# Caltech's Physics 237-2002

# Gravitational Waves

## PART A: GRAVITATIONAL-WAVE THEORY AND SOURCES

# *Course Outline*

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I. ***Overview of Gravitational-Wave Science*** [Lectures by Kip]
   A. The nature of gravitational waves [GW's]: **Week 1, Lecture 1, slides 1 - 18**
   B. The GW spectrum: HF, LF, VLF, ELF bands
   C. Detection techniques:
      1. resonant-mass detectors
      2. interferometers: LIGO and its partners
      3. LIGO details, noise curves, technology: **Week 1, Lecture 2, slides 19 - 37**
      4. LISA
   D. GW data analysis
   E. GW sources and science
      1. Inspiral of compact body into supermassive hole
      2. Binary black hole mergers
      3. Neutron-star / black-hole mergers: **Week 2, Lecture 3 - Part 1, slides 38 - 47**
      4. Neutron-star / neutron-star inspiral
      5. Spinning neutron stars
      6. Neutron-star births
      7. Binaries in our galaxy
      8. The very early universe

II. ***Introduction to General Relativity*** [Lectures by Kip]
   A. Tidal gravity in Newtonian theory: **Week 2, Lecture 3 - Part 2**
      1. Motivation: tidal gravity as spacetime curvature
      2. The Newtonian tidal gravity tensor
      3. Relative acceleration of freely falling particles
   B. The mathematics underlying general relativity: **Week 2, Lecture 4**

1. Vectors, tensors, tensor algebra
2. Differentiation of tensors, connection coefficients
3. Commutators, coordinate and noncoordinate bases: **Week 3, Lecture 5**
4. Spacetime curvature: the Riemann and Ricci tensors
5. Relativistic tidal gravity; geodesic deviation

C. The Einstein field equations

1. Motivation via tidal gravity
2. "Derivation" of the Einstein equations; number of equations and number of unknowns; contracted Bianchi identitity

III. ***Weak Gravitational Waves [GW's] in Flat Spacetime*** **[Lectures by Kip]**: **Week 4, Lecture 6**

A. Wave equation for Riemann tensor
B. Transverse-traceless [TT] GW field; + and x polarizations
C. A GW's tidal forces (relative motion of freely falling particles)
D. Metric perturbations; TT gauge and other gauges
E. Proper reference frame of an observer: **Week 4, Lecture 7**
F. Physical measurements of GW's in a proper reference frame
G. Generation of GW's: The linearized Einstein field equations
H. Projecting out the TT GW field
I. Slow-motion, weak-stress approximation for GW sources: **Week 5, Lecture 8**
J. The quadrupole formula for GW generation

1. Derivation in slow-motion, weak-stress approximation
2. Validity  for slow-motion sources with strong internal gravity and arbitrary stresses

IV. ***Propagation of GW's Through Curved Spacetime*** **[Lectures by Kip]**

A. Short wavelength approximation; two-lenghscale expansion
B. Curved-spacetime wave equation for Riemann tensor
C. Solution of wave equation via eikonal approximation (geometric optics) - Foundations
D. Geometric optics - Details: **Week 5, Lecture 9**

1. gravitons and their propagation; graviton conservation
2. rays as graviton world lines; propagation of + and x GW fields along rays
3. + and x polarizations and fields, rays and transport of waves along rays
4. gravitational focusing of GW's, e.g. by the sun; diffraction at the focus
5. stress-energy tensor for GW's; nonlocalizability of GW energy
6. conservation of GW energy and momentum
7. conseervation of a graviton's energy and momentum

E. Propagation of GW's through homogeneous matter:  **Week 6, Lecture 10**

1. impact of matter on the waves is always negligible
2. propagation through dust, perfect fluid, viscous fluid, elastic medium
3. propagation through a cloud of neutron stars

V. ***Generation of GW's by Slow-Motion Sources in Curved Spacetime*** **[Lectures by Kip]**: **Week 6, Lecture 11**
   A. Strong-field region, weak-field near zone, local wave zone, distant wave zone
   B. Multipolar expansions of metric perturbation in weak-field near zone and local wave zone
      1. influence of source's mass and angular momentum
      2. mass quadrupolar component of GW's; current quadrupolar component
      3. rates of emission of energy, linear momentum, and angular momentum
   C. Application to a binary star system with circular orbit
      1. inspiral rate and timescale
      2. chirp waveform; chirp mass

VI. ***Astrophysical Phenomenology of Binary-Star GW Sources***
   A. GW's from Binary Star Systems: **Week 7, Lecture 12** **[by E. Sterl Phinney]**
      1. GW-driven inspiral of a single binary [review]
      2. Inspiral evolution of a steady-sstate population of many binaries
      3. Types of stars: main-sequence stars, white dwarfs [WD], neutron stars [NS], black holes [BH]; their masses and radii
      4. Binary systems observable by LIGO (and its partners), and by LISA
   B. Issues relevant to estimating numbers of binary GW sources and their merger rates
      1. Cosmology: parameters describing the universe as a whole
      2. Our Milky Way galaxy: its star-formation history, stellar populations and binary populations
      3. Use of blue light to extrapolate from rates in Milky Way to rates in the distant universe
   C. Estimates of numbers of binary GW sources and inspiral/merger rates: preview of next lecture
      1. NS/NS rates based on binary-pulsar statistics and blue-light extrapolation
      2. Population synthesis as foundation for estimates
   D. Estimates of numbers of binary GW sources [for LISA] and inspiral/merger rates [for LIGO]: **Week 8, Lecture 13** **[by E. Sterl Phinney]**
      1. Estimates based on observed numbers in our galaxy
         a. pitfalls
         b. NS/NS; WD/WD, WD/NS

2. Population synthesis
   a. Foundations for population synthesis:
      i. stellar structure and evolution
      ii. binary evolution: mass transfers etc.
   b. Estimates of binary numbers for LISA
3. Estimates of NS/NS, NS/BH, and BH/BH numbers for LIGO -- **Week 8, Lecture 14 - Part 1** **[by Kip]**

VII. ***Binary Inspiral: Post-Newtonian Gravitational Waveforms for LIGO and Its Partners --***
   A. Matched-filtering data analysis to detect inspiral waves
   B. Foundations for post-Newtonian approximations to General Relativity
      1. Mathematical foundations
      2. Physical effects at various orders
   C. Post-Newtonian inspiral waveforms for circular orbits and vanishing spins -- **Week 8, Lecture 14 - Part 2** **[by Alessandra Buonanno]**
   D. Expansion parameter v = (pi M f)^1/3
   E. Phase evolution governed by energy balance
   F. Waveform in time domain
   G. Waveform in frequency domain, via stationary-phase approximation
   H. Influence of spin-orbit and spin-spin coupling: Orbital and spin precession; waveform modulation
      1. NS/BH binary
      2. BH/BH binary
   I. Innermost stable circular orbit (ISCO) and transition from inspiral to plunge
   J. The IBBH problem: failure of post-Newtonian waveforms in late inspiral; methods to deal with this:
      1. Pade resummation
      2. Effective one-body formalism
      3. Search templates designed to deal with uncertainties in our knowledge of the waveforms

VIII. ***Supermassive Black Holes [SMBH's] and their Gravitational Waves [for LISA/ --*** **Week 9, Lecture 15** **[by E. Sterl Phinney]**
   A. Astrophysical phenomenology of SMBH's in galactic nuclei
      1. Evidence for their existence
      2. Measurement of SMBH masses via cusp in stellar velocity dispersion (for masses above 10^6 Msun)
      3. Correlation of SMBH masses with velocity dispersion in galactic bulges
      4. Number of SMBH's per unit volume in universe; their distribution of masses (for masses above 10^6 Msun)
      5. Observed quasar and other electromagnetic emission from SMBH's;

quiescence of most SMBH's

B. Mergers of galaxies
    1. Statistics of mergers: observational data; predictions of CDM simulations
    2. Physics of mergers
    3. Dynamical friction on SMBH's, SMBH binary formation
C. Evolution of SMBH binary
    1. Interaction with stars; loss cone
    2. Hangup and ways to overcome it: repopulation of loss cone; effect of binary motion in galaxy core; effect of ellipticity of galactic potential; interaction with gas
    3. Gravitational radiation reaction
    4. SMBH merger rates
D. Capture and inspiral of  stars by a SMBH
    1. Loss cone and its repopulation
    2. Tidal disruption of main-sequence stars
    3. Capture of compact stars [WD, NS, small BH] into highly elliptical orbits
    4. Evolution of orbital ellipticity during inspiral
    5. Event rate estimates for captures
E. Gravitational waves from SMBH binary inspiral, as measured by LISA -- **Week 9, Lecture 16** **[by Kip]**
    1. Frequency evolution, signal-to-noise ratios
    2. Cosmological influences on waves: gravitational redshift; gravitational lensing
    3. Observables: redshifted masses, luminosity distance, inclination angle
F. GW's from inspiral of a compact star (or BH) into a SMBH
    1. Frequency evolution, signal to noise ratios
    2. Loss of signal strength due to non-optimal signal processing - caused by complexity of inspiral orbits and resulting complexity of waveforms
        a. Implications for event rates
        b. Implications for specifying the level of LISA's noise floor

IX. ***GW's from Big Bang: Amplification of Vacuum Fluctuations by Inflation***

A. Basic idea: same as parametric amplification of classical waves
B. Mathematical details
    1. Background cosmological metric
    2. Geometric optics propagation of GW's at "late times'
    3. Wave equation for GW's at all times
    4. Frozen and decaying solutions when wavelength is much larger than background radius of curvature
    5. Matching solutions together: resulting wave amplification

X. ***GW's from Neutron-Star Rotation and Pulsation --*** **Week 10, Lecture 17** **[by Lee Lindblom]**
   A. GW's from a structurally deformed, rotating NS
      1. Deformations maintained by a solid crust
      2. Deformations maintained by stress of a strong internal magnetic field
      3. Deformations due to temperature anisotropy induced by accretion of gas onto NS [low-mass X-ray binaries; LMXB's]
      4. Magnitudes of deformation (ellipticities) detectable by LIGO-I and LIGO-II
   B. GW's from pulsations in a rotating NS
      1. Types of pulsational [bar-mode] instabilities: dynamical; secular
      2. beta = T/W as diagnostic for instabilities
      3. Instabilities in uniform-density Newtonian stars [Maclaurin Spheroids]
      4. Mechanisms for forming rapidly rotating NS's:
         a. Collapse of degenerate stellar cores
         b. Accretion-induced collapse of a white dwarf
         c. Spinup by accretion
         d. Merger of a low-mass NS/NS binary
      5. NS's formed by collapse: differential rotation, values of beta,bar-mode instabilities, numerical evolution of unstable stars
         a. realistic models
         b. models with extreme differential rotationg: instability at small beta

XI. ***Numerical Relativity as a Tool for Computing GW Generation --*** **Week 10, Lecture 18** **[by Marc Scheel]**
   A. Motivation: Sources that require numerical relativity for their analysis
      1. Binary black hole mergers
         a. Relevance to LIGO & partners, and to LISA
         b. Estimated event rates for LIGO-I, LIGO-II and LISA
         c. Inspiral, merger, and ringdown; estimated wave strengths from each
         d. Rich physics expected in mergers: strong, nonlinear effects; spin-spin and spin-orbit coupling; angular-momentum hangup
         e. Importance of simulating mergers as foundation for interpreting observations
      2. Tidal disruption of NS by a BH companion
         a. Estimated event rate for LIGO-II
         b. Information carried by waves: NS structure and equation of state
         c. Possible connection to gamma ray bursts
         d. Importance of simulations for interpreting observations
      3. Some other sources: NS/NS mergers, cosmic string vibrations, brane

excitations in early universe
    4. The necessity to use numerical relativity in simulations of these sources
- B. Mathematical underpinnings of numerical relativity
    1. 3+1 decomposition of spacetime into space plus time
    2. Initial data must satisfy "constraint equations"
    3. Evolve via "dynamical Einstein equations"
    4. Gauge freedom
    5. Analogy with electromagnetic theory
- C. Mathematical details
    1. Spacetime slicing; lapse, shift, and 3-metric; extrinsic curvature
    2. Hamiltonian constraint equation
    3. Momentum constraint equations
    4. Dynamical equations
    5. Choices of lapse and shift
- D. Current state of the art in numerical relativity; current efforts on BH/BH inspiral & merger

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***Links to this course's other web pages:***

Course home page
Course description
Outlines of Part B:
Part B: Gravitational-Wave Detection: original outline
Part B: Gravitational-Wave Detection: alternative outline, with the order of the lectures made more logical
Course Materials (videos of lectures, reading, homework, solutions)

Ph 237a: Gravitational Waves 9 January 2001

## WEEK 1: OVERVIEW

**Recommended Reading:**

Note: Almost all readings will be available for downloading on the web; the url will be given at the end of each reference.

1. Scott A. Hughes, Szabolcs Marka, Peter L. Bender and Craig J. Hogan, "New physics and astronomy with the new gravitational-wave observatories", to be published in Proceedings of the 2001 Snowmass Meeting, *http://xxx.lanl.gov/abs/astro-ph/0110349*.

**Possible Supplementary Reading [listed in reverse chronological order]:**

2. Kip S. Thorne, "The scientific case for advanced LIGO interferometers", LIGO Document Number P-000024-00-D, *available as a pdf file on the Ph237 web site.*
3. Barry C. Barish and Rainer Weiss, "LIGO and the detection of gravitational waves", Physics Today, 52, 44 (October 1999), *not available electronically*
4. Barry C. Barish, "The detection of gravitational waves with LIGO", Proceedings of DPF'99, *http://xxx.lanl.gov/abs/gr-qc/9905026*
5. Bernard F. Schutz, "Gravitational wave astronomy", Classical and Quantum Gravity, 16, A131-A156 (1999), *http://xxx.lanl.gov/abs/gr-qc/9911034*
6. Eanna E. Flanagan, "Sources of gravitational radiation and prospects for their detection", Proceedings of GR15, *http://xxx.lanl.gov/abs/gr-qc/9804024*
7. Kip S. Thorne, "Probing black holes and relativistic stars with gravitational waves", in Black Holes and Relativistic Stars, Proceedings of a Conference in Memory of S. Chandrasekhar, ed. R. M. Wald (University of Chicago Press, Chicago, 1998), pp. 41-78. *http://xxx.lanl.gov/abs/gr-qc/9706079*

**Assignment, to be turned in at beginning of class on Wednesday 16 January by students registered in the course:**

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.
   iv. Seek private tutoring from a knowledgable person about this week's topic.
   v. Pursue some other method of learning about this week's topic, and state what you have done.

**EXERCISES**

Note: Work only those exercises that are useful for you!

1. **Multipolar expansion of the gravitational-wave field**
   Fill in the details of the argument sketched on slide 10 of Kip's lectures. In particular, show that each of the terms in the expansion of $h$ is dimensionless, and explain why the terms could not have any other form.

2. **Strengths of the waves for various multipoles**
   a. Give an order of magnitude formula for the contribution of each multipole to the gravitational-wave field $h$ — a formula analogous to that derived on slide 11 for the mass quadrupole moment. Express your answer in terms of the source's mass and internal velocity, and the distance to the source.
   b. Assuming that the internal velocity is generated by the source's own gravitational field, give an alternative answer in terms of the source's mass and radius, and the distance to the source.
   c. Compare with similar order of magnitude formulas for the contributions of the electric multipoles and magnetic multipoles to electromagnetic radiation.
   d. What is the magnitude (a dimensionless number) of the multipole's contribution to $h$ in the case of colliding black holes at a distance of 100Mpc (about 300 million light years)? In the case of a neutron-star binary (with neutron-star masses equal to 1.4 that of the sun) at 100Mpc distance when the emitted waves have the frequency at which LIGO's noise is smallest?
3. **How LIGO works**
   On slide 13 of Kip's lectures there is a graph in the lower left corner that he did not discuss. Explain what it means and use it to explain in some detail how a LIGO interferometer works.
4. **Standard Quantum Limit for advanced LIGO interferometers**
   On slide 25 of Kip's lectures there is a statement that the advanced interferometers in LIGO will monitor the motions of their 40kg sapphire mirrors with an accuracy about the half width of the mirror's quantum (Schroedinger) wave function. This accuracy is called the "standard quantum limit" because, for conventional interferometer designs (such as that of the first LIGO interferometers), quantum mechanics prevents an accuracy better than this from being achieved.
   a. Derive a formula for the half width of the wave function in terms of Planck's constant, the mass of the mirror, and the frequency of the gravitational waves (which is approximately $f \sim 0.5/$(time during which each measurement of the mirror's center-of-mass position is made).
   b. Evaluate your formula numerically, for the frequency of the minimum of the advanced-detector noise curve, and thereby show that the detector does, indeed, operate near the standard quantum limit.
5. **Relative motions of LISA's spacecraft**
   LISA's spacecraft have an orbit shown in slide 27 of Kip's lectures. This orbit is disturbed by the gravitational fields of the planets, most especially Jupiter. Estimate

the relative motion of the spacecraft induced by Jupiter, and compare with the relative motion quoted in Kip's slide 27.

## Problem 1

It's natural to have $G$ in front of the terms, and $1/r$ is required for waves. Suppose each term has to be products of $G/r$ with time derivatives of the multipole moments and some powers of $c$, we have terms of the form

$$\frac{G}{r}c^l\left(\frac{d}{dt}\right)^k M_n\,, \tag{1}$$

and

$$\frac{G}{r}c^l\left(\frac{d}{dt}\right)^k S_n\,. \tag{2}$$

Noticing that,

$$\begin{aligned} G &\sim [M^{-1}L^3T^{-2}]\,, &(3)\\ r &\sim [L]\,, &(4)\\ c &\sim [LT^{-1}]\,, &(5)\\ M_n &\sim [M\,L^n]\,, &(6)\\ S_n &\sim [M\,L^{n+1}T^{-1}]\,, &(7)\end{aligned}$$

we have

$$\begin{aligned} \frac{G}{r}c^l\left(\frac{d}{dt}\right)^k M_n &\sim [L^{n+l+2}\,T^{-(k+l+2)}]\,, &(8)\\ \frac{G}{r}c^l\left(\frac{d}{dt}\right)^k S_n &\sim [L^{n+l+3}\,T^{-(k+l+3)}]\,. &(9)\end{aligned}$$

In order to have dimensionless quantities, we must require $k = n$ and $l = -n-2$, and all the possible terms are

$$\frac{G}{rc^{n+2}}\left(\frac{d}{dt}\right)^n M_n \tag{10}$$

and

$$\frac{G}{rc^{n+3}}\left(\frac{d}{dt}\right)^n S_n\,. \tag{11}$$

These are just the terms that appear on slide 10:

$$\begin{matrix} (G/c^2)(M_0/r) & (G/c^3)(\dot{M}_1/r) & (G/c^4)(\ddot{M}_2/r) & \ldots \\ & (G/c^4)(\dot{S}_1/r) & (G/c^5)(\ddot{S}_2/r) & \ldots \end{matrix} \tag{12}$$

**Problem 2**

(a) For the mass and mass-current multipole moments $M_n$ and $S_n$, we have

$$M_n \sim ML^n\,, \tag{13}$$

and

$$S_n \sim MvL^n\,, \tag{14}$$

where $M$, $L$ and $v$ are the characteristic mass, size and internal velocity of the system. For their rates of change, we have

$$\left(\frac{d}{dt}\right)^n M_n \sim ML^n/P^n \sim Mv^n\,, \tag{15}$$

and

$$\left(\frac{d}{dt}\right)^n S_n \sim MvL^n/P^n \sim Mv^{n+1}\,, \tag{16}$$

where $P \sim L/v$ is the characteristic motion period of the system.

Inserting the above into the expansion terms [see Problem 1,] we have,

$$h[M_n] \sim \frac{G}{rc^{n+2}}\left(\frac{d}{dt}\right)^n M_n \sim \frac{GM}{c^2 r}\left(\frac{v}{c}\right)^n\,, \tag{17}$$

and

$$h[S_n] \sim \frac{G}{rc^{n+3}}\left(\frac{d}{dt}\right)^n S_n \sim \frac{GM}{c^2 r}\left(\frac{v}{c}\right)^{n+1}\,. \tag{18}$$

This means the magnitude of higher-order multipole contributions decrease by powers of $v/c$.

(b) Here we just have to recall that for self-generated internal motion,

$$v \sim \sqrt{\frac{GM}{L}} \tag{19}$$

Inserting this relation into part (a), we have

$$h[M_n] \sim \frac{GM}{c^2 r}\left(\frac{v}{c}\right)^n \sim \frac{GM}{c^2 r}\left(\frac{GM}{c^2 L}\right)^{\frac{n}{2}} \tag{20}$$

and

$$h[S_n] \sim \frac{GM}{c^2 r}\left(\frac{v}{c}\right)^{n+1} \sim \frac{GM}{c^2 r}\left(\frac{GM}{c^2 L}\right)^{\frac{n+1}{2}}. \tag{21}$$

(c) From electromagnetism, we have [in the Gaussian unit system]

$$\mathbf{E},\, \mathbf{B} \sim \frac{1}{rc^2}\left(\frac{d}{dt}\right)^2 \mathcal{E}_1 \;\;\&\;\; \frac{1}{rc^3}\left(\frac{d}{dt}\right)^3 \mathcal{E}_2 \ldots \tag{22}$$

$$\frac{1}{rc^2}\left(\frac{d}{dt}\right)^2 \mathcal{M}_1 \;\;\&\;\; \frac{1}{rc^3}\left(\frac{d}{dt}\right)^3 \mathcal{M}_2 \ldots \tag{23}$$

$\mathcal{E}_n \sim QL^n$ and $\mathcal{M}_n \sim Qv/cL^n$, $n = 1, 2, 3, \ldots$ are the electric and magnetic multipoles, where $Q$, $L$ and $v$ are the characteristic charge, size and velocity of the radiating system. Inserting these into Eqs. (22) and (23), we obtain order-of-magnitude expressions similar to those in part (a):

$$\mathbf{E},\, \mathbf{B} \sim \frac{Q}{rL}\left(\frac{v}{c}\right)^2 \;\;\&\;\; \frac{Q}{rL}\left(\frac{v}{c}\right)^3 \ldots \tag{24}$$

$$\frac{Q}{rL}\left(\frac{v}{c}\right)^3 \;\;\&\;\; \frac{Q}{rL}\left(\frac{v}{c}\right)^4 \ldots \tag{25}$$

where $Q$, $L$ and $v$ are the characteristic charge, size and velocity of the radiating system.

(d) For the case of colliding binary BH's, the leading contribution is

$$h[M_2] \sim \frac{G}{c^2 r}\frac{Mv^2}{c^2} \sim \frac{G}{c^2 r}(\text{m}ass\ equivalent\ of\ kinetic\ energy). \tag{26}$$

By assuming that the mass equivalent of kinetic energy to be $M_{\text{sun}}$, [which could be achieved in a typical situation where the mass of the system is several $M_{\text{sun}}$ and the internal velocity is a nontrivial fraction of $c$,] we have

$$h[M_2] \sim 5 \cdot 10^{-22}. \tag{27}$$

Since the velocity might be a significant portion of $c$, higher multipoles might also have significant contributions.

For a NS-NS binary with 1.4 solar mass each, we have

$$\omega_{\text{orbital}} \sim \left(\frac{GM_{\text{total}}}{R^3}\right)^{\frac{1}{2}}, \tag{28}$$

where $R$ is the separation of the binary, and thus

$$v \sim \omega_{\text{orbital}} \frac{R}{2} \sim \frac{1}{2} (G M_{\text{total}} \omega_{\text{orbital}})^{\frac{1}{3}} \,. \tag{29}$$

For the leading quadrupole contribution, we have

$$\omega_{\text{GW}} = 2\omega_{\text{orbital}}, \tag{30}$$

since the components of the quadrupole moment of the binary return to their original values after only half an orbital cycle. Therefore, we have

$$v \sim \frac{1}{2} \left( \frac{G M_{\text{total}} \omega_{\text{GW}}}{2} \right)^{\frac{1}{3}} \sim 0.1c \,, \tag{31}$$

and

$$h_{\text{quad}} \sim \frac{G M_{\text{total}}}{c^2 r} \left( \frac{v}{c} \right)^2 \sim 10^{-23} \,, \tag{32}$$

where $\omega_{\text{GW}}$ is chosen at $2\pi \times 160\,\text{Hz}$. The contributions of higher order multipoles will decrease as powers of $v/c \sim 0.1$.

