Circles

vs

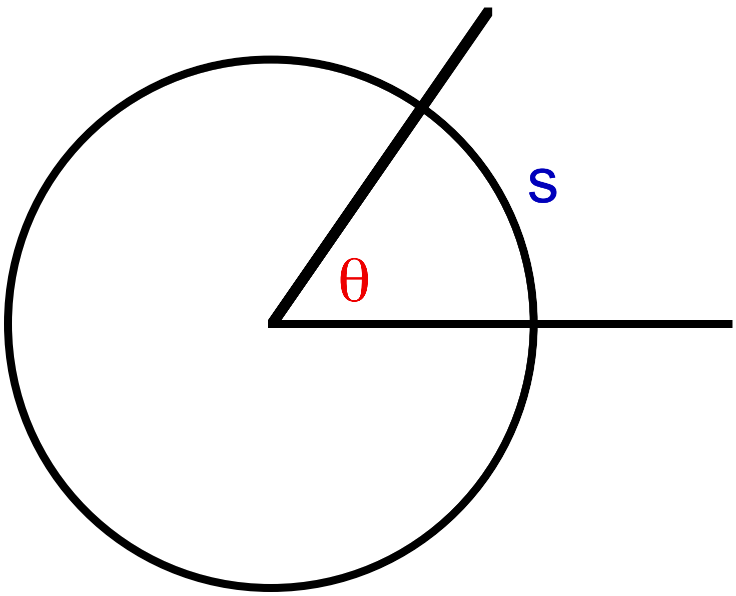
Hyperbolas

Geometries

Trigonometries

Calcului

Define Euclidian angles



$$\theta = s$$

The angle is equal to the arc length

Euclidian (Circle)
Geometry and Trigonometry

Distance (Pythagoras)

$$d^2 = x^2+y^2$$

Fundamental Identity

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Adding Angles (add arcs)

