CHAPTER 46

THE LORENTZ TRANSFORMATION

The hypothesis [that the dimensions of solid bodies are slightly altered by their motion through the aether] certainly looks rather startling at first sight, but we can scarcely escape from it, so long as we persist in regarding the aether as immovable. We may, I think, go so far as to say that, on this assumption, Michelson's experiment *proves* the changes of dimensions in question. . . .

H. A. Lorentz, *The Theory of Electrons,* page 196 (Lectures delivered at Columbia University in 1906, published in 1909 and revised in 1915)

46.1 INTERPRETING THE MICHELSON – MORLEY EXPERIMENT

The Michelson–Morley experiment demonstrated that light could travel at the same speed in all directions according to all inertial observers. This seemed to conflict with Maxwell's magnificent theory of electromagnetism, which assumed the existence of an aether to propagate light. Several physicists put forth theories that attempted to preserve the aether and at the same time explain the results of the Michelson–Morley experiment. One of these was George Fitzgerald of Ireland, who proposed in 1889 that all moving bodies were foreshortened in the direction of their motion through the aether.

Specifically, Fitzgerald conjectured that if a linear object like a rod has length L_0 when it is at rest, then its length contracts to $L_0\sqrt{1 - v^2/c^2}$ when it moves with speed v through the aether. This contraction is negligible for ordinary objects moving at speeds that are small compared to c, the speed of light. Even the earth moving rapidly around the sun would only be contracted by an amount equal to the length of a blade of grass. But Fitzgerald's contraction is exactly the amount needed to explain the result of the Michelson–Morley experiment. One arm of the Michelson interferometer would be contracted just enough so that the light beams in the experiment would travel the same time along both arms, and consequently no interference fringe shift would be observed. Despite the ingenuity of the idea, most of Fitzgerald's scientific friends scoffed at it.

In 1892 the Dutch physicist Hendrik Lorentz independently proposed the same idea. Lorentz, who was the world's greatest expert on Maxwell's theory of electromagnetism, had a reputation that commanded attention. He attributed the contraction to interactions of electrons with the aether in which they were imbedded.

Figure 46.1 Photograph of Lorentz and Einstein. (From AIP Neils Bohr Library.)

(Originally the name "electron" applied to both positive and negative charges inside atoms; later the positive charges were found to be different from electrons and became known as protons. Little was known about electrons in the 1890s.) Lorentz's ideas represented a radical departure from traditional dynamics. For example, he proposed that the mass of an electron would vary with its motion through the aether and that its dimensions would contract in the direction of motion because of a modification of molecular forces in that direction. He reasoned that if electrons have these properties and if electric forces bind matter, then macroscopic bodies would exhibit the same properties.

In particular, measuring sticks, used to measure distance, and clocks, used to measure time, would exhibit these properties in their motion through the aether: measuring sticks would contract in the direction of their motion and clocks would run more slowly. To determine precisely how to relate measurements of distance and time so that motion through the aether can't be detected, Lorentz derived a set of equations that relate the measurements of two observers. These equations, known as the *Lorentz transformation*, are the focus of this chapter. They became the basic core of the special theory of relativity later developed by Einstein, even though Lorentz's ideas about electron reactions with the aether did not survive. Lorentz firmly believed in the aether frame and an absolute time scale, but Einstein later argued that those concepts were incorrect, and there was no need to assume the existence of aether.

46.2 THE POSTULATES OF THE SPECIAL THEORY OF RELATIVITY

The French mathematician, Henri Poincaré, objected to the ad hoc explanation of Lorentz and others for the null result of the Michelson–Morley experiment. He believed there must be a central principle to explain the results of the experiment, and proposed the principle of relativity, which states that absolute motion is impossible to detect. In a widely heralded address made at the St. Louis exhibition in 1904, he announced "... the principle of relativity, according to which the laws of physical phenomena should be the same, whether for an observer fixed, or for an observer carried along in a uniform movement of translation; so that we have not and could not have any means of discerning whether or not we are carried along in such a motion."

Poincaré suggested that electromagnetic theory be revised to conform to the principle of relativity. Although he was one of the first to advocate that the relativity of mechanics should be generalized into a universal principle encompassing all physical laws, Poincaré himself did not construct such a theory. On the one hand, he thought that the aether might vanish from the theory of light much like the ideas of caloric and electric fluids vanished into the backwaters of the history of physics. On the other hand, he apparently felt that some kind of medium was necessary for the propagation of light. In any case, Poincaré did not develop the theory of relativity and its radical consequences. That revolution in thought had to await the genius of Albert Einstein.

In 1905 Einstein published three remarkable papers in the same volume of the highly respected physics journal *Annalen der Physik*. The first, entitled "Generation and Transformation of Light," proposed that light can be regarded as a stream of photons. For the results in this paper he was awarded the Nobel prize in 1921. The second, "Motion of Suspended Particles in the Kinetic Theory," was concerned with a phenomenon called Brownian motion and helped to establish the existence of atoms, and in the third, "On the Electrodynamics of Moving Bodies," he introduced a bold new treatment of the theoretical problems that had been plaguing Lorentz, Poincaré, and others. This paper is now regarded as the cornerstone of the theory of special relativity.

Einstein's goal was not to explain the null result of the Michelson–Morley experiment – by Einstein's own account he had no knowledge of that experiment when he wrote his paper. Instead, his goal was to establish relativity as a fundamental universal principle for all of physics. Although he capitalized boldly on its consequences, Einstein did not attempt to account for the principle itself in terms of other hypotheses. He elevated the principle of relativity to the status of a fundamental postulate not requiring further explanation. This is now known as the *first postulate of the special theory of relativity*, which, in Einstein's own words, is stated as follows:

> The same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good.

Einstein's *second postulate* concerns the speed of light:

> Light is always propagated in empty space with a definite velocity c, which is independent of the state of motion of the emitting body.

Taken together, these postulates imply that the speed of light (one of the "laws" of electrodynamics) must be independent of the state of motion of the observer. Although Einstein did not realize it at the time, the Michelson–Morley experiment provided experimental verification of those postulates. Einstein proceeded to show that these two postulates implied that the fundamental concepts of length and time must be regarded as *relative* quantities rather than *absolute* quantities.