## Problem 3

[From a more general point of view, by Kip]

From elementary physics, we know that the phase of the driven oscillation of a harmonic oscillator with eigenfrequency $\omega_0$ by a force with frequency $\omega$, or the excitation phase, is a function of $\omega - \omega_0$, which looks like the graph in slide 13.

More specifically, this relation can be derived by looking at the complex amplitude of the driven motion,

$$A \sim \frac{F}{\omega - \omega_0 + i\epsilon}, \tag{33}$$

and extracting its phase

$$\Phi = \arg \frac{1}{\omega - \omega_0 + i\epsilon}. \tag{34}$$

Here $\epsilon$ is the depth of the resonance. It should be noted that the phase of excitation is most sensitive to the change of $\omega - \omega_0$ when the latter is 0, i.e. right on resonance. Also, the higher the Q, the smaller is the depth $\epsilon$ and the more sensitive is the phase of excitation to $\omega - \omega_0$.

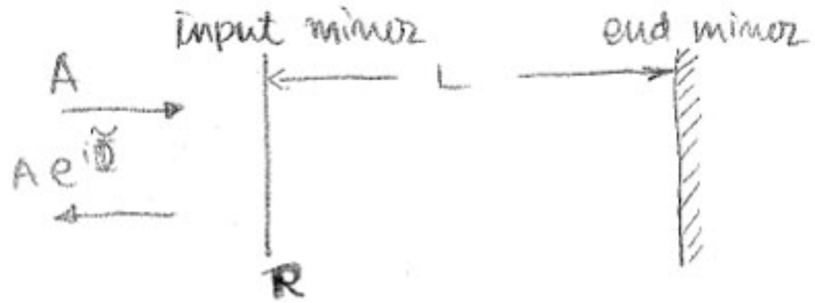

In a LIGO interferometer, the light is exciting a normal mode of the optical cavity ("Fabry-Perot cavity") formed by the corner and end mirrors of each LIGO arm. This normal mode is a harmonic oscillator with eigenfrequency $\omega = \pi Nc/L$, where N is an integer. The phase of the light inside the cavity excited by the laser depends on $\omega_{\rm laser} - \omega_{\rm cavity}$ in the same way as shown as shown on slide 13 and in the graph above. The light that leaks out of the arm cavity and arrive at the beamsplitter will carry this $\omega_{\text{cavity}}$ sensitive phase with it.

Since $\omega_{\text{cavity}}$ is directly determined by the length of the arm cavity, the phase of the two beams that arrive at the beamsplitter from the two arms will depend on the lengths of each arm cavity. Therefore, the output signal, which is determined by the phase difference of the two beams, will be sensitive to the differential motion of the two arms.

[From optics]

From a calculation in optics, we can see in more detail how the phase of the output light depend on the arm-cavity length.

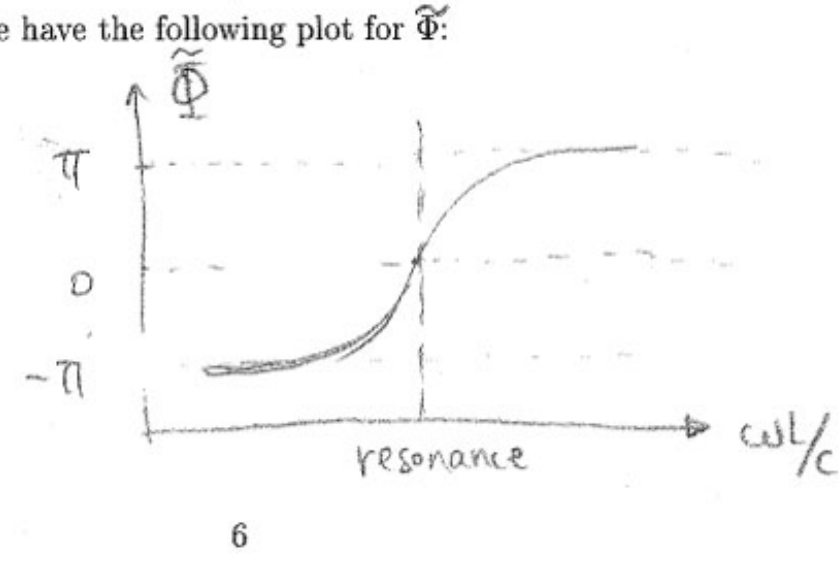
Let us consider a Fabry-Perot (FP) cavity with length $L$, input-mirror reflectivity $R$, and a perfect end mirror. Suppose a monochromatic light with (angular) frequency $\omega$ enters the cavity.




This light will come out with the same absolute amplitude but will gain a phase of

$$e^{i\widetilde{\Phi}} = \frac{e^{2i\omega L/c} - \sqrt{R}}{1 - \sqrt{R}\, e^{2i\omega L/c}}. \tag{35}$$

As a function of $L$, we have the following plot for $\widetilde{\Phi}$:



Now if the length of the cavity change by $\Delta L$, we can find the corresponding change in $\widetilde{\Phi}$ from the plot. It should be noted that the phase change is most sensitive to length change when the cavity is initially on resonant with the laser.

# Standard Quantum Limit for Advanced LIGO Interferometers

January 15, 2002

## 1 Part a

A gravitational wave going through the interferometer will cause the mirrors to oscilate with an amplitude

$$x = Lh$$

and a maximum speed

$$v = Lh\Omega$$

where:
$L =$ length of the interferometer's arm
$h =$ amplitude of the graviational wave
$\Omega =$ frequency of the GW
Therefore, for a mirror of mass $m$ the momentum is:

$$p = mLh\Omega$$

The wavelength can be found using the de Broglie relation:

$$\lambda = \frac{h_{\mathrm{Planck}}}{p}$$

to get

$$\lambda = \frac{h_{\mathrm{Planck}}}{mLh\Omega}$$

or using

$$\Omega = 2\pi f$$

we have:

$$\lambda = \frac{h_{\mathrm{Planck}}}{2\pi f m L h}$$

Later in the course we will derive this by more sophysticated techniques.

## 2 Part b

At the minimum of the Advanced Detectors noise curve (Slide 24)

$$h_{\text{rms}} = 3 * 10^{-24}$$

$$f = 200\text{Hz}$$

and

$$h_{\text{rms}} = h\sqrt{f}$$

So

$$h = \frac{3 * 10^{-24}}{\sqrt{200}}$$

If we approximate $\sqrt{2}$ by 1.5 we get

$$h = 2 * 10^{-25}$$

By plugging this into the expresion for $\lambda$ derived in Part a we get

$$\lambda = 0.6 * 10^{-19}\text{cm}$$

The displacement produced by the gravitational waves will be $x = Lh$. That is $0.8 * 10^{-19}$ cm

It's easy to see that advanced LIGO will operate in the Standard quantun limit.

# Relative motion of LISA's spacecraft

January 15, 2002

If we denote Jupiter's mass by $M_J$ and radius of the orbits of Jupiter and the Spacecraft by $r_J$ and $r_S$ the average acceleration will be:

$$a = \frac{GM_J}{r_J^2}$$

The relative acceleration of two spacecraft can be approximated by

$$a = \frac{GM_J}{(r_J)^2} - \frac{GM_J}{(r_J + d)^2}$$

Where $d$ is the diference in the distances berween the two spacecraft and Jupitrer. This can be approximated by one half of the armlength of LISA. If we neglect terms of higher orders in $\frac{d}{r_J}$ we get

$$a = \frac{2GM_J d}{r_J^3} = \frac{GM_J L}{r_J^3}$$

Now we plug in the nmumbers:
$M_J = 1.9 * 10^{30}\text{g}$
$R = 7.8 * 10^{13}\text{cm}$
$L = 5 * 10^{11}\text{cm}$
$G = 6.6 * 10^{-8}\frac{\text{cm}^3}{\text{gs}^2}$
to get something like

$$a = 1.3 * 10^{-7}\frac{\text{cm}}{\text{s}^2}$$

The typical time this acceleration acts on the spacecrafts is 6 months. That is 15552000s. Therefore the maximum relative speed will be:

$$v = 2\text{cm/s}$$

This means the length of LISA's arms will change at a rate of $2 * 10^4$ times the wavelength of the light used each second.

This is a lot less than the total spacecraft relative velocity LISA will to have to deal with. [1]

Of course there is always room for errors in the numerical calculation. Please help us find the right answer!

[1] see http://lisa.jpl.nasa.gov/documents/ppa2-09.pdf, page 138

Ph 237a: Gravitational Waves 16 January 2001

**WEEK 2: THE MATHEMATICS UNDERLYING GENERAL RELATIVITY**

**Recommended Reading:**

1. Roger D. Blandford and Kip S. Thorne, *Applications of Classical Physics* [cited henceforth as "Blandford and Thorne"], available on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html
   a. Sections 23.1–23.3 of Chapter 23, "From Special to General Relativity" (version 0023.2)
   b. Equation (24.39) and Section 24.6 of Chapter 24, "The Field Equations of General relativity" (version 0024.2).

   Note: this introduction to the mathematics underlying general relativity assumes some prior familiarity with special relativity from a particular point of view, which is presented in Chapter 1 of Blandford and Thorne, "Physics in Flat Spacetime: Geometric Viewpoint" (available at above url). Some readers may wish to consult that chapter as they study Chapters 23 and 24; there are extensive cross references to it.

**Possible Supplementary Reading:**

2. Blandford and Thorne, Chapter 1, "Physics in Flat Spacetime: Geometric Viewpoint". See note on item 1 above.
3. Bernard F. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1985 & 1990), Chapters 2, 3, 5, 6. Not available on the web; a few copies are on sale at the bookstore, and one copy is on reserve in Millikan Library.
4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Chapters 2, 3, 5, 8. Not available on the web; a few copies are on sale at the bookstore, and one copy is on reserve in Millikan Library. Warning: This textbook is terribly out of date, as are *all* other advanced textbooks on general relativity. However, that does not affect the introduction to general relativity; only the applications are out of date (black holes, gravitational waves, cosmology, experimental tests, ...).

**Assignment, to be turned in at beginning of class on Wednesday 23 January by students registered in the course:**

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.

iv. Seek private tutoring from a knowledgable person about this week's topic.
v. Pursue some other method of learning about this week's topic, and state what you have done.

**EXERCISES**

Note: There are more exercises here than any single person is expected to work. Work only those exercises that are useful for you!

**Exercises filling in the gaps in Kip's Wednesday lecture**

1. **Computation of components of a tensor**
From the duality relation $\vec{e}^{\mu} \cdot \vec{e}_{\nu} = \delta^{\mu}_{\nu}$ and expansions of a tensor $\mathsf{T}(\_,\_,\_)$ in terms of basis vectors, e.g. $\mathsf{T} = T^{\alpha\beta}{}_{\mu} \vec{e}_{\alpha} \otimes \vec{e}_{\beta} \otimes \vec{e}^{\mu}$, deduce that the components of a tensor can be computed by inserting basis vectors into its slots and lining up the indices, e.g. $T^{\alpha\beta}{}_{\mu} = \mathsf{T}(\vec{e}^{\alpha}, \vec{e}^{\beta}, \vec{e}_{\mu})$.

2. **Raising and lowering of indices**
From the properties of tensors discussed in exercise 1 and the definition of the metric in terms of the inner product, $\mathsf{g}(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}$, show that indices on tensors can be raised and lowered using the metric components, e.g. $T^{\alpha\beta}{}_{\mu} = g^{\alpha\rho} g_{\mu\sigma} T_{\rho}{}^{\beta\sigma}$.

3. **Directional derivatives of bases**
From the duality relation for bases and the definition $\nabla_{\vec{e}_{\alpha}} \vec{e}_{\beta} \equiv \Gamma^{\mu}{}_{\beta\alpha} \vec{e}_{\mu}$ of the connection coefficients, show that $\nabla_{\vec{e}_{\alpha}} \vec{e}^{\rho} \equiv -\Gamma^{\rho}{}_{\nu\alpha} \vec{e}^{\nu}$.

4. **Connection coefficients for the orthonormal basis associated with circular polar coordinates, and their use**
In Euclidean 2-space (a flat sheet of paper) construct circular polar coordinates $(r, \phi)$.
   a. Show that the basis
$$\mathbf{e}_{\hat{r}} \equiv \frac{\partial}{\partial r}, \quad \mathbf{e}_{\hat{\phi}} \equiv \frac{1}{r}\frac{\partial}{\partial \phi}$$
is orthonormal; i.e. in this basis the components of the metric are the Kronecker delta.
   b. By drawing pictures, deduce the values of all the connection coefficients for this basis.
   c. Let $A^{\hat{\alpha}}$ be the components of a vector field $\mathbf{A}$ in this basis. Using your connection coefficients, derive a formula for the divergence of $\mathbf{A}$, $\boldsymbol{\nabla} \cdot \mathbf{A} = A^{\hat{\alpha}}{}_{;\hat{\alpha}}$ in terms of partial derivatives of the components of $\mathbf{A}$. Your answer should be the familiar formula
$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (r A^{\hat{r}})}{\partial r} + \frac{1}{r}\frac{\partial A^{\hat{\phi}}}{\partial \phi} \; .$$

5. **Components of gradient of a tensor in terms of connection coefficients**
By the same technique as is used in Eq. (23.29) of Blandford and Thorne, derive an expression for $F^{\alpha}{}_{\beta;\mu}$ in terms of $F^{\alpha}{}_{\beta,\mu}$ and the components of $\mathsf{F}$ and the connection coefficients of the chosen basis.

6. **Components of commutator of two vector fields**
   Let $\vec{A}(\mathcal{P})$ and $\vec{B}(\mathcal{P})$ be two vector fields. In an arbitary basis (which might or might not be a coordinate basis), derive a formula for the components of the commutator $[\vec{A}, \vec{B}]$ in terms of the components $A^\alpha$ and $B^\beta$, their derivatives along the basis vectors, $A^\alpha{}_{,\mu}$ and $B^\beta{}_{,\nu}$, and the basis's commutation coefficients $c_{\alpha\beta\gamma}$. In a coordinate basis your result should reduce to the one given by Kip in his lecture; cf. Eq. (23.24) of Blandford and Thorne.

7. **Formula for components of Riemann tensor in an arbitrary basis**
   Exercise 24.8 of Blandford and Thorne

8. **Formula for components of Riemann tensor in a local Lorentz frame**
   Show that in a local Lorentz frame in spacetime, the components of the Riemann tensor are given by Eq. (24.51) of Blandford and Thorne.

**Additional Exercises**

8. **Practice with frame-independent tensors**
   Exercise 23.3 of Blandford and Thorne.

9. **Practice with index shuffling**
   Let $\mathbf{F}$ be a second-rank tensor that is antisymmetric under interchange of its slots, i.e. $F_{\alpha\beta} = -F_{\beta\alpha}$, and that satisfies the relation

   $$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \ .$$

   Define $J^\alpha$ by $F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$, and define

   $$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \ .$$

   (Actually, the electric and magnetic fields can be embodied in an antisymmetric field tensor; if $\mathbf{F}$ is that tensor, then $\vec{J}$ is the charge-current 4-vector and $\mathbf{T}$ is the electromagnetic stress-energy tensor.) Show that

   $$T^{\alpha\beta}{}_{;\beta} = -F^{\alpha\beta}J_\beta \ .$$

   Actually, this equation describes the the rate at which energy and momentum are transfered between the electromagnetic field and the charge-current distribution with which it interacts.

Ph 237 Solutions to Problem Set 2 23 Jan 2002

1. $\mathbb{T}(\vec{e}^{\alpha}, \vec{e}^{\beta}, \vec{e}_{\mu}) = T^{\mu\nu}{}_{\lambda}\, \vec{e}_{\mu} \otimes \vec{e}_{\nu} \otimes \vec{e}^{\lambda} (\vec{e}^{\alpha}, \vec{e}^{\beta}, \vec{e}_{\mu})$

$$= T^{\mu\nu}{}_{\lambda} \underbrace{(\vec{e}_{\mu} \cdot \vec{e}^{\alpha})}_{\delta_{\mu}{}^{\alpha}} \underbrace{(\vec{e}_{\nu} \cdot \vec{e}^{\beta})}_{\delta_{\nu}{}^{\beta}} \underbrace{(\vec{e}^{\lambda} \cdot \vec{e}_{\mu})}_{\delta^{\lambda}{}_{\mu}} = T^{\alpha\beta}{}_{\mu} \quad \underline{\underline{QED}}$$

2. $T^{\alpha\beta}{}_{\mu} = \mathbb{T}(\vec{e}^{\alpha}, \vec{e}^{\beta}, \vec{e}_{\mu}) = T_{\rho}{}^{\lambda\sigma}\, \vec{e}^{\rho} \otimes \vec{e}_{\lambda} \otimes \vec{e}_{\sigma} (\vec{e}^{\alpha}, \vec{e}^{\beta}, \vec{e}_{\mu})$

$$= T_{\rho}{}^{\lambda\sigma} \underbrace{(\vec{e}^{\rho} \cdot \vec{e}^{\alpha})}_{= g(\vec{e}^{\rho}, \vec{e}^{\alpha}) = g^{\rho\alpha}} \underbrace{(\vec{e}_{\lambda} \cdot \vec{e}^{\beta})}_{\delta_{\lambda}{}^{\beta}} \underbrace{(\vec{e}_{\sigma} \cdot \vec{e}_{\mu})}_{= g(\vec{e}_{\sigma}, \vec{e}_{\mu}) = g_{\sigma\mu}}$$

$$= T_{\rho}{}^{\beta\sigma} g^{\rho\alpha} g_{\sigma\mu} \quad \underline{\underline{QED}}$$

3. a. $\nabla_{\vec{e}_{\alpha}} \underbrace{(\vec{e}^{\beta} \cdot \vec{e}_{\gamma})}_{\delta^{\beta}{}_{\gamma}} = 0$ since $\delta^{\beta}{}_{\gamma}$ is a constant, independent of location in spacetime

$$\parallel$$

$$(\nabla_{\vec{e}_{\alpha}} \vec{e}^{\beta}) \cdot \vec{e}_{\gamma} + \underbrace{\vec{e}^{\beta} \cdot \underbrace{(\nabla_{\vec{e}_{\alpha}} \vec{e}_{\gamma})}_{\Gamma^{\mu}{}_{\gamma\alpha} \vec{e}_{\mu}}}_{\delta^{\beta}{}_{\mu} \Gamma^{\mu}{}_{\gamma\alpha} = \Gamma^{\beta}{}_{\gamma\alpha}}$$

b. Thus: $(\nabla_{\vec{e}_{\alpha}} \vec{e}^{\beta}) \cdot \vec{e}_{\gamma} = -\Gamma^{\beta}{}_{\gamma\alpha}$

c. Now $\nabla_{\vec{e}_{\alpha}} \vec{e}^{\beta}$ is a vector so it can be expanded in terms of our basis as $\vec{\nabla}_{\vec{e}_{\alpha}} \vec{e}^{\beta} = S^{\beta}{}_{\mu\alpha} \vec{e}^{\mu}$ for some $S^{\beta}{}_{\mu\alpha}$.

d. Insert this into b. The result is

$$\underbrace{(S^\beta{}_{\mu\alpha}\,\underbrace{\vec{e}^{\mu})\cdot \vec{e}_\gamma}_{\delta^\mu{}_\gamma}}_{S^\beta{}_{\gamma\alpha}} = -\Gamma^\beta{}_{\gamma\alpha}$$

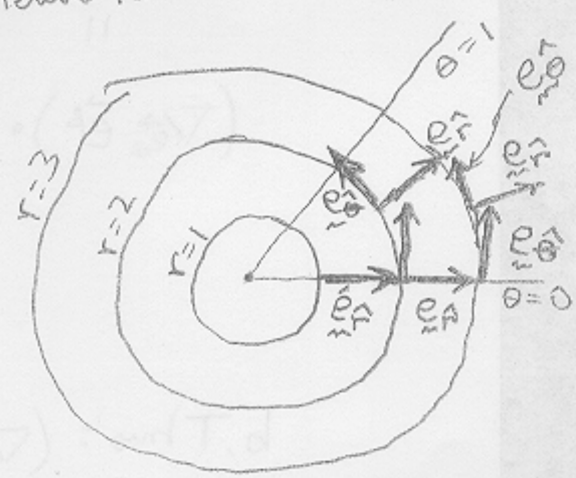
Thus $S^\beta{}_{\gamma\alpha} = -\Gamma^\beta{}_{\gamma\alpha}$, so combining with c. we get

$$\nabla_{\vec{e}_\alpha} \vec{e}^\beta = -\Gamma^\beta{}_{\mu\alpha}\,\vec{e}^\mu . \qquad \underline{QED}$$

4. a. Since $\underset{\sim}{e}_{\hat r} = \frac{\partial}{\partial r}$ and $\underset{\sim}{e}_{\hat\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}$ both differentiate with respect to <u>physical</u> distance, they must have unit length; and since they point in the $r$ & $\theta$ directions, they are orthogonal; i.e.

$$\underset{\sim}{e}_{\hat j}\cdot \underset{\sim}{e}_{\hat k} = \delta_{jk}$$

b. In flat 2-D space, parallel transport is defined in the obvious way. From the picture it should be clear that




i. $\nabla_{\underset{\sim}{e}_{\hat r}} \underset{\sim}{e}_{\hat r} = \nabla_{\underset{\sim}{e}_{\hat r}} \underset{\sim}{e}_{\hat\theta} = 0$, so

$$\boxed{\begin{array}{l}\Gamma^{\hat r}{}_{\hat r\hat r} = \Gamma^{\hat\theta}{}_{\hat r\hat r} = 0 \\ \Gamma^{\hat r}{}_{\hat\theta\hat r} = \Gamma^{\hat\theta}{}_{\hat\theta\hat r} = 0\end{array}}$$

ii. $\nabla_{\underset{\sim}{e}_{\hat\theta}} \underset{\sim}{e}_{\hat r} = \frac{1}{r}\underset{\sim}{e}_{\hat\theta}$

$\nabla_{\underset{\sim}{e}_{\hat\theta}} \underset{\sim}{e}_{\hat\theta} = -\frac{1}{r}\underset{\sim}{e}_{\hat r}$, so

$$\boxed{\begin{array}{l}\Gamma^{\hat\theta}{}_{\hat r\hat\theta} = \frac{1}{r},\ \Gamma^{\hat r}{}_{\hat r\hat\theta} = 0 \\ \Gamma^{\hat r}{}_{\hat\theta\hat\theta} = -\frac{1}{r},\ \Gamma^{\hat\theta}{}_{\hat\theta\hat\theta} = 0\end{array}}$$

4C  3

$$A^{\hat{\alpha}}{}_{;\hat{\alpha}} = A^{\hat{\alpha}}{}_{,\hat{\alpha}} + \Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}} A^{\hat{\mu}}$$

$$= \frac{\partial A^{\hat{r}}}{\partial r} + \frac{1}{r}\frac{\partial A^{\hat{\theta}}}{\partial \theta} + \underbrace{\Gamma^{\hat{r}}{}_{\hat{\theta}\hat{r}}}_{0} A^{\hat{\theta}} + \underbrace{\Gamma^{\theta}{}_{\theta\theta}}_{0} A^{\theta}$$

$$+ \underbrace{\Gamma^{\hat{r}}{}_{\hat{r}\hat{r}}}_{0} A^{\hat{r}} + \underbrace{\Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}}}_{\frac{1}{r}} A^{\hat{r}}$$

$$= \frac{\partial}{\partial r} A^{\hat{r}} + \frac{1}{r}\frac{\partial}{\partial \theta} A^{\hat{\theta}} + \frac{1}{r} A^{\hat{r}}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r A^{\hat{r}}\right) + \frac{1}{r}\frac{\partial}{\partial \theta} A^{\theta}.$$

Ph237 Solution from Yanbei

#5. Components of gradiant of a tensor in terms of connection coefficients.