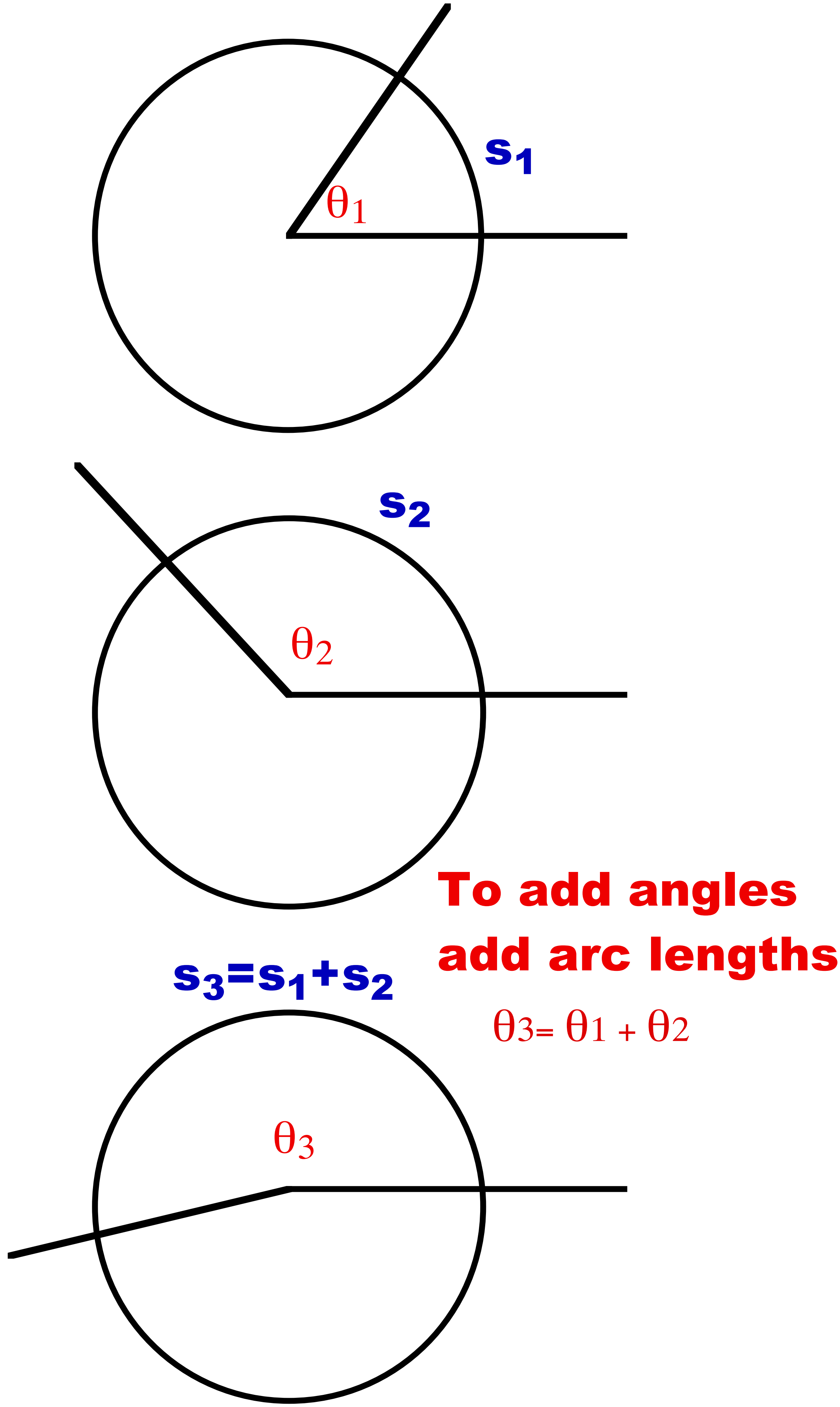
$$\theta_1 = s_1$$

$$\theta_2 = s_2$$

$$\theta_3 = \theta_1 + \theta_2 = s_1 + s_2 = s_3$$

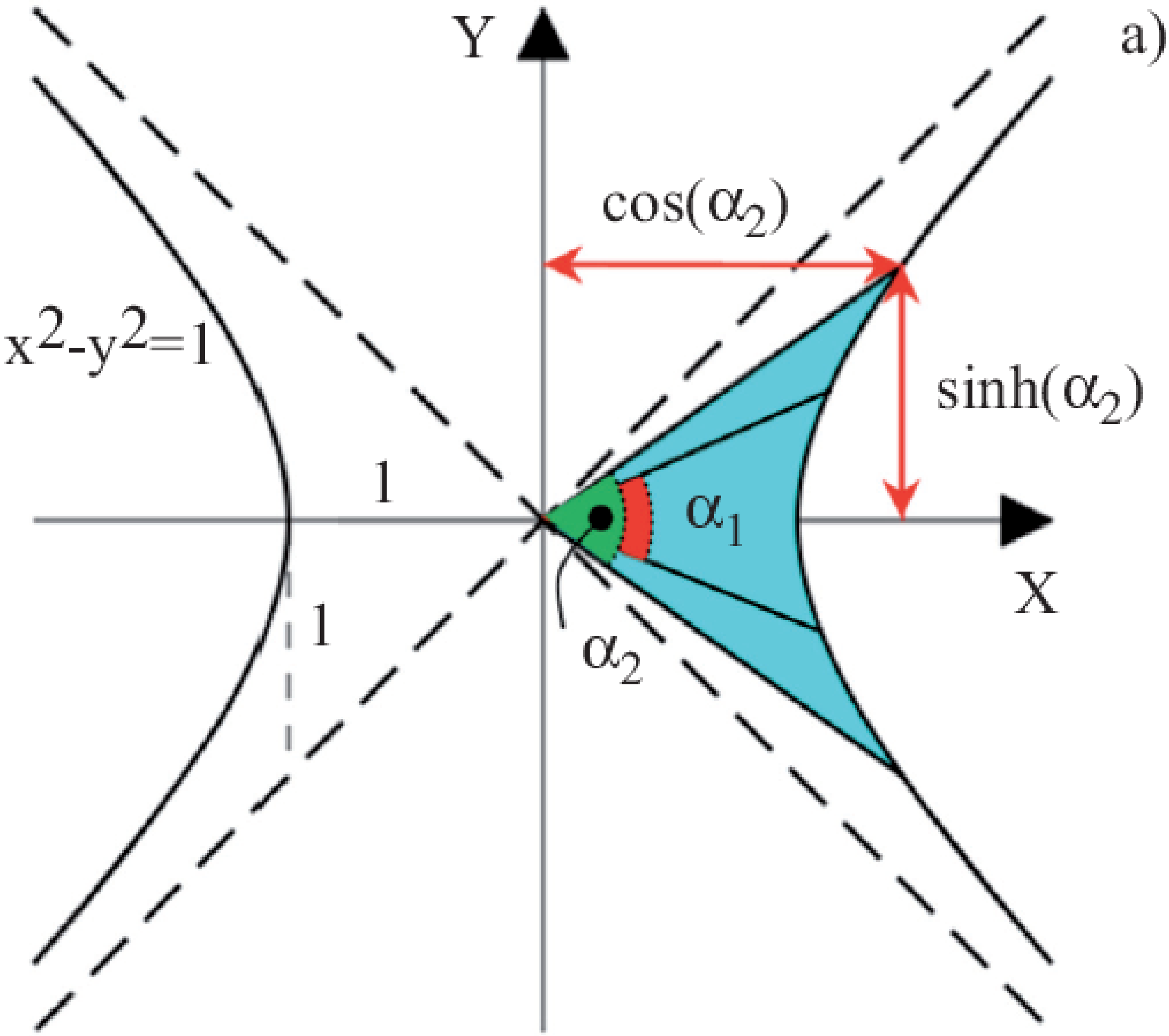
Trig Identities

$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)$$

