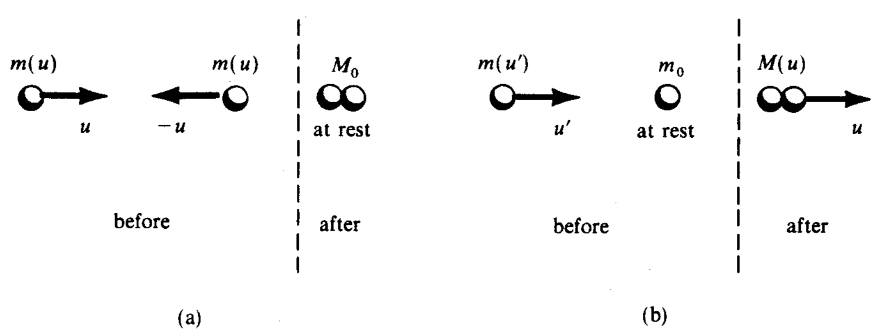


Figure 48.2 Relativistic momentum is conserved in both frames.

In frame S', before collision we have one body of mass m_0 at rest and one with relativistic mass $m(u')$ moving with velocity u', as indicated in Fig. 48.2b. Therefore, the total relativistic momentum before collision is

$$m(u')u'. \tag{48.5}$$

After collision the composite body has relativistic mass $M(u)$, say, and travels at velocity u in frame S', so its relativistic momentum is

$$M(u)u. \tag{48.6}$$

To conserve relativistic momentum in frame S' we need

$$m(u')u' = M(u)u, \tag{48.7}$$

and to conserve the total relativistic mass of the system we have

$$M(u) = m(u') + m_0. \tag{48.8}$$

Substituting this into Eq. (48.7) we find

$$m(u')u' = (m(u') + m_0)u,$$

which, when solved for $m(u')$, gives us

$$m(u') = \frac{um_0}{u' - u} = \frac{m_0}{u'/u - 1}. \tag{48.9}$$

Now the velocities u' and u are related by Eq. (48.1).

$$u' = \frac{2u}{1 + u^2/c^2}. \tag{48.1}$$

Simple algebraic manipulation, which is described below in Example 1, shows that Eq. (48.1) implies the relation

$$u'/u - 1 = \sqrt{1 - (u'/c)^2}, \tag{48.10}$$

so Eq. (48.9) becomes

$$m(u') = \frac{m_0}{\sqrt{1 - (u'/c)^2}}.$$

Thus we see that conservation of mass and momentum in each frame implies that the relativistic mass function $m(u)$ must be given by Eq. (48.2).

For the example of inelastic collision of two bodies, relativistic mass and momentum are conserved both in frame S and in frame S'. More generally, it can also be shown that the definition of relativistic mass in Eq. (48.2) together with the definition of relativistic momentum in Eq. (48.4) implies that the laws of conservation of relativistic mass and momentum hold true under the Lorentz transformation for any system of bodies.

The fact that mass depends on speed should not be viewed with alarm, because from the outset mass was a somewhat mysterious quantity. Mass, a numerical quantity associated with a body, is a measure of how difficult it is to change the inertia of the body. Relativistic mass still serves this purpose, but now the quantity

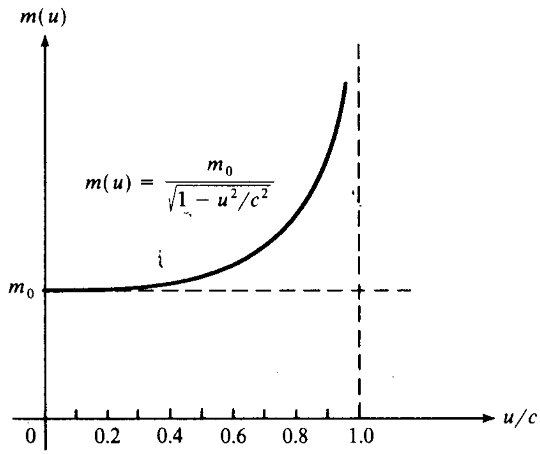


Figure 48.3 Relativistic mass $m(u)$ as a function of the ratio u/c.

changes with speed and becomes very large if the speed is greatly increased. In other words, as the speed of a body gets larger, it becomes increasingly difficult to further increase the body's speed. Figure 48.3 shows the graph of relativistic mass $m(u)$ as a function of the ratio u/c.

Example 1

Verify that Eq. (48.1) implies Eq. (48.10).

From Eq. (48.1) we find

$$\frac{u'}{u} = \frac{2}{1 + u^2/c^2} = \frac{2c^2}{c^2 + u^2},$$

and therefore

$$\frac{u'}{u} - 1 = \frac{2c^2}{c^2 + u^2} - 1 = \frac{c^2 - u^2}{c^2 + u^2}. \tag{48.11}$$

On the other hand, Eq. (48.1) also gives us

$$\frac{u'}{c} = \frac{2uc}{c^2 + u^2},$$

and hence

$$1 - \left(\frac{u'}{c}\right)^2 = 1 - \frac{4u^2c^2}{(c^2 + u^2)^2} = \left(\frac{c^2 - u^2}{c^2 + u^2}\right)^2.$$

Taking the positive square root of both members we get

$$\sqrt{1-\left(\frac{u'}{c}\right)^2} = \frac{c^2-u^2}{c^2+u^2}.$$

Comparing this with Eq. (48.11) we obtain Eq. (48.10).

Example 2

Refer to the collision in Fig. 48.2.