By definition,

$$F^{\alpha}{}_{\beta;\mu}\, \vec{e}_\alpha \otimes \vec{e}^\beta = \nabla_\mu \left( F^\alpha{}_\beta\, \vec{e}_\alpha \otimes \vec{e}^\beta \right)$$

distributing $\nabla_\mu$, we have

$$\nabla_\mu \left[ F^\alpha{}_\beta\, \vec{e}_\alpha \otimes \vec{e}^\beta \right] = \left( \nabla_\mu F^\alpha{}_\beta \right) \vec{e}_\alpha \otimes \vec{e}^\beta + F^\alpha{}_\beta \left( \nabla_\mu \vec{e}_\alpha \right) \otimes \vec{e}^\beta + F^\alpha{}_\beta\, \vec{e}_\alpha \otimes \left( \nabla_\mu \vec{e}^\beta \right)$$

noticing that

$$\begin{cases} \nabla_\mu \vec{e}_\alpha = \Gamma^\gamma{}_{\alpha\mu}\, \vec{e}_\gamma \\ \nabla_\mu \vec{e}^\beta = -\Gamma^\beta{}_{\gamma\mu}\, \vec{e}^\gamma \end{cases}.$$

$$\Rightarrow F^\alpha{}_{\beta;\mu}\, \vec{e}_\alpha \otimes \vec{e}^\beta = \nabla_\mu F^\alpha{}_\beta\, \vec{e}_\alpha \otimes \vec{e}^\beta + F^\alpha{}_\beta \Gamma^\gamma{}_{\alpha\mu}\, \vec{e}_\gamma \otimes \vec{e}^\beta - F^\alpha{}_\beta \Gamma^\beta{}_{\gamma\mu}\, \vec{e}_\alpha \otimes \vec{e}^\gamma$$

$$= \left( F^\alpha{}_{\beta,\mu} + F^\gamma{}_\beta \Gamma^\alpha{}_{\gamma\mu} - F^\alpha{}_\gamma \Gamma^\gamma{}_{\beta\mu} \right) \vec{e}_\alpha \otimes \vec{e}^\beta$$

$$\Rightarrow \quad F^\alpha{}_{\beta;\mu} = F^\alpha{}_{\beta,\mu} + F^\gamma{}_\beta \Gamma^\alpha{}_{\gamma\mu} - F^\alpha{}_\gamma \Gamma^\gamma{}_{\beta\mu}$$

6. $[\vec{A}, \vec{B}] = [A^\alpha \partial_{\vec{e}_\alpha}, B^\beta \partial_{\vec{e}_\beta}]$

↑ not a partial derivative unless basis is a coordinate basis

$$= A^\alpha \partial_{\vec{e}_\alpha}\left(B^\beta \partial_{\vec{e}_\beta}\right) - B^\beta \partial_{\vec{e}_\beta}\left(A^\alpha \partial_{\vec{e}_\alpha}\right)$$

$$= A^\alpha \underbrace{\left(\partial_{\vec{e}_\alpha} B^\beta\right)}_{B^\beta{}_{,\alpha}} \underbrace{\partial_{\vec{e}_\beta}}_{=\vec{e}_\beta} + A^\alpha B^\beta \partial_{\vec{e}_\alpha} \partial_{\vec{e}_\beta}$$

$$- B^\beta \underbrace{\left(\partial_{\vec{e}_\beta} A^\alpha\right)}_{A^\alpha{}_{,\beta}} \underbrace{\partial_{\vec{e}_\alpha}}_{=\vec{e}_\alpha} - B^\beta A^\alpha \partial_{\vec{e}_\beta} \partial_{\vec{e}_\alpha}$$

$$= A^\alpha B^\beta{}_{,\alpha}\, \vec{e}_\beta \underbrace{- B^\beta A^\alpha{}_{,\beta}\, \vec{e}_\alpha}_{\substack{\text{rename indices} \\ \alpha \to \beta,\ \beta \to \alpha \\ -B^\alpha A^\beta{}_{,\alpha}\, \vec{e}_\beta}} + A^\alpha B^\beta \underbrace{\left[\partial_{\vec{e}_\alpha}, \partial_{\vec{e}_\beta}\right]}_{\substack{= [\vec{e}_\alpha, \vec{e}_\beta] \\ = C_{\alpha\beta}{}^\mu\, \vec{e}_\mu}}$$

$$\boxed{\begin{aligned} &= \left(A^\alpha B^\beta{}_{,\alpha} - B^\alpha A^\beta{}_{,\alpha}\right)\vec{e}_\beta + A^\alpha B^\beta C_{\alpha\beta}{}^\mu\, \vec{e}_\mu \\ &= [\vec{A}, \vec{B}] \end{aligned}}$$

Problems 7 & 9 "Tanbe"

#7. Formula for components of Riemann tensor in an arbitrary basis [Ex. 24.8 of Blandford and Thorne]

For a vector field $p^\alpha$

$$p^\alpha{}_{;\gamma\delta} = (p^\alpha{}_{;\gamma})_{;\delta} = (p^\alpha{}_{;\gamma})_{,\delta} + \Gamma^\alpha{}_{\mu\delta} p^\mu{}_{;\gamma} - \Gamma^\mu{}_{\gamma\delta} p^\alpha{}_{;\mu}$$

$$= (p^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\mu\gamma} p^\mu)_{,\delta} + \Gamma^\alpha{}_{\mu\delta} p^\mu{}_{,\gamma} + \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\nu\gamma} p^\nu - \Gamma^\mu{}_{\gamma\delta} p^\alpha{}_{,\mu} - \Gamma^\mu{}_{\gamma\delta}\Gamma^\alpha{}_{\nu\mu} p^\nu$$

$$= p^\alpha{}_{,\gamma\delta} + \Gamma^\alpha{}_{\mu\gamma,\delta} p^\mu + \Gamma^\alpha{}_{\mu\gamma} p^\mu{}_{,\delta} + \Gamma^\alpha{}_{\mu\delta} p^\mu{}_{,\gamma} + \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\nu\gamma} p^\nu - \Gamma^\mu{}_{\gamma\delta} p^\alpha{}_{,\mu} - \Gamma^\mu{}_{\gamma\delta}\Gamma^\alpha{}_{\nu\mu} p^\nu$$

$$\Rightarrow p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma}$$

$$= (p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma}) + (\Gamma^\alpha{}_{\mu\gamma,\delta} - \Gamma^\alpha{}_{\mu\delta,\gamma}) p^\mu + (\Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\nu\gamma} - \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\nu\delta}) p^\nu - (\Gamma^\mu{}_{\gamma\delta} - \Gamma^\mu{}_{\delta\gamma})(p^\alpha{}_{,\mu} + \Gamma^\alpha{}_{\nu\mu} p^\nu)$$

- $p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma} = \vec{e}_\delta\cdot\vec{\nabla}(\vec{e}_\gamma\cdot\vec{\nabla} p^\alpha) - \vec{e}_\gamma\cdot\vec{\nabla}(\vec{e}_\delta\cdot\vec{\nabla} p^\alpha)$

$$= [(\vec{e}_\delta\cdot\vec{\nabla})\vec{e}_\gamma]\cdot\vec{\nabla} p^\alpha - [(\vec{e}_\gamma\cdot\vec{\nabla})\vec{e}_\delta]\cdot\vec{\nabla} p^\alpha$$

$$= [\vec{e}_\delta, \vec{e}_\gamma]\cdot\vec{\nabla} p^\alpha = c_{\delta\gamma}{}^\mu p^\alpha{}_{,\mu}$$

- $\Gamma^{\mu}{}_{\gamma\delta} - \Gamma^{\mu}{}_{\delta\gamma}$

since $\begin{cases} \nabla_\delta \vec{e}_\gamma = \Gamma^{\mu}{}_{\gamma\delta}\, \vec{e}_\mu \\ \nabla_\gamma \vec{e}_\delta = \Gamma^{\mu}{}_{\delta\gamma}\, \vec{e}_\mu \end{cases}$

$$\underbrace{\nabla_\delta \vec{e}_\gamma - \nabla_\gamma \vec{e}_\delta}_{[\vec{e}_\delta,\, \vec{e}_\gamma] \,=\, C_{\delta\gamma}{}^{\mu}\, \vec{e}_\mu} = \left(\Gamma^{\mu}{}_{\gamma\delta} - \Gamma^{\mu}{}_{\delta\gamma}\right) \vec{e}_\mu$$

$$\Rightarrow \quad \Gamma^{\mu}{}_{\gamma\delta} - \Gamma^{\mu}{}_{\delta\gamma} = C_{\delta\gamma}{}^{\mu}$$

$$\Rightarrow \quad p^{\alpha}{}_{;\gamma\delta} - p^{\alpha}{}_{;\delta\gamma} = \cancel{C_{\delta\gamma}{}^{\mu}\, p^{\alpha}{}_{,\mu}} + \left(\Gamma^{\alpha}{}_{\mu\gamma,\delta} - \Gamma^{\alpha}{}_{\mu\delta,\gamma}\right) p^{\mu}$$
$$+ \left(\Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\nu\gamma} - \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\nu\delta}\right) p^{\nu}$$
$$- C_{\delta\gamma}{}^{\mu}\left(\cancel{p^{\alpha}{}_{,\mu}} + \Gamma^{\alpha}{}_{\nu\mu}\, p^{\nu}\right)$$

$$= \left[\Gamma^{\alpha}{}_{\nu\gamma,\delta} - \Gamma^{\alpha}{}_{\nu\delta,\gamma} + \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\nu\gamma} - \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\nu\delta} - \Gamma^{\alpha}{}_{\nu\mu} C_{\delta\gamma}{}^{\mu}\right] p^{\nu}$$

$$= -\underbrace{\left[\Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\mu} C_{\gamma\delta}{}^{\mu}\right]}_{R^{\alpha}{}_{\beta\gamma\delta}} p^{\beta}$$

this agrees w/ Eq. (24.57)

# 8. Formula for components of Riemann tensor in a Local Lorentz frame.

In a local Lorentz frame, all connection coefficients vanish (at the spatial origin). and

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} \quad \left(\text{by Eq. (24.57)}\right)$$

$$\Gamma^{\alpha}{}_{\beta\delta} = g^{\alpha\mu}\,\frac{1}{2}\left[g_{\mu\beta,\delta} + g_{\mu\delta,\beta} - g_{\beta\delta,\mu}\right]$$

$$\Gamma^{\alpha}{}_{\beta\delta,\gamma} = g^{\alpha\mu}\,\frac{1}{2}\left[g_{\mu\beta,\delta\gamma} + g_{\mu\delta,\beta\gamma} - g_{\beta\delta,\mu\gamma}\right] + \underbrace{g^{\alpha\mu}{}_{,\gamma}\,\frac{1}{2}\left[g_{\mu\beta,\delta} + g_{\mu\delta,\beta} - g_{\beta\delta,\mu}\right]}_{\text{vanishes}}$$

$$= \frac{1}{2} g^{\alpha\mu}\left[g_{\mu\beta,\delta\gamma} + g_{\mu\delta,\beta\gamma} - g_{\beta\delta,\mu\gamma}\right]$$

$$\Gamma^{\alpha}{}_{\beta\gamma,\delta} = \frac{1}{2} g^{\alpha\mu}\left[g_{\mu\beta,\gamma\delta} + g_{\mu\gamma,\beta\delta} - g_{\beta\gamma,\mu\delta}\right]$$

$$\Rightarrow \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} = \frac{1}{2} g^{\alpha\mu}\Big[g_{\mu\beta,\delta\gamma} + g_{\mu\delta,\beta\gamma} - g_{\beta\delta,\mu\gamma} - g_{\mu\beta,\gamma\delta} - g_{\mu\gamma,\beta\delta} + g_{\beta\gamma,\mu\delta}\Big]$$

$$\Rightarrow R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left[g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma} - g_{\alpha\gamma,\beta\delta}\right]$$

[we work in a coordinate basis]

Additional Exercise #8

(a) $(\mathbb{A}+\mathbb{B})(\_,\_) \equiv \mathbb{A}(\_,\_)+\mathbb{B}(\_,\_)$

for $\alpha, \beta, \vec{u}, \vec{v}, \vec{w}$

$$(\mathbb{A}+\mathbb{B})(\alpha\vec{u}+\beta\vec{v},\vec{w}) = \mathbb{A}(\alpha\vec{u}+\beta\vec{v},\vec{w})+\mathbb{B}(\alpha\vec{u}+\beta\vec{v},\vec{w})$$
$$= \alpha\,\mathbb{A}(\vec{u},\vec{w})+\alpha\,\mathbb{B}(\vec{u},\vec{w})$$
$$+\beta\,\mathbb{A}(\vec{v},\vec{w})+\beta\,\mathbb{B}(\vec{v},\vec{w})$$
$$= \alpha(\mathbb{A}+\mathbb{B})(\vec{u},\vec{w})+\beta(\mathbb{A}+\mathbb{B})(\vec{v},\vec{w})$$

similarly

$$(\mathbb{A}+\mathbb{B})(\vec{w},\alpha\vec{u}+\beta\vec{v}) = \alpha(\mathbb{A}+\mathbb{B})(\vec{w},\vec{u})$$
$$+\beta(\mathbb{A}+\mathbb{B})(\vec{w},\vec{v})$$

(b) $\mathbb{A}\otimes\mathbb{B}(\underset{1}{\_},\underset{2}{\_},\underset{3}{\_},\underset{4}{\_}) \equiv \mathbb{A}(\underset{1}{\_},\underset{2}{\_})\cdot\mathbb{B}(\underset{3}{\_},\underset{4}{\_})$

for $\alpha, \beta, \vec{u}, \vec{v}, \vec{w}_1, \vec{w}_2, \vec{w}_3$,

for example

$$\mathbb{A}\otimes\mathbb{B}(\vec{w}_1,\alpha\vec{u}+\beta\vec{v},\vec{w}_2,\vec{w}_3)$$
$$= \mathbb{A}(\vec{w}_1,\alpha\vec{u}+\beta\vec{v})\cdot\mathbb{B}(\vec{w}_2,\vec{w}_3)$$
$$= \alpha\,\mathbb{A}(\vec{w}_1,\vec{u})\cdot\mathbb{B}(\vec{w}_2,\vec{w}_3)+\beta\,\mathbb{A}(\vec{w}_1,\vec{v})\cdot\mathbb{B}(\vec{w}_2,\vec{w}_3)$$
$$= \alpha\,\mathbb{A}\otimes\mathbb{B}(\vec{w}_1,\vec{u},\vec{w}_2,\vec{w}_3)+\beta\,\mathbb{A}\otimes\mathbb{B}(\vec{w}_1,\vec{v},\vec{w}_2,\vec{w}_3)$$

(c). We can first write $\mathbb{A}$ and $\mathbb{B}$ both as sums of direct products of vectors:

$$\begin{cases} \mathbb{A} = \sum_k \vec{A}^{(1)}_k \otimes \vec{A}^{(2)}_k \\ \mathbb{B} = \sum_k \vec{B}^{(1)}_k \otimes \vec{B}^{(2)}_k \end{cases}$$

then $\mathbb{A} \otimes \mathbb{B}$ is

$$\mathbb{A} \otimes \mathbb{B} = \sum_{i,j} \vec{A}^{(1)}_i \otimes \vec{A}^{(2)}_i \otimes \vec{B}^{(1)}_j \otimes \vec{B}^{(2)}_j$$

and the contraction between 1 & 4 slots is

$$\sum_{i,j} \left(\vec{A}^{(1)}_i \cdot \vec{B}^{(2)}_j\right) \vec{A}^{(2)}_i \otimes \vec{B}^{(1)}_j$$

This is obviously a 2nd rank tensor because

$$\vec{A}^{(2)}_i \otimes \vec{B}^{(2)}_j$$

are 2nd rank tensors, and so is their linear combination.

(d). $\mathbb{A} \to A_{\alpha\beta}$

$\mathbb{B} \to B_{\gamma\delta}$

$\mathbb{A} \otimes \mathbb{B} \to A_{\alpha\beta} B_{\gamma\delta}$

$[\mathbb{A} \otimes \mathbb{B}]_{14\,\text{contraction}} \to A^{\alpha}{}_{\beta} B_{\gamma\alpha}$

$[\mathbb{A} \otimes \mathbb{B}]_{23\,\text{contraction}} \to A_{\alpha\beta} B^{\beta}{}_{\delta}$

Additional Exercise #9.
Practice with Index Shuffling.

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]$$

$$T^{\mu\nu}{}_{;\nu} = \frac{1}{4\pi}\Big\{F^{\mu\alpha}{}_{;\nu}F^{\nu}{}_{\alpha} + F^{\mu\alpha}F^{\nu}{}_{\alpha;\nu}$$

$$- \frac{1}{4}g^{\mu\nu}F_{\alpha\beta;\nu}F^{\alpha\beta} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}{}_{;\nu}\Big\}$$

(note that $g^{\mu\nu}{}_{;\delta} = 0$ )

(Also, $F_{\alpha\beta;\nu}F^{\alpha\beta} = F^{\alpha\beta}{}_{;\nu}F_{\alpha\beta}$)

(and, $F^{\nu}{}_{\alpha;\nu} = -F_{\alpha}{}^{\nu}{}_{;\nu} = -4\pi J_{\alpha}$ )

$$= \frac{1}{4\pi}\left\{-4\pi J_{\alpha}F^{\mu\alpha} + F^{\mu\alpha}{}_{;\nu}F^{\nu}{}_{\alpha} - \frac{1}{2}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}{}_{;\nu}\right\}$$

$$= -F^{\mu\alpha}J_{\alpha} + \frac{1}{4\pi}g^{\mu\nu}\left\{F_{\nu}{}^{\alpha}{}_{;\beta}F^{\beta}{}_{\alpha} - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}{}_{;\nu}\right\}$$

However, $F_{\nu}{}^{\alpha}{}_{;\beta}F^{\beta}{}_{\alpha} - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}{}_{;\nu}$

$$= F_{\alpha\nu;\beta}F^{\alpha\beta} - \frac{1}{2}F^{\alpha\beta}F_{\alpha\beta;\nu}$$

$$= \left(F_{\alpha\nu;\beta} - F_{\beta\nu;\alpha}\right)F^{\alpha\beta}\frac{1}{2} - \frac{1}{2}F^{\alpha\beta}F_{\alpha\beta;\nu}$$

$$= F^{\alpha\beta}\frac{1}{2}\left[F_{\alpha\nu;\beta} + F_{\nu\beta;\alpha} + F_{\beta\alpha;\nu}\right]$$

$$= 0$$

So, $\quad T^{\mu\nu}{}_{;\nu} = -F^{\mu\alpha}J_{\alpha}$

i.e. $\quad T^{\alpha\beta}{}_{;\beta} = -F^{\alpha\beta}J_{\beta}$

Ph 237a: Gravitational Waves 23 January 2001

# WEEK 3: INTRODUCTION TO GENERAL RELATIVITY & GRAVITATIONAL WAVES

**Recommended Reading:**

1. Blandford and Thorne, *Applications of Classical Physics*, [available on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html]: the following sections of Chapter 23 (version 0023.2) and Chapter 24 (version 0024.2).
   a. Section 23.4, "The Stress-Energy Tensor Revisited"; also item 2 in Possible Supplementary Reading, below, if you are not already familiar with the stress-energy tensor.
   b. Sections 24.1 — 24.8. Pay special attention to Section 24.4 (in which, if you wish, you can regard the particle as having a finite rest mass $m$ and a 4-momentum $\vec{p} \equiv m\vec{u}$ where $\vec{u}$ is its 4-velocity, so the geodesic equation $\nabla_{\vec{p}}\vec{p} = 0$ is equivalent to the one Kip gave in his Wednesday lecture, $\nabla_{\vec{u}}\vec{u} = 0$, and so $\zeta = \tau/m$). Also pay special attention to Sections 24.5, 24.6 and 24.8.

**Possible Supplementary Reading:**

2. Blandford and Thorne, Chapter 1, "Physics in Flat Spacetime: Geometric Viewpoint": those portions cross referenced in the recommended reading, most especially
   a. The discussion of the Levi-Civita tensor in Section 1.9
   b. Section 1.11 "Volumes, Integration, and the Gauss and Stokes Theorems",
   c. Section 1.12 "The Stress-Energy Tensor and Conservation of 4-Momentum".
3. Bernard F. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1985 & 1990), Sections 6.4–6.7 and Chapter 8.
4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Section 8.7, Chapter 11 and Chapter 17.

**Assignment, to be turned in at beginning of class on Wednesday 30 January by students registered in the course:**

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.
   iv. Seek private tutoring from a knowledgable person about this week's topic.
   v. Pursue some other method of learning about this week's topic, and state what you have done.

**EXERCISES**

Note: There are more exercises here than any single person is expected to work. **You may learn useful things by reading all the exercises**, but work only those that are useful for you.

1. **Transformation of Coordinates; Fixing Four Components of the Metric**
   a. Suppose that one knows the components of a tensor $T_{\alpha\beta}$ and $T^{\mu\nu}$ in the coordinate basis associated with a coordinate system $x^\alpha(\mathcal{P})$. Introduce some other "primed" coordinate system $x^{\sigma'}(\mathcal{P})$. Then the primed coordinates can be written as functions of the unprimed coordinates, $x^{\sigma'}(x^0, x^1, x^2, x^3)$ or in abbreviated notation $x^{\sigma'}(x^\alpha)$; and similarly the unprimed coordinates can be written as functions of the primed ones, $x^\alpha(x^{\sigma'})$. Show that the components of the tensor $\mathbf{T}$ in the primed coordinate basis are related to those in the original, unprimed coordinate basis by

$$T_{\sigma'\rho'} = \frac{\partial x^\alpha}{\partial x^{\sigma'}} \frac{\partial x^\beta}{\partial x^{\rho'}} T_{\alpha\beta} \; , \quad T_{\alpha\beta} = \frac{\partial x^{\sigma'}}{\partial x^\alpha} \frac{\partial x^{\rho'}}{\partial x^\beta} T_{\sigma'\rho'} \; .$$

   Note that the indices in these equations line up in the usual automatic way.

   b. Suppose that one knows the components $g_{\alpha\beta}$ of the metric in the coordinate basis associated with a coordinate system $x^\mu(\mathcal{P})$, and one wants to find a coordinate transformation $x^{\sigma'}(x^\alpha)$ [and its inverse $x^\mu(x^{\sigma'})$] to a new coordinate system $x^{\sigma'}(\mathcal{P})$, in which four of the metric components have the following special values: $g_{0'0'} = -1$, $g_{0'j'} = 0$ with $j' = 1', 2', 3'$. Exhibit a set of four differential equations for the four functions $x^\mu(x^{\sigma'})$ which, if satisfied, will guarantee that these special values are achieved. It is possible, quite generally, to find solutions to these four equations for the four unknowns $x^\mu(x^{\sigma'})$. It is this possibility of fixing four components of the metric however one wishes (e.g. so $g_{0'0'} = -1$, $g_{0'j'} = 0$) that forced Einstein to abandon his original guess $R_{\mu\nu} = 4\pi G T_{\mu\nu}$ for the field equation: that guess gave 10 differential equations for the 10 components of the metric, leaving no freedom to adjust 4 components at will. Note: A coordinate system in which $g_{0'0'} = -1$ and $g_{0'j'} = 0$ is called a *synchronous coordinate system.*

2. **Vacuum Einstein Equations**
   Show that in vacuum the Einstein field equations $G_{\alpha\beta} = 0$ (where $\mathbf{G}$ is the Einstein tensor) are equivalent to $R_{\alpha\beta} = 0$ (where $\mathbf{R}$ is the Ricci tensor, i.e. the contraction of the Riemann tensor on its first and third slots); i.e. show that the vacuum Einstein equations reduce to

$$R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta} = 0 \; . \tag{1}$$

3. **Vacuum Wave Equation for Riemann Tensor when Curvature is Weak**
   Consider the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ describing a very weak warpage of spacetime in vacuum — e.g. a gravitational wave propagating through intergalactic space. When the curvature is ignored (zero-order approximation), we can introduce a global Lorentz frame (global Minkowski coordinates) in which the metric coefficients

are $g_{\alpha\beta} = \eta_{\alpha\beta}$ and the connection coefficients vanish. At first order in the curvature, we can treat $R_{\alpha\beta\gamma\delta}$ as a linear field living in this global Lorentz frame. It then has the following properties discussed in the reading and mentioned briefly in Kip's lecture: (i) It satisfies the Bianchi identity

$$R_{\alpha\beta\gamma\delta,\epsilon} + R_{\alpha\beta\delta\epsilon,\gamma} + R_{\alpha\beta\epsilon\gamma,\delta} = 0 \; ; \tag{2}$$

here the derivatives would normally be spacetime gradients ("covariant derivatives", replace comma by semicolon), but in our global Lorentz frame they reduce to partial derivatives with respect to the coordinates and thus are written with commas rather than semicolons. (ii) It satisfies the vacuum Einstein equations (1), in which the $\mu$ index is raised using the flat-space metric $\eta^{\mu\nu}$ since our reference frame is globally Lorentz. (iii) It satisfies the symmetries

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \; , \quad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \; , \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \; . \tag{3}$$

a. By contracting the Bianchi identity on its first and fifth slots and combining with the vacuum Einstein equation, show that the Riemann tensor is divergence-free on its first slot:

$$R^{\mu}{}_{\beta\gamma\delta,\mu} = 0 \; . \tag{4}$$

b. By invoking Riemann's symmetries, show that it is divergence-free on all four slots.

c. By taking the divergence of the Bianchi identity on its last slot and using the fact that Riemann is divergence-free, derive the wave equation

$$R_{\alpha\beta\gamma\delta,\mu\nu}\eta^{\mu\nu} = \left[ -\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 \right] R_{\alpha\beta\gamma\delta} = 0 \; . \tag{5}$$

Kip will use this wave equation as a starting point for his analysis of gravitational waves next week.