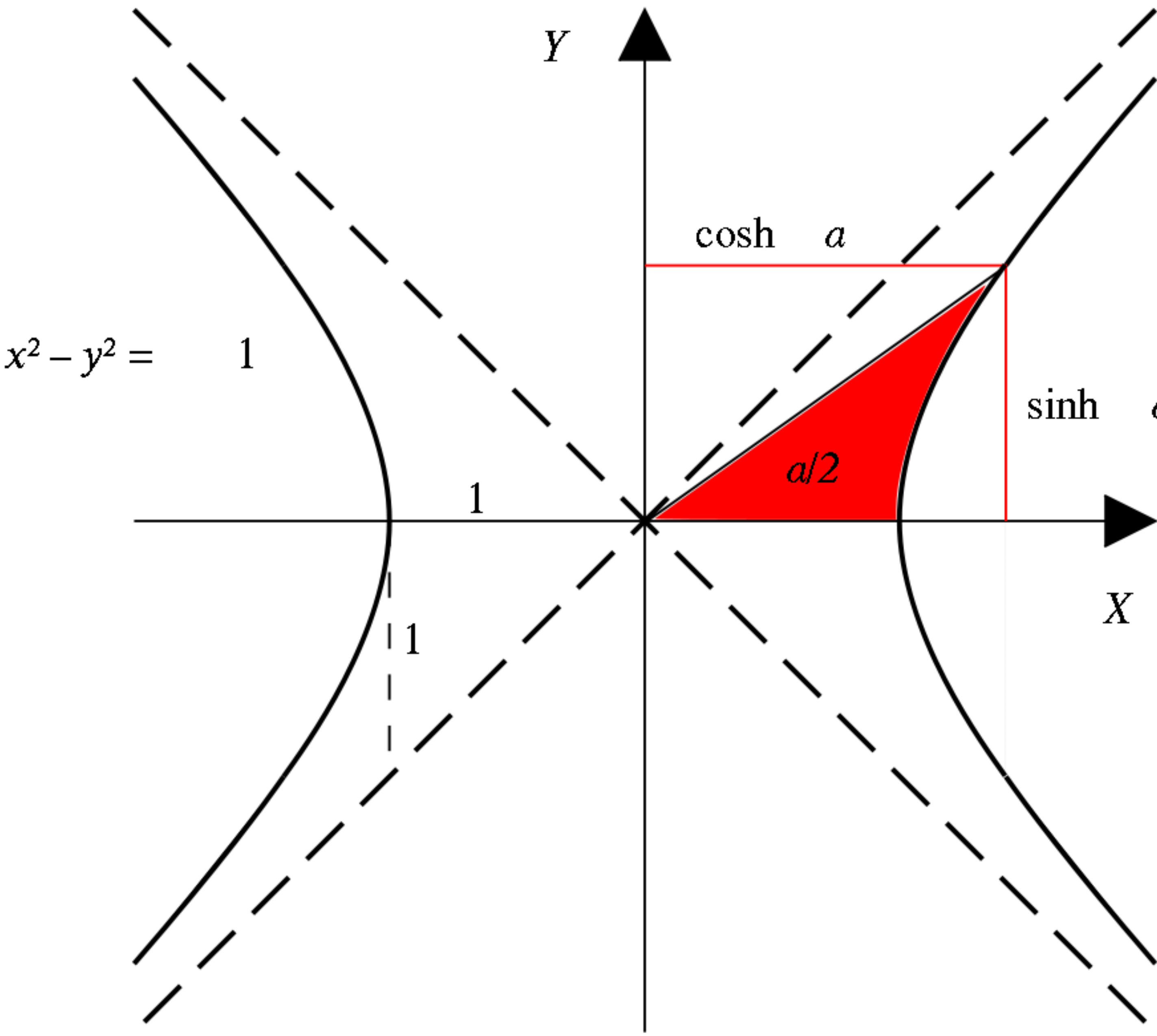


s_1
θ_1
s_2
θ_2
$s_3 = s_1 + s_2$
To add angles
add arc lengths
$\theta_3 = \theta_1 + \theta_2$
θ_3

Define Hyperbolic angles



The angle is equal to the arc length



Y
cosh a
$x^2 - y^2 = 1$
sinh a
a/2
1
1
X

Hyperbolic Geometry and Trigonometry

Distance (Pythagoras)

$$d^2 = t^2 - x^2$$

Fundamental Identity

$$\cosh^2(\theta) - \sinh^2(\theta) = 1$$

Adding Angles (add arcs)