(a) Show that the rest mass M_0 of the composite body in frame S is not $2m_0$, as might be expected, but rather

$$M_0 = 2\gamma(u)m_0.$$

(b) Show that the relativistic mass of the composite body in frame S' is given by

$$M(u) = \gamma(u)M_0.$$

(a) In frame S, the composite body is formed from two bodies, each with relativistic mass $m(u)$, so $M_0 = 2m(u)$. But $m(u) = \gamma(u)m_0$, hence $M_0 = 2\gamma(u)m_0$.

(b) In frame S' we have

$$M(u) = m_0 + m(u') = m_0(1+\gamma(u')).$$

But from the last equation of Example 1 we find that

$$\gamma(u') = \frac{1+u^2/c^2}{1-u^2/c^2},$$

hence

$$1+\gamma(u') = 1 + \frac{1+u^2/c^2}{1-u^2/c^2} = \frac{2}{1-u^2/c^2} = 2\gamma^2(u),$$

so

$$M(u) = 2m_0\gamma^2(u) = \gamma(u)(2m_0\gamma(u)) = \gamma(u)M_0.$$

Note that when $u = 0$ we get $M(0) = \gamma(0)M_0 = M_0$, so the rest mass $M(0)$ in frame S' is equal to the rest mass M_0 in frame S. This tells us that the rest mass of an object is an intrinsic and invariant property of the object, independent of the motion of the object. By contrast, the relativistic mass depends on the reference frame from which it is viewed. Increasing the speed in that frame increases its relativistic mass in that frame.

Figure 48.4 shows a graph of $m(v)v$, the magnitude of relativistic momentum, plotted as a function of the ratio v/c. In Newtonian mechanics, with mass assumed

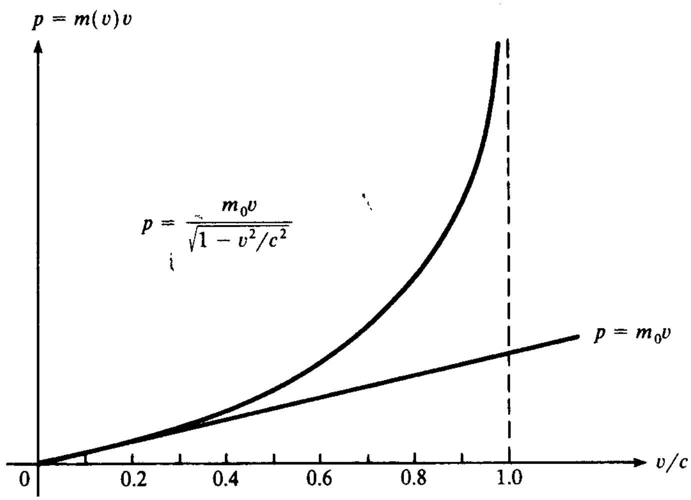


Figure 48.4 Relativistic momentum as a function of the ratio v/c.

constant, the graph of the momentum is just a straight line $p = m_0 v$ of constant slope. But the relativistic momentum behaves differently. At low speeds it very nearly obeys Newtonian mechanics and its graph is nearly a straight line of constant slope. At speeds exceeding about $0.3c$ the graph starts to bend upward, and as $v \to c$ it becomes asymptotic to the vertical line $v/c = 1$. So as you push a particle you keep increasing its momentum, but instead of only moving faster it also becomes more massive.

In Newtonian mechanics the force $\mathbf{F}$ with which you have to push the particle is related to its momentum $\mathbf{p}$ by the equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}). \tag{48.12}$$

In relativistic mechanics this equation is also adopted as the *definition* of force, with the understanding that $\mathbf{p}$ is relativistic momentum. This preserves Newton's second law of classical mechanics and it turns out to be a useful definition in relativistic mechanics as well.

When the mass m is not constant, the product rule for derivatives shows that the derivative of the product $m\mathbf{v}$ consists of two terms, so Eq. (48.12) becomes

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt}. \tag{48.13}$$

In other words, the force required to change an object's momentum consists of two components, one parallel to the acceleration $d\mathbf{v}/dt$, due to the change in velocity, and another parallel to the velocity $\mathbf{v}$, due to the change in mass. For an alternate form of Eq. (48.13), see Question 7 at the end of this section.

Example 3

If $\gamma = (1 - v^2/c^2)^{-1/2}$, show that

$$v\frac{dv}{d\gamma} = \frac{c^2}{\gamma^3}. \tag{48.14}$$

The definition of γ implies

$$\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1. \tag{48.15}$$

Differentiate both members of this equation with respect to γ, using the product rule, to obtain

$$2\gamma\left(1 - \frac{v^2}{c^2}\right) + \gamma^2\left(-\frac{2v}{c^2}\right)\frac{dv}{d\gamma} = 0.$$

Multiply by $\gamma/2$ and use Eq. (48.15) to rewrite this as

$$1 = \frac{\gamma^3 v}{c^2}\frac{dv}{d\gamma}.$$

This is equivalent to Eq. (48.14).

Example 4

If $\mathbf{p} = m\mathbf{v}$, where $m = \gamma m_0$ and $\gamma = (1 - v^2/c^2)^{-1/2}$, show that

$$\mathbf{v}\cdot\frac{d\mathbf{p}}{d\gamma} = m_0c^2. \tag{48.16}$$

This equation, which shows that the dot product $\mathbf{v}\cdot d\mathbf{p}/d\gamma$ is constant, will be used in the discussion of relativistic kinetic energy in the next section.