4. **Vacuum Wave Equation for Riemann Tensor when Curvature is Strong**
Suppose that the spacetime curvature is strong, so one cannot introduce, as a zero-order approximation, a global Lorentz frame. Then repeat the analysis of the previous exercise to obtain a wave equation of the form

$$R_{\alpha\beta\gamma\delta;\mu\nu}g^{\mu\nu} = (\text{terms involving products of the Riemann tensor with itself}). \tag{6}$$

[Note: in order to derive this equation you will have to generalize the defining equation $A^{\alpha}{}_{;\beta\gamma} - A^{\alpha}{}_{;\gamma\beta} = -R^{\alpha}{}_{;\mu\beta\gamma}A^{\mu}$ for the Riemann tensor, to get a formula for what happens when you interchange gradient slots on a tensor of higher rank than one, e.g. a formula for $B_{\mu\nu\lambda\rho;\beta\gamma} - B_{\mu\nu\lambda\rho;\gamma\beta}$ for a fourth rank tensor $\mathbf{B}$.] Kip will use the nonlinear vacuum wave equation (6) as a starting point, later in this class, for analyzing the propagation of gravitational waves through a strongly curved "background" spacetime.

5. **Local Lorentz Frame in Friedman Universe**
   a. Exercise 24.2 of Blandford and Thorne. Note: This exercise illustrates the fact that, in a local Lorentz frame, the metric components have their flat spacetime form in the vicinity of the spatial origin for all time, aside from corrections that are second order in the spatial distance from the origin [Eq. (24.15) of Blandford and Thorne].

6. **Second Time Derivative in a Local Lorentz Frame**
   In his lecture on Wednesday, Kip asserted that, when one evaluates the equation of geodesic deviation in the local Lorentz frame of one of the two freely falling particles, the left-hand side of the equation, $\nabla_{\vec{u}}\nabla_{\vec{u}}\vec{\xi}$ (where $\vec{u}$ is the 4-velocity of the particle whose local Lorentz frame one is using and $\vec{\xi}$ is the separation vector between particles) reduces to $(\partial/\partial t)^2\xi^\alpha$. Show that this is true using the result of the previous exercise.

7. **Stress-Energy Tensor for a Perfect Fluid**
   Exercise 23.7(a) of Blandford and Thorne.

8. **Orders of Magnitude of the Radius of Curvature of Spacetime**
   Exercise 24.7 of Blandford and Thorne.

1a

$$T_{\sigma'\rho'} = \mathbb{T}\left(\frac{\partial}{\partial x^{\sigma'}}, \frac{\partial}{\partial x^{\rho'}}\right)$$

by elementary calculus,

$$\frac{\partial}{\partial x^{\sigma'}} = \frac{\partial x^{\alpha}}{\partial x^{\sigma'}} \frac{\partial}{\partial x^{\alpha}}$$

$$\frac{\partial}{\partial x^{\rho'}} = \frac{\partial x^{\beta}}{\partial x^{\rho'}} \frac{\partial}{\partial x^{\beta}}$$

Therefore,

$$T_{\sigma'\rho'} = \mathbb{T}\left(\frac{\partial x^{\alpha}}{\partial x^{\sigma'}} \frac{\partial}{\partial x^{\alpha}}, \frac{\partial x^{\beta}}{\partial x^{\rho'}} \frac{\partial}{\partial x^{\beta}}\right)$$

by linearity of $\mathbb{T}$ in its slots

$$T_{\sigma'\rho'} = \left(\frac{\partial x^{\alpha}}{\partial x^{\sigma'}}\right)\left(\frac{\partial x^{\beta}}{\partial x^{\rho'}}\right) \underbrace{\mathbb{T}\left(\frac{\partial}{\partial x^{\alpha}}, \frac{\partial}{\partial x^{\beta}}\right)}_{\equiv T_{\alpha\beta}}$$

by the method of computing components in the unprimed basis

(1 b)

$$g_{\alpha'\beta'} = \frac{\partial x^{\gamma}}{\partial x^{\alpha'}} \frac{\partial x^{\delta}}{\partial x^{\beta'}} \, g_{\gamma\delta} \qquad (1)$$

$$g_{0'0'} = -1 = \frac{\partial x^{\gamma}}{\partial x^{0'}} \frac{\partial x^{\delta}}{\partial x^{0'}} \, g_{\gamma\delta} \qquad // \text{ 1 equation (2)}$$

$$g_{0'j'} = 0 = \frac{\partial x^{\gamma}}{\partial x^{0'}} \frac{\partial x^{\delta}}{\partial x^{j'}} \, g_{\gamma\delta} \qquad // \text{ 3 equations (3)} \ (j = 1, 2, 3)$$

As long as $X^{\gamma}$ and $X^{\delta}$ satisfy the above equations the metric components will have the required values.

We can, of course, write (2) and (3) as:

$$-1 = g^{0'0'} = \frac{\partial X^{0'}}{\partial X^{\gamma}} \frac{\partial X^{0'}}{\partial X^{\delta}} \, g^{\gamma\delta}$$

$$0 = g^{0'j'} = \frac{\partial X^{0'}}{\partial X^{\gamma}} \frac{\partial X^{j'}}{\partial X^{\delta}} \, g^{\gamma\delta}$$

where we have inferred that $g^{0'0'} = -1$ and $g^{0'j'} = 0$ from the fact that, viewed as matrices,

$$\| g^{\alpha'\beta'} \| = \left\| \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{g^{j'k'}} & \\ 0 & & & \end{matrix} \right\| \text{ is the inverse of } g_{\mu'\nu'} = \left\| \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{a'b'} & \\ 0 & & & \end{matrix} \right\|$$

(2)

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$$

Einstein Eq in vacuum are

$$G_{\alpha\beta} = 0$$

That is

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = 0 \qquad (*)$$

Contract $(*)$ on $\alpha\beta$ to get:

$$\underbrace{R^{\alpha}{}_{\alpha}}_{R} - \frac{1}{2} \underbrace{g^{\alpha}{}_{\alpha}}_{\equiv \delta^{\alpha}{}_{\alpha} \equiv 4} R = 0$$

$$\Rightarrow R - 2R = 0 \Rightarrow R = 0$$

Insert this back into $(*)$ to get $R^{\alpha\beta} = 0$

or equivalently

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = 0.$$

③ ⓐ

$$R^{\alpha}{}_{\beta\gamma\delta,\varepsilon} + R^{\alpha}{}_{\beta\delta\varepsilon,\gamma} + R^{\alpha}{}_{\beta\varepsilon\gamma,\delta} = 0$$

By contracting the above eq with $g^{\varepsilon}{}_{\alpha} = \delta^{\varepsilon}{}_{\alpha}$ we get

$$R^{\mu}{}_{\beta\gamma\delta,\mu} + R^{\mu}{}_{\beta\delta\mu,\gamma} + R^{\mu}{}_{\beta\mu\gamma,\delta} = 0$$

$R^{\mu}{}_{\beta\delta\mu} = -R^{\mu}{}_{\beta\mu\delta} \equiv 0$ from EE in vacuum

$\equiv 0$ from EE in vacuum

$$\Rightarrow R^{\mu}{}_{\beta\gamma\delta,\mu} = 0$$

## 3b

$$0 = R^{\mu}{}_{\beta\gamma\delta,\mu} = R_{\mu\beta\gamma\delta}{}^{,\mu}$$
$$= -R_{\beta\mu\gamma\delta}{}^{,\mu}$$
$$= R_{\gamma\delta\mu\beta}{}^{,\mu}$$
$$= -R_{\gamma\delta\beta\mu}{}^{,\mu}$$

## 3c

$$R_{\alpha\beta\gamma\delta,\varepsilon} + R_{\alpha\beta\delta\varepsilon,\gamma} + R_{\alpha\beta\varepsilon\gamma,\delta} = 0$$

$$R_{\alpha\beta\gamma\delta,\varepsilon}{}^{,\varepsilon} + R_{\alpha\beta\delta\varepsilon,\gamma}{}^{,\varepsilon} + R_{\alpha\beta\varepsilon\gamma,\delta}{}^{,\varepsilon} = 0$$

Partial derivatives commute, so we get

$$R_{\alpha\beta\gamma\delta,\varepsilon}{}^{,\varepsilon} + \underbrace{R_{\alpha\beta\delta\varepsilon}{}^{,\varepsilon}{}_{,\gamma}}_{\equiv 0} + \underbrace{R_{\alpha\beta\varepsilon\gamma}{}^{,\varepsilon}{}_{,\delta}}_{\equiv 0}$$

Since $R_{\alpha\beta\gamma\delta}$ is divergence-free

$$\Rightarrow R_{\alpha\beta\gamma\delta,\varepsilon}{}^{,\varepsilon} = 0$$

or, $R_{\alpha\beta\gamma\delta,\mu\nu}\,\eta^{\mu\nu} = 0$

This is $\left[-\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2\right] R_{\alpha\beta\gamma\delta} = 0$

4. Vacuum wave equation for Riemann tensor when curvature is strong.

(1) From Bianchi identity

$$R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\delta\epsilon;\gamma} + R_{\alpha\beta\epsilon\gamma;\delta} = 0$$

$$\Rightarrow \quad 0 = g^{\alpha\epsilon}\left[R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\delta\epsilon;\gamma} + R_{\alpha\beta\epsilon\gamma;\delta}\right]$$

$$= R^{\mu}{}_{\beta\gamma\delta;\mu} + \underbrace{R^{\mu}{}_{\beta\delta\mu;\gamma} + R^{\mu}{}_{\beta\mu\gamma;\delta}}_{\sim R_{\alpha\beta;\gamma} = 0}$$

$= 0$, for vacuum.

$\because$ since $\quad R^{\mu}{}_{\beta\gamma\delta} = -R_{\beta}{}^{\mu}{}_{\gamma\delta} = R_{\gamma\delta\beta}{}^{\mu} = -R_{\gamma\delta}{}^{\mu}{}_{\beta}$

we have

$$R^{\mu}{}_{\beta\gamma\delta;\mu} = R_{\beta}{}^{\mu}{}_{\gamma\delta;\mu} = R_{\gamma\delta\beta}{}^{\mu}{}_{;\mu} = R_{\gamma\delta}{}^{\mu}{}_{\beta;\mu} = 0$$

(2) Let's consider the following tensor:

$$T^{\mu\nu\lambda\rho} = f A^{\mu} B^{\nu} C^{\lambda} D^{\rho}$$

where $f$ is a scalar; $A^{\mu}, B^{\nu}, C^{\lambda}, D^{\rho}$ are vector fields.

$$T^{\mu\nu\lambda\rho}{}_{;\beta\gamma} - T^{\mu\nu\lambda\rho}{}_{;\gamma\beta}$$

$$= \left(f A^{\mu} B^{\nu} C^{\lambda} D^{\rho}\right)_{;\beta\gamma} - \left(f A^{\mu} B^{\nu} C^{\lambda} D^{\rho}\right)_{;\gamma\beta}$$

$$= \cancelto{0}{(f_{;\beta\gamma} - f_{;\gamma\beta})} A^{\mu} B^{\nu} C^{\lambda} D^{\rho}$$

$$+ f \left(A^{\mu}{}_{;\beta\gamma} - A^{\mu}{}_{;\gamma\beta}\right) B^{\nu} C^{\lambda} D^{\rho}$$

$$+ f A^{\mu} \left(B^{\nu}{}_{;\beta\gamma} - B^{\nu}{}_{;\gamma\beta}\right) C^{\lambda} D^{\rho}$$

$$+ f A^{\mu} B^{\nu} \left(C^{\lambda}{}_{;\beta\gamma} - C^{\lambda}{}_{;\gamma\beta}\right) D^{\rho}$$

$$+ f A^{\mu} B^{\nu} C^{\lambda} \left(D^{\rho}{}_{;\beta\gamma} - D^{\rho}{}_{;\gamma\beta}\right)$$

$$= f A^{\alpha} B^{\nu} C^{\lambda} D^{\rho} \left(-R^{\mu}{}_{\alpha\beta\gamma}\right)$$

$$+ f A^{\mu} B^{\alpha} C^{\lambda} D^{\rho} \left(-R^{\nu}{}_{\alpha\beta\gamma}\right)$$

$$+ f A^{\mu} B^{\nu} C^{\alpha} D^{\rho} \left(-R^{\lambda}{}_{\alpha\beta\gamma}\right)$$

$$+ f A^{\mu} B^{\nu} C^{\lambda} D^{\alpha} \left(-R^{\rho}{}_{\alpha\beta\gamma}\right)$$

$$= - T^{\alpha\nu\lambda\rho} R^{\mu}{}_{\alpha\beta\gamma} - T^{\mu\alpha\lambda\rho} R^{\nu}{}_{\alpha\beta\gamma} - T^{\mu\nu\alpha\rho} R^{\lambda}{}_{\alpha\beta\gamma} - T^{\mu\nu\lambda\alpha} R^{\rho}{}_{\alpha\beta\gamma}$$

Since any tensor $T^{\mu\nu\lambda\rho}$ can be written as a sum of terms like $f A^{\mu} B^{\nu} C^{\lambda} D^{\rho}$, this relation is true in general.

(3) Let's take the Bianchi identity, do one more differentiation, and contract.

$$0 = \left[R_{\alpha\beta\gamma\delta;\mu\nu} + R_{\alpha\beta\delta\mu;\gamma\nu} + R_{\alpha\beta\mu\gamma;\delta\nu}\right] g^{\mu\nu}$$

$$= R_{\alpha\beta\gamma\delta;\mu\nu} g^{\mu\nu} + \left(R_{\alpha\beta\delta\mu;\gamma\nu} - R_{\alpha\beta\delta\mu;\nu\gamma}\right) g^{\mu\nu} + \cancelto{0}{R_{\alpha\beta\delta\mu;\nu\gamma} g^{\mu\nu}}$$

$$+ \left(R_{\alpha\beta\mu\gamma;\delta\nu} - R_{\alpha\beta\mu\gamma;\nu\delta}\right) g^{\mu\nu} + \cancelto{0}{R_{\alpha\beta\mu\gamma;\nu\delta} g^{\mu\nu}}$$

$$(R_{\alpha\beta\delta\mu;\gamma\nu} - R_{\alpha\beta\delta\mu;\nu\gamma})g^{\mu\nu}$$

$$= -\left[R_{\xi\beta\delta\mu} R_\alpha{}^\xi{}_{\gamma\nu} + R_{\alpha\xi\delta\mu} R_\beta{}^\xi{}_{\gamma\nu} + R_{\alpha\beta\xi\mu} R_\delta{}^\xi{}_{\gamma\nu} + R_{\alpha\beta\delta\xi} R_\mu{}^\xi{}_{\gamma\nu}\right] g^{\mu\nu}$$

$$= -\left(R_{\xi\beta\delta\mu} R_\alpha{}^\xi{}_\gamma{}^\mu + R_{\alpha\xi\delta\mu} R_\beta{}^\xi{}_\gamma{}^\mu + R_{\alpha\beta\xi\mu} R_\delta{}^\xi{}_\gamma{}^\mu + \cancelto{0}{R_{\alpha\beta\delta\xi}} \underbrace{R_\mu{}^\xi{}_\gamma{}^\mu}_{0}\right)$$

$$= + R_{\alpha\xi\gamma\mu} R_\beta{}^\xi{}_\delta{}^\mu - R_{\alpha\xi\delta\mu} R_\beta{}^\xi{}_\gamma{}^\mu - R_{\alpha\beta\xi\mu} R_\delta{}^\xi{}_\gamma{}^\mu$$

$$(R_{\alpha\beta\mu\gamma;\delta\nu} - R_{\alpha\beta\mu\gamma;\nu\delta})g^{\mu\nu}$$

$$= -(R_{\alpha\beta\gamma\mu;\delta\nu} - R_{\beta\gamma\mu;\nu\delta})g^{\mu\nu}$$

$$= \left(R_{\xi\beta\gamma\mu} R_\alpha{}^\xi{}_\delta{}^\mu + R_{\alpha\xi\gamma\mu} R_\beta{}^\xi{}_\delta{}^\mu + R_{\alpha\beta\xi\mu} R_\gamma{}^\xi{}_\delta{}^\mu\right)$$

$$= -R_{\alpha\xi\delta\mu} R_\beta{}^\xi{}_\gamma{}^\mu + R_{\alpha\xi\gamma\mu} R_\beta{}^\xi{}_\delta{}^\mu - R_{\alpha\beta\xi\mu} R_\delta{}^\xi{}_\gamma{}^\mu$$

$$\Rightarrow \quad 0 = R_{\alpha\beta\gamma\delta;\mu\nu} g^{\mu\nu} + 2R_{\alpha\xi\gamma\mu} R_\beta{}^\xi{}_\delta{}^\mu - 2R_{\alpha\xi\delta\mu} R_\beta{}^\xi{}_\gamma{}^\mu - 2R_{\alpha\beta\xi\mu} R_\delta{}^\xi{}_\gamma{}^\mu$$

$$\Rightarrow \quad R_{\alpha\beta\gamma\delta;\mu\nu} g^{\mu\nu} = 2R_{\alpha\beta\xi\mu} R_\delta{}^\xi{}_\gamma{}^\mu + 2R_{\alpha\xi\delta\mu} R_\beta{}^\xi{}_\gamma{}^\mu - 2R_{\alpha\xi\gamma\mu} R_\beta{}^\xi{}_\delta{}^\mu$$

Problem 5. Local Lorentz Frame in Friedman Universe

(a) Ex. 24.2 of Blandford & Thorne.

From (24.12).

$$\begin{cases} t = \int_0^\eta a(\eta')d\eta' + \frac{1}{2}\chi^2 \frac{da}{d\eta} \\ x = a\chi \sin\theta \cos\phi \\ y = a\chi \sin\theta \sin\phi \\ z = a\chi \cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} dt = a\,d\eta + \chi \frac{da}{d\eta} d\chi + \frac{1}{2}\chi^2 \frac{da^2}{d\eta^2} d\eta \\ dx = a\chi(\cos\theta\cos\phi\, d\theta - \sin\theta \sin\phi\, d\phi) \\ \qquad + \sin\theta\cos\phi\left(a\,d\chi + \chi\frac{da}{d\eta}d\eta\right) \\ dy = a\chi(\cos\theta \sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi) \\ \qquad + \sin\theta\sin\phi\left(a\,d\chi + \chi\frac{da}{d\eta}d\eta\right) \\ dz = -a\chi\sin\theta\, d\theta + \cos\theta\left(a\,d\chi + \chi\frac{da}{d\eta}d\eta\right) \end{cases}$$

$$\Rightarrow \quad -dt^2 + dx^2 + dy^2 + dz^2$$

$$= a^2(-d\eta^2 + d\chi^2 + \chi^2 d\theta^2 + \chi^2\sin^2\theta\, d\phi^2)$$

$$+ \left(\left(\frac{da}{d\eta}\right)^2 - a\frac{d^2a}{d\eta^2} - \frac{1}{4}\left(\frac{d^2a}{d\eta^2}\right)^2\chi^2\right)\chi^2 d\eta^2 - \frac{da}{d\eta}\frac{d^2a}{d\eta^2}\chi^3 d\chi\, d\eta$$

i.e.

$$ds^2 = a^2(-d\eta^2 + d\chi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2\theta\, d\phi^2)$$

$$= -dt^2 + dx^2 + dy^2 + dz^2$$

$$+ \left[\left(\frac{da}{d\eta}\right)^2 - a\frac{d^2a}{d\eta^2} - \frac{1}{4}\left(\frac{d^2a}{d\eta^2}\right)^2 \chi^2\right] \chi^2 d\eta^2$$

$$- \frac{da}{d\eta}\frac{d^2a}{d\eta^2} \chi^3 d\chi\, d\eta$$

As we can also see,

$$\begin{cases} d\eta = \frac{1}{a} dt + O(\chi) \\ d\chi = dr + O(\chi) \end{cases} \qquad r = \sqrt{x^2+y^2+z^2}$$

so

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$+ \left[\left(\frac{1}{a}\frac{da}{d\eta}\right)^2 - \frac{1}{a}\frac{d^2a}{d\eta^2}\right](x^2+y^2+z^2)\, dt^2$$

$$+ (\text{higher orders in } x, y, z)\, dx^\alpha dx^\beta$$

i.e. the metric deviates from the Lorentz metric by $O\left(\frac{\delta_{jk} x^j x^k}{\mathcal{R}^2}\right)$

where $\frac{1}{\mathcal{R}^2} \sim O\left(\left(\frac{1}{a}\frac{da}{d\eta}\right)^2, \frac{1}{a}\frac{d^2a}{d\eta^2}\right)$

Problem 6. Second Time Derivative in a Local Lorentz Frame

Along the trajectory of one of the free-falling particles, we can build a frame similar to the one in problem 5, i.e, a frame in which the metric differs from the Lorentz metric by square the spatial distance to the spatial origins:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + O\left(\frac{\delta_{ij}x^i x^j}{\mathcal{R}^2}\right) \qquad \text{where } i,j \sim 1,2,3$$

in this frame. $\Gamma^\alpha{}_{\beta\gamma} = 0$ at spatial origin.

Moreover, since $g_{\alpha\beta,0}$ is still $O\left(\frac{\delta_{ij}x^i x^j}{\mathcal{R}^2}\right)$, we also have $\Gamma^\alpha{}_{\beta\gamma,0} = 0$ at spatial origin.

Now, for $\nabla_{\vec{u}}\nabla_{\vec{u}}\vec{\xi}$, at spatial origin

$$u^\alpha \nabla_\alpha\left(u^\beta \nabla_\beta \xi^\mu\right) = \underbrace{\left(u^\alpha \nabla_\alpha u^\beta\right)}_{0} \nabla_\beta \xi^\mu + u^\alpha u^\beta \nabla_\alpha \nabla_\beta \xi^\mu$$

$$= u^\alpha u^\beta \nabla_\alpha\left(\xi^\mu{}_{,\beta} + \Gamma^\mu{}_{\lambda\beta}\xi^\lambda\right)$$

using $\Gamma^\alpha{}_{\beta\gamma}=0$

$$= u^\alpha u^\beta \xi^\mu{}_{,\beta\alpha} + u^\alpha u^\beta \Gamma^\mu{}_{\lambda\beta,\alpha}\xi^\lambda$$

$$= \xi^\mu{}_{,00} + \underbrace{\Gamma^\mu{}_{\lambda 0,0}}_{0}\xi^\lambda = \xi^\mu{}_{,00} = \frac{\partial^2 \xi^\mu}{\partial t^2}$$

Problem 7. Stress-Energy Tensor for a Perfect Fluid

- In the rest frame of the perfect fluid

$$T^{00} = \rho. \quad T^{jk} = P\delta^{jk}, \quad T^{0j} = 0 = T^{j0}$$

we also have

(1) $$u^0 = 1. \quad u^j = 0$$

and thus

$$u^0u^0 = 1. \quad u^0u^j = 0 = u^ju^0, \quad u^ju^k = 0$$

(2) $$g^{00} = -1 \quad g^{jk} = \delta^{jk}, \quad g^{0j} = 0 = g^{j0}$$

combining (1) and (2) by $(\rho+P)(1) + P(2)$, we have

$$(\rho+P)u^0u^0 + Pg^{00} = \rho$$
$$(\rho+P)u^0u^j + Pg^{0j} = 0 = (\rho+P)u^ju^0 + Pg^{j0}$$
$$(\rho+P)u^ju^k + Pg^{jk} = P\delta^{jk}$$

this means the relation [for the components]

(*) $$(\rho+P)u^\alpha u^\beta + Pg^{\alpha\beta} = T^{\alpha\beta} \qquad \alpha,\beta = 0,1,2,3$$

holds in this particular frame.