To illustrate the relative nature of time and length we shall describe a thought experiment that is suggested by the Michelson–Morley experiment. Imagine a moving trolley car with an observer in the center of the car, as illustrated in Fig. 46.2a. The observer turns on a light source and measures the time t' it takes for the light to travel vertically upward to a mirror on the ceiling and reflect back down. If the mirror is at height h above the observer, the light travels a total distance $2h$ with constant speed c, hence $2h = ct'$ or

$$t' = \frac{2h}{c}.$$

Now we ask an observer on the ground to make the same observation. He sees things differently because the trolley car is moving past him. He sees the light move along the two legs of an isosceles triangle as shown in Fig. 46.2b, a path whose length is greater than $2h$. But the speed of light is the same for both observers, so,

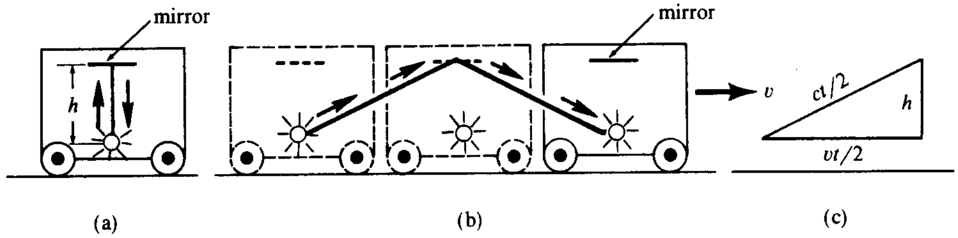


Figure 46.2 A thought experiment with a moving trolley car. (a) Path of light according to an observer in the car. (b) Path according to an observer on the ground. (c) Right triangle used to calculate time t.

relative to the ground observer, it takes a greater time to travel the greater distance. Thus we see that Einstein's postulates imply that time is a *relative* quantity, not an absolute quantity. The observer in the moving car would say that the clock of the observer on the ground runs more slowly than the clock on the train.

It's easy to calculate the exact relation between the two times. Suppose that the car is moving with a horizontal speed v relative to the ground. Then the base of the isosceles triangle has length vt, where t is the time measured by the observer on the ground. Applying the theorem of Pythagoras to the triangle in Fig. 46.2c we find

$$(ct/2)^2 = (vt/2)^2 + h^2.$$

If $v^2 < c^2$ we can solve for t to obtain

$$t = \frac{2h}{\sqrt{c^2 - v^2}} = \frac{2h}{c}\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{t'}{\sqrt{1 - v^2/c^2}}.$$

The factor multiplying t' is usually denoted by γ,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}},$$

and is called the *time dilation* factor. The time taken by the light for the back and forth trip in the moving system appears to the stationary observer to be longer than to the moving observer by the factor γ. It is the same factor we saw in connection with the Michelson–Morley experiment in Chapter 45. Note that $\gamma \geq 1$, with $\gamma = 1$ only if $v = 0$.

The same thought experiment can be used to illustrate that length is also a relative quantity. Suppose the same observers are asked to measure the length of a horizontal rod. Specifically, suppose the rod is on the ground and joins the two points directly below the base of the isosceles triangle in Fig. 46.2b, which is shown again in Fig. 46.3. The observer on the ground measures this distance with a measuring stick and finds it equal to L_0, say. This distance L_0 is also equal to vt, as previously noted. But the observer in the car cannot measure the rod directly because he is in motion relative to the rod. Instead, he calculates the length of the rod by measuring the time it takes to move from one endpoint of the rod to the

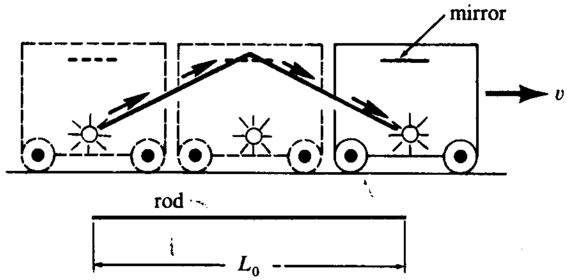


Figure 46.3 A rod measured by an observer on the ground has length L_0. An observer in the moving car determines its length to be L', where $L' < L_0$.

other, and multiplying this by the speed with which he is traveling relative to the rod. As previously seen, the time is t', and the speed is v, so he calculates the length of the rod to be $L' = vt'$. If $v \neq 0$, then $\gamma > 1$ and $t' < t$, so $vt' < vt$ or $L' < L_0$. In other words, the length of the rod is smaller when measured by a moving observer than when measured by a fixed observer. Thus, Einstein's postulates imply that length is a relative quantity, and, in fact, the contraction factor for length is exactly $1/\gamma$, the reciprocal of the time dilation factor. The dilation factor γ will appear again in the Lorentz transformation to be derived in the next section.

Questions

1. Suppose that spacecraft A is moving at $1/3$ the speed of light, $c/3$, relative to a space station. Spacecraft B travels in the opposite direction at $1/2$ the speed of light, $c/2$. If an observer in spacecraft A turns on a flashlight pointed toward the front of the craft, what does an observer in B measure for the speed of light?

2. If you were traveling near the speed of light and looked at yourself in a mirror you were carrying, would you appear "contracted"? Explain your reasoning.

46.3 THE LORENTZ TRANSFORMATION

In the year before Einstein's momentous paper was published, Lorentz presented, as part of his electron theory, a set of relativistic transformation equations that relate the measurements of two inertial observers. Although Lorentz's electron theory was wrong, his transformation equations are correct and they are the basic relations of the special theory of relativity. In this section we shall derive the Lorentz transformation equations from Einstein's two postulates for the special theory of relativity.

Recall that an event is something that happens at some point in space at some instant of time as measured by an observer. In one reference frame S, which we regard as stationary, we label each event with an ordered quadruple of numbers (x, y, z, t), where (x, y, z) represents the point in space and t the time. In another reference frame S', which we regard as moving relative to S, the same event will be represented by another quadruple of numbers, say (x', y', z', t'). The Lorentz transformation is a set of equations relating the quantities x, y, z, t with x', y', z', t'.

We consider, as Einstein did in his original paper, a fixed frame S and another frame S' that coincides with S at time $t = t' = 0$ and that moves with constant velocity v parallel to the x axis of S. Because there is no relative motion along the y or z directions, it seems reasonable to assume that

$$y' = y \tag{46.1}$$

and

$$z' = z, \tag{46.2}$$

so we only need to find the relations between x, t and x', t'. We assume throughout that $v \neq 0$.