Write $\mathbf{p} = \gamma m_0\mathbf{v}$ and differentiate with respect to γ, using the product rule. This gives us

$$\frac{d\mathbf{p}}{d\gamma} = m_0\mathbf{v} + \gamma m_0\frac{d\mathbf{v}}{d\gamma}.$$

Dot multiplication by $\mathbf{v}$ transforms this to

$$\mathbf{v}\cdot\frac{d\mathbf{p}}{d\gamma} = m_0v^2 + \gamma m_0\mathbf{v}\cdot\frac{d\mathbf{v}}{d\gamma} \tag{48.17}$$

because

$$\mathbf{v}\cdot\mathbf{v} = v^2.$$

Differentiating this last equation with respect to γ we find

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{d\gamma} + \frac{d\mathbf{v}}{d\gamma} \cdot \mathbf{v} = 2v\frac{dv}{d\gamma}$$

or

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{d\gamma} = v\frac{dv}{d\gamma}.$$

Using this in the last term of Eq. (48.17) we obtain

$$\mathbf{v} \cdot \frac{d\mathbf{p}}{d\gamma} = m_0 v^2 + \gamma m_0 v \frac{dv}{d\gamma}.$$

Because of Eq. (48.14) in Example 3, this becomes

$$\mathbf{v} \cdot \frac{d\mathbf{p}}{d\gamma} = m_0 v^2 + m_0 \frac{c^2}{\gamma^2} = m_0\left(v^2 + \frac{c^2}{\gamma^2}\right). \tag{48.18}$$

But Eq. (48.15) implies

$$v^2 + \frac{c^2}{\gamma^2} = c^2,$$

so Eq. (48.18) reduces to (48.16).

Questions

1. An electron's speed cannot be greater than c. Is there an upper limit to its momentum? Explain why or why not.
2. A proton of rest mass 1.67×10^{-27} kg has a relativistic mass which is three times its rest mass. What is its speed?
3. At what speed will an object's mass be 1% greater than its rest mass?
4. Derive a formula that gives the density of an object as a function of speed.
5. An electron used in an experiment at the Stanford Linear Accelerator (SLAC) has $\gamma(u) = 10^4$. What fraction of the speed of light is its speed u?
6. A proton ($m_0 = 1.67 \times 10^{-27}$ kg) travels at a speed of $0.8c$.

   (a) Determine its relativistic momentum.
   (b) Compare its relativistic momentum to its Newtonian momentum.
7. Let $\mathbf{a} = d\mathbf{v}/dt$, $m = \gamma m_0$, and $\gamma = (1 - v^2/c^2)^{-1/2}$. Show that:

   (a) $d\gamma/dv = v\gamma^3/c^2$.
   (b) $v(dv/dt) = \mathbf{v} \cdot \mathbf{a}$.
   (c) $dm/dt = m\gamma^2(\mathbf{a} \cdot \mathbf{v})/c^2$.

(d) Use part (c) to show that Eq. (48.13) can be written in the form

$$\mathbf{F} = m\mathbf{a} + \frac{m\gamma^2(\mathbf{a} \cdot \mathbf{v})}{c^2}\mathbf{v}.$$

This equation shows that the velocity component of **F** is zero if $\mathbf{v} \cdot \mathbf{a} = 0$, and Eq. (48.13) shows that this component is zero if the mass is constant.

48.3 RELATIVISTIC KINETIC ENERGY

In Chapter 45 we learned that the law of conservation of kinetic energy is preserved under the Galilean transformation. That is, if the Newtonian energy $mv^2/2$ of a system (with m constant) is conserved in an inertial frame S, then it is also conserved in any other frame S' related to S by the Galilean transformation. But conservation of Newtonian kinetic energy doesn't survive under the Lorentz transformation whether m is taken as the rest mass m_0 or as the relativistic mass $m(v) = \gamma(v)m_0$. This can be seen by considering examples involving elastic collision of two identical masses from the point of view of two different observers. Such examples show that a new definition of kinetic energy is needed in relativistic mechanics.

In Newtonian mechanics, energy is related to work. If a force **F** moves a particle from point A to point B then the work done by this force is equal to the change in its kinetic energy $K_B - K_A$, or, in other words,

$$K_B - K_A = \int_A^B \mathbf{F} \cdot d\mathbf{r}, \tag{48.19}$$

where **r** is the vector function describing the path of the particle from A to B. We will use this integral to define the work in relativistic mechanics as well, and this will suggest a natural definition for relativistic kinetic energy. For the force **F** we take the rate of change of relativistic momentum, given by Eq. (48.12),

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{48.12}$$

In a moment we will show that line integration of this force yields a surprisingly simple result. When the force **F** brings a particle from rest to a final speed v the work done along any path turns out to be

$$(m - m_0)c^2, \tag{48.20}$$

where m_0 is the rest mass and $m = \gamma m_0$ is the relativistic mass. We then use this quantity as the *definition* of relativistic kinetic energy. Although this seems to be markedly different from the Newtonian formula $m_0v^2/2$ for kinetic energy, in Example 5 we show that it reduces to the Newtonian formula when the speed v is small compared to c.

To derive (48.20) from Eq. (48.19) we write

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt,$$

where $\mathbf{r}(a) = A$ and $\mathbf{r}(b) = B$. Using $\mathbf{v} = d\mathbf{r}/dt$ and Eq. (48.12) for $\mathbf{F}$ we find

$$\begin{aligned}\int_A^B \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \frac{d\mathbf{p}}{dt} \cdot \mathbf{v}\, dt \\ &= \int_a^b \mathbf{v} \cdot \frac{d\mathbf{p}}{d\gamma} \frac{d\gamma}{dt}\, dt.\end{aligned} \tag{48.21}$$

But in Example 4 we showed that

$$\mathbf{v} \cdot \frac{d\mathbf{p}}{d\gamma} = m_0 c^2, \tag{48.16}$$

a constant, so Eq. (48.21) becomes

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = m_0 c^2 \int_a^b \frac{d\gamma}{dt}\, dt.$$

The last integral can be evaluated by the second fundamental theorem of calculus, giving us

$$\begin{aligned}\int_A^B \mathbf{F} \cdot d\mathbf{r} &= m_0 c^2 (\gamma(b) - \gamma(a)) \\ &= (m_B - m_A) c^2,\end{aligned}$$

where m_B is the relativistic mass at B and m_A the relativistic mass at A. In particular, if the particle is at rest at A and has speed v at B, the value of the integral reduces to (48.20).