- Since the components of the two tensors are the same in this frame, the tensors themselves, i.e. the frame-independent geometric objects, when expanded in the basis vectors, must be equal. In this particular case.

$$\mathbb{T} = T^{\alpha\beta}\,\vec{e}_\alpha \otimes \vec{e}_\beta = [(\rho+P)u^\alpha u^\beta + Pg^{\alpha\beta}]\,\vec{e}_\alpha \otimes \vec{e}_\beta = (\rho+P)\vec{u}\otimes\vec{u} + Pg$$

8. Orders of magnitude of the Radius of Curvature of Spacetime.

$$\mathcal{R} \sim \left(\frac{r^3}{GM}\right)^{\frac{1}{2}} = \left(\frac{c^2 r^3}{GM}\right)^{1/2}$$

- near the earth's surface

$M_{earth} = 6 \times 10^{24}\,kg \qquad r_{earth} = 6 \times 10^6\,m$

$\mathcal{R} \sim 2 \times 10^{11}\,m \approx 1\,AU$

- near the sun's surface

$M_{sun} = 2 \times 10^{30}\,kg \qquad r_{sun} = 7 \times 10^8\,m$

$\mathcal{R} \sim 5 \times 10^{11}\,m$

- near the surface of a white-dwarf star.

$M_{WD} \sim M_{sun} \qquad r_{WD} \sim 10^{-2}\, r_{sun}$

$\mathcal{R} \sim 5 \times 10^8\,m \sim r_{sun}$

- near the surface of a neutron star

$M_N \sim M_{sun} \qquad r_N \sim 10^{-5}\, r_{sun} \sim 10\,km$

$\mathcal{R} \sim 15\,km$ only a little larger than the star itself!

- near the surface of a one-solar-mass BH

for BH's, $r \sim GM/c^2 \Rightarrow \mathcal{R} \sim r \sim \frac{GM_{sun}}{c^2} \sim 2\,KM$

• In intergalactic space

$$\mathcal{R} \sim \left(\frac{3c^2}{4\pi G\rho}\right)^{1/2}$$

where $\rho \sim$ density of the universe $\sim 10^{-26}\,kg\,m^{-3}$

$$\Rightarrow \mathcal{R} \sim 10^{26}\,m \sim \text{Hubble radius}.$$

Ph 237a: Gravitational Waves 30 January 2002

## WEEK 4: WEAK GRAVITATIONAL WAVES IN OTHERWISE FLAT SPACETIME

**Recommended Reading:**

1. Blandford and Thorne, *Applications of Classical Physics*, [available on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html]: the following sections of Chapter 24 (version 0024.2) and Chapter 26 (version 0026.2).
   a. Section 24.9, "Weak Gravitational Fields"; especially Sec. 24.9.2 on "Linearized Theory".
   b. Sections 26.3.1, 26.3.2, 26.3.3, 26.3.7 on gravitational waves. Note: These sections are written from a more sophisticated viewpoint than we have taken as yet: they are treating gravitational waves that propagate through curved spacetime, not flat. However, they make extensive use of local Lorentz frames of the curved background spacetime through which the waves are propagating, and if one replaces these local Lorentz frames by global Lorentz frames of a flat background spacetime, one obtains much of the material that Kip covered in class. This replacement entails replacing every subscript "—" (gradient or covariant derivative with respect to the flat background) by a subscript comma (gradient in our background global Lorentz frame, i.e. partial derivative).

**Possible Supplementary Reading:**

3. Bernard F. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1985 & 1990), Sections 8.3, 9.1 and 9.2
4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Chapter 18 on the linearized approximation to general relativity, and Sections 35.1 to 35.6 on gravitational waves.

**Assignment, to be turned in at beginning of class on Wednesday 6 February by students registered in the course:**

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.
   iv. Seek private tutoring from a knowledgable person about this week's topic.
   v. Pursue some other method of learning about this week's topic, and state what you have done.

## EXERCISES

Note: There are more exercises here than any single person is expected to work. Work only those exercises that are useful for you!

### Exercises filling in the gaps in Kip's lectures

1. **Electromagnetic Analogs of $h^{\rm TT}_{jk}$, $h_+$ and $h_\times$**

   The gravitational-wave analysis given in Kip's lectures and in the exercises that follow is closely analogous to the following treatment of electromagnetic waves.

   Consider a plane electromagnetic wave propagating in the $z$ direction through a Lorentz frame of flat spacetime. The wave has an antisymmetric electromagnetic field tensor $F_{\mu\nu}(t-z)$ whose components are related to those of the electric and magnetic field by $F_{j0} = -F_{0j} = E_j$ and $(F_{23}, F_{31}, F_{12}) = (B_1, B_2, B_3)$.

   a. Use Maxwell's equations to verify that $\mathbf{E}$ and $\mathbf{B}$ are transverse (have vanishing $z$ components), and that all components of $F_{\mu\nu}$ can be expressed in terms of $F_{j0}$; in other words, the magnetic field can be expressed in terms of the electric field. This is analogous to the transversality of the tidal forces in a gravitational wave, and to the fact that all components of the Riemann tensor for a gravitational wave are expressable in terms of $R_{j0k0}$.

   b. Define $A^{\rm T}_j$ by $E_j \equiv -A^{\rm T}_{j,t}$. Here and throughout we use the notation that subscripts $0, 1, 2, 3$ are equivalent to subscripts $t, x, y, z$. This $A^{\rm T}_j$ is the analog of $h^{\rm TT}_{jk}$ for a gravitational wave. Since the electromagnetic wave is transverse, the only nonzero components of $A^{\rm T}_j$ are $A^{\rm T}_x$ (the analog of $h_+$) and $A^{\rm T}_y$ (the analog of $h_\times$).

   c. Now introduce the 4-vector potential $A_\mu$ (not to be confused with $A^{\rm T}_j$), from which the electromagnetic field tensor can be constructed via $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. Show that in Lorenz gauge, where $A_\mu{}^{,\mu} = 0$, Maxwell's equations reduce to the wave equation for $A_\mu$, and thence (since we are considering a plane wave propagating in the $z$ direction), $A_\mu$ is a function only of $t-z$. This is the analog of the trace reversed metric peturbation for a plane gravitational wave being a function of $t-z$ in gravitational Lorenz gauge (discussed by Kip in his lectures).

   d. Find a specific gauge-change generator $\Psi(t-z)$ that brings $A_\mu$ into a special Lorenz gauge in which $A_0 = A_z = 0$ so the that $A_\mu$ is *transverse*. Show that in this special Lorenz gauge, the spatial components of the vector potential are $A_j = A_j{}^{\rm T}$. We call this Transverse gauge or T gauge. It is the electromagnetic analog of TT gauge for a gravitational wave.

   e. Show that the T-gauge fields $A_x{}^{\rm T}$ and $A_y{}^{\rm T}$ can be obtained from the vector potential in any gauge where $A_\mu = A_\mu(t-z)$ by simple projection — i.e., by throwing way the temporal and longitudinal components of $A_\mu$ and setting $A^{\rm T}_x = A_x$ and $A^{\rm T}_y = A_y$. This is the analog of computing the components of $h^{\rm TT}_{jk}$ by projection, in any gauge where the metric perturbation is a function of $t-z$.

   f. The fields $A_x{}^{\rm T}$ and $A_y{}^{\rm T}$ depend on one's choice of reference frame. Show that when one rotates the frame's basis vectors in the transverse plane in the manner of Eq. (26.51), ${A_x}^{\rm T}$ and ${A_y}^{\rm T}$ change by

$$({A_x}^{\rm T} + i{A_y}^{\rm T})_{\rm new} = ({A_x}^{\rm T} + i{A_y}^{\rm T})_{\rm old} e^{i\psi} \ .$$

g. Show that, when one performs a boost along the $z$ axis to a new reference frame moving at speed $\beta$ with respect to the old one, the fields $A_x^{\rm T}$ and $A_y^{\rm T}$ (which are defined in terms of the electric fields measured in the two frames), are unchanged at a fixed location in spacetime; i.e. they behave like scalars. This is not true of the electric field itself! [Hint: use the result of part e.]

2. **For a Weak, Plane Gravitational Wave, All Components of Riemann are Determined by $R_{j0k0}$, Which is TT**

Consider a solution of the gravitational wave equation for plane waves propagating in the $z$ direction through a global Lorentz frame,

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(t - z) \ . \tag{6}$$

a. By integrating the Bianchi identity with respect to time for such a gravitational wave, derive Eqs. (26.39) of Blandford and Thorne.

b. Use these relations and Riemann's symmetries to show explicitly that all components of the Riemann tensor for these waves can be expressed in terms of $R_{j0k0}$. This is the analog of Exercise 1.a for the electromagnetic field.

c. Then use the vacuum Einstein equations to show explicitly that $R_{j0k0}$ is spatially transverse and trace-free (TT), i.e. it satisfies Eqs. (26.40) of Blandford and Thorne, and $R_{x0x0} + R_{y0y0} = 0$; cf. Eqs. (26.41); cf. the transversality of the electric field, Exercise 1.a.

3. **Gravitational Gauge Changes; Transformation to Lorenz Gauge**

Exercise 24.13 of Blandford and Thorne. This is analogous to Exercise 1.c for the electromagnetic field.

4. **Transformation to TT Gauge**

Consider a plane gravitational wave propagating in the $z$ direction and analyzed in any gauge in which $h_{\alpha\beta}$ is a function only of $t - z$ (e.g. in any Lorenz gauge).

a. Exhibit a gauge transformation that brings this wave into TT gauge, so $h^{\rm new}_{\alpha\beta} = h^{\rm TT}_{\alpha\beta}$. This is analogous to Exercise 1.d.

b. Show that this gauge transformation can be achieved by TT projection — i.e., by simply throwing away the time-time and time-space and longitudinal components of $h_{alpha\beta}(t - z)$, keeping only the spatial and transverse components (those in the $x$-$y$ plane, and removing the trace of these components, so

$$h_+ \equiv h^{\rm TT}_{xx} = h_{xx} - \frac{1}{2}(h_{xx} + h_{yy}) = \frac{1}{2}(h_{xx} - h_{yy}) \ ,$$

$$-h_+ = h^{\rm TT}_{yy} = h_{yy} - \frac{1}{2}(h_{xx} + h_{yy}) = -\frac{1}{2}(h_{xx} - h_{yy}) \ ,$$

$$h_\times \equiv h^{\rm TT}_{xy} = h^{\rm TT}_{yx} = h_{xy} = h_{yx} \ .$$

This is analogous to Exercise 1.e.

5. **Behavior of $h_+$ and $h_\times$ Under Rotations and Boosts**
   a. For a weak plane wave propagating in the $z$ direction of a background Lorentz frame, show that $h_+$ and $h_\times$ transform under rotations through an angle $\psi$ about the propagation direction in the manner of Eq. (26.51) of Blandford and Thorne. Rewrite this transformation law in terms of $\cos 2\psi$ and $\sin 2\psi$, and thereby recover the formulas that Kip gave in his Monday lecture. This is analogous to Exercise 1.f.
   b. For this same weak, plane wave, show that at any event in spacetime $h_+$ and $h_\times$ are invariant under boosts along the waves' propagation direction ($z$ direction). [This is analogous to Exercise 1.g.] One way to show this is: (i) apply a Lorentz transformation to the components of the Riemann tensor, and then (ii) in each of the two reference frames construct $h^{\rm TT}_{jk}$. This is a hard way with pitfalls. A much simpler way is to use a result from Exercise 4.b that, in any gauge where the metric perturbation has the speed-of-light-propagation form $h_{\alpha\beta}(t-x)$, one can compute the gravitational wave field ${h_{jk}}^{\rm TT}$ by projection. The idea, then, is to begin in TT gauge of one of the two frames, with $h_{\alpha\beta}(t-z)$ equal to that frame's TT field, perform a Lorentz transformation of that field to take its components to the other reference frame, then use projection to extract the second frame's TT field.
6. **Motion of a Free Particle in TT Gauge**
   Consider a gravitational wave as described in TT gauge, so the spacetime metric has the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h^{\rm TT}_{\alpha\beta}(t-z)$. Consider a free particle that is at rest in this coordinate system before the wave arrives. Use the geodesic equation to show that the particle remains always at rest in this coordinate system, even while the wave is interacting with it.

II-7

1a) Proof using Maxwell's Eqs. in vacuum

$$\begin{cases} F_{\mu\nu}{}^{,\nu} = 0 & (1) \\ F_{\alpha\beta}{}^{,\gamma} + F_{\beta\gamma}{}^{,\alpha} + F_{\gamma\alpha}{}^{,\beta} = 0 \end{cases}$$

→ This is also known as $F^{*}{}_{\alpha\beta}{}^{,\beta} = 0$

$F^{*}_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}$ SEE NOTE

Using that $F_{\mu\nu} = F_{\mu\nu}(t-z)$ and $\frac{\partial}{\partial z} F_{\mu\nu}(t-z) = -\frac{\partial}{\partial t} F_{\mu\nu}(t-z)$ (1) gives

$$\left.\begin{aligned} F_{03}{}^{,0} &= 0 \\ F_{30}{}^{,0} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} B_z &= \text{constant} \\ E_z &= \text{constant} \end{aligned}$$

They must be 0 since we don't have an EM field constant in all space.

$$\left.\begin{aligned} F_{10}{}^{,0} - F_{13}{}^{,0} &= 0 \\ F_{20}{}^{,0} - F_{23}{}^{,0} &= 0 \end{aligned}\right\} \to \begin{cases} \dot{E}_x + \dot{B}_y = 0 \\ \dot{E}_y - \dot{B}_x = 0 \end{cases}$$

We have 2 Eqs that constrain the 4 non-0 independent components of $F_{\mu\nu}$ that give

$$F_{13}{}^{,0} = F_{10}{}^{,0}$$

$$\Rightarrow \quad F_{20}{}^{,0} = F_{23}{}^{,0}$$

→ All components of $F_{\mu\nu}$ can be expressed as functions of $F_{10}$ and $F_{20}$.

NOTE ON $F^{*}_{\alpha\beta}$ ON NEXT PAGE



Ph237 HW4 Solution, Feb 2002

NOTE $F^*_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}$

$\varepsilon_{\alpha\beta\gamma\delta}$ is the total antisymmetric tensor

$$\varepsilon_{\alpha\beta\gamma\delta} = \begin{cases} 0 & \text{if 2 indices are equal} \\ (-1)^{(\text{signature of the permutation } \alpha\beta\gamma\delta)} & \text{otherwise.} \end{cases}$$

Ex: $\varepsilon_{0012} = 0$
$\varepsilon_{0123} = 1$
$\varepsilon_{0132} = -1$

(12) - another partial proof:
Electromagnetic waves propagate in the direction of the Poynting vector

$$\vec{S} = \vec{E} \times \vec{B}$$

which is perpendicular on both E and B

$\Rightarrow$ E and B are transverse fields.

b) $A^T_{j,t} = -E_j$

$E_0 = 0$ $E_3 = 0$ since the waves are transverse $\Rightarrow$ ~~the only non~~

$\Rightarrow A^T_{0,t} = 0$
$A^T_{3,0} = 0$

This specifies $A^T_0$ and $A^T_3$ up to a constant which we may chose to be 0. (2)

c)

$F_{\mu\nu}{}^{,\nu} = 0$ becomes

$$A_{\nu,\mu}{}^{,\nu} - A_{\mu,\nu}{}^{,\nu} = 0$$

by commutativity of the partial derivatives

$$= A_{\nu}{}^{,\nu}{}_{,\mu} = 0 \quad \text{because of the Lorentz Gauge}$$

$$\Rightarrow A_{\mu,\nu}{}^{,\nu} = 0 \quad \text{or} \quad \boxed{\Box A_\mu = 0}$$

That is $$-A_{\mu,0}{}^{,0} + A_{\mu,3}{}^{,3} = 0$$

since we are considering plane waves propagating in the z direction

(d) Gauge $A_\mu \to A_\mu - \partial_\mu \Psi(t-z)$

Lorentz Gauge $\partial_\mu A^\mu = 0$ gives

$$\partial_0 A^0 + \partial_3 A^3 = 0$$

$$\partial_0 A^0 - \partial_0 A^3 = 0$$

$$\Rightarrow A^0 = A^3 \Rightarrow A_0 = - A_3 \text{ up to a constant}$$

and the gauge we need is

$$\Psi = \int_0^{t-z} A_0(u)\, du$$

(e) This will cancel $A^0$ and $A^3$ up to a constant.
and should not change $A_1$ and $A_2$ since
$\partial_1 \Psi = 0 = \partial_2 \Psi$ because $\Psi$ depends only on $t$ and $z$.

Under a rotation of angle $\varphi$ in the xy plane,

f) $(e_x + i e_y)_{new} = (e_x + i e_y)_{old}\, e^{i\varphi}$

$$\left(A_x^T + i A_y^T\right)_{new} = A^T \cdot (e_x + i e_y)_{new} =$$

$$= A^T e^{i\varphi} (e_x + i e_y)_{old} = \left(A_x^T + i A_y^T\right)_{old} \cdot e^{i\varphi}$$

(8)

- $A^T$ is not a coordinate independent object. It is defined only after you choose a reference frame.
- The vector potential $\vec{A}$ is frame independent but gauge dependent
- In this problem we have plane waves propagating in the $z$ direction. For this problem there exists a gauge in the initial reference frame in which

$$A_t = A_z = 0$$
$$A_x = A_x^T(t-z)$$
$$A_y = A_y^T(t-z)$$

- In this reference frame you can perform a boost in the $z$ direction to a primed frame on the 4-vector $A_\mu$.

Since $A_z = A_0 = 0$ and $t - z = \gamma(1+\beta)(t'-z')$ the result is

$$A_x' = A_x = A_x^T(t-z) = A_x^T(\gamma(1+\beta)(t'-z'))$$
$$A_y' = A_y = A_y^T(t-z) = A_y^T(\gamma(1+\beta)(t'-z'))$$
$$A_z' = A_{t'} = 0$$

Now we find $A_{x'}^T$ and $A_y^T$ in the new frame by projection - and the projection is trivial

$$A_{x'}^T = A_{x'} = A_x^T(\gamma(1+\beta)(t'-z')) \; ; \; A_{y'}^T = A_{y'} = A_y^T(\gamma(1+\beta)(t'-z'))$$

So $A_x^T$ is not changed at a fixed event, but its dependence on the coordinates shows a Doppler shift.

(5)

#2. For a weak, Plane GW, All components of Riemann are Determined by $R_{j0k0}$. Which is TT.

(a)

$$R_{\alpha\beta\gamma\delta,\epsilon} + R_{\alpha\beta\delta\epsilon,\gamma} + R_{\alpha\beta\epsilon\gamma,\delta} = 0$$

$$\Rightarrow \begin{cases} R_{\alpha\beta 12,0} + R_{\alpha\beta 20,1} + R_{\alpha\beta 01,2} = 0 & \Rightarrow R_{\alpha\beta 12,0} = 0 \\ R_{\alpha\beta 23,0} + R_{\alpha\beta 30,2} + R_{\alpha\beta 02,3} = 0 & \Rightarrow R_{\alpha\beta 23,0} + R_{\alpha\beta 02,3} = 0 \\ R_{\alpha\beta 13,0} + R_{\alpha\beta 30,1} + R_{\alpha\beta 01,3} = 0 & \Rightarrow R_{\alpha\beta 13,0} + R_{\alpha\beta 01,3} = 0 \end{cases}$$

$$R_{\alpha\beta 12,3} + R_{\alpha\beta 23,1} + R_{\alpha\beta 31,2} = 0 \Rightarrow R_{\alpha\beta 12,3} = 0$$

these gives

$$R_{\alpha\beta 12} = 0$$

$$R_{\alpha\beta 23} = R_{\alpha\beta 02} = -R_{\alpha\beta 20}$$

$$R_{\alpha\beta 13} = R_{\alpha\beta 01} = -R_{\alpha\beta 10}$$

$$R_{\alpha\beta 12} = 0$$

[Apart from constants, which vanish since it's a wave.]

