Relativistic generalizations of mass

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Abstract. It is shown mathematically that there are two distinct ways of generalizing the concept of mass in Newtonian mechanics to Einstein's theory of relativity. We discuss the logical structure of the theory in terms of this.

Zusammenfassung. Es wird mathematisch nachgewiesen, daß sich das Konzept der Masse in Newtons Mechanik auf zwei Weisen zu Einsteins Relativitätstheorie verallgemeinert werden kann. Die logische Struktur der Theorie wird mit Hinsicht auf diese Tatsache diskutiert.

1. Introduction

The subject of relativistic mass and how we teach it has long been a topic for debate (Brehme 1968, Whitaker 1976, Adler 1987, Okun 1989a, b, Rindler 1990, Sandin 1991, Strnad 1991). A central issue is whether it is better to consider mass as depending on velocity or whether it should be regarded as an invariant. Whatever one's preference, any serious student of relativity needs to be acquainted with both of the conventions used in the literature and so the teacher who wants to avoid confusion needs to have a clear understanding of the matter. Unfortunately most of the (often quite ardent) discussion is, in our opinion, overly embedded in ill-defined semantics, frequently at cross purposes, and circumvents an interesting issue: why are there two possibilities in the first place?

It is our goal here to answer this question—to explain how it is that the logical structure of special relativity permits two different types of mass. It is not our purpose to advocate one convention or another. We do though offer a few comments on teaching aspects of mass in relativity based on the insight gained. We also note an intimate connection with the perennial issue of whether or not mass can be converted to energy (Baierlein 1991).

2. Newtonian mass

Part of the difficulty in discussing this subject stems from the fact that the term 'mass' in classical physics means many different things (Bohm 1989) such as inertia, gravitational mass, etc. Let us avoid all of this by concentrating on the mathematics and considering the quantity m that appears in classical equations such as $p = mv$, $F = ma$, $E = \frac{1}{2}mv^2$ and others. This is something that can be measured and represents a property of a material body. Without further ado we call it the Newtonian mass and note that under (non-relativistic) changes of inertial reference frame Newtonian physics expects the mass, m, to remain unchanged. We can see this explicitly by considering the so-called Galilean (boost) transformations of Newtonian mechanics:

$$x' = x + ut \qquad y' = y$$
$$z' = z \qquad t' = t \tag{1}$$

where u is the relative speed of the two inertial frames, the relative motion being taken as in the x direction. Thus $v = \mathrm{d}x/\mathrm{d}t$ changes to $v' = v + u$, and, for example, the momentum p changes to $p' = mv'$ with m unchanged. It is evident then that in Newtonian (non-relativistic) physics mass is an invariant.

3. Generalizing an invariant

The dichotomy of mass in the relativistic theory arises because there is more than one way in which a quantity which is invariant in the non-relativistic limit can behave. In the relativistic theory the Galilean transformations of equation (1) must be replaced by the Lorentz transformations:

$$x' = \gamma(x - \beta ct) \qquad y' = y$$
$$z' = z \qquad ct' = \gamma(ct - \beta x) \tag{2}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and $\beta = u/c$. These reduce to the Galilean transformations in the limit that c becomes infinite. Since the Lorentz transformations hold for any c they *incorporate* the Galilean transformations and any quantity which is invariant under the Lorentz transformations, such as the spacetime interval $(ct)^2 - x^2 - y^2 - z^2$, must also be invariant under the Galilean transformations. (To verify this it helps to divide the spacetime interval by c^2 before applying the Galilean transformations.) However, it is clear that any quantity that transforms in the same way as the time variable will also be invariant in the Galilean limit (since time is unchanged by Galilean transformations). Hence we have two distinct and mathematically acceptable ways of generalizing a Galilean invariant such as the Newtonian mass. One way is to insist that mass be a Lorentz invariant (i.e. a scalar) and the other is to let it transform as the time component of a four-vector. The latter possibility is often referred to as relativistic mass but both masses are of course relativistic in the sense that they are compatible with the special theory of relativity. We shall refer to them as scalar mass and vector mass in analogy with standard terminology for relativistic potentials.

The usual way of introducing a vector mass (i.e. what is often referred to as relativistic mass) in special relativity is via the formula for the momentum of a *free* particle. In Newtonian mechanics this is $\boldsymbol{p} = m\boldsymbol{v}$ and one argues by any of several methods (e.g. analysis (Lewis and Tolman 1909, Tolman 1912, 1934, Pauli 1921, Peters 1986) of colliding billiard balls and the requirements of momentum conservation, or simply by arguing (Eddington 1921, Taylor and Wheeler 1992) that $m\,\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ should be replaced by $m\,\mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ where $\tau = t/\gamma_v$ is the proper time) that in special relativity one must have instead

$$\boldsymbol{p} = \gamma_v m\boldsymbol{v} \tag{3}$$

where γ_v is the Lorentz boost factor of equation (2) *but calculated using the speed v of the particle.* This immediately suggests defining mass in relativity as

$$m^r = \gamma_v m. \tag{4}$$

The rest frame value is just (taking $v = 0$)

$$m_0^r = m \tag{5}$$

and is called the rest mass. It is of course just the Newtonian mass. Note though that we introduce new notation for m^r because in general it is not the same as the Newtonian mass but is a bold generalization of it. Indeed, not only is m^r not equal to the Newtonian mass but it changes value from one inertial frame to another. This is a concept quite foreign to Newtonian physics and, as we shall see shortly, m^r corresponds to what we have called the vector mass.