$$\theta_1 = s_1$$

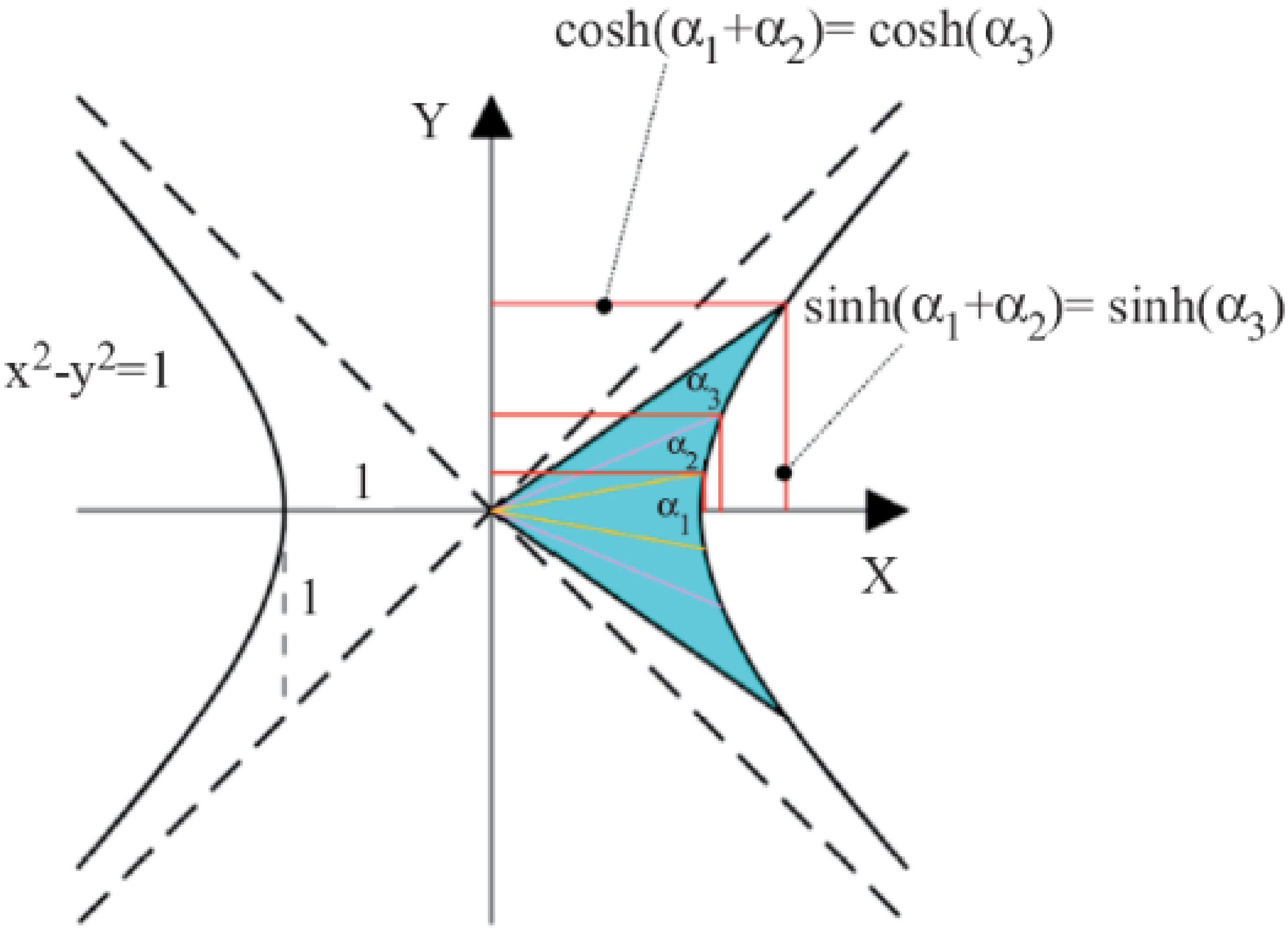
$$\theta_2 = s_2$$

$$\theta_1 + \theta_2 = s_1 + s_2$$

Trig Identities

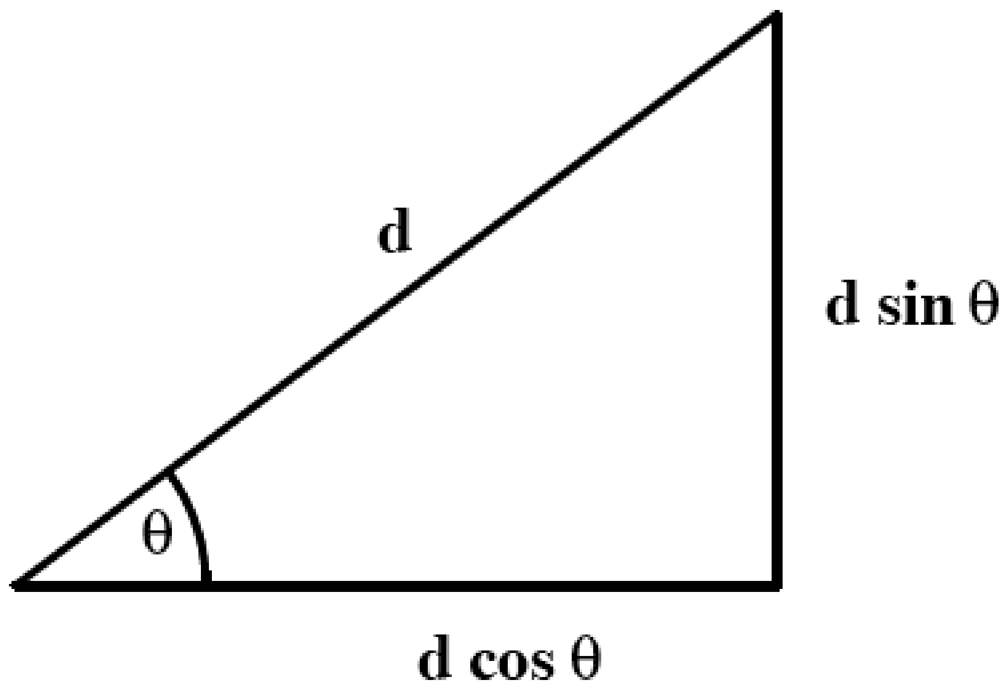
$$\sinh(\theta+\phi) =$$
$$\sinh(\theta)\cosh(\phi)+\cosh(\theta)\sinh(\phi)$$