(b) All the independent components are

$R_{0i0j}$

$R_{0ijk}$: related by part (a) to $R_{0i0l}$ (or 0)

$$R_{lmij} : \xrightarrow{\text{part (a)}} R_{lm0k} \ (\text{or } 0)$$

$$\begin{gathered} \| \\ R_{0klm} \\ \downarrow \text{part (a)} \\ R_{0k0p} \ (\text{or } 0) \end{gathered}$$

6

(c) Einstein eq., $R_{\alpha\beta\gamma}{}^{\beta}=0$

$R_{00}$ : $R_{0i0}{}^{i} + \cancel{R_{000}{}^{0}} = 0$

$$\Rightarrow R_{0101} + R_{0202} + R_{0303} = 0$$

$R_{0i}$ : $\cancel{R_{00i}{}^{0}} + R_{0ki}{}^{k} = 0$

$$\begin{cases} R_{011}{}^{1} + R_{021}{}^{2} + R_{031}{}^{3} = 0 \Rightarrow R_{0313} = 0 \\ \qquad \Rightarrow R_{0301} = 0 = R_{1030} \\ R_{012}{}^{1} + R_{022}{}^{2} + R_{032}{}^{3} = 0 \Rightarrow R_{0323} = 0 \\ \qquad \Rightarrow R_{0302} = 0 = R_{2030} \\ R_{013}{}^{1} + R_{023}{}^{2} + R_{033}{}^{3} = 0 \Rightarrow R_{0101} + R_{0202} = 0 \end{cases}$$

$R_{ij}$ : $R_{102}{}^{0} + R_{112}{}^{1} + R_{122}{}^{2} + R_{132}{}^{3} = 0$

$\Rightarrow -R_{1020} + R_{1320} = 0$

$\Rightarrow -R_{1020} - R_{2013} = 0$

$\Rightarrow -R_{1020} + R_{2010} = 0$ automatically true

$R_{103}{}^{0} + R_{113}{}^{1} + R_{123}{}^{2} + R_{133}{}^{3} = 0$

$\Rightarrow R_{1030} = 0$

$R_{203}{}^{0} + R_{213}{}^{1} + R_{223}{}^{2} + R_{233}{}^{3} = 0$

$\Rightarrow R_{2030} = 0$

[Already from $R_{00}$ and $R_{0i}$] $\Rightarrow R_{1030} = R_{3010} = R_{2030} = R_{3020} = R_{3030} = 0$

$R_{1010} + R_{2020} = 0$

i.e. $R_{i0j0}$ is TT

#3. Gravitational Gauge Changes; Transformation to Lorenz Gauge. [Ex 24.13 of B&T]

(a) $$x^\alpha_{new}(x^\mu_{old}) = x^\alpha_{old} + \xi^\alpha(x^\mu_{old})$$

$$\Rightarrow \quad \frac{\partial x^\alpha_{new}}{\partial x^\beta_{old}} = \delta^\alpha{}_\beta + \xi^\alpha{}_{,\beta}$$

and

$$\frac{\partial x^\alpha_{old}}{\partial x^\beta_{new}} = \delta^\alpha{}_\beta - \xi^\alpha{}_{,\beta} + O(\xi^2)$$

so.
$$g^{new}_{\mu\nu} = \frac{\partial x^\alpha_{old}}{\partial x^\mu_{new}} \frac{\partial x^\beta_{old}}{\partial x^\nu_{new}} g^{old}_{\alpha\beta}$$

$$\Rightarrow \eta_{\mu\nu} + h^{new}_{\mu\nu} = \left(\delta^\alpha{}_\mu - \xi^\alpha{}_{,\mu}\right)\left(\delta^\beta{}_\nu - \xi^\beta{}_{,\nu}\right)\left(\eta_{\alpha\beta} + h^{old}_{\alpha\beta}\right)$$

$$= \eta_{\mu\nu} + h^{old}_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + O(h^2, \xi^2)$$

$$\Rightarrow \quad h^{new}_{\mu\nu} = h^{old}_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu}$$

(b) $$\bar{h}^{new}_{\mu\nu} = h^{new}_{\mu\nu} - \frac{1}{2}\eta^{\alpha\beta} h^{new}_{\alpha\beta}\,\eta_{\mu\nu}$$

$$= h^{old}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} - \frac{1}{2}\eta^{\alpha\beta}\left[h^{old}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}\right]\eta_{\mu\nu}$$

$$= \bar{h}^{old}_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + \xi^\alpha{}_{,\alpha}\,\eta_{\mu\nu}$$

and $$\bar{h}^{new}_{\mu\nu}{}^{,\nu} = \bar{h}^{old}_{\mu\nu}{}^{,\nu} - \cancel{\xi_{\nu,\mu}{}^{\nu}} - \xi_{\mu,\nu}{}^{\nu} + \cancel{\xi^\alpha{}_{,\alpha}}{}\,\eta_{\mu\nu}$$

$$= \bar{h}^{old}_{\mu\nu}{}^{,\nu} - \xi_{\mu,\nu}{}^{\nu}$$

we just have to impose

$$\Box \xi_\mu = \bar{h}^{old}_{\mu\nu}{}^{,\nu}$$ [which is solvable by inverting "$\Box$", e.g. using Green function]

to get $\bar{h}^{new}_{\mu\nu}{}^{,\nu} = 0$

(c) Given $\bar{h}_{\mu\nu}{}^{,\nu} = 0$, Let's look at (4.102) again

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} + \bar{h}_{\mu\alpha,\nu}{}^{\alpha} + \bar{h}_{\nu\alpha,\mu}{}^{\alpha} = 16\pi T_{\mu\nu}$$

$$\begin{cases} \bar{h}_{\alpha\beta}{}^{,\alpha\beta} = \left[\bar{h}_{\alpha\beta}{}^{,\beta}\right]^{,\alpha} = 0 \\ \bar{h}_{\mu\alpha,\nu}{}^{\alpha} = \left[\bar{h}_{\mu\alpha,}{}^{\alpha}\right]_{,\nu} = 0 \\ \bar{h}_{\nu\alpha,\mu}{}^{\alpha} = \left[\bar{h}_{\nu\alpha,}{}^{\alpha}\right]_{,\mu} = 0 \end{cases}$$

$$\Rightarrow \quad -\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 16\pi T_{\mu\nu}$$

4. Transformation to TT Gauge.

(a) $$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}(t-z)$$

Let's look at the trace-reversed metric perturbation,

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta}$$

- The Lorenz gauge condition on $\bar{h}_{\alpha\beta}$ yields:

$$\bar{h}_{\alpha\beta}{}^{,\beta} = 0 \Rightarrow \bar{h}_{00}{}^{,0} + \bar{h}_{03}{}^{,3} = 0 \Rightarrow \bar{h}_{00} + \bar{h}_{03} = 0$$

$$\bar{h}_{10}{}^{,0} + \bar{h}_{13}{}^{,3} = 0 \Rightarrow \bar{h}_{10} + \bar{h}_{13} = 0$$

$$\bar{h}_{20}{}^{,0} + \bar{h}_{23}{}^{,3} = 0 \Rightarrow \bar{h}_{20} + \bar{h}_{23} = 0$$

$$\bar{h}_{30}{}^{,0} + \bar{h}_{33}{}^{,3} = 0 \Rightarrow \bar{h}_{30} + \bar{h}_{33} = 0$$

(where uninterested constants are assumed to be 0)

- The Einstein Eq. is already satisfied since the dependence on $t$ & $z$ is $t-z$.

For a vector field $\xi_\alpha$,

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \xi^{\alpha}{}_{,\alpha}\eta_{\mu\nu}$$

And

$$\bar{h}_{00} \rightarrow \bar{h}_{00} - \dot{\xi}_0 + \dot{\xi}_3$$

$$\begin{cases} \bar{h}_{01} \rightarrow \bar{h}_{01} - \dot{\xi}_1 \\ \bar{h}_{02} \rightarrow \bar{h}_{02} - \dot{\xi}_2 \\ \bar{h}_{03} \rightarrow \bar{h}_{03} + \dot{\xi}_0 - \dot{\xi}_3 \end{cases} \qquad \begin{cases} \bar{h}_{12} \rightarrow \bar{h}_{12} \\ \bar{h}_{13} \rightarrow \bar{h}_{13} + \dot{\xi}_1 \\ \bar{h}_{23} \rightarrow \bar{h}_{23} + \dot{\xi}_2 \end{cases}$$

$$\begin{cases} \bar{h}_{11} \to \bar{h}_{11} - \dot{\xi}_0 - \dot{\xi}_3 \\ \bar{h}_{22} \to \bar{h}_{22} - \dot{\xi}_0 - \dot{\xi}_3 \\ \bar{h}_{33} \to \bar{h}_{33} + \dot{\xi}_3 - \dot{\xi}_0 \end{cases}$$

"transverse" requires:

$$\begin{cases} \bar{h}_{00} - \dot{\xi}_0 + \dot{\xi}_3 = 0 \\ \bar{h}_{01} - \dot{\xi}_1 = 0 \quad , \quad \bar{h}_{13} + \dot{\xi}_1 = 0 \\ \bar{h}_{02} - \dot{\xi}_2 = 0 \quad , \quad \bar{h}_{23} + \dot{\xi}_2 = 0 \\ \bar{h}_{03} + \dot{\xi}_0 - \dot{\xi}_3 = 0 \\ \bar{h}_{33} + \dot{\xi}_3 - \dot{\xi}_0 = 0 \end{cases}$$

this set of equations can be satisfied if we set

$$\begin{cases} \xi_1(u) = \int^u \bar{h}_{01}(u')du' \\ \xi_2(u) = \int^u \bar{h}_{02}(u')du' \\ \xi_0(u) - \xi_3(u) = \int^u \bar{h}_{00}(u')du' \end{cases}$$

[Although there are 7 equations, the Lorenz gauge conditions reduce the # to 3]

the "tracelessness" of $\bar{h}_{\alpha\beta}$ [and thus of $h_{\alpha\beta}$] requires

$$\bar{h}_{11} + \bar{h}_{22} - 2\dot{\xi}_0 - 2\dot{\xi}_3 = 0$$

this can be satisfied by setting

$$\xi_0(u) + \xi_3(u) = \frac{1}{2}\int^u \left[\bar{h}_{11}(u') + \bar{h}_{22}(u')\right]du'$$

After using this $\xi_\alpha$, we have

$$\begin{cases} \bar{h}_{00}^{new} = \bar{h}_{0j}^{new} = 0. \quad \bar{h}_{zj}^{new} = 0 \\ \bar{h}_{xy}^{new} = \bar{h}_{yx}^{new} = \bar{h}_{xy} = \bar{h}_{yx} = h_{xy} = h_{yx} \\ \bar{h}_{xx}^{new} = \bar{h}_{xx} - \frac{1}{2}(\bar{h}_{xx} + \bar{h}_{yy}) \\ \bar{h}_{yy}^{new} = \bar{h}_{yy} - \frac{1}{2}(\bar{h}_{xx} + \bar{h}_{yy}) \end{cases}$$

using the fact that $\bar{h}^{new} = 0$ [and $h_{\alpha\beta}^{new} = \bar{h}_{\alpha\beta}^{new}$] and

$$\begin{cases} \bar{h}_{xx} = h_{xx} - \frac{1}{2}h \\ \bar{h}_{yy} = h_{yy} - \frac{1}{2}h \end{cases}$$

we have

$$\begin{cases} h_{00}^{new} = h_{0j}^{new} = h_{zj}^{new} = 0 \\ h_{xy}^{new} = h_{yx}^{new} = h_{xy} = h_{yx} \\ h_{xx}^{new} = h_{xx} - \frac{1}{2}(h_{xx} + h_{yy}) = \frac{1}{2}(h_{xx} - h_{yy}) \\ h_{yy}^{new} = h_{yy} - \frac{1}{2}(h_{xx} + h_{yy}) = -\frac{1}{2}(h_{xx} - h_{yy}) \end{cases}$$

(b) It's clear that the above gauge transformation is in the end equivalent to a projection.

(b)

(1) Following the first way of Riemann tensor:
for $i', j' = x', y'$, we have, in the second frame,

$$R_{i'0'j'0'} = \mathbb{R}(\vec{e}_{i'}, \vec{e}_{0'}, \vec{e}_{j'}, \vec{e}_{0'})$$

$$\boxed{\text{where } \begin{cases} \vec{e}_{i'} = \vec{e}_i \\ \vec{e}_{0'} = \gamma\vec{e}_0 - \beta\gamma\vec{e}_z \\ \vec{e}_{z'} = -\beta\gamma\vec{e}_0 + \gamma\vec{e}_z \end{cases} \qquad \begin{matrix} x' = x,\ y' = y \\ \begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} \end{matrix}}$$

$$= R_{i0j0} - R_{i0jz}\,\beta\gamma^2 - R_{izj0}\,\beta\gamma^2 + \beta^2\gamma^2 R_{izjz}$$

using Eqs.(26,39) $\xrightarrow{} \; \frac{1+\beta}{1-\beta} R_{i0j0}$

i.e.
$$R_{i'0'j'0'}(t', z') = \frac{1+\beta}{1-\beta} R_{i0j0}(t-z)$$

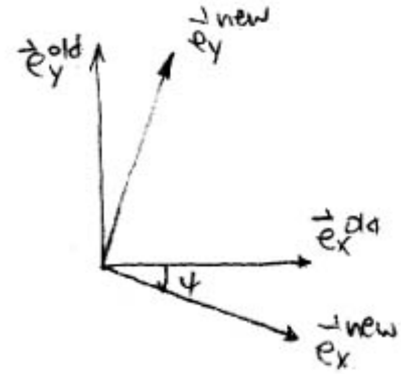
$$= \frac{1+\beta}{1-\beta} R_{i0j0}\left[\sqrt{\frac{1+\beta}{1-\beta}}\,(t'-z')\right]$$

i.e.
$$\ddot{h}'_{+,x}(t'-z') = \frac{1+\beta}{1-\beta}\,\ddot{h}_{+,x}\left[\sqrt{\frac{1+\beta}{1-\beta}}\,(t'-z')\right]$$

i.e.
$$\ddot{h}'_{+,x}(u') = \frac{d^2}{du'^2}\left\{h_{+,x}\left[\sqrt{\frac{1+\beta}{1-\beta}}\,u'\right]\right\}$$

#5 Behavior of $h_\times$ & $h_+$ under Rotations and Boosts

(a)




$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix}^{new} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix}^{old}$$



For both cases. [new and old]

$$\mathbb{h}(\vec{e}_x \pm i\vec{e}_y, \vec{e}_x \pm i\vec{e}_y) = h_{xx} - h_{yy} \pm i(h_{xy} + h_{yx})$$

$$= h_+ \pm i h_\times$$

Also noticing that,

$$(\vec{e}_x \pm i\vec{e}_y)^{new} = e^{\pm i\psi}(\vec{e}_x \pm i\vec{e}_y)^{old}$$

We have

$$(h_+ \pm i h_\times)^{new} = \mathbb{h}\left[(\vec{e}_x \pm i\vec{e}_y)^{new}, (\vec{e}_x \pm i\vec{e}_y)^{new}\right]$$

$$= e^{\pm 2i\psi}\, \mathbb{h}\left[(\vec{e}_x \pm i\vec{e}_y)^{old}, (\vec{e}_x \pm i\vec{e}_y)^{old}\right]$$

$$= e^{\pm 2i\psi}(h_+ \pm i h_\times)^{old}$$

$$\Rightarrow \quad h'_{+,x}(u') = h_{+,x}\left(\sqrt{\tfrac{1+\beta}{1-\beta}}\,u'\right) + \underbrace{Au'+B}$$

these represent the same event

uninteresting Integration consts, vanish for waves

$$\Rightarrow \begin{cases} h'_{+,x}(u') = h_{+,x}(u) \\ u' = \sqrt{\frac{1-\beta}{1+\beta}}\,u \end{cases}$$

(b) Following the second way.

In our first frame, $h_{00} = h_{0j} = 0$, $h_{zj} = 0$

The components of $h_{\alpha\beta}$ in the second frame is related to the first one by

$$h'_{\alpha\beta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu h_{\mu\nu}$$

where $\Lambda_\alpha{}^\beta$ is nonzero only for $\alpha$ or $\beta$ equal to $t$ or $z$. This means,

$$h'_{xx} = h_{xx}, \quad h'_{yy} = h_{yy}, \quad h'_{xy} = h'_{yx} = h_{xy} = h_{yx}$$

Because the TT components $h_{ij}^{TT}$ only involve $h'_{xx}$, $h'_{xy}$ and $h'_{yy}$, we immediately conclude that

$$h'^{TT}_{ij} = h^{TT}_{ij}$$

[this proof is analogous to Problem 1 g.]

#6. Motion of a free particle in TT gauge.

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h^{TT}_{\alpha\beta}(t-z)$$

A free particle initially at rest has a 4-velocity of $\vec{u} = \partial/\partial t$. [Note that it's normalized because $h^{TT}_{00} = 0$ ]

Instead of solving for the motion, we just have to check that $\vec{u} = \partial/\partial t$ satisfies the geodesic equation, i.e.

$$u^\alpha \nabla_\alpha u^\beta = 0$$

This is easy:

$$\begin{aligned} u^\alpha \nabla_\alpha u^\beta &= u^\alpha u^\beta{}_{,\alpha} + \Gamma^\beta{}_{\gamma\alpha} u^\alpha u^\gamma \\ &= \quad 0 \qquad + \Gamma^\beta{}_{00} \\ &= \eta^{\beta\mu} \tfrac{1}{2} \left( h^{TT}_{0\mu,0} + h^{TT}_{\mu 0,0} - h^{TT}_{00,\mu} \right) = 0 \end{aligned}$$

# WEEK 5: THE QUADRUPOLE FORMULA FOR GW GENERATION, PROPAGATION OF GW's THROUGH CURVED SPACETIME AND THE GW STRESS-ENERGY TENSOR

**Lectures 8 and 9**

**Recommended Reading:**
Note: All of this material is on the course web site.

1. Derivation of the quadrupole formula for gravitational-wave generation (beginning of Lecture 8): Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Sections 36.9 and 36.10. [A copy of this will be put on the class's web site.]
   Note: The treatment given in this section includes effects of self gravity in the source's interior, using the approach of Landau and Lifshitz, which Kip briefly discussed in his lecture. The analysis given in Kip's lecture corresponds to neglecting self gravity and correspondingly setting $t^{\mu\nu} = 0$ in the MTW analysis.
2. Wave propagation through curved spacetime (the remainder of Lecture 8 and most of lecture 9): Kip S. Thorne, "The Theory of Gravitational Radiation: an Introductory Review," in *Gravitational Radiation*, eds. N. Dereulle and T. Piran (North Holland, Amsterdam, 1983), pp. 1–57: Sections 1.2, 2.4.1, 2.4.2, and 2.5 and 2.6. 2.4.5.
3. The gravitational-wave stess-energy tensor (remainder of lecture 9): Kip S. Thorne, "The Theory of Gravitational Radiation: an Introductory Review," in *Gravitational Radiation*, eds. N. Dereulle and T. Piran (North Holland, Amsterdam, 1983), pp. 1–57: Section 2.4.5.

**Possible Supplementary Reading:**

4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973): Sections 35.7–35.15, including the exercises at the end of the chapter. This covers wave propagation through curved spacetime and the gravitational-wave stress-energy tensor. [This material is *not* on the course web site.]

**Assignment, to be turned in at beginning of class on Wednesday 13 February by students registered in the course:**

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.

iii. Carry out further reading and state what you have done.
iv. Seek private tutoring from a knowledgable person about this week's topic.
v. Pursue some other method of learning about this week's topic, and state what you have done.

**EXERCISES**

Note: There are more exercises here than any single person is expected to work. Work only those exercises that are useful for you!

**Exercises filling in the gaps in Kip's lectures**

1. **Derivation of Quadrupole-moment formula**
   Carry out the full details of the derivation of the quadrupole-moment formula for a source with negligible self gravity — i.e. a source whose internal accelerations are produced by non-gravitational forces. In particular:
   a. In a Lorentz frame in flat spacetime, use the energy-momentum conservation law $T^{\mu\nu}{}_{,\nu} = 0$ to show that

   $$T^{00}{}_{,00} x^j x^k = 2T^{jk} + (T^{lm} x^j x^k)_{,ml} - 2(T^{lj} x^k + T^{lk} x^j)_{,l} \; . \tag{1}$$

   b. Use this result to show that, in the slow-motion approximation, the standard retarded-integral formula for the gravitational-wave field

   $$h^{\mathrm{TT}}_{jk} = 4 \left[ \frac{T_{jk}(\mathbf{x}'; t' = t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} d^3x' \right]^{\mathrm{TT}} \tag{2}$$

   reduces to

   $$h^{\mathrm{TT}}_{jk} = \frac{2}{r} \left[ \ddot{I}_{jk}(t - r) \right]^{\mathrm{TT}} = \frac{2}{r} \left[ \ddot{\mathcal{I}}_{jk}(t - r) \right]^{\mathrm{TT}} \; , \tag{3}$$

   where $I_{jk}$ is the second moment of the source's mass distribution, $\mathcal{I}_{jk}$ is the source's mass quadrupole moment, the dots denote time derivatives, and $r$ is the distance from the source's center of mass to the observer. **NOTE: IN HIS LECTURE, KIP MISSED THE FACTOR 4 IN EQ. (2) AND THEREBY WROTE DOWN THE WRONG FORM FOR EQ. (3): HE WROTE $1/2r$ INSTEAD OF $2/r$.**
2. **Derivation of Geometric Optics Equations for GW Propagation**
   Carry out the full details of the derivation of the geometric-optics equations for gravitational-wave propagation. In particular, begin by expressing the Lorenz-gauge trace-reversed metric perturbation, in curved spacetime, in the form

   $$\bar{h}_{\alpha\beta} = \Re(A_{\alpha\beta} e^{i\varphi}) \; , \tag{4}$$

   where $\Re$ means take the real part, $\varphi$ is the waves' phase which varies on the very short lengthscale of the waves' reduced wavelength $\lambda\!\!\!^{-}$, and $A_{\alpha\beta}$ is the waves' amplitude which varies on a much longer lengthscale $\mathcal{L}$ (the smaller of the radius of curvature of the waves' phase fronts and the radius of curvature of spacetime). Motivated by the form $\varphi = \omega(z - t)$ of the phase for plane waves propagating in the $z$ direction of a local Lorentz frame, define the wave vector by $\vec{k} = \vec{\nabla}\varphi$ (so in the local Lorentz frame $k^0 = k^z = \omega$).

a. Using an argument in the local Lorentz frame, explain why $\vec{k}$ must change on the long lengthscale $\mathcal{L}$ and not on the short lengthscale $\lambda\!\!\!^{-}$.

b. Show that at leading order in the small parameter $\lambda\!\!\!^{-}/\mathcal{L} \ll 1$, the Lorenz gauge condition $\bar{h}^{\alpha\beta}{}_{|\beta} = 0$ reduces to the transversality condition

$$\bar{h}_{\alpha\beta} k^{\beta} = 0 \; . \tag{5}$$

c. Show that at the leading order in $\lambda\!\!\!^{-}/\mathcal{L}$, the gravitational wave equation $\bar{h}_{\alpha\beta|\mu}{}^{\mu} = 0$ reduces to the statement that the wave vector is null $\vec{k} \cdot \vec{k} = 0$, and that the gradient of $\vec{k} \cdot \vec{k} = 0$ implies that $\vec{k}$ *is the tangent to a null geodesic* (the waves' *ray*).

d. Show that at the next order in $\lambda\!\!\!^{-}/\mathcal{L}$, the gravitational wave equation reduces to the following transport law for the trace-reversed metric perturbation:

$$\bar{h}_{\alpha\beta|\mu} k^{\mu} = -\frac{1}{2} k^{\mu}{}_{|\mu} \bar{h}_{\alpha\beta} \; . \tag{6}$$

Note that in his lecture, Kip wrote this equation in terms of $A_{\alpha\beta}$. Explain explicitly why it can be written equally well in terms of $\bar{h}_{\alpha\beta}$ and $A_{\alpha\beta}$.

3. **Propagation Laws for $h_+$, $h_\times$, and their polarization tensors**
Express the trace-reversed metric perturbation in the form

$$\bar{h}_{\alpha\beta} = h_+ e^{+}_{\alpha\beta} + h_\times e^{\times}_{\alpha\beta} \; , \tag{7}$$

where $\mathbf{e}^+$ and $\mathbf{e}^\times$ are polarization tensors that are defined to be parallel propagated along the rays, $\nabla_{\vec{k}} \mathbf{e}^J = 0$ (for $J = +, \times$), and that in a local Lorentz frame of the source, near the source, have the usual components: $e^+_{xx} = -e^+_{yy} = 1$, $e^\times_{xy} = e^\times_{yx} = 1$, all other components vanish. (Here, on any chosen ray, we have oriented the coordinates so the ray points spatially in the $z$ direction.) We do not yet know that the $h_+$ and $h_\times$ in Eq. (7) are the usual gravitational-wave fields measured by observers; we shall show that this is so below.

a. Show that in the local Lorentz frame of the source, our Lorenz-gauge, trace-reversed metric perturbation $\bar{h}_{\alpha\beta}$ is trace free, and therefore is equal to the metric perturbation itself, $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$.

b. Use the curved-spacetime wave equation for $\bar{h}_{\alpha\beta}$ to show that it remains trace-free as it propagates, so everywhere $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$. We did not have to choose our gauge so this is true, but it was convenient to do so.

c. Show, from the parallel-transport law for the polarization tensors, that $e^J_{\alpha\beta}$ always remains trace free and always satisfies $e^J_{\alpha\beta} e^{J\alpha\beta} = 2$.

d. Consider an observer far from the source, whom the waves pass. Introduce the observer's local Lorentz frame and orient its axes so the waves are propagating in the $z$ direction, and the $+$ and $\times$ polarization axes are oriented in the usual way. Show that, by virtue of the transversality relation (5) and the relation $e^J_{\alpha\beta}e^{J\alpha\beta} = 2$, the observer's TT projection of the polarization tensors will have the usual form

$$(e^+_{xx})^{\rm TT} = -(e^+_{xx})^{\rm TT} = 1 \ , \quad (e^\times_{xy})^{\rm TT} = -(e^\times_{yx})^{\rm TT} = 1 \ , \tag{8}$$

all other components vanish. Explain why this, together with Eq. (7), implies that, as seen in the local Lorentz frame of any observer, the $h_+$ and $h_\times$ of Eq. (7) are the usual gravitational wave fields.

e. By inserting Eq. (7) into the propagation law (6) for $\bar{h}_{\alpha\beta}$, derive the following law for propagation of the gravitational-wave fields along the waves' rays:

$$\nabla_{\vec{k}} h_+ = -\frac{1}{2}(\vec{\nabla}\cdot\vec{k})h_+ \ , \quad \nabla_{\vec{k}} h_\times = -\frac{1}{2}(\vec{\nabla}\cdot\vec{k})h_\times \ . \tag{9}$$

4. **Gravitons**

a. In Ref. 3 of the suggested reading (above) there is a written version of the derivation Kip gave in his Lecture 9, of the Isaacson stress-energy tensor for gravitational waves. The final answer for $T^{\rm GW}_{\mu\nu}$ is given in three different forms in Eq. (2.47). Explain why the first of these forms reduces to the second in trace-free Lorenz gauge (the gauge used in Exercise 3), and reduces to the third in the local Lorentz frame of any observer. Show that the third reduces to

$$T^{{\rm GW}\,\mu\nu} = \frac{1}{16\pi}\langle h_+^{|\mu} h_+^{|\nu} + h_\times^{|\mu} h_\times^{|\nu}\rangle \ . \tag{10}$$

b. Show that in the geometric optics limit, Isaacson's gravitational-wave stress-energy tensor reduces to a sum over contributions from the two polarizations, each of which has the form

$$T^{\rm GW\ J}_{\alpha\beta} = \frac{1}{16\pi}\langle h_J^2\rangle k_\mu k_\nu \ . \tag{11}$$

Here as above, $J = +$ or $\times$.

c. These waves are carried by gravitons, each of which has a 4-momentum $\vec{p} = \hbar\vec{k}$. This means that the energy density and energy flux for gravitons with polarization $J$ can be written as

$$T^{00}_{\rm GW\ J} = N^0_J p^0 \ , \quad T^{i0}_{\rm GW\ J} = N^i_J p^0 \ , \tag{12}$$

where $N^0_J$ is the graviton number density and $N^i_J$ is the graviton flux. Write down, similarly, the momentum density and the momentum flux in terms of $p^\mu$ and $N^\nu_J$.

d. Show, from Eq. (11), that the graviton number-flux 4-vector is given by

$$N^\mu_J = \frac{1}{16\pi\hbar}\langle h_J^2\rangle k^\mu \ . \tag{13}$$

e. Show that the equations of geometric optics imply that the gravitons parallel transport their 4-momenta along their world lines, $\nabla_{\vec{p}}\vec{p} = 0$. Since their 4-momenta are tangent to their world lines and are null, this means they move along null geodesics.

f. Show that the transport law for the gravitational-wave field, Eq. (9), is equivalent to the statement that gravitons are conserved, $N^{\mu}_{J\,|\mu} = 0$.

g. Show that graviton conservation and the geodesic motion of the gravitons together guarantee conservation of energy and momentum, $T^{\mu\nu}_{\mathrm{GW}J\,|\nu} = 0$.

h. Show that graviton conservation implies that $h_J$ decreases as $1/\sqrt{\mathcal{A}}$, where $\mathcal{A}$ is the cross sectional area of a bundle of rays along which the waves are propagating. Hint: perform the calculation in a local Lorentz frame.

i. Show that graviton conservation implies that $h_J$ decreases as $1/r$, where $r$ is the radius of curvature of the waves' phase fronts. Hint: perform the calculation in a local Lorentz frame.