The foregoing calculation suggests that we define the relativistic kinetic energy K by the formula

$$K = (m - m_0) c^2. \tag{48.22}$$

Note that this assigns zero kinetic energy to a body at rest. In Example 5 we will show that it assigns Newtonian kinetic energy $m_0 v^2/2$ to a body moving with a speed v that is small compared to c.

Example 5

Show that when the speed v is small compared to c, the relativistic kinetic energy K in Eq. (48.22) reduces to the Newtonian kinetic energy $m_0 v^2/2$.

Let $\delta = 1/\gamma$ so that $\delta^2 = 1/\gamma^2 = 1 - v^2/c^2$ and $1 - \delta^2 = v^2/c^2$. Then Eq. (48.22) can be written as

$$K = m_0 c^2 (\gamma - 1) = m_0 c^2 \left(\frac{1}{\delta} - 1\right) = m_0 c^2 \left(\frac{1 - \delta}{\delta}\right). \tag{48.23}$$

But

$$\frac{1 - \delta}{\delta} = \frac{1 - \delta}{\delta} \frac{1 + \delta}{1 + \delta} = \frac{1 - \delta^2}{\delta(1 + \delta)} = \frac{v^2/c^2}{\delta(1 + \delta)},$$

so (48.23) becomes

$$K = m_0 \frac{v^2}{\delta(1 + \delta)}.$$

But if v/c is small, then γ is nearly 1, so δ is nearly 1, and $1 + \delta$ is nearly 2, hence K is nearly equal to $m_0 v^2/2$, the Newtonian kinetic energy.

For those familiar with infinite series there is an alternate derivation using the binomial expansion

$$\frac{1}{\sqrt{1 - x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots,$$

valid for $|x| < 1$, where $+ \cdots$ refers to terms involving higher powers of x. Taking $x = v^2/c^2$ we find

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots.$$

Neglecting higher powers we obtain

$$\gamma - 1 \approx \tfrac{1}{2}v^2/c^2 \quad \text{and} \quad K = m_0 c^2(\gamma - 1) \approx \tfrac{1}{2}m_0 v^2.$$

The discussion that led to our definition of kinetic energy in Eq. (48.22) shows how an applied force increases both the mass and the speed of a body and thereby increases its kinetic energy. The difference $m - m_0$ is the increase in the mass over the rest mass m_0, and $(m - m_0)c^2$ is the corresponding increase in kinetic energy. Now the quantity mc^2 itself has units of energy, and when the mass increases from m_0 to m the quantity mc^2 changes by an amount equal to the kinetic energy acquired. Einstein called the quantity mc^2 the *total energy* E associated with a particle,

$$E = mc^2. \tag{48.24}$$

This is the famous Einstein equation, which says that mass and energy are equivalent. The quantity

$$E_0 = m_0 c^2 \tag{48.25}$$

is called the *rest-mass energy* or simply the *rest energy*. Equation (48.22) states that the kinetic energy K represents the change in the total energy of a particle,

$$K = E - E_0. \tag{48.26}$$

The equation $E = mc^2$ is one of the most important discoveries in twentieth century physics. When Einstein first proposed this equation in 1905, he wrote

> It is not impossible that with bodies whose energy content is variable to a high degree (e.g. with radium salts) the theory may be successfully put to the test.

The theory has indeed been abundantly put to the test. We'll discuss specific examples in the next section. Let us review briefly the line of reasoning verified by the successful tests.

The equation $E = mc^2$ arose out of our assumptions that relativistic momentum and relativistic mass are conserved. The final conclusion was that, in relativity, mass and energy are equivalent quantities, related by a constant of conversion, just as miles and kilometers are equivalent quantities, related by a constant of conversion. Thus, when we assumed that the total relativistic mass would be conserved, it turned out that this assumption was logically equivalent to assuming that energy would be conserved. So the underlying principle is the law of conservation of energy.

This point is not trivial, especially in the example we used – an *inelastic* collision, one in which kinetic energy is not conserved in any frame of reference. In classical mechanics that kind of collision is described by saying that all the kinetic energy is converted into heat or some other internal energy of the system. In relativity all energy, including heat, contributes to an increase in the rest mass of the impacted objects. All forms of energy, potential, kinetic, thermal, and every other kind of energy contribute to the relativistic mass of a body. The law of conservation of mass–energy states that the total energy in the universe is constant.