Add Hyperbolic angles

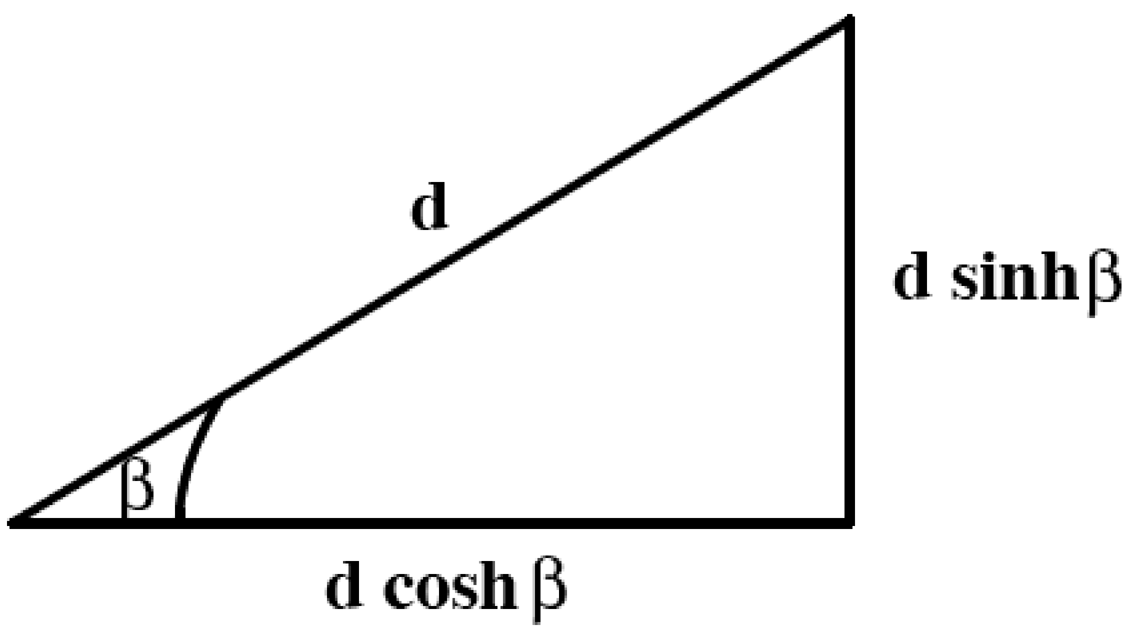


The angle is equal to the arc length

Euclidian Triangle Trig



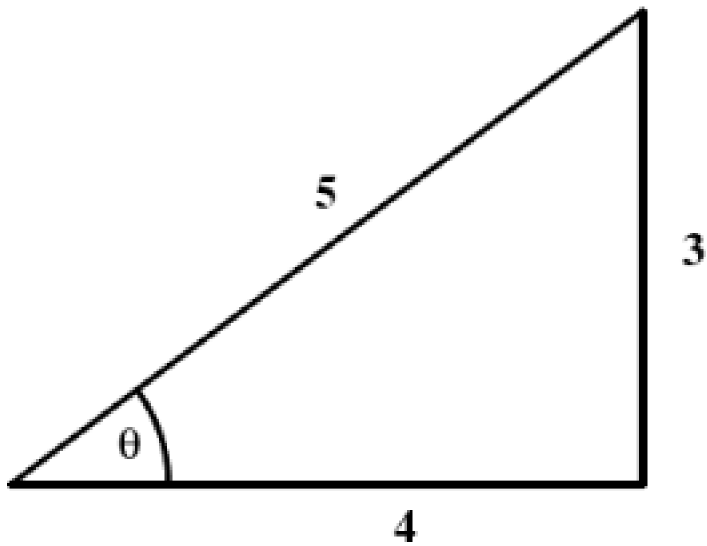
Hyperbolic Triangle Trig



Euclidian Triangles

$$\cos^2(\theta) = [\ 1 + \tan^2(\theta)\ ]^{-1}$$