In his original paper, Einstein claimed that the equations relating x and t with x' and t' must be linear "on account of the properties of homogeneity which we attribute to space and time," but Einstein did not explain what he meant by "homogeneity." Quite possibly he meant that every straight line in the xt plane corresponds to a straight line in the $x't'$ plane, in which case it is known that this does indeed require linear relations. The approach we take here is simply to write down a pair of linear relations,

$$x = Ax' + Bt', \tag{46.3}$$

$$t = Cx' + Dt', \tag{46.4}$$

and show that the coefficients A, B, C, D can be determined in terms of the constant velocity v of the moving frame and the constant velocity c of light. We also want the equations to be solvable for x' and t' in terms of x and t. This requires that $AD - BC$ must be different from zero, or in other words, that $AD \neq BC$. We'll see later that this requirement reduces to $v^2 \neq c^2$.

We now proceed to determine the coefficients A, B, C, D. First we use the fact that the origin of S' is moving at constant velocity v relative to S. The origin in S' corresponds to taking $x' = 0$, so in frame S the origin of S' has coordinates $x = Bt'$ and $t = Dt'$, which means $Dx = Bt$. Differentiating this equation with respect to t and using the fact that the velocity of the moving origin of S' is $dx/dt = v$, we find that

$$Dv = B.$$

This equation determines B in terms of D.

Now the observer in the S' frame thinks he is at rest and that the origin of the S frame is moving relative to S' with constant velocity $-v$. (This is where we use the first postulate of special relativity.) The origin of S corresponds to $x = 0$. Taking $x = 0$ in Eq. (46.3) we find

$$Ax' + Bt' = 0.$$

Differentiating this equation with respect to t' and using

$$-v = dx'/dt',$$

we find $-Av + B = 0$ or

$$B = Av.$$

Comparing this with the previous equation for B we see that $D = A$. Therefore, we can put $B = Av$ and $D = A$ in the transformation equations (46.3) and (46.4) to get

$$x = A(x' + vt'), \qquad t = Cx' + At'. \tag{46.5}$$

The last equation can also be written as

$$t = A(Ex' + t'), \tag{46.6}$$

where $E = C/A$. (Note that $A \neq 0$, otherwise Eq. (46.5) would imply that we would always have $x = 0$, hence $v = dx/dt = 0$, which we have excluded.)

We now have only two coefficients to determine, A and E. The numbers A and E depend only on the two constant velocities v and c and not on the particular type of event being observed. Therefore, to determine A and E we shall examine a very special event.

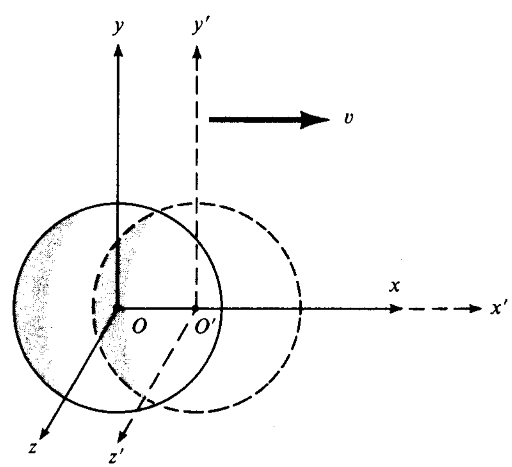
Consider a light source at rest at the origin O of frame S, and suppose there is a stationary observer in S. At any time t after a light pulse is emitted from O the observer sees a wavefront of light spreading outward from O as a sphere. Because light travels at a constant velocity c, the radius of the sphere is ct. Therefore, any point (x, y, z) on the surface of this light sphere must satisfy the equation

$$x^2 + y^2 + z^2 = (ct)^2, \tag{46.7}$$

because each side of this equation is the square of the radius of the sphere.

Now consider a second observer in the moving frame S', and assume that frame S' coincides with S at the instant the light pulse is emitted. The second observer sees the light spreading from O' as a different sphere, illustrated in Fig. 46.4. At time t' this sphere has radius ct', so any point (x', y', z') on its surface satisfies the equation

$$x'^2 + y'^2 + z'^2 = (ct')^2. \tag{46.8}$$



Light spheres spreading outward from sources O and O'.

Equation (46.7) can be written as

$$y^2 + z^2 = c^2t^2 - x^2,$$

whereas (46.8) can be written as

$$y'^2 + z'^2 = c^2t'^2 - x'^2.$$

But we also have $y = y'$ and $z = z'$, as suggested by Eqs. (45.5) and (45.6), so the last two equations imply

$$c^2t^2 - x^2 = c^2t'^2 - x'^2. \tag{46.9}$$

This means that the general form of the Lorentz transformation in (46.5) and (46.6) must be consistent with (46.9) in the special case of light spheres. It may be of interest to note that the Galilean transformation derived in the previous chapter is *not* consistent with (46.9). In this case, the Galilean transformation becomes

$$x = x' + vt,$$
$$t = t'.$$

Using $t = t'$ in (46.9) we find $x^2 = x'^2$, and this cannot be satisfied if $x = x' + vt$ because $v \neq 0$.

Returning once again to the transformations, we substitute (46.5) and (46.6) into Eq. (46.9) and obtain

$$c^2A^2(Ex' + t')^2 - A^2(x' + vt')^2 = c^2t'^2 - x'^2.$$

When $t' = 0$, this equation must hold for all x', in particular for $x' = 1$, which implies that the coefficients of the x'^2 terms must be equal on both sides, giving us

$$c^2A^2E^2 - A^2 = -1. \tag{46.10}$$

Similarly, the coefficients of the t'^2 terms must be equal,

$$c^2A^2 - A^2v^2 = c^2. \tag{46.11}$$

Equation (46.11) determines A^2 only if $c^2 \neq v^2$, and it gives us

$$A^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - (v/c)^2}. \tag{46.12}$$

When this value of A is inserted in Eq. (46.10) we find that $E = v/c^2$. If we want the positive x axis to correspond to the positive x' axis we take the positive square root and obtain

$$A = \frac{1}{\sqrt{1 - (v/c)^2}}. \tag{46.13}$$

Therefore, we have determined all the coefficients A, B, C, D in terms of v and c. Coefficient A is given by (46.13) and for the others we have

$$B = vA,$$
$$C = EA = (v/c^2)A,$$
$$D = A.$$

The condition $AD \neq BC$, which is needed in order to be able to solve for x' and t' in terms of x and t, now becomes $A^2 \neq (v/c)^2 A^2$ or

$$v^2 \neq c^2,$$

the same condition that allowed us to determine A. In other words, the coefficients in the Lorentz transformation can be determined only if the velocity v of the moving frame is different from the velocity of light. And Eq. (46.13) shows that we need $v^2 < c^2$ to obtain a real value for A.