## Some Applications

5. **Gravitational Waves from an Equal-Mass Binary Star System with Circular Orbit**

Consider a binary system made of two identical stars, each with mass $m$ and radius $R \gg m$, separated by a distance $a$ large compared to their radii.

a. Show that the binary satisfies the slow-motion assumption (internal velocity small compared to the speed of light) and has weak gravity, $|h_{\mu\nu}| \ll 1$, so the quadrupole formula should be valid (thanks to the Landau-Lifshitz-type derivation that includes self gravity). Weak gravity and slow motion also imply that Newtonian theory is quite accurate, which means that Kepler's laws should be satisfied: the orbital angular velocity is $\Omega = \sqrt{2m/a^3}$.

b. Place the binary's center at the origin of a Cartesian coordinate system with the orbit in the $x$-$y$ plane and the stars on the $x$ axis at time $t = 0$. Show that the second moment of the mass distribution has as its only nonzero components

$$I_{xx} = 2ma^2\cos^2\Omega t = ma^2(1+\cos 2\Omega t)\ , \quad I_{yy} = 2ma^2\sin^2\Omega t = ma^2(1-\cos 2\Omega t),$$

$$I_{xy} = I_{yx} = 2ma^2\cos\Omega t\sin\Omega t = \sin 2\Omega t\ ; \tag{14}$$

and thence that the second time derivative of this second moment is

$$\ddot{I}_{xx} = -\ddot{I}_{yy} = -4m(a\Omega)^2\cos 2\Omega t\ , \quad \ddot{I}_{xy} = \ddot{I}_{yx} = -4m(a\Omega)^2\sin 2\Omega t\ . \tag{15}$$

c. Introduce a spherical polar coordinate system $(r,\theta,\phi)$ related to the Cartesian coordinates in the usual way, and denote by $\mathbf{e}_{\hat{\theta}}$ and $\mathbf{e}_{\hat{\phi}}$ the unit vectors pointing along the $\phi$ and $\theta$ directions. For an observer at location $(r,\theta,\phi)$, use these basis vectors as the polarization axes, so that

$$h_+ = \frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat{\theta}\hat{\theta}}(t-r) = -\frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat{\phi}\hat{\phi}}(t-r)\ , \quad h_\times = \frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat{\theta}\hat{\phi}}(t-r) = -\frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat{\phi}\hat{\theta}}(t-r)\ . \tag{16}$$

By computing from (15) the $\hat{\theta}$ and $\hat{\phi}$ components of $\ddot{I}_{jk}$ and then removing the trace, obtain the TT components of $\ddot{I}_{jk}$, and thereby conclude that the gravitational-wave fields have the following forms. *These forms are written in a way that turns out to remain valid for a circular binary with unequal masses.*

$$h_+ = 2(1+\cos^2\theta)\frac{\mu}{r}(\pi M f)^{2/3}\cos(2\pi f t) \qquad h_\times = 4\cos\theta\frac{\mu}{r}(\pi M f)^{2/3}\sin(2\pi f t)\ . \tag{17}$$

Here $f = 2(\Omega/2\pi) = \Omega/\pi$ is the waves' frequency, $\mu = m/2$ is the binary's reduced mass, $M = 2m$ is the binary's total mass, and $(\pi M f)^{2/3} = (a\Omega)^2$.

d. Show that these waveforms agree with the result, derived by dimensional analysis in Kip's introductory lectures, that the gravitational-wave amplitude has a magnitude equal to $1/c^2$ times the Newtonian gravitational potential produced by the mass equivalent of the source's internal kinetic energy.

6. **Theorem: Conservation Laws Associated with Symmetries of the Metric**

   a. Consider a particle that moves along a geodesc though curved spacetime. Parametrize the geodesic by a parameter $\zeta$ defined such that $d/d\zeta = \vec{p}$, where $\vec{p}$ is the particle's 4-momentum. Show that if the particle has finite rest mass $m$, then $\zeta$ is related to its proper time by $\zeta = \tau/m$. If the particle is a photon or graviton and so has vanishing rest mass, $m$ vanishes. Show that there also is no proper time lapse along the particle's world line, so $\tau$ is undefined. For such a particle $\zeta$ is a valid parameter along its world line but $\tau$ is not. Show that the geodesic equation for such a particle takes the form

$$\frac{d^2x^\mu}{d\zeta^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\zeta}\frac{dx^\beta}{d\zeta} = 0. \tag{18}$$

   b. Suppose spacetime has a metric which, in some carefully chosen coordinate system, is independent of the time coordinate, so $g_{\alpha\beta,0} = 0$. Show from the geodesic equation that the component $p_0 = g_{0\mu}p^\mu$ of the particle's 4-momentum is conserved. [Similarly, if $g_{\alpha\beta,j} = 0$ for some specific $j = 1, 2, 3$, then $p_j$ is conserved.]

7. **Gravitational Redshift of Gravitational Waves**

   Consider gravitational waves traveling through the spacetime of a nonspinning black hole. In appropriate coordinates $(t, r, \theta, \phi)$ the spacetime metric has the Schwarzschild form

$$ds^2 = -(1-2M/r)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{19}$$

   Here $M$ is the hole's mass and the radial location $r = 2M$ is the hole's horizon. Far from the hole, $r \gg 2M$, the metric becomes that of flat spacetime in spherical polar coordinates.

   a. Consider a family of observers who are at rest with respect to the black hole so their 4-velocities $\vec{u}$ all point along the time direction. Show that

$$\vec{u} = \vec{e}_{\hat{0}} = \frac{1}{\sqrt{1-2M/r}}\frac{\partial}{\partial t} \tag{20}$$

b. Let the gravitational waves have a reduced wavelength small compared to the hole's size, $\lambdabar \ll 2M$, and small compared to the radii of curvature of their phase fronts. Then geometric optics is valid. Consider a graviton moving along a ray of the waves. The at-rest observers measure the graviton's energy as it passes. Explain why the energy they measure is $E = -\vec{p} \cdot \vec{u} = -\vec{p} \cdot \vec{e}_{\hat{0}} = -p_{\hat{0}}$.

c. Show that the measured energy is

$$E = \frac{-p_0}{\sqrt{1 - 2M/r}} \, . \tag{21}$$

Show that $p_0$ is conserved by virtue of Exercise 6. This means that as the gravitons travel to larger and larger radii $r$, the graviton energy measured by the at-rest observers grows smaller and smaller, i.e. it gets gravitationally redshifted by the black hole's spacetime curvature.

c. Show that, if the waves are traveling precisely radially through the black-hole spacetime, then the amplitudes of their wave fields will decrease as $1/r$, where $r$ is the radial coordinate. Hint: consider the cross sectional area of a bundle of rays.

d. Assume that these radially traveling waves are monochromatic. Show that their phase must have the form $\varphi = \sigma(r_* - t)$, where $r_* = r + 2M \ln(r/2M - 1)$. Hint: show that the gradient of this phase function is null and has $k_0 = p_0/\hbar$ constant. Explain why this proves the desired result.

e. What is the energy $E$ of a graviton for these waves, measured by an at-rest observer, in terms of the constant $\sigma$? What is the frequency that the observer measures?

e. Combining the results of (c) and (d), show that the radially traveling, monochromatic waves have the form

$$h_J = \frac{A_J \cos[\sigma(r_* - t) + \delta_J]}{r} \, , \tag{22}$$

where $\delta_J$ is some arbitrary constant phase factor and $A_J$ is a constant amplitude.

# Ph 237 HW5 Solution, Winter 2002

## Problem 1, Derivation of Quadrupole Formula

(a) We have to notice:

- $(x^i T^{jk})_{,k} = \delta^i_k T^{jk} + x^i T^{jk}{}_{,k} = T^{ij} - x^i T^{j0}{}_{,0}$
- $$(x^i x^j T^{kl})_{,lk} = \left[(\delta^i_l x^j + x^i \delta^j_l) T^{kl} + x^i x^j T^{kl}{}_{,l}\right]_{,k}$$
$$= \left[T^{ki} x^j + T^{kj} x^i - x^i x^j T^{k0}{}_{,0}\right]_{,k}$$
$$= x^j T^{ki}{}_{,k} + \delta^j_k T^{ki} + T^{kj}{}_{,k} x^i + T^{kj} \delta^i_k$$
$$- \delta^i_k x^j T^{k0}{}_{,0} - x^i \delta^j_k T^{k0}{}_{,0} - x^i x^j T^{k0}{}_{,0k}$$
$$= -x^j T^{i0}{}_{,0} + T^{ij} - x^i T^{j0}{}_{,0} + T^{ij}$$
$$- x^j T^{i0}{}_{,0} - x^i T^{j0}{}_{,0} + x^i x^j T^{00}{}_{,00}$$
$$= 2T^{ij} - 2(\underbrace{x^i T^{j0}{}_{,0}}_{[T^{ij} - (x^i T^{jk})_{,k}]} + \underbrace{x^j T^{i0}{}_{,0}}_{[T^{ij} - (x^j T^{ik})_{,k}]}) + x^i x^j T^{00}{}_{,00}$$

inserting the above equation

$$= -2T^{ij} + 2(x^i T^{jk} + x^j T^{ik})_{,k} + x^i x^j T^{00}{}_{,00}$$

$$\Rightarrow T^{00}{}_{,00}\, x^i x^j = 2T^{ij} - 2(x^i T^{jk} + x^j T^{ik})_{,k} + (x^i x^j T^{kl})_{,kl}$$

(b)
$$h^{TT}_{jk} = \left[4 \int d^3\underline{x}' \, \frac{T_{jk}(\underline{x}', t - |\underline{x} - \underline{x}'|)}{|\underline{x} - \underline{x}'|}\right]^{TT}$$
$$\approx 4\left[\frac{1}{r} \int d^3\underline{x}' \, T_{jk}(\underline{x}', t - r)\right]^{TT}$$

$$= \frac{4}{r}\left\{ \int d^3\underline{x}' \left[ \frac{1}{2} T^{00}{}_{,00}(\underline{x}', t-r)\, x_j' x_k' \right.\right.$$

$$+ \left[ x_j' T_{kl}(\underline{x}', t-r) + x_k' T_{jl}(\underline{x}', t-r) \right]_{,l}$$

$$\left.\left. + \left[ x_k' x_j' T^{lm}(\underline{x}', t-r) \right]_{,lm} \right] \right\}^{TT}$$

$$= \left[ \frac{2}{r} \int d^3\underline{x}'\; T^{00}{}_{,00}(\underline{x}', t-r)\, x_j' x_k' \right]^{TT}$$

Last two terms vanish

$$= \frac{2}{r} \frac{d^2}{dt^2} \left[ \int d^3\underline{x}'\; T^{00}(\underline{x}', t-r)\, x_j' x_k' \right]^{TT}$$

$$= \frac{2}{r} \left[ \ddot{I}_{jk}(t-r) \right]^{TT}$$

[since $\left(\ddot{I}_{jk}(t-r)\right)$ and $\left(\ddot{\mathcal{I}}_{jk}(t-r)\right)$ only differ by a trace, ]

$$= \frac{2}{r} \left[ \ddot{\mathcal{I}}_{jk}(t-r) \right]^{TT}$$

2. Derivation of Geometric Optics Equations for GW Propagation

(a) We have, $\lambda \ll \mathcal{L} \ll \mathcal{R}$.

A Local Lorentz frame differ from a true Lorentz frame at length scale $\sim \mathcal{R} \gg \mathcal{L}$. Therefore, the wave propagation within the scale of $\mathcal{L}$ will be the same as that in a flat spacetime. Moreover, since the amplitude changes at a length scale of $\mathcal{L} \gg \lambda$ the wave could be regarded as a constant amplitude one within $\mathcal{L}$, i.e.

$$h \sim e^{i\varphi}$$

Also because the phase fronts have radii of curvature $\gg \mathcal{L}$, we now have

$$h \sim e^{i\varphi(t-z)}$$

using the flat spacetime wave equation, we get the usual plane wave solutions:

$$h \sim e^{-i\omega(t-z)}$$

for which both $\omega$ & $\vec{k}$ are constants.
— This will be true until the length scale reaches $\mathcal{L}$

(b) $$0 = \bar{h}^{\alpha\beta}{}_{|\beta} = \mathrm{Re}\left[\underbrace{A^{\alpha\beta}{}_{|\beta}}_{A/\mathcal{L}} e^{i\varphi} + i \underbrace{A^{\alpha\beta} k_\beta}_{A/\lambda} e^{i\varphi}\right]$$

$$\xrightarrow[\text{order}]{\text{Leading}} \quad A^{\alpha\beta} k_\beta = 0 \qquad \text{i.e} \quad \bar{h}^{\alpha\beta} k_\beta = 0$$

(c) $$0 = \bar{h}^{\alpha\beta}{}_{|\mu}{}^{\mu}$$

$$= \mathrm{Re}\left[ A^{\alpha\beta}{}_{|\mu} e^{i\varphi} + i k_\mu A^{\alpha\beta} e^{i\varphi} \right]^{|\mu}$$

$$= \mathrm{Re}\Big[ \underbrace{A^{\alpha\beta}{}_{|\mu}{}^{\mu}}_{A^{\alpha\beta}/\mathcal{L}^2} e^{i\varphi} + 2 i k_\mu \underbrace{A^{\alpha\beta|\mu}}_{A^{\alpha\beta}/\mathcal{L}} e^{i\varphi}$$
$$+ i \underbrace{k_\mu{}^{|\mu}}_{} A^{\alpha\beta} e^{i\varphi} - k_\mu k^\mu A^{\alpha\beta} e^{i\varphi} \Big] \qquad (*)$$

because $k_\mu$ varies in lengthscale $\mathcal{L}$, as argued in part (a), $k_\mu{}^{|\mu} \sim k/\mathcal{L}$

$$\Rightarrow [\text{Leading order}] \quad k_\mu k^\mu = 0$$

$$\nabla_\nu (k_\mu k^\mu) = 0 \Rightarrow 2 k^\mu \nabla_\nu k_\mu = 0$$
$$\Rightarrow 2 k^\mu \nabla_\nu \nabla_\mu \varphi = 0$$
$$\Rightarrow 2 k^\mu \nabla_\mu \nabla_\nu \varphi = 0$$
$$\Rightarrow 2 k^\mu \nabla_\mu k_\nu = 0 \qquad [\text{Geodesic equation}]$$

(d) The next-to-Leading order. we have
[see equation (*) in last page]

$$k_\mu{}^{|\mu} A^{\alpha\beta} + 2 k_\mu A^{\alpha\beta|\mu} = 0$$

Therefore, for the propagation of $\bar{h}^{\alpha\beta}$, we have,

$$k_\mu{}^{|\mu} \bar{h}^{\alpha\beta} + 2 k_\mu \bar{h}^{\alpha\beta|\mu}$$

$$= k_\mu{}^{|\mu} A^{\alpha\beta} e^{i\varphi} + 2 k_\mu A^{\alpha\beta|\mu} e^{i\varphi} + \underbrace{2 k_\mu A^{\alpha\beta} i k^\mu}_{0}$$

$$= \left[k_\mu{}^{|\mu} A^{\alpha\beta} + 2 k_\mu A^{\alpha\beta|\mu}\right] e^{i\varphi} = 0$$

Problem 3. Propagation Laws for $h_+$, $h_\times$ and their polarization tensors

(a) Since $e^{+}_{\alpha\beta}$ and $e^{\times}_{\alpha\beta}$ are both trace free,

$\bar{h}_{\alpha\beta} = h_+ e^{+}_{\alpha\beta} + h_\times e^{\times}_{\alpha\beta}$ is also trace free.

Since $h = -\bar{h}$, $h_{\alpha\beta}$ is also trace free.

$$\Rightarrow \quad h_{\alpha\beta} = \bar{h}_{\alpha\beta}$$

(b) from Eq. (6)

$$k^\mu \bar{h}_{\alpha\beta|\mu} = -\tfrac{1}{2} k^\mu{}_{|\mu} \bar{h}_{\alpha\beta}$$

$$\Rightarrow \quad g^{\alpha\beta}_{B} k^\mu \bar{h}_{\alpha\beta|\mu} = -\tfrac{1}{2} k^\mu{}_{|\mu} \bar{h}$$

$$\Rightarrow \quad k^\mu \bar{h}_{|\mu} = -\tfrac{1}{2} k^\mu{}_{|\mu} \bar{h}$$

thus, since $\bar{h}$ is initially 0, it'll remain 0

$$\Rightarrow \quad \bar{h}_{\alpha\beta} = h_{\alpha\beta} \text{ everywhere}$$

(c)
$$k^\mu e^{J}{}_{\alpha\beta|\mu} = 0$$

$$\Rightarrow \quad g^{\alpha\beta}_{B} k^\mu e^{J}_{\alpha\beta|\mu} = 0 \Rightarrow k^\mu \left(g^{\alpha\beta}_{B} e^{J}_{\alpha\beta}\right)_{|\mu} = 0$$

$\Rightarrow$ $g^{\alpha\beta}_{B} e^{J}_{\alpha\beta}$, i.e. the trace of $e^J$, remains constant, i.e. 0

$$k^\mu [e^J_{\alpha\beta} e^{J'\alpha\beta}]_{|\mu}$$

$$= k^\mu e^J_{\alpha\beta|\mu} e^{J'\alpha\beta} + e^{\alpha\beta J} k^\mu e^{J'}_{\alpha\beta|\mu} = 0$$

$\Rightarrow e^J_{\alpha\beta} e^{J'\alpha\beta}$ is conserved i.e. $2\delta^{JJ'}$

(d) first, let's notice that

$$(e^J_{\alpha\beta})^{TT} (e^{J'\alpha\beta})^{TT} = e^J_{\alpha\beta} e^{J'\alpha\beta}$$

$$e^J_{\alpha\beta} e^{J'\alpha\beta} = e^J_{\alpha 0} e^{J'\alpha 0} + e^J_{0\alpha} e^{J'0\alpha} + e^J_{\alpha z} e^{J'\alpha z} + e^J_{z\alpha} e^{J'z\alpha} - e^J_{00} e^{J'00} - e^J_{zz} e^{J'zz} + e^J_{\hat{i}\hat{j}} e^{J'\hat{i}\hat{j}}$$

($\hat{i}, \hat{j}$ take values 1 & 2)

since $e^J_{\alpha\beta}$ transverse. $e^J_{0\alpha} = -e^J_{z\alpha}$, $e^J_{\alpha 0} = -e^J_{\alpha z}$

$e^{J0\alpha} = e^{Jz\alpha}$, $e^{J\alpha 0} = e^{J\alpha z}$

and in particular, $e^J_{00} = -e^J_{z0} = -e^J_{0z} = e^J_{zz}$

$$\Rightarrow e^J_{\alpha\beta} e^{J'\alpha\beta} = e^J_{\hat{i}\hat{j}} e^{J'\hat{i}\hat{j}}$$

Also, since $e^{J\alpha}_{\alpha} = 0$, and $e^J_{00} = e^J_{zz}$, $e^J_{\hat{i}\hat{i}} = 0$

this means: $(e^J_{\hat{i}\hat{j}})^{TT} = e^J_{\hat{i}\hat{j}}$

and that $(e^J_{\hat{i}\hat{j}})^{TT} (e^{J'\hat{i}\hat{j}})^{TT} = 2\delta^{JJ'}$

we can always orient the coordinates to have

$$[e^{+TT}_{\hat{i}\hat{j}}] = \begin{pmatrix} A & \\ & -A \end{pmatrix} \quad \text{since} \quad (e^+_{\hat{i}\hat{j}})^{TT} (e^{+\hat{i}\hat{j}})^{TT} = 2,$$

we have

$$[e^{+\hat{i}\hat{j}TT}_{\hat{i}\hat{j}}] = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

then, since $(e^{\times}_{\hat{i}\hat{j}})^{TT}(e^{+\hat{i}\hat{j}})^{TT} = 0$, we have

$$(e^{\times}_{\hat{i}\hat{j}})^{TT} = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \xrightarrow{(e^{\times}_{\hat{i}\hat{j}})^{TT}(e^{\times\hat{i}\hat{j}})^{TT}=2} \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

[by flipping the direction of $\vec{e}_x$ alone, we can change the sign of $(e^{\times}_{\hat{i}\hat{j}})^{TT}$ without changing $(e^{+}_{\hat{i}\hat{j}})^{TT}$ ]

Now since the TT projection of $e^{+}_{\alpha\beta}$ & $e^{\times}_{\alpha\beta}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the metric perturbation can be projected to

$$h^{TT}_{\alpha\beta} = \bar{h}^{TT}_{\alpha\beta} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so, $h_+$ & $h_\times$ are the + & × polarization of the GW.