On the other hand, scalar mass in relativity must, by its Lorentz invariant nature, be the same as the Newtonian mass and no new notation need be introduced. Proponents of scalar mass need simply carry along the product γm wherever proponents of vector mass would use m^r.

4. What about energy?

If this were all, then the choice between the two would be purely a matter of taste. However one should consider further the implications of generalizing the concept of momentum as in equation (3). In a frame moving with velocity u with respect to that of equation (3) the momentum is given by $\boldsymbol{p}' = m\,\mathrm{d}\boldsymbol{x}'/\mathrm{d}\tau$ and using the Lorentz transformations we obtain (noting that γ_u is *necessarily* constant for two inertial frames, unlike γ_v which *may* vary with time)

$$\begin{aligned} p'_x &= \gamma_u(p_x - \beta\gamma_v mc) \\ p'_y &= p_y \qquad p'_z = p_z. \end{aligned} \tag{6}$$

This looks just like the Lorentz transformations of a space vector (equation (2)) if we introduce a new quantity, the time component

$$p^0 = \gamma_v mc = mc\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{7}$$

of a four-vector p. One easily checks that p^0 transforms as

$$(p^0)' = \gamma_u(p^0 - \beta p_x) \tag{8}$$

just as one would expect. It remains to interpret p^0. Making a binomial expansion for γ in the low velocity limit we find

$$p^0 = \frac{1}{c}(mc^2 + \tfrac{1}{2}mv^2 + \cdots) \tag{9}$$

in which we recognize $\frac{1}{2}mv^2$ as the Newtonian kinetic energy. Clearly p^0c has units of energy and we may write

$$p^0 = E^r/c. \tag{10}$$

Note that we have written E^r with a superscript r to emphasize that E^r is not the same energy E that appears in Newtonian physics but rather is a generalization of it. The important *new* element is that even at rest (and in the absence even of potentials) the particle has a *rest energy*,

$$E_0^r = mc^2 = m_0^r c^2. \tag{11}$$

Thus, unlike $\boldsymbol{p}$ which reduces to $m\boldsymbol{v}$ in the nonrelativistic limit, E^r does not reduce to the nonrelativistic $\frac{1}{2}mv^2$ in that same limit. Combining equations (10) and (7) we obtain

$$E^r = \gamma_v mc^2 = m^r c^2. \tag{12}$$

Thus we see that the new quantities E^r and m^r are essentially the same thing, differing only by a dimensionality constant, c^2. Put another way, in generalizing both energy and mass we have merely obtained a duplication of what has turned out to be the same physical quantity. Thus the equality expressed by equation (12) does not rank with, for example, Maxwell's unification of light and electromagnetism. That distinction belongs to equation

(11) which tells us that the *previously known* entity, Newtonian mass, is to be interpreted as a form of energy.

To emphasize this point consider how m^r transforms under changes of inertial reference frame. We can of course always determine its value in the rest frame and then use equation (4). However, consider a direct transformation from one arbitrary inertial frame to another arbitrary inertial frame. Since $m^r = p^0/c$ we immediately see that it transforms according to equation (8) in the form

$$(m^r)' = \gamma_u(m^r - \beta m_x) \tag{13}$$

where, for convenience only, we have introduced $\boldsymbol{m} = \boldsymbol{p}/c$. It is clear now that m^r transforms as the time component of a four-vector, as advertized. But note that there is nothing profound in dividing equation (8) by c and just as nobody would claim anything profound in our $\boldsymbol{m} = \boldsymbol{p}/c$ so we should avoid teaching that there is something profound in equation (12). What is profound is the recognition, expressed in equation (11), that we should associate with each body a rest energy and that energy is essentially its *Newtonian* mass. This relationship was unrecognized in Newtonian physics and only after this step has been made can we introduce E^r.

Of course there is nothing new in our observation that equation (12) is a mere tautology. What we have succeeded in showing is that this is a *fundamental* issue associated with the logical structure of the theory and inextricably connected with the very mathematical freedom that permits us to use either a scalar or vector mass. However, unless logical leanness is one's sole objective, this does not mean that a vector mass has no useful role. Many books on relativity state that with an appropriate choice of units one could measure time in metres. As elegant as the idea may seem we persist in measuring time in seconds. Likewise one may continue to use both E^r and m^r. What we must not do though is confuse a choice of units with physics.