It is common practice to use the symbol γ for the coefficient A in (46.13):

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}. \tag{46.14}$$

This is the same dilation factor $\gamma > 1$ that we encountered earlier in Section 46.2. The four linear equations relating x, y, z, t and x', y', z', t' now become

$$x = \gamma(x' + vt'), \quad (46.15x)$$

$$y = y', \quad (46.15y)$$

$$z = z', \quad (46.15z)$$

$$t = \gamma(t' + vx'/c^2). \quad (46.15t)$$

This is the *Lorentz transformation*; it relates the coordinates of an event in one inertial frame with those in another inertial frame that is moving with relative velocity v.

The inverse transformation, which gives the coordinates in S' in terms of those in S, is listed in Table 46.1. It can be obtained by solving the foregoing system for x' and t' in terms of x and t. Note that the inverse transformation can also be obtained by simply replacing v with $-v$ and interchanging primed and unprimed quantities.

Table 46.1 Lorentz Transformation for Relative Motion in the x Direction

$x = \gamma(x' + vt')$	$x' = \gamma(x - vt)$
$y = y'$	$y' = y$
$z = z'$	$z' = z$
$t = \gamma(t' + vx'/c^2)$	$t' = \gamma(t - vx/c^2)$

Example 1

In one inertial frame an explosion occurs at the coordinates $(x, y, z, t) = (6\text{ m}, 0, 0, 2 \times 10^{-8}\text{ s})$. Determine the coordinates of this event according to an observer who is moving along the positive x axis with a relative speed of $0.8c$, assuming that at $t = t' = 0$ the origins coincide.

To find the coordinates in frame S' we use the Lorentz transformation. A relative speed of $0.8c$ indicates that

$$\gamma = (1 - 0.8^2)^{-1/2} = \tfrac{5}{3}.$$

Therefore,

$$x' = \gamma(x - vt) = (\tfrac{5}{3})(6 - 4.8) = 2 \text{ m},$$

$$t' = \gamma(t - vx/c^2) = (\tfrac{5}{3})(2 \times 10^{-8} - (0.8)6/(3 \times 10^8)),$$

$$t' = 0.67 \times 10^{-8} \text{ s}.$$

Of course, $y' = 0$ and $z' = 0$. Therefore, the coordinates of the event in the S' frame are $(2 \text{ m}, 0, 0, 0.67 \times 10^{-8} \text{ s})$.

In the Lorentz transformation equations the x and t coordinates of an event in one frame become commingled with the relative velocity to give the coordinates x' and t' of the *same* event in the moving frame. In other words, space and time are no longer completely independent of one another. An event that happens at a certain place and time according to one observer happens at different space–time coordinates according to a moving observer. The Lorentz transformation gives us a precise mathematical relation connecting the coordinates of an event measured by two inertial observers. This transformation is the relativistic generalization of the Galilean transformation to which it reduces when $v \to 0$ with c fixed, or when $c \to \infty$ with v fixed (see Question 9).

Example 2

In a certain inertial reference frame, the time between two events occurring in the same position is 10 s. According to an observer moving relative to that frame the events are separated in time by 12 s. What is the relative speed of the two frames?

From the equation for t' in Table 46.1, the time $t_2' - t_1'$ between two events is given by

$$t_2' - t_1' = \gamma\left[(t_2 - t_1) - v(x_2 - x_1)/c^2\right].$$

In the case at hand we have $t_2' - t_1' = \Delta t' = 12$ s, $t_2 - t_1 = \Delta t = 10$ s, and $x_2 = x_1$, so $\Delta t' = \gamma \Delta t$, hence $1/\gamma = \Delta t/\Delta t' = 5/6$. Solving for v in the equation $\gamma = 1/\sqrt{1 - v^2/c^2}$, we get

$$v = c\sqrt{1 - (1/\gamma)^2} = 0.55c.$$

Questions

Questions 3 through 6 refer to the following situation. An observer S' is in a train whose rest length is 100 m. A flashbulb is placed at the center of the train.

Clocks located at the front and rear of the train are arranged to begin ticking when light from the flashbulb reaches them. The train is moving at a speed of $0.6c$ relative to an observer S on the ground, who is directly opposite the flashbulb when it flashes. At that instant this observer sets his clock to begin ticking.

3. What is the length of the train according to observer S?

4. Show that the clock of observer S reads $25/c$ s when the light flash reaches the rear of the train.

5. Determine the time on the clock of observer S when the light flash reaches the front of the train.

6. According to S, calculate the amount of time by which the clock at the rear of the train leads the one at the front.

7. If event A occurs before event B in some inertial frame, is there a moving frame in which event B occurs before event A?

8. Two events are simultaneous in one frame and are separated by a distance Δx. Are the events simultaneous in a frame moving with speed v? If not, find the time difference in terms of Δx, v, and c.

9. Show that the Galilean transformation is a limiting case of the Lorentz transformation when (a) $v \to 0$ with c fixed or (b) $c \to \infty$ with v fixed.

10. Show that the Lorentz transformation implies

$$x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2.$$

This is described by saying that the quadratic form $x^2 + y^2 + z^2 - (ct)^2$ is an invariant of the Lorentz transformation. It has the same value in any inertial frame.

46.4 LENGTH CONTRACTION

We return once again to the thought experiment described in Section 46.2 in which an observer on a moving trolley car determined the length of a rod on the ground to be smaller than that measured by an observer on the ground. We will show that the Lorentz transformation also describes length contraction.