(f) $$\bar{h}_{\alpha\beta|\mu}k^\mu = (h_+ e^+_{\alpha\beta} + h_\times e^\times_{\alpha\beta})_{|\mu} k^\mu = k^\mu h_{+|\mu} e^+_{\alpha\beta} + k^\mu h_{\times|\mu} e^\times_{\alpha\beta}$$

$$\|$$

$$-\tfrac{1}{2} k^\mu{}_{|\mu} \bar{h}_{\alpha\beta} = -\tfrac{1}{2} k^\mu{}_{|\mu} h_+ e^+_{\alpha\beta} - \tfrac{1}{2} k^\mu{}_{|\mu} h_\times e^\times_{\alpha\beta}$$

by taking $e^{+\alpha\beta}(*)$ and $e^{\times\alpha\beta}(*)$ of the above two objects, we have,

$$k^\mu h_{J|\mu} = -\tfrac{1}{2} k^\mu{}_{|\mu} h_J, \quad J = +, \times$$

4. Gravitons

(a)

- First $\Rightarrow$ Second

$$\bar{h} = 0 \quad \text{and} \quad \bar{h}^{\alpha\beta}{}_{|\beta} = 0$$

- Second $\Rightarrow$ third ;

$$\langle \bar{h}_{\alpha\beta|\mu} \bar{h}^{\alpha\beta}{}_{|\nu} \rangle \to \langle \bar{h}_{\alpha\beta,\mu} \bar{h}^{\alpha\beta}{}_{,\nu} \rangle \; (\text{LLF}, \Gamma = 0)$$

$$\to \langle h^{TT}_{jk,\mu} h^{TT}_{jk,\nu} \rangle \; (\text{Gauge invariance})$$

$$\langle h^{TT}_{jk,\mu} h^{TT}_{jk,\nu} \rangle = \langle 2h_{+,\mu} h_{+,\nu} + 2h_{\times,\mu} h_{\times,\nu} \rangle$$

$$h^{TT}_{jk} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

go back to general frame this is

$$2 \langle h_{+|\mu} h_{+|\nu} + h_{\times|\mu} h_{\times|\nu} \rangle$$

$$\Rightarrow T^{GW\,\mu\nu} = \frac{1}{16\pi} \langle h_+{}^{|\mu} h_+{}^{|\nu} + h_\times{}^{|\mu} h_\times{}^{|\nu} \rangle$$

(b) In geometric optics limit

$$\bar{h}_{\alpha\beta|\mu} = (A_{\alpha\beta} e^{i\varphi})_{|\mu} \underset{\substack{\uparrow \\ \text{leading} \\ \text{order}}}{=} i k_\mu A_{\alpha\beta} e^{i\varphi} = i k_\mu \bar{h}_{\alpha\beta}$$

$$\Rightarrow \langle \bar{h}_{\alpha\beta|\mu} \bar{h}^{\alpha\beta}{}_{|\nu} \rangle = k_\mu k_\nu \langle \bar{h}_{\alpha\beta} \bar{h}^{\alpha\beta} \rangle = 2 k_\mu k_\nu \langle h_+^2 + h_\times^2 \rangle$$

$$\Rightarrow T_{\mu\nu}^{GW J} = \frac{1}{16\pi} \langle h_J^2 \rangle k_\mu k_\nu$$

(c)
$$\begin{cases} T^{0i}_{GWJ} = N^0_J p^i & \text{— momentum density} \\ T^{ij}_{GWJ} = N^i_J p^j & \text{— momentum flux} \end{cases}$$

(d)
$$N^\mu_J p^0 = T^{\mu 0}_{GWJ} \quad \text{— Eq. (12)}$$

$$\Rightarrow N^\mu_J \hbar k^0 = \frac{1}{16\pi} \langle h_J^2 \rangle k^\mu k^0$$

$$\Rightarrow N^\mu_J = \frac{1}{16\pi\hbar} \langle h_J^2 \rangle k^\mu$$

(e) since $\vec{p} = \hbar \vec{k}$

$$\nabla_{\vec{k}} \vec{k} = 0 \Rightarrow \nabla_{\vec{p}} \vec{p} = 0$$

(f) $$N_J{}^\mu{}_{|\mu} = \frac{1}{16\pi\hbar} \left[ \langle h_J^2 \rangle k^\mu \right]_{|\mu}$$

here, in geometric optics limit.

$$\underline{h}_J = A_J e^{i\varphi}$$

and $$\langle h_J^2 \rangle = A_J^2$$

similar to problem 2.

$$k^\mu h_{J|\mu} = k^\mu A_{J|\mu} e^{i\varphi} + \cancel{k^\mu A_J i k_\mu e^{i\varphi}}$$

$$\|$$

$$-\frac{1}{2} k^\mu{}_{|\mu} h_J = -\frac{1}{2} k^\mu{}_{|\mu} A_J e^{i\varphi}$$

$$\Rightarrow \quad k^\mu A_{J|\mu} = -\frac{1}{2} k^\mu{}_{|\mu} A_J$$

Jow we have

$$N_J{}^{\mu}{}_{|\mu} = \frac{1}{16\pi\hbar}\left[\langle h_J^2\rangle k^\mu\right]_{|\mu}$$

$$= \frac{1}{16\pi\hbar}\left(A_J^2 k^\mu\right)_{|\mu}$$

$$= \frac{1}{16\pi\hbar}\left[2 k^\mu \underset{\substack{\uparrow\\ 0}}{A_{J|\mu} + A_J k^\mu{}_{|\mu}}\right] A_J$$

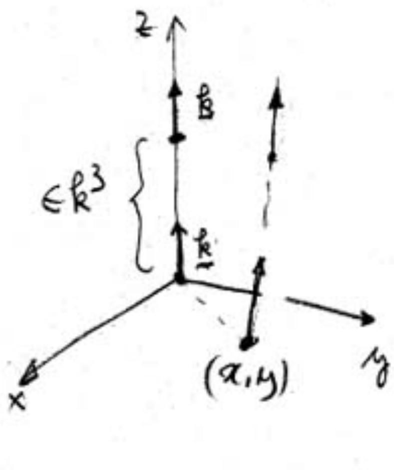
$$= 0$$

(g) $$T_{\mu\nu}^{GWJ} = \frac{1}{16\pi}\langle h_J^2\rangle k_\mu k_\nu = \frac{1}{16\pi} A_J^2 k_\mu k_\nu$$

$$T_{\mu\nu}^{GWJ\,|\nu} = \frac{1}{16\pi}\Big[\underbrace{(A_J^2)^{|\nu} k_\mu k_\nu + \left(A_J^2 k_\mu{}^{|\mu}\right) k_\nu}_{0,\text{ by propagation eq for } A_J} + \underset{\substack{\uparrow\\ 0,\text{ geodesic eq}}}{A_J^2 k_\mu k_\nu{}^{|\mu}}\Big]$$

$$= 0$$

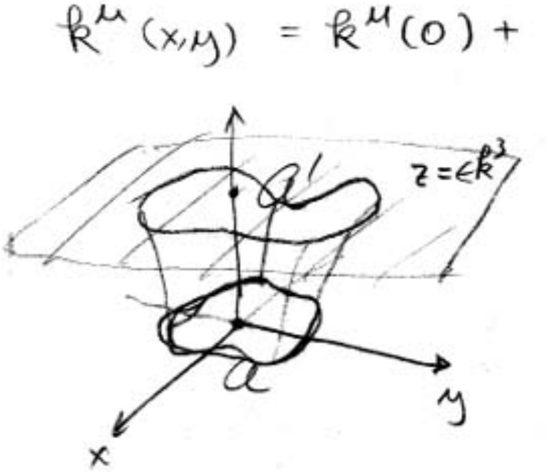
(h) Let's choose a Local Lorentz frame, in which the wave is propagating in the z direction




Let's follow the central for $\epsilon$, since $\vec{k}$ satisfies $k^\mu \nabla_\mu k^\nu = 0$, or $k^\mu k^\nu{}_{,\mu} = 0$ in our frame, the change in $k^\mu$ will be $O(\epsilon^2)$

All other rays will also follow their

$$k^\mu(x,y) = k^\mu(0) + \partial k^\mu/\partial x \cdot x + \partial k^\mu/\partial y \cdot y$$




Let's imagin having a number of fiducial gravitons incident from below, along these rays, within a very short interval $\Delta t \ll \epsilon k^0, \epsilon k^3$

the Number of gravitons entering is

$$\int_a dx\,dy\; N^3_J(t=0,x,y,z=0)\;\Delta t$$

$$\approx a\; A_J^2(t=0,z=0)\; k^3(t=0,z=0)\;\Delta t$$

these gravitons must go out. from the upper area, i.e. the cross section of the bundle by $z=\epsilon k^3$.

at Leading order, this number is

$$a'\; A_J^2(t=\epsilon k^0, z=\epsilon k^3)\; k^3(t=\epsilon k^0, z=\epsilon k^3)\;\Delta t$$

this is equal to $k^3(t=0,z=0)$

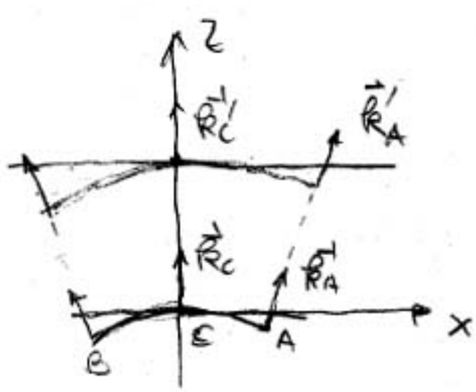
so we have

$a\,A_J^2$ is conserved.

i.e. $A_J \sim 1/\sqrt{a}$

(ii) It's easier to relate $r$ to $a$.
Let's still take the graph of last part

in the two instants that we consider,
i.e. $(t=0, z=0)$ & $(t=\epsilon k^0, z=\epsilon k^3)$.
the two regions are to the second order
equal-phase surfaces. By virtue of parallel
transport. $\vec{k}_{A'}$, $\vec{k}_A$;




$\vec{k}_C'$, $\vec{k}_C$, $\vec{k}_B'$, $\vec{k}_B$ are also the
same (to second order)

Therefore, the (solid) angle spanned by the
bundle rays remain constant.

$$\Rightarrow \quad r'^2 = \frac{a'}{\Delta\Omega}, \quad r^2 = \frac{a}{\Delta\Omega}$$

and $\quad r'/r = \sqrt{a'/a}$

$\Rightarrow \quad r A_J$ conserved

or, $\quad A_J \sim 1/r$

However, since $\Delta\Omega$ (thus $r$) is not invariant when
the observers are different [unlike $a$!],
this relation is not very general

# Applications.

## #5. Gravitational Waves from an Equal-Mass Binary Star system with Circular Orbit.

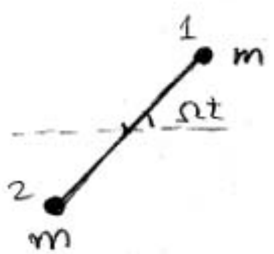
(a) $v = \sqrt{m/R} \ll 1$ (as assumed)

$\Rightarrow$ slow motion

$h_{\mu\nu} \sim m/R \ll 1$

$\Rightarrow$ weak gravity

(b)



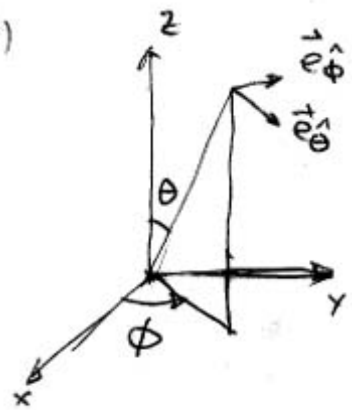
$$I_{xx} = \int d^3\underline{x}\ \rho(x)\, x^2$$
$$= m x_1^2 + m x_2^2$$
$$= m\left(\frac{a}{2}\cos\Omega t\right)^2 \cdot 2$$
$$= \frac{1}{4} m a^2 [1 + \cos(2\Omega t)]$$

similarly $I_{yy} = \frac{1}{4} m a^2 [1 - \cos(2\Omega t)]$

$$I_{xy} = I_{yx} = m x_1 y_1 + m x_2 y_2 = \left(\frac{a}{2}\right)^2 m \sin\Omega t \cos\Omega t \cdot 2$$
$$= \frac{1}{4} m a^2 \sin(2\Omega t)$$

$$\Rightarrow \ddot{I}_{xx} = -\ddot{I}_{yy} = m a^2 \Omega^2 \cos(2\Omega t)$$
$$\ddot{I}_{xy} = \ddot{I}_{yx} = -m a^2 \Omega^2 \sin(2\Omega t)$$

(c)




$$\begin{cases} \vec{e}_{\hat\theta} = \vec{e}_x \cos\theta\cos\phi + \vec{e}_y \cos\theta\sin\phi - \vec{e}_z \sin\theta \\ \vec{e}_{\hat\phi} = -\vec{e}_x \sin\phi + \vec{e}_y \cos\phi \end{cases}$$

$$\Rightarrow \ddot{I}_{\hat\theta\hat\theta} = \ddot{I}_{xx}\cos^2\theta\cos^2\phi + \ddot{I}_{yy}\cos^2\theta\sin^2\phi + 2\ddot{I}_{xy}\cos^2\theta\sin\phi\cos\phi$$

$$= -ma^2\Omega^2\cos(2\Omega t)\cos^2\theta\cos 2\phi$$

$$- ma^2\Omega^2\sin(2\Omega t)\cos^2\theta\sin 2\phi$$

$$\ddot{I}_{\hat\phi\hat\phi} = \ddot{I}_{xx}\sin^2\phi + \ddot{I}_{yy}\cos^2\phi - 2\ddot{I}_{xy}\cos\phi\sin\phi$$

$$= ma^2\Omega^2\cos(2\Omega t)\cos 2\phi + ma^2\Omega^2\sin(2\Omega t)\sin 2\phi$$

$$\Rightarrow h_+ = \frac{2}{r}\left[\left(\ddot{I}_{\hat\theta\hat\theta} - \ddot{I}_{\hat\phi\hat\phi}\right)/2\right] = \Big[-ma^2\Omega^2\cos[2\Omega(t-r)]\cos 2\phi\ (1+\cos^2\theta)$$
$$- ma^2\Omega\sin[2\Omega(t-r)]\sin 2\phi\ (1+\cos^2\theta)\Big]/r$$

$$= -\frac{ma^2\Omega^2}{r}(1+\cos^2\theta)\cos[2\Omega(t-r)-2\phi]$$

$$\Omega = \sqrt{\frac{2m}{a^3}} \Rightarrow a = \left(\frac{2m}{\Omega^2}\right)^{1/3}$$

$$\Rightarrow h_+ = -\frac{m}{r}(2m\Omega)^{2/3}(1+\cos^2\theta)\cos[2\Omega(t-r)-2\phi]$$

$$= -\frac{2\mu}{r}(M\pi f)^{2/3}(1+\cos^2\theta)\cos[2\pi f(t-r)-2\phi]$$

here $\mu = m/2$, $M = 2m$, $f = \Omega/\pi$.

note that GW frequency is twice the orbital freq.

similarly

$$\ddot{I}_{\hat\theta\hat\phi} = -\cos\theta\sin\phi\cos\phi\,\ddot{I}_{xx} + \cos\theta\cos 2\phi\,\ddot{I}_{xy}$$
$$+\cos\theta\sin\phi\cos\phi\,\ddot{I}_{yy}$$
$$= ma^2\Omega^2\cos[2\Omega(t-r)]\sin 2\phi\cos\theta - ma^2\Omega^2\sin[2\Omega(t-r)]\cos 2\phi\cos\theta$$
$$= -ma^2\Omega^2\cos\theta\ \sin[2\Omega(t-r)]$$

$$\Rightarrow h_\times = \frac{2}{r}\ddot{I}_{\hat\theta\hat\phi} = -\frac{2\mu}{r}(\pi M f)^{2/3}(2\cos\theta)\sin[2\pi f(t-r)-2\phi]$$

(d) From previous part,

$$h_{+,\times} \sim \frac{ma^2\Omega^2}{r} = \frac{Gma^2\Omega^2}{r c^4}$$
$$= \frac{G}{c^2 r}\left[\frac{ma^2\Omega^2}{c^2}\right]$$

mass-equivalent
of internal
kinetic energy

6a) $\vec{P} = \frac{d}{d\zeta} = m\vec{u} = m\frac{d}{d\tau}$

$$\Rightarrow \frac{d\tau}{m} = d\zeta \rightarrow \tau = m\zeta + \text{constant}$$

For a photon, the trajectory is a null geodesic. So $d\tau^2 = -g_{\alpha\beta}dx^\alpha dx^\beta = 0$ along the world line

Geodesic Equation:

$\vec{P}$ is parallel transported along itself

$$\nabla_{\vec{P}}\vec{P} = 0$$

$$p^\alpha{}_{;\beta}\,p^\beta = 0$$

$$p^\alpha{}_{,\beta}\,p^\beta + \Gamma^\alpha{}_{\mu\beta}\,p^\mu p^\beta = 0$$

But $p^\alpha = \frac{dx^\alpha}{d\zeta}$ so

$$\left(\frac{dx^\alpha}{d\zeta}\right)_{,\beta}\left(\frac{dx^\beta}{d\zeta}\right) + \Gamma^\alpha{}_{\mu\beta}\,p^\mu p^\beta = 0$$

That is $\frac{d^2x^\alpha}{d\zeta^2} + \Gamma^\alpha{}_{\mu\beta}\,p^\mu p^\beta = 0$

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For indices down, $\nabla_{\vec{p}}\,\vec{p} = 0$ says

$$0 = p_{\alpha;\beta}\,p^{\beta} = \left(p_{\alpha,\beta} - \Gamma^{\mu}{}_{\alpha\beta}\,p_{\mu}\right)p^{\beta} = 0$$

For $\alpha = 0$ we have

$$p_{0,\beta}\,p^{\beta} = \Gamma^{\mu}{}_{0\beta}\,p_{\mu}\,p^{\beta} =$$

$$= \Gamma_{\mu 0 \beta}\,p^{\mu}p^{\beta} =$$

$$= \frac{1}{2}\Big(\cancelto{0}{g_{\mu\beta,0}} + \underbrace{g_{\mu 0,\beta} - g_{0\beta,\mu}}_{\text{Antisymmetric in } \mu\beta}\Big)\overbrace{p^{\mu}p^{\beta}}^{\text{Symmetric in } \mu\beta}$$

$= 0$ since it is antisymmetric in $\mu\beta$ and $\mu\beta$ are summed over.

7 a)

$\vec{u}$ points in the time direction

$$\vec{u} = u^0 \frac{\partial}{\partial t}$$

Since $\vec{u} \cdot \vec{u} = -1$ we have

$$-1 = u^\alpha u^\alpha g_{\alpha\beta} = u^0 u^0 g_{00} =$$

$$= (u^0)^2 \left(-\left(1 - \frac{2M}{r}\right)\right)$$

$$\Rightarrow u^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

$$u = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \frac{\partial}{\partial t}$$

This is equal to the basis vector $\vec{e}_{\hat{0}}$ along the time direction.

b) For any observer with 4-velocity $\vec{u}$ in that observer's reference frame

$$\vec{u} = \vec{e}_{\hat{0}} = \text{unit vector in time direction.}$$

Therefore, in that observer's reference frame, the energy of the particle is

$$E = p^{\hat{0}} = -p_{\hat{0}} = -\vec{p}\,\underbrace{\vec{e}_{\hat{0}}}_{\vec{u}}$$

So $E = -\vec{p} \cdot \vec{u}$

c) Evaluate $E = -\vec{p}\,\vec{u}$, <u>not</u> in observer's frame but in Schwarzschild coordinates where

$$\vec{u} = \frac{1}{\sqrt{1-\frac{2M}{r}}} \frac{\partial}{\partial t}$$

$$\text{So } E = -\vec{p}\,\vec{u} = -p_0 \frac{1}{\sqrt{1-\frac{2M}{r}}}$$

d) Introduce a local Lorentz frame of an observer who is momentarily at rest but freely falling at radius $r$. Consider a bundle of graviton rays propagating radially and substanding a thin solid angle $\Delta\Omega = \sin\theta \, \Delta\varphi \, \Delta\theta$

The crossectional area of the bundle is $A = r^2 \Delta\Omega$ As rays propagate,

$$h_j \sim \frac{1}{\sqrt{A}} \sim \frac{1}{\sqrt{r^2 \Delta\Omega}} \sim \frac{1}{r}$$

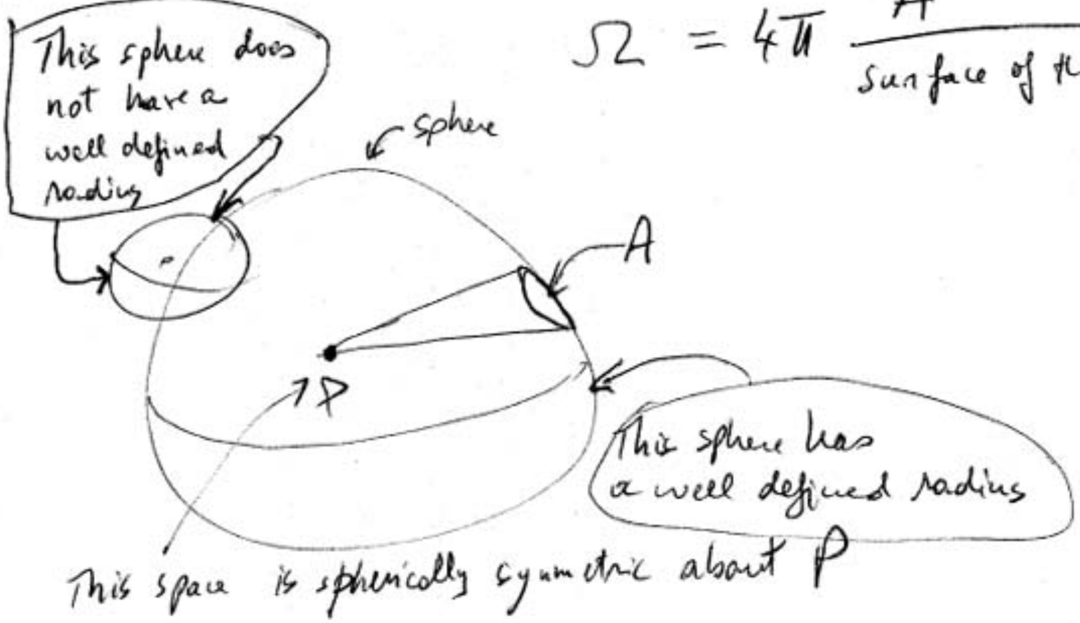
since $\Delta\Omega$ = constant.

<u>Note.</u> In curved space times there is no obvious way to define the radius of a sphere and the solid angle.

In spherically symmetric curved spaces the radius of a sphere centered in the point the space is spherically symmetric about is defined as $r = \sqrt{\dfrac{\text{Surface of the sphere}}{4\pi}}$

and the solid angle is

$$\Omega = 4\pi \, \frac{A}{\text{surface of the sphere}}$$



e) what's so special about $r^*$?

$$\frac{dr^*}{dr} = 1 + \frac{(2M)\frac{1}{2M}}{\frac{r}{2M} - 1} = 1 + \frac{1}{\frac{r}{2M}-1} = \frac{1}{1-\frac{2M}{r}}$$

and the line given by $r^* = t$ has a tangent vector in Schwarzschild coordinates $(r,t)$ that is null.

The phase is given by $\phi = \sigma(t - r^*)$

So $k_0 = \varphi_{,0} = \sigma$

$$k_r = \varphi_{,r} = -\sigma \frac{dr^*}{dr} = -\frac{\sigma}{1-\frac{2M}{r}}$$

Thus $$k_\alpha k_\beta \underbrace{g^{\alpha\beta}}_{(g_{\alpha\beta})^{-1}} = k_\alpha k_\beta (g_{\alpha\beta})^{-1} =$$

$$= -\left(\frac{1}{1-\frac{2M}{r}}\right)\sigma^2 + \left(1-\frac{2M}{r}\right)\frac{1}{\left(1-\frac{2M}{r}\right)^2}\sigma^2 = 0$$

$\Rightarrow$ The gradient of $\varphi$ is null.

Now the GW's travel at the speed of light. So there is no proper time elapsed between 2 points along the world line of a photon, so there can be no change in the graviton's phase. $\Rightarrow$ the ~~p~~ surfaces of constant phase propagate at light speed too.

All we have left to do is find a null radial vector. That was suggested by Kip by ~~[illegible]~~ giving us $r^*$.

So we can write $\Phi = \sigma(t - r^*)$.
it's easy to see that it does not change along null radial vectors with $r^* - t =$ constant.

f) $E = -\frac{p_0}{\sqrt{1-\frac{2M}{r}}} = -\frac{\hbar k_0}{\sqrt{1-\frac{2M}{r}}}$

$$k_0 = \varphi_{,0} = -\sigma$$

$$E = \frac{\hbar \sigma}{\sqrt{1-\frac{2M}{r}}}$$

g) The phase has the form

$$\varphi = \sigma(r_* - t)$$

and the amplitude is $A \sim \frac{1}{r}$

$\Rightarrow$ The waves have the form

$$h_{ij} = \frac{A_{ij} \cos(\sigma(r_* - t) + \delta_{ij})}{r}$$