According to Newtonian mechanics, the nonrelativistic kinetic energy $m_0v^2/2$ of a body increases with the square of the speed, as shown on the lower curve in Fig. 48.5, reaching the value $m_0c^2/2$ when $v = c$. The total relativistic energy of a particle is plotted in the upper curve in Fig. 48.5. Its dependence on v/c is given by

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}}.$$

At zero speed the total energy is $E_0 = m_0c^2$, and as the speed increases towards c

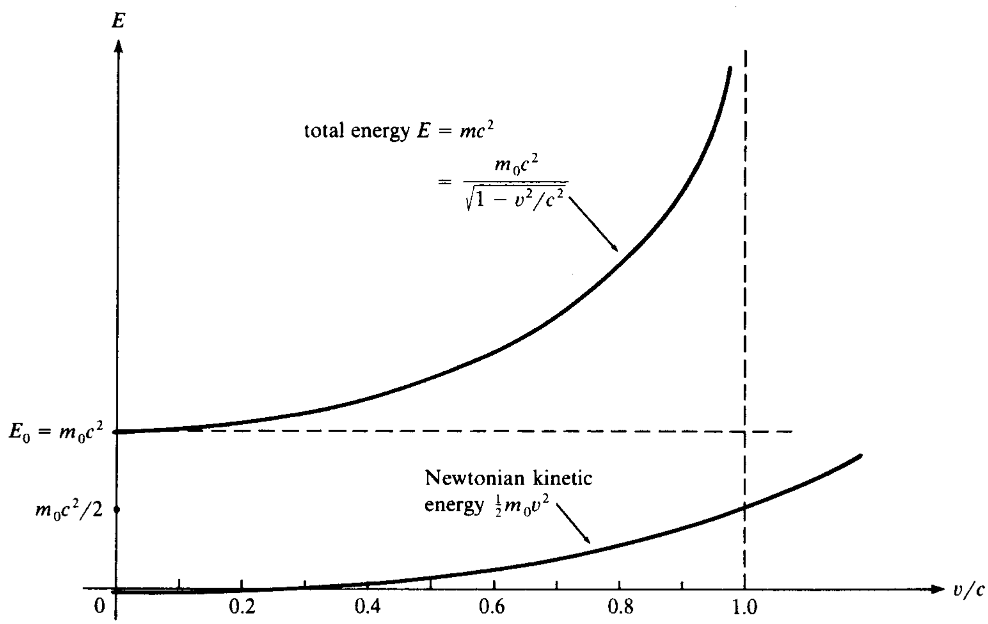


Figure 48.5 Comparison of Newtonian kinetic energy (lower curve) and relativistic total energy (upper curve) as functions of v/c.

the total energy becomes arbitrarily large. This indicates that to propel an object to the speed of light would require giving the object an infinite amount of energy. At small speeds the upper and lower curves have nearly the same shape because $E = E_0 + K$ and, as shown in Example 5, $K \approx m_0 v^2/2$ for small v, so $E - m_0 v^2/2 \approx E - K = E_0$, a constant.

We can gain greater insight by expressing the total energy E in terms of the magnitude of momentum, $p = mv$. Start with the equation $m = \gamma m_0$, square both members, solve for m_0^2, and use $1/\gamma^2 = 1 - v^2/c^2$ to get

$$m_0^2 = \left(1 - v^2/c^2\right) m^2.$$

Multiplication by c^4 gives us

$$m_0^2 c^4 = m^2 c^4 - m^2 v^2 c^2.$$

Substituting $p = mv$ and $E = mc^2$, we obtain

$$m_0^2 c^4 = E^2 - p^2 c^2$$

or

$$E^2 = (pc)^2 + \left(m_0 c^2\right)^2. \tag{48.27}$$

This result gives a relationship between energy and momentum that does not involve the speed. It shows that part of the energy is associated with the momentum and part with the rest mass. The graph of E as a function of p is shown in Fig. 48.6. This graph is compared with the corresponding relation from classical mechanics, $E - E_0 = m_0 v^2/2$, which can be written in terms of p as

$$E = E_0 + \frac{1}{2m_0} p^2.$$

The graph reveals what happens in the two extreme cases of very low speeds and very high speeds.

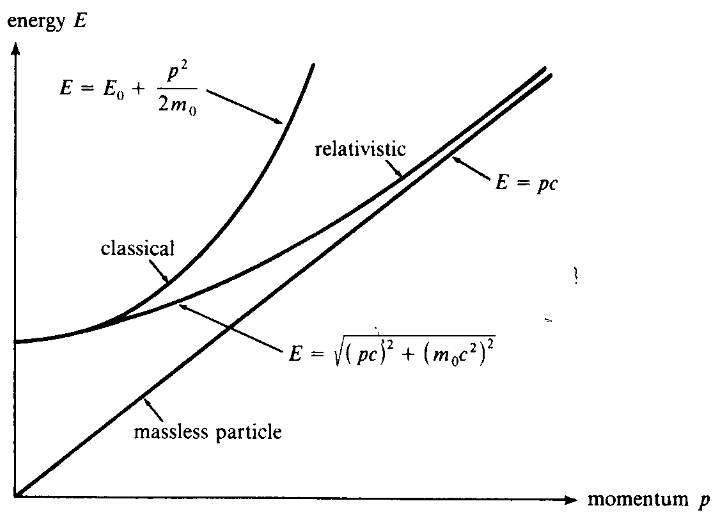


Figure 48.6 Total relativistic energy as a function of momentum.

For $v \ll c$ we have the classical region of a nonrelativistic particle. The relativistic equations for mass, momentum, and kinetic energy reduce to

$$m \approx m_0, \qquad p \approx m_0 v, \qquad K \approx \tfrac{1}{2} m_0 v^2.$$

In this region the kinetic energy is much less than the rest energy, as is seen by examining the ratio

$$\frac{K}{E_0} = \frac{\frac{1}{2} m_0 v^2}{m_0 c^2} = \frac{1}{2}\frac{v^2}{c^2} \ll 1.$$

At the opposite extreme, when v is close to c, p^2c^2 is nearly m^2c^4 and, because m is large compared to m_0, the term $m_0^2c^4$ in Eq. (48.27) is small compared to p^2c^2 and can be ignored. Then the relationship between energy and momentum is $E \approx pc$. At this extreme we also have $m \gg m_0$, $E \gg E_0$, and $K \approx E$.