The observer on the ground is in one frame S. He measures the position of the rear end of the rod at time t_1 and obtaines the value x_1. He measures the position of the forward end of the rod at time t_2 and obtains the value x_2. These measurements can be regarded as labels (x_1, t_1) and (x_2, t_2) for two events in S. The observer in the car is in another frame S' and he describes the same events with labels (x_1', t_1') and (x_2', t_2'). In other words, measuring the length of the rod requires two events to be observed in each frame, and, of course, the labels for these events are related by the Lorentz transformation. By applying the Lorentz transformation for x' and t' twice and subtracting we obtain

$$x_2' - x_1' = \gamma(x_2 - x_1) - \gamma v(t_2 - t_1), \tag{46.16}$$

$$t_2' - t_1' = \gamma(t_2 - t_1) - \gamma(v/c^2)(x_2 - x_1). \tag{46.17}$$

Now if the rod is at rest in the S frame, it doesn't matter when the observer on the ground makes his measurements because the rod is not moving. But in order to get an accurate, honest measurement of its length in the S' frame we must insure that the two ends are marked *simultaneously* in S'. Otherwise, relative to the observer in S', the rod will move between the marking of one end and the marking of the other, and the measured distance will not be correct. This means we require that $t_2' = t_1'$ in S'. Using this in Eq. (46.17) we determine the time difference $t_2 - t_1$ in terms of $x_2 - x_1$:

$$t_2 - t_1 = \frac{v}{c^2}(x_2 - x_1).$$

But $x_2 - x_1 = L_0$, the rest length of the rod as measured by the observer on the ground, so this last equation gives us

$$t_2 - t_1 = \frac{vL_0}{c^2}.$$

Therefore, if the observer in the S frame makes his measurements at times differing by exactly the amount vL_0/c^2, these events will be simultaneous in frame S'. Substituting this value of $t_2 - t_1$ into Eq. (46.16) we find

$$x_2' - x_1' = \gamma\left(L_0 - \frac{v^2L_0}{c^2}\right) = \gamma L_0\left(1 - \frac{v^2}{c^2}\right) = \frac{L_0}{\gamma}. \tag{46.18}$$

This shows that the length of the rod as measured in S' is contracted by the factor $1/\gamma$, just as we found by a different argument in Section 46.2.

This contraction in the direction of motion was exactly what Fitzgerald had called for to explain the Michelson–Morley experiment. Of course it should not be surprising that Lorentz's result does the trick; it was specifically designed for that purpose. But Lorentz's purpose was not merely to explain the result of the Michelson–Morley experiment. Instead, he was trying to determine what it would take to make absolute motion undetectable. Although his electron theory was wrong, it had caused him to ask the right question, and he obtained the right answer.

We could, of course, reverse the process by placing a rod in the trolley car in the S' frame and ask an observer in the S frame to measure its length. In this case we use the formulas in the Lorentz transformation that express x and t in terms of x' and t'. Using these formulas twice and subtracting we find

$$x_2 - x_1 = \gamma(x_2' - x_1') + \gamma v(t_2' - t_1').$$

$$t_2 - t_1 = \gamma(t_2' - t_1') + (\gamma v/c^2)(x_2' - x_1').$$

Now the rod is at rest in the S' frame so it does not matter when an observer in S' measures its length. But for the observer in S to get an accurate length, he must make his two measurements at the same time, that is, when $t_1 = t_2$. Setting $t_2 - t_1 = 0$ in the second equation we find

$$t_2' - t_1' = -(v/c^2)(x_2' - x_1').$$

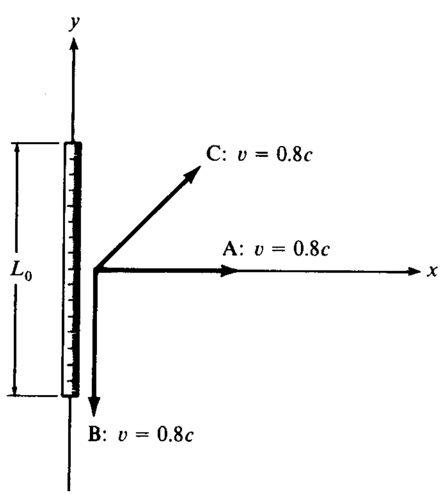
Therefore, if the observer in the S' frame makes his marks at times differing by exactly this amount, the measurements will be simultaneous in frame S. When this is used in the equation for $x_2 - x_1$ we get

$$x_2 - x_1 = \gamma(x_2' - x_1')(1 - v^2/c^2) = (x_2' - x_1')/\gamma.$$

Thus we see that the rod of length $x_2' - x_1'$ in S' appears contracted by the factor $1/\gamma$ in frame S, just as in the earlier discussion it appeared contracted by $1/\gamma$ in frame S'. In other words, if the rod is at rest in one frame, it appears to be contracted by the factor $1/\gamma$ when measured in a frame that is moving relative to the rod. The observations are completely symmetric. The measured length of a rod at rest in a given frame is often called the *proper length*. The shorter length measured in the moving frame is called the *contracted length*.

Example 3

A meter stick is oriented along the y axis as shown. Three observers measure the length of the stick: (A) one is traveling along the positive x axis at a speed $v = 0.8c$, (B) one is traveling along the negative y axis at $0.8c$, and (C) one is traveling at speed $0.8c$ at an angle of 45° from the x axis. What lengths are measured by each of these observers?



(A) Because the velocity of observer A is perpendicular to the moving meter stick, that observer measures the same length for the meter stick, namely $L_A = 1$ m.

(B) Observer B is traveling parallel to the stick, so according to Eq. (46.18) this observer measures a contracted length given by

$$L_B = L_0\sqrt{1 - v^2/c^2} = (1\text{ m})\sqrt{1 - 0.8^2} = 0.6\text{ m}.$$

(C) Observer C has a component of velocity parallel to the stick, $v \cos 45°$, so this observer will measure a contracted length given by Eq. (46.18) with $v \cos 45°$ as the relative speed:

$$L_C = L_0\sqrt{1 - (v \cos 45°)^2} = (1 \text{ m})\sqrt{1 - (0.8 \cos 45°)^2} = 0.8 \text{ m}.$$

Questions

11. A rocket ship passes by you at $0.6c$. If you measure its length to be 100 m, what is its proper length?

12. Suppose you decided to travel to a star that is 4 light years away. How fast would you have to travel so that while you are moving the distance would appear to you to be only be 2 light years? (One light year is the distance light travels in 1 year.)

13. Suppose you are in a spaceship when another spacecraft passes you at $0.4c$. An observer in the spacecraft claims that his craft is 120-m long and that yours is 150 m.