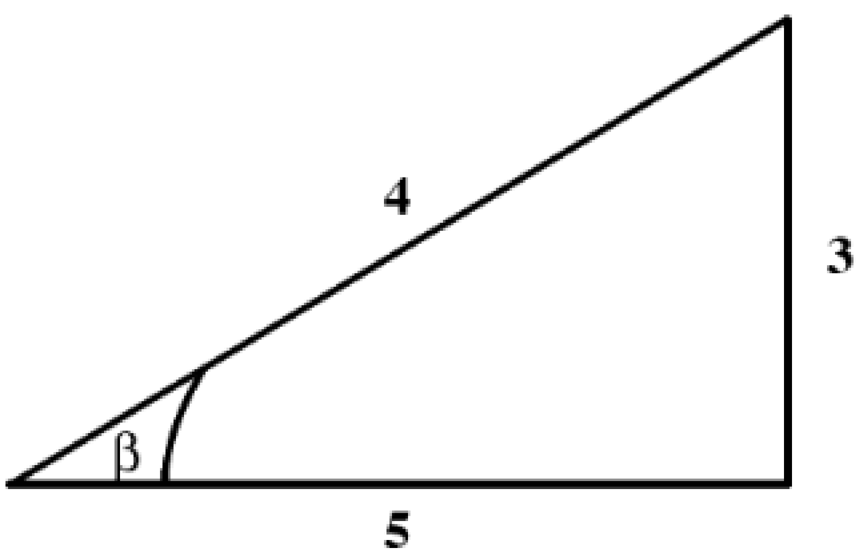
$$3^2 + 4^2 = 5^2$$



Hyperbolic Triangles

$$\cosh^2(\theta) = [\ 1 - \tanh^2(\theta)\ ]^{-1}$$

$$5^2 - 3^2 = 4^2$$



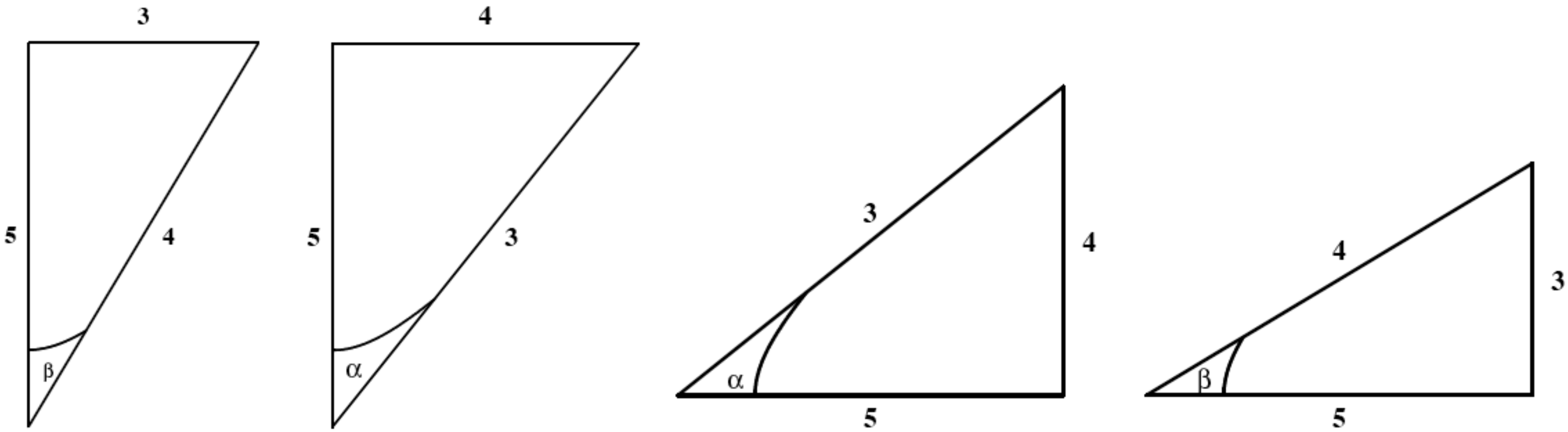


Figure 5.3: Some hyperbolic right triangles.