There's a third interesting case – that of a particle with *zero* rest mass, $m_0 = 0$. Particles of this type include photons (particles of light) and neutrinos, subatomic particles produced by the weak nuclear force. For such a particle, energy and momentum make no sense in the classical viewpoint, but they are meaningful in relativity theory. From Eq. (48.27) we see that a particle with zero rest mass has total energy given by

$$E = pc, \tag{48.28}$$

which is also equal to its kinetic energy, $K = E$. Moreover, the equation $E = mc^2$ becomes $E = (p/v)c^2$, so the speed of such a particle is given by

$$v = pc^2/E, \tag{48.29}$$

which, when combined with Eq. (48.28) gives $v = c$. In other words, particles that have zero rest mass must travel at the speed of light.

One additional note regarding units. A convenient unit of energy for subatomic particles is the mega (million) electron volt, denoted by MeV, where

$$1 \text{ MeV} = 10^6 \text{ eV}.$$

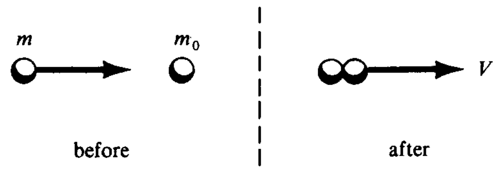
We recall that $1 \text{ eV} = 1.6 \times 10^{-19}$ J, so $1 \text{ MeV} = 1.6 \times 10^{-13}$ J. Another unit is the giga electron volt, GeV, where

$$1 \text{ GeV} = 1000 \text{ MeV} = 10^9 \text{ eV} = 1.6 \times 10^{-10} \text{ J}.$$

The Einstein relation $E = mc^2$ implies $m = E/c^2$, which gives a convenient measure of mass in terms of energy because c is a universal constant. A convenient unit of mass is MeV/c^2. In these units an electron has a mass of $0.511 \text{ MeV}/c^2$, and a proton has mass $938 \text{ MeV}/c^2$. To convert to kg we use the fact that $1 \text{ eV}/c^2$ equals 1.78×10^{-36} kg. Similarly, Eq. (48.29) implies $p = Ev/c^2 = (v/c)E/c$, so a convenient unit of momentum is that of energy divided by the speed of light, such as MeV/c or GeV/c.

Example 6

A particle of rest mass m_0 and kinetic energy $5m_0c^2$ strikes a stationary particle of rest mass m_0. The resulting composite particle has rest mass M_0 and moves with a speed V, as shown.



(a) What is the total momentum of the system before the collision?
(b) What is the total energy of the system before the collision?
(c) What is the speed V of the composite particle?
(d) What is the mass M_0 in terms of m_0?

(a) The total energy of the moving particle is related to its momentum by the equation

$$E^2 = (pc)^2 + (m_0c^2)^2, \quad (48.27)$$

so the momentum of the moving particle before the collision is

$$p = \frac{1}{c}\sqrt{E^2 - (m_0c^2)^2},$$

where E, the total energy of the particle, is given by

$$E = K + m_0c^2 = 6m_0c^2.$$

Using this in the expression for p we find that $p = \sqrt{35}\, m_0c$. But the other particle is not moving before collision, so this is also the total momentum p_T of the system before the collision.

$$p_T = \sqrt{35}\, m_0c.$$

(b) The total energy E_T of the system before the collision is the sum of the total energies of the two particles.

$$E_T = 6m_0c^2 + m_0c^2 = 7m_0c^2.$$

(c) After collision, the composite particle has momentum p_T and energy E_T. According to Eq. (48.29), the speed of the particle is given by

$$V = p_Tc^2/E_T = (\sqrt{35}\, m_0c)c^2/(7m_0c^2) = 0.85c.$$

(d) Applying Eq. (48.27) to the composite particle and solving for M_0 we have

$$M_0c^2 = \sqrt{E_T^2 - (p_Tc)^2}.$$

Substituting for E_T and p_T we find $M_0 = \sqrt{14}\, m_0$. Note that $M_0 \neq 2m_0$, so rest mass is not conserved. Instead, kinetic energy of the incident particle has been converted into mass of the final particle.

Questions

8. If mass is a form of energy, does a spring have more mass when it is compressed than when it is relaxed? Explain.

9. Does the equivalence of mass and energy depend on the nature of the forces involved? That is, does $E = mc^2$ hold for nuclear, electric, and gravitational forces? Explain.

10. How many grams of rest mass are needed to produce 1 J of energy if all the rest mass is transformed?

11. An electron ($m_0 = 9.1 \times 10^{-31}$ kg) travels at a speed of $0.6c$. What is its kinetic energy? How does this compare with the Newtonian kinetic energy at the same speed?

12. An electron has a total energy that is five times its rest-mass energy. Determine (a) its speed and (b) its momentum.

13. A particle of rest mass m_0 initially moves at a speed of $0.4c$.

(a) If its speed is doubled, by what factor does its kinetic energy increase?
(b) If the total energy is increased by the factor 100, by what factor is its speed increased?

14. Show that the momentum p of a particle is related to its kinetic energy K and rest mass m_0 by the equation $pc = \sqrt{K^2 + 2Km_0c^2}$.

15. A particle has a total energy of 5 GeV and a momentum of 3 GeV/c according to measurements in a certain frame.

(a) What is the energy of the particle in a frame in which its momentum is 5 GeV/c?
(b) What is the rest mass of the particle?
(c) What is the relative speed of the two frames?

16. **(a)** Show that in classical mechanics a particle having Newtonian kinetic energy K and momentum p has speed $v = dK/dp$.
(b) Show that the speed of a relativistic particle is given by dE/dp, where E is the total energy.