(a) Describe precisely what events are necessary for you to measure the length of the other spacecraft.
(b) What do you obtain for these measurements?

14. A spaceship is traveling relative to a nearby star at $0.86c$. Observers inside the ship find that it takes 4×10^{-8} s to pass a marker that is at rest relative to the star.

(a) According to observers in the spaceship, what is the length of their craft?
(b) According to observers at rest relative to the star, what is the length of the spaceship?

15. A mathematician has a right triangle with angles θ, $90° - \theta$, and $90°$. Another mathematician flies by at relative speed v in a direction parallel to the shortest edge of the triangle.

(a) Show that the triangle also is measured to be a right triangle by the moving mathematician.
(b) Find the relation between the angle θ and the corresponding angle θ' measured by the moving mathematician.
(c) Prove that the hypotenuse r of one triangle is related to the hypotenuse r' of the other by the equation

$$r' = r\left(1 - (v \cos \theta)^2/c^2\right)^{1/2}.$$

16. Electrons leave an accelerator with a speed of $0.9c$ and travel through an evacuated tube that has a proper length of 2.0 m before reaching their target.

(a) How long does it take an electron to travel the length of the tube according to an observer in the lab?

(b) How long does it take an electron to travel the length of the tube according to an observer traveling along with the electron?

(c) What is the length of the tube as measured by someone traveling with the electron?

17. Suppose that you are standing near the center of a long horizontal rod, which you perceive to fall with both ends hitting the ground simultaneously. What will a runner who is dashing by at a speed $3c/4$ say about how the rod hits the ground?

46.5 SPACE-TIME DIAGRAMS

The Lorentz transformation gives algebraic relations between the coordinates (x, y, z, t) of an event in one inertial frame S and the coordinates (x', y', z', t') of the same event in another inertial frame S'. Because we assumed that $y = y'$ and $z = z'$ we focus our attention on the two Lorentz equations relating (x, t) and (x', t'):

$$x = \gamma(x' + vt'), \tag{46.15x}$$

$$t = \gamma(t' + vx'/c^2). \tag{46.15t}$$

There are various ways to represent these equations geometrically. For example, we could use two separate sets of rectangular coordinate axes, one for frame S and another for frame S', plotting an event as a point (x, t) in the S-frame coordinate system and as a point (x', t') in the S'-frame coordinate system. A collection of events, say the history of a moving particle, would then be represented by a curve C in the xt system and by another curve C' in the $x't'$ system. Such curves are called *world lines*. In this type of representation, geometric relations between the two world lines C and C' due to the special nature of the Lorentz equations are not readily apparent. There are other types of diagrams that are specifically designed to reveal the geometric nature of the Lorentz transformation.

One of these was introduced by Hermann Minkowski in 1908 and is called a space–time diagram, or a Minkowski diagram. We already encountered one type of space–time diagram in Chapter 45 in connection with Galilean relativity. This section describes a similar type of space–time diagram that is suited to special relativity.

Before we explain how to construct Minkowski diagrams, we introduce some notation that gives the Lorentz equations a simpler form. First, we use the variables

$$r = ct \quad \text{and} \quad r' = ct'$$

instead of the variables t and t'. This makes sense from a physical point of view because each of ct and ct' represents a distance, being a product of speed and time, and comparing two distances such as r and x or r' and x' on a graph seems more natural than comparing distance with time. Therefore, we replace t by r/c and t' by r'/c in the Lorentz equations (46.15x) and (46.15t) to obtain

$$x = \gamma(x' + r'v/c),$$

$$r = \gamma(r' + x'v/c).$$

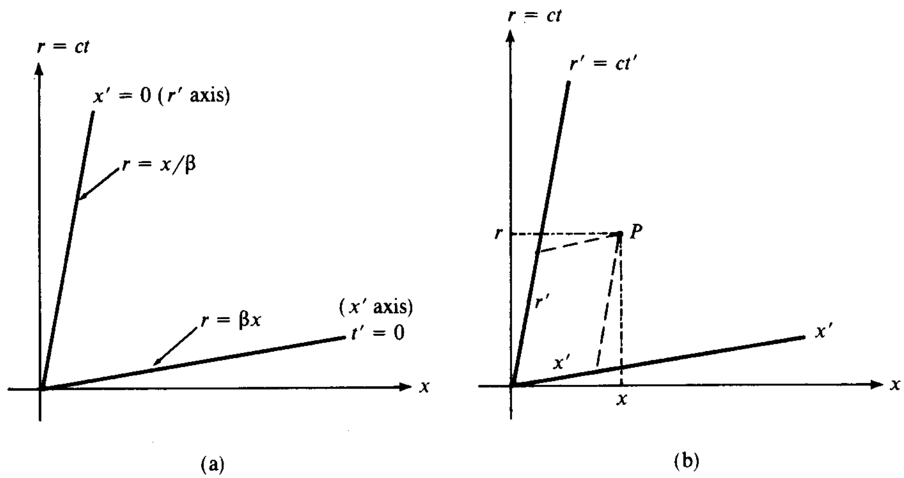


Figure 46.5 Space–time diagram.

The presence of v/c in each equation suggests a further replacement. The quotient v/c is the ratio of the velocity of the moving frame to the velocity of light. We replace this ratio by a new symbol β,

$$\beta = v/c,$$

and the Lorentz equations suddenly appear much simpler:

$$x = \gamma(x' + \beta r'), \tag{46.19x}$$

$$r = \gamma(r' + \beta x'). \tag{46.19r}$$

The constant γ is related to β as follows:

$$1/\gamma = \sqrt{1 - \beta^2},$$

so $\beta^2 = 1 - (1/\gamma)^2$, hence $|\beta| = \sqrt{1 - (1/\gamma)^2}$. The constant β can be positive or negative, but $|\beta| < 1$ because $\gamma > 1$.