Regardless of the choice of mass made one should take great care in teaching the subject that one does not mislead. We have already discussed a common fallacy associated with equation (12). Another example would be the famous experiments (Kaufmann 1901, Bucherer 1909) on the motion of a charged particle in a uniform magnetic field. The formula for the radius of the orbit

$$R = \frac{p}{qB} \tag{14}$$

is valid for both relativistic and non-relativistic motion. Measurements of R versus velocity, v, are frequently cited as evidence for an increase of mass with velocity. Actually, all the experiments show is that p is not linearly dependent on v, for v approaching c. One may, if one wishes, *interpret* this as due to an increase in m^r; or one may simply interpret it as evidence for the γ factor in equation (3).

A related issue is whether mass can be converted to energy or not. It is surprising to see how much debate has occurred on this point without a clear appreciation that the answer depends on which version of relativistic mass one is using and whether one is talking about the system as a whole or its parts. (We set aside the semantics issue (Ellse 1988, Beynon 1994) of whether energy is 'converted' or 'transferred'.) Consider a neutral pion at rest decaying into two photons. Initially both schools of thought agree that the pion has a mass of approximately 140 MeV/c^2. Conservation of energy dictates that both before and after decay the system has an energy of 140 MeV. Initially this energy is in the form of rest energy or mass of the pion. After decay it is in the form of radiation. This is where the debate starts. The scalar mass proponents consider the photons to be massless and are horrified at any talk of photon mass (even though they are quite prepared to talk of the mass of this particular two photon system taken as a whole, since it has zero momentum in the pion rest frame). Thus mass has been converted to energy according to this school of thought and moreover it is energy that can be made to do work, e.g. by having the photons impinge on a pair of solar cells. (This further negates somewhat the extraneous argument that since the system as a whole still has mass—the same mass as before—there has been no conversion of mass to energy. At some stage it makes little sense to continue thinking of a single system.) Thus mass is to be regarded as a form of energy and it can be converted to other forms in the same sense that we speak of potential energy being converted to kinetic energy (which also only occurs *within* a system). On the other hand, proponents of vector mass are quite happy to say that each photon has a mass of 70 MeV/c^2 and so mass has not been converted to energy; both mass and energy are equivalent concepts and conversion of one to the other is merely a mathematical operation devoid of physical meaning. What really matters is that one is consistent and so the teacher is urged not to carelessly adopt the catch phrases of one school when they are teaching the curriculum of the other.

5. Specious arguments

It is not our goal here to consider the merits of the many various arguments that have been put forward for preferring one type of mass over another. We would though like to warn against unsophisticated arguments. One example is the frequent claim that in relativity, the centre of mass generalizes to

$$x_{\rm cm} = \sum_i m_i^r x_i \Big/ \sum_i m_i^r \tag{15}$$

and so it is better to use vector mass, as then equation

(15) is of the same form as the non-relativistic result. However, such remarks ignore that the concept of centre of mass is inherently non-relativistic and that there are many possible generalizations (Pryce 1948). The above definition, while satisfactory for some purposes, has the undesirable property that it is not independent of reference frame. Furthermore the components of x_{cm} do not commute with each other (i.e. their Poisson brackets are non-zero in classical mechanics (Pryce 1948)) so this definition poses difficulties in quantum mechanics (cf Newton and Wigner 1949). Similar weaknesses can be found in other arguments.

6. Other generalizations?

One final point is worth making concerning the possible relativistic generalizations of mass. We introduced above the four-vector (m',m) for the sake of argument. In the rest frame it reduces to the Galilean four-vector $(m'_0, 0)$. This is not a single number like the Newtonian mass and so we do not regard our relativistic four-vector as a generalization of mass. Only scalars or components will do. Consider now the generalization of Newton's second law for a particular of *fixed* Newtonian mass. One finds that

$$F = Ma \tag{16}$$

where $M = [m^i_j]$ is, in general, a non-diagonal matrix. (The form of M is entirely analogous to that for the moment of inertia I for an object being rotated about other than one of its principal axes.) M has eigenvalues $\gamma_v m$ (twice) and $\gamma_v^3 m$, the so-called transverse and longitudinal masses. Even though M becomes diagonal in the Galilean limit it remains a matrix and so we prefer to consider equation (16) as a generalization of Newton's law rather than M as a generalization of mass. (Indeed, introducing the (Minkowski) 4-force and 4-acceleration (see, for example, Brehme 1968) allows one to write this law as $\mathscr{F}^\mu = \mathrm{d}p^\mu/\mathrm{d}\tau = mA^\mu$ without any modification at all of the mass.)

7. Conclusions

Thus we conclude by noting that in answering the elementary question of why two different masses are allowed in relativity, one obtains a clearer picture of the subject—a picture that is rooted in mathematics and logic rather than semantics and opinion.

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