48.4 APPLICATIONS OF CONSERVATION OF RELATIVISTIC ENERGY AND MOMENTUM

The interconversion of mass and energy was a stunning prediction of relativity theory. In the decades since this prediction was made by Einstein, the equivalence of mass and energy has been confirmed by a host of experiments. The interconversion is most easily detected in processes involving subatomic particles

For example, consider the decay of an elementary particle known as a neutral pion, or pi zero (π^0). This subatomic particle, which has a rest mass of 135 MeV/c^2 and an average proper decay time of 0.83×10^{-16} s, decays into two photons. If the decay occurs in a frame in which the pion is at rest, what are the energies of the photons? We analyze the decay by applying conservation of relativistic momentum and energy.

In the rest frame of π^0 the particle is at rest and has zero momentum. Conservation of momentum requires that the total momentum of the photons in that frame must be zero. Therefore, the photons have momenta that are equal in magnitude but have opposite directions. But the momentum pc and energy E_γ of a photon are related by Eq. (48.28), $E_\gamma = pc$, so the photons also have equal energies.

According to the conservation of mass–energy, the mass of the pion appears entirely as energy of the photons because they have zero rest mass. If m_π is the rest mass of the pion and E_γ is the energy of either photon, conservation of mass–energy requires that

$$m_\pi c^2 = 2E_\gamma,$$

so that each photon has an energy

$$E_\gamma = m_\pi c^2/2,$$

a result that has been confirmed by experiment.

The reverse process – the conversion of energy into mass – is also commonly observed by nuclear physicists. Under certain conditions, photons passing near nuclei disappear, with the simultaneous creation of particles with rest mass.

On a larger scale, the energy produced in nuclear reactors is the result of the conversion of the rest mass of uranium nuclei into energy in a process called fission. On an even grander scale, the sunlight received on Earth results from fusion reactions taking place in the core of the sun, where, through a series of reactions, hydrogen is converted into helium; the difference in rest mass between the initial reactants and the final products appears as energy. Consequently, the mass of the sun is actually decreasing as it radiates energy into space.

Example 7

An elementary particle called the neutral K meson, or kaon, decays into two neutral pions. The rest mass of a kaon is 498 MeV/c^2 and that of a neutral pion is 135 MeV/c^2.

(a) If a kaon decays at rest in the laboratory frame, what is the kinetic energy of the resulting pions?

(b) What is the speed of each pion?

(a) Conservation of energy requires that the energy of the kaon, which is entirely rest-mass energy, be equal to the total energy of the resulting pions. Thus, we have

$$m_K c^2 = K_T + 2m_\pi c^2,$$

where K_T is the total kinetic energy of the pions. Solving for K_T and substituting for the rest masses, we find $K_T = 228$ MeV.

(b) Because the initial kaon is at rest and has zero momentum, conservation of momentum requires that the pions have equal and opposite momenta, and because the particles have identical rest masses, it follows that they also are produced with equal kinetic energies. Therefore, the kinetic energy of either pion is $K = \frac{1}{2}K_T =$

114 MeV. Applying Eq. (48.27) to a single pion, we find

$$p = \sqrt{E^2 - \left(m_\pi c^2\right)^2}/c = 209 \text{ MeV}/c,$$

where we used $E = K + m_\pi c^2$. Then by Eq. (48.29), we find the speed to be $v = pc^2/E = 0.84c$.

Questions

17. Would it be possible for a pi zero at rest to decay into two identical particles of rest mass 70 MeV/c^2? Would the decay be possible if the pi zero is moving? Explain.

18. Radiation from the sun reaches the earth at a rate of 1400 W/m^2 at the equator at noon on the first day of spring.

(a) What is the loss in mass of the sun each second?
(b) What is the percentage decrease in the mass of the sun over 10 billion years? (The distance from the earth to the sun is 1.5×10^{11} m and the present mass of the sun is 2.0×10^{30} kg.)

19. An experimenter observes a proton and an antiproton (a particle having the same rest mass as a proton, 938 MeV/c^2, but opposite charge) approach each other from opposite directions. Each particle has kinetic energy equal to twice its rest-mass energy. The particles collide and annihilate each other, and two photons are created. In the frame of the experimenter, determine each of the following:

(a) the momentum of the proton;
(b) the speed of the proton;
(c) the total energy of each photon;
(d) the momentum of each photon.
(e) Can the proton and antiproton annihilate each other and in the process create only a single photon?

20. A particle of rest mass m_0 and kinetic energy $3m_0c^2$ strikes and becomes bound to an initially stationary particle of rest mass $5m_0$. The composite particle moves off with a speed V.

(a) Find the speed of the composite particle in terms of the speed of light.
(b) Determine the rest mass of the composite particle in terms of m_0.
(c) Find the speed of the incoming particle.

21. A pi meson of rest mass m_π decays at rest into a muon (rest mass m_μ) and a neutrino (of zero rest mass). Show that the kinetic energy of the muon is given by $K(m_\pi - m_\mu)^2c^2/(2m_\pi)$.

22. The Σ^+ particle is an unstable elementary particle with a rest mass of 1.19 GeV/c^2 and a mean proper lifetime of 0.8×10^{-10} s. What is the minimum kinetic energy such a particle must have to travel a distance of 10 cm before decaying?

23. Show that it is dynamically impossible for a single photon to strike a stationary electron and give up all its energy to the electron.