The inverse transformation is given by

$$x' = \gamma(x - \beta r), \tag{46.20x'}$$

$$r' = \gamma(r - \beta x). \tag{46.20r'}$$

Now we draw a set of rectangular coordinate axes in the xr plane, as shown in Fig. 46.5a. Suppose that frame S' moves relative to S at a constant velocity v parallel to the x axis. The origin O' of frame S' corresponds to $x' = 0$, so by Eq. (46.20x′) the path of O' according to an observer in S is a straight line of slope $1/\beta$ through the origin:

$$r = x/\beta.$$

If $\beta > 0$ this line has slope greater than 1. The example is shown in Fig. 46.5a is the world line of O' and gives a geometric representation of the r' axis from the point of view of S. Similarly, a geometric representation of the x' axis, from the point of

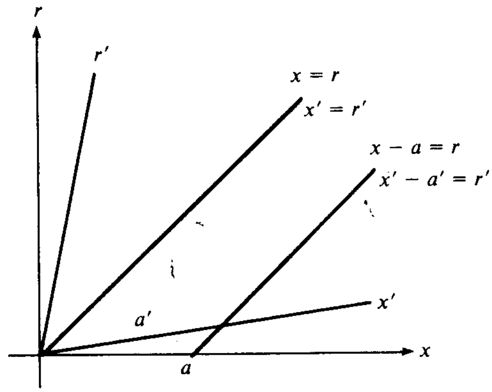


Figure 46.6 World lines of two light signals.

view of S, is obtained by setting $r' = 0$ in Eq. (46.20r'). This gives us a straight line with slope β:

$$r = \beta x.$$

Because the slopes β and $1/\beta$ are reciprocals, the angle between the x and x' axes is equal to that between the r and r' axes.

Minkowski's idea was to use perpendicular axes for the S frame and then to use the lines $r = \beta x$ and $r = x/\beta$ as oblique x' and r' axes for the S' frame on the same diagram, as shown in Fig. 46.5b. Then any given event can be represented by a single point P. From the diagram we can read off its coordinates (x, r) in the S frame and its coordinates (x', r') in the S' frame by dropping lines parallel to the corresponding axes as indicated in the Fig. 46.5b. Once we have numerical values for r and r' it is a simple matter to convert to the values $t = r/c$ and $t' = r'/c$.

A sequence of events (a world line) is represented by a curve on the space–time diagram. That curve serves as the world line for both the xr system and the $x'r'$ system. For example, the world line of a light signal that at time $t = 0$ has $x = 0$ would be represented by the line $r = x$, a line of slope 1. This is because light travels in vacuum with speed c according to any inertial observer, so in time t it travels a distance $x = ct = r$. The world line of light bisects the angle between the x and r axes, as shown in Fig. 46.6.

The very same line is the world line of light in the $x'r'$ system because in time t' it travels a distance $x' = ct' = r'$. If another light beam starts at time $t = 0$ with $x = a$, then it travels a distance $x - a = ct = r$ in time t, so its world line is parallel to the world line through the origin. The world line of any particle moving with speed less than c would be a curve whose tangent line at any point is inclined at an angle with the r axis less than 45°.

In constructing a space–time diagram we could just as well have taken the x' and r' axes perpendicular and used oblique axes for x and r. Putting $x = 0$ and $t = 0$ in Eqs. (46.19x) and (46.19t) we find that the line with equation $r' = -x'/\beta$ represents the r axis, while the line $r' = -\beta x'$ represents the x axis, as shown in Fig. 46.7.

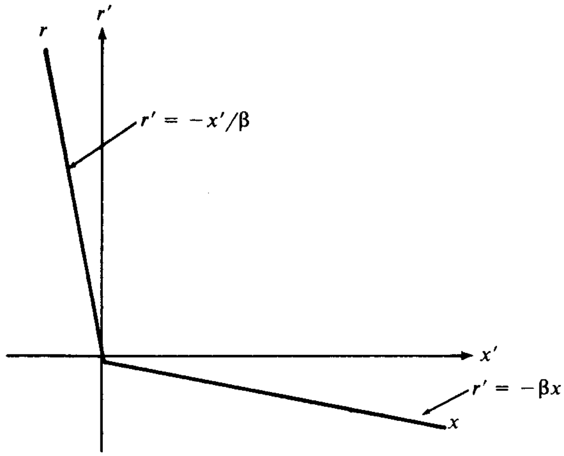


Figure 46.7 Space–time diagram with rectangular $x'r'$ axes.

Now that we understand how space–time diagrams are constructed we can see how they reveal some basic features of special relativity. Figure 46.8a shows two events E_1 and E_2 that occur at the same time $t_1 = t_2$ in frame S. Then $r_1 = ct_1 = ct_2 = r_2$, so in the xr diagram these events lie on a horizontal line $r = r_1$. The x' coordinates of these events are obtained by dropping lines parallel to the r' axis, as shown. The figure suggests that $x_2' - x_1' > x_2 - x_1$, and this is readily confirmed by using Eq. (46.20x′) twice and subtracting to get

$$x_2' - x_1' = \gamma(x_2 - x_1).$$

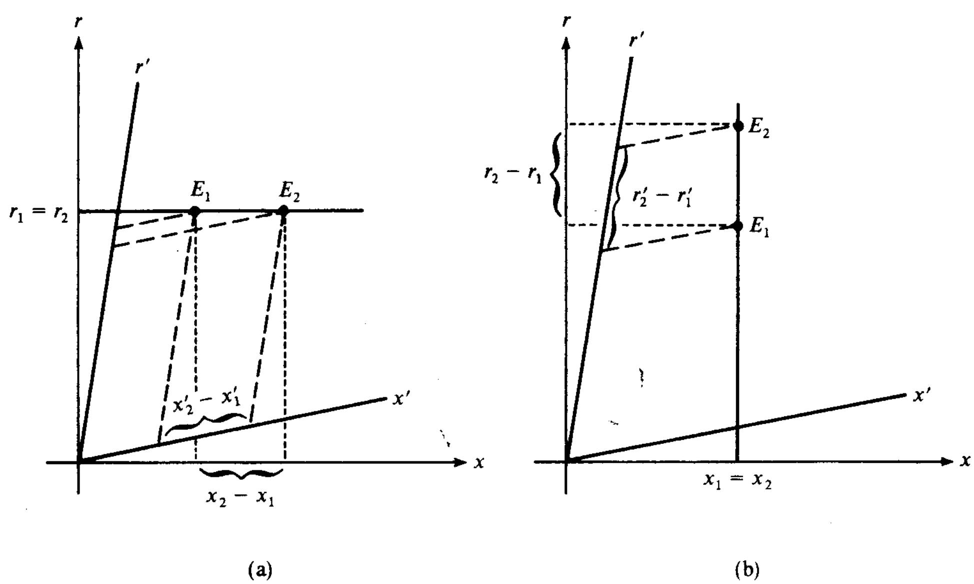


Figure 46.8 Relativity of events illustrated on a space–time diagram.