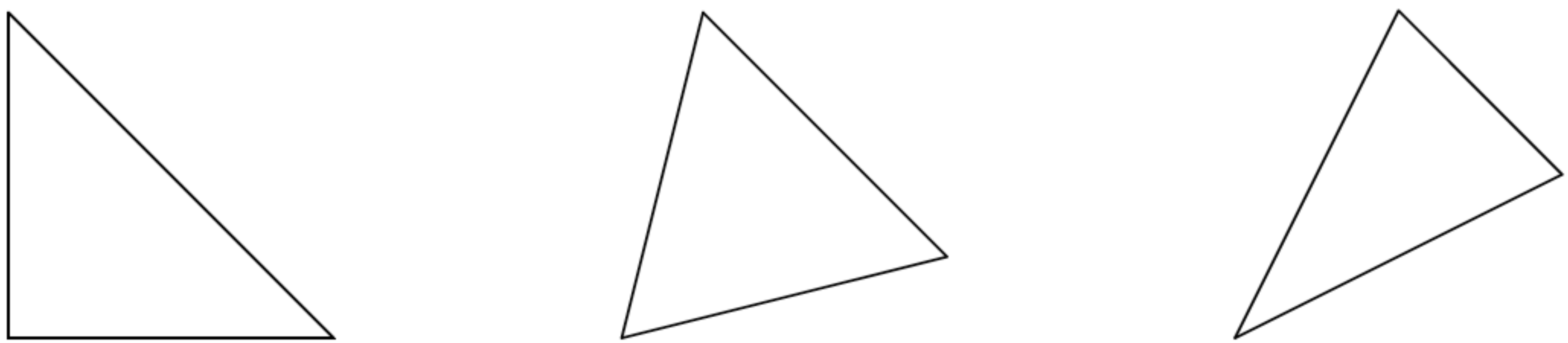
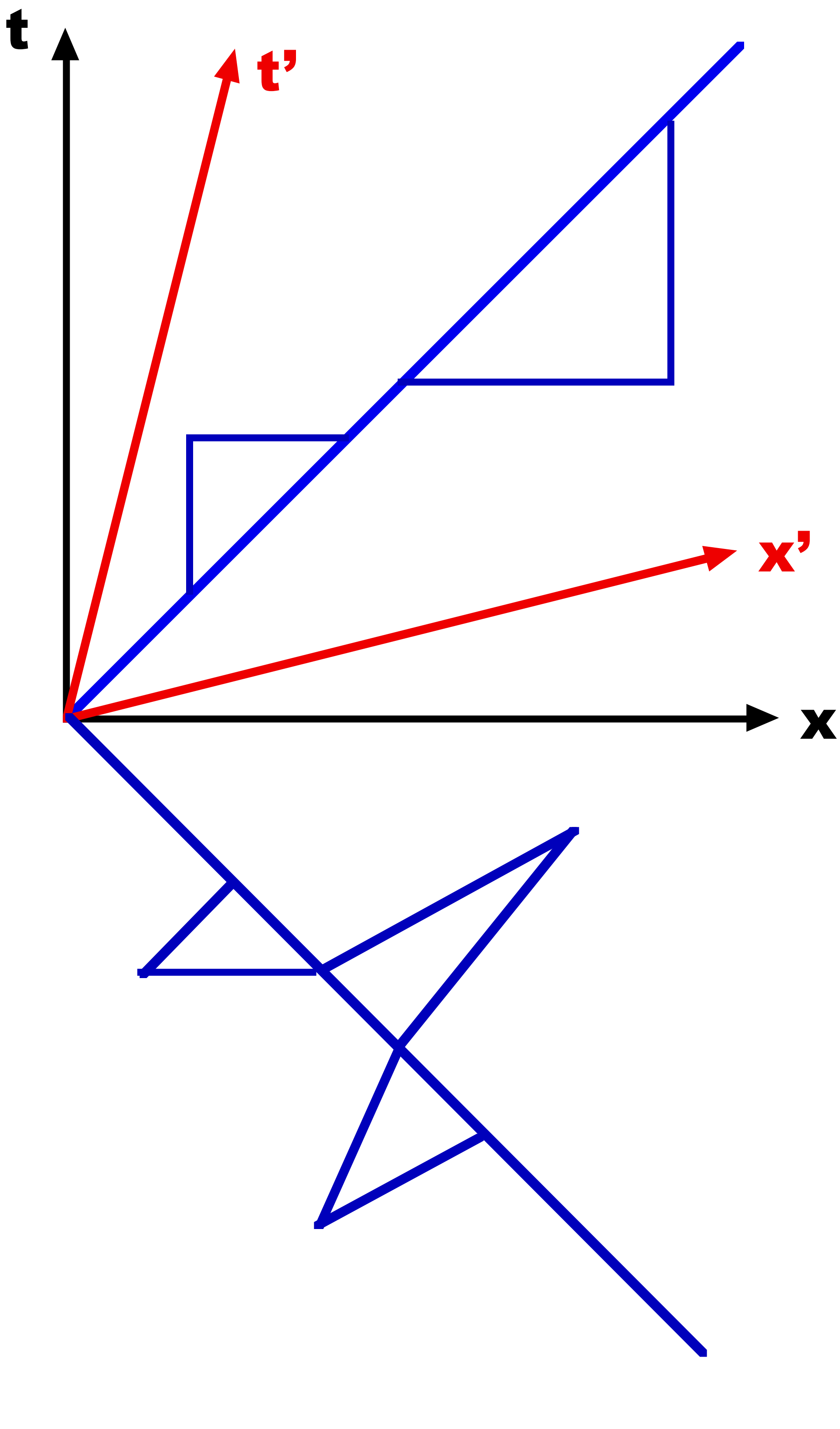