48.5 A FINAL WORD

When Einstein wrote his famous paper on the theory of relativity in 1905, he was only 26 years old and was employed as a patent clerk in Berne, Switzerland. The paper changed the meaning of space and time, and superseded Newtonian mechanics, which had been the basis of all physics, and indeed of all philosophy, for 200 years. What sort of man was this patent clerk who had the audacity to throw out everything that was known about the world and invent a complex theory that contained no internal contradictions? This may be an appropriate time to say a few words about his background.

Einstein was born in 1879 and died in 1955. His father, an unsuccessful Munich businessman, was an owner of an electrochemical plant that failed. The family eventually left Germany, spending one year in Milan, Italy, and then moving to Switzerland, where Einstein finished high school. As a child, Albert was a slow learner and didn't learn to speak until after he was 3 years old. Although he was an indifferent student, he finished high school in Switzerland and went to the famous Swiss Federal Polytechnic (ETH) in Zurich, from which he graduated in 1900.

From 1902 to 1909 he was employed as a patent clerk in Berne, and during that period laid some of the foundations for twentieth century physics. His contributions were not only to relativity. During the epic year 1905 he published three landmark papers in the same volume of the journal *Annalen der Physik*. For the first of these, on the photoelectric effect, he was awarded the Nobel prize in physics in 1921. The second paper, an outgrowth of his Ph.D. thesis for the University of Zurich, was an analysis of the phenomenon known as Brownian motion. If he had done nothing else, this work by itself would have made him famous. The third paper presented the special theory of relativity. During his years as a patent clerk he also invented the quantum theory of solid state physics.

The period from 1902 to 1909, and especially the year 1905, can be compared to the period Newton spent on his farm in Lincolnshire in the plague years 1665–6. It was an immensely rich and productive period in Einstein's life. By 1909 the academic world had heard of him, and he was appointed Associate Professor of Physics at the University of Zurich where he had done his graduate work. In 1911 he moved to Prague, then back to his earlier alma mater, ETH, in 1912, and in 1914 he was appointed director of the Kaiser Wilhelm Institute in Berlin.

From 1909 to 1916 he tried to extend his theory of relativity to encompass the phenomenon of gravity, and by 1916 he succeeded, and produced what is known today as the general theory of relativity.

In a letter to George Ellery Hale of Caltech dated 1913, Einstein states, "...some simple theoretical considerations have led me to believe that...light passing near the sun would be deflected by the sun." That is, the gravitational pull

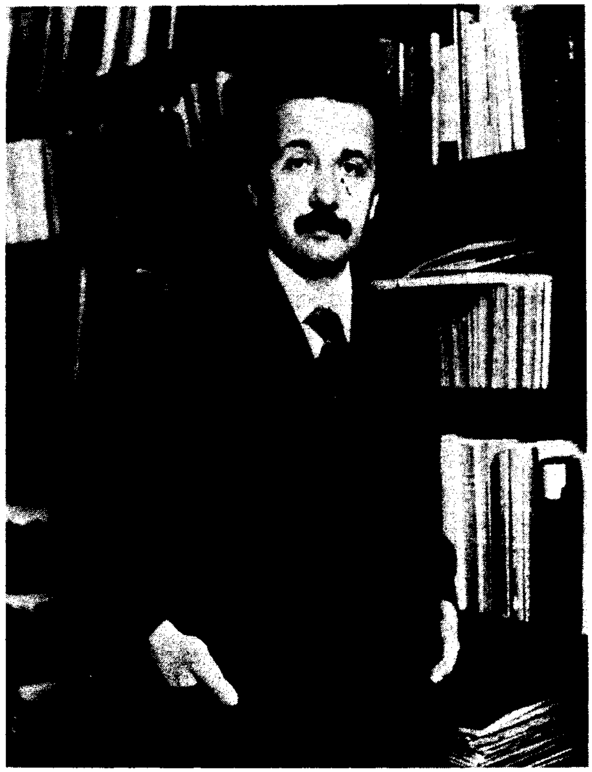


Figure 48.7 Photograph of Einstein circa 1916. (By permission of the Hebrew University of Jerusalem, Israel.)

of the sun would deflect a light beam passing near it. Einstein predicted the angle of deflection as 0.84 s of arc. It turned out that 0.84 was the wrong numerical value. There are two possible results: one deflection derived from special relativity and another twice as large predicted from general relativity neither of which is 0.84 s of arc. Einstein knew there would be a deflection, but arithmetic was not one of his strong points and he made an arithmetical error in his calculation. He asked Hale to use the Mt. Wilson Observatory telescope (the world's largest telescope at the time) to look toward the sun and see if stars near the sun seem to be displaced a little bit.

Hale replied, in essence, that they wouldn't point the telescope at the sun because it might cost the observer his eyesight. He also suggested that the ideal time to conduct such an experiment is during a solar eclipse. Because of World War I, the experiment could not be carried out until 1919. Eclipses are visible only in certain regions, and in 1919 two expeditions were sent by the Royal Astronomical Society, one to northern Brazil, and one to West Africa. Both found the displacement of light near the sun to be close to what Einstein's general theory had predicted. Sir Arthur Eddington, leader of the African expedition, cabled the news to Einstein and to the Society. The announcement had an effect that is very difficult to understand today. It was not only the scientific community that was impressed, but the world at large. Although it was an obscure, arcane discovery, which very few people understood, the newspapers and all the media gave it wide publicity and, as a result, Einstein immediately became an international folk hero. In later years the experiment was refined and repeated many times. The average deflection observed at 11 different eclipses agrees with the predicted value to within one part in 500.

It's hard to explain the sudden surge of public adulation of Einstein. Perhaps the public, weary from World War I, craved good news and was desperate for a nonpolitical hero. Or perhaps it was because Einstein resembled everyone's favorite uncle. In any case, Einstein's name became a household word and he became a legend in his own time.