In other words, the difference $x_2' - x_1'$ is greater than $x_2 - x_1$ by the dilation factor γ. A unit distance along the x axis corresponds to a distance γ along the x' axis.

Figure 46.8b shows two events E_1 and E_2 that occur at the same place $x_1 = x_2$ in the S frame but at different times $t_1 < t_2$. These events are on the same vertical line in the xr plane with $r_1 = ct_1 < ct_2 = r_2$. Figure 46.8b suggests that $r_2' - r_1' > r_2 - r_1$, and this is easily confirmed by using Eq. (46.20r′) twice and subtracting:

$$r_2' - r_1' = \gamma(r_2 - r_1).$$

This shows that the difference $r_2' - r_1'$ is greater than $r_2 - r_1$ by the dilation factor γ. A unit distance along the r axis corresponds to a distance γ along the r' axis. The magnification factor γ is the same along both the x' and r' axes.

Questions

18. Use a space–time diagram to illustrate that if event A occurs before event B in one inertial frame, A precedes B in any other inertial frame (provided light from A can reach B in the time interval between events).

19. Using a space–time diagram, show that events that are simultaneous in one frame are not simultaneous in another.

20. An inertial frame S' is moving with speed $0.6c$ relative to another inertial frame S.

- **(a)** Draw a space–time diagram with the xr axes for S perpendicular and with the $x'r'$ axes for S' oblique.
- **(b)** Plot each of the following events on the diagram of (a): (A) $x = 1,\ r = 1$; (B) $x' = 1,\ r' = 1$; (C) $x' = 3,\ r' = 0$; (D) $x = 0,\ r = 2$.
- **(c)** Determine the coordinates (x, t) in S and (x', t') in S' for each of the events in (b).
- **(d)** Draw another space–time diagram with the $x'r'$ axes for S' perpendicular and the xr axes oblique and plot each of the events in (b) on this new diagram.
- **(e)** Determine the coordinates (x, t) in S and (x', t') in S' for each of the events in (b).

46.6 A FINAL WORD

Poincaré enunciated the principle of relativity and Lorentz published his transformation equations before Einstein wrote his 1905 paper. In the second volume of Sir Edmund Whittaker's *The History of Theories of Aether and Electricity*, published in 1953, there is a chapter on relativity, pointedly entitled, "The Relativity Theory of Poincaré and Lorentz." Einstein is mentioned for the first time in a paragraph on the thirteenth page, which is famous for what it says and what it doesn't say:

> In the autumn of the same year (1905) Einstein published a paper which set forth the relativity theory of Poincaré and Lorentz with some amplification, and which attracted much attention.

Whittaker continues,

> He asserted as a fundamental principle the constancy of the velocity of light, i.e., that the velocity of light in a vacuum is the same in all systems of reference which are moving relatively to each other, an assertion which at the time was widely accepted. In this paper Einstein gave modifications which must now be introduced into the formulae for aberration and for the Doppler effect.

Aberration has to do with the direction in which you point a telescope to see a star if you're on a moving platform. The earth is such a moving platform and the aberration of a star is slightly displaced because of relativistic effects. The Doppler effect concerns the observation that light seen from a moving source has its color changed (because of a frequency change) although the velocity is not changed. So Whittaker is willing to concede only these two minor points as Einstein's contribution to relativity theory.

Is it true that Einstein made only minor contributions to the theory of relativity? It is certainly true that he made only minor contributions to the equations of relativity, because Lorentz had derived his transformation equations before Einstein published his 1905 paper.

If we think of the theory of relativity as being only a description of the behavior of moving rods and clocks, then Einstein had little to do with that theory – all the key results were worked out before 1905. But these earlier results were developed with one purpose in mind – to explain the embarrassing Michelson–Morley experiment. The motivation for Einstein's work was something completely different. He decided that the principle of relativity and the constancy of the speed of light in all frames of reference were fundamental laws of the world, and these became his two postulates. He didn't assert that these were needed to explain the Michelson–Morley experiment. Instead, for philosophical reasons that are very deep, he said that the world must satisfy these postulates.

To understand the significance of his ideas it's useful to think of a simpler analogy – the principle of inertia. Suppose for a moment that you are Galileo and you have just discovered that a body in motion tends to stay in motion in a straight line. There are various things you could do once you've made this discovery. For example, you could try to find a mechanism that explains it. Do bodies have little motors inside them that keep them moving? What keeps them going so they don't come to rest? You would soon find that this type of scientific inquiry leads you nowhere, because you are asking the wrong questions. Moreover, you are speculating on matters that are impossible to test by experiment. A more fruitful approach is to accept inertia as a fundamental principle governing the way the universe works, and try to explain other phenomena using this principle. This is exactly what Einstein did. He said that the universe is governed by the principle of relativity and the constancy of the speed of light. Everything else comes about as a logical consequence of these principles.

Poincaré had made an attempt in the same direction, believing that the relativity principle should be the fundamental principle from which other things are derived. But Einstein understood that by adopting these principles we would have to change our intuitive notions of space and time. When he explained that in his paper, he did so in the simplest and most direct way you can imagine. He said,

> If we wish to describe the motion of a material point, we give the values of its coordinates as functions of time. We must bear carefully in mind that a mathematical description of this kind acquires no physical meaning unless we are quite clear as to what we understand by "time." We have to take into account that all our judgments in which time plays a part are only judgments of simultaneous events. If, for instance, I say, that a train arrives here at 7 o'clock, I mean something like this. The pointing of the small hand of my watch to seven and the arrival of the train are simultaneous events.

That's part of the definition of time. Next he goes into a discussion of how to establish whether two events are simultaneous if they don't take place at the same position in space. Einstein shows that the only way one can establish that is to make some way of synchronizing clocks by exchanging light signals back and forth. But once you exchange light signals, you're stuck with the constancy of the speed of light as seen by all observers and it necessarily means that observers in different frames moving at different velocities will make different decisions about whether events are simultaneous or not. What Einstein did was to yank us loose from our absolute notions of the meaning of space and time. It was an act as profound and revolutionary as Copernicus yanking the earth loose from its stationary position at the center of the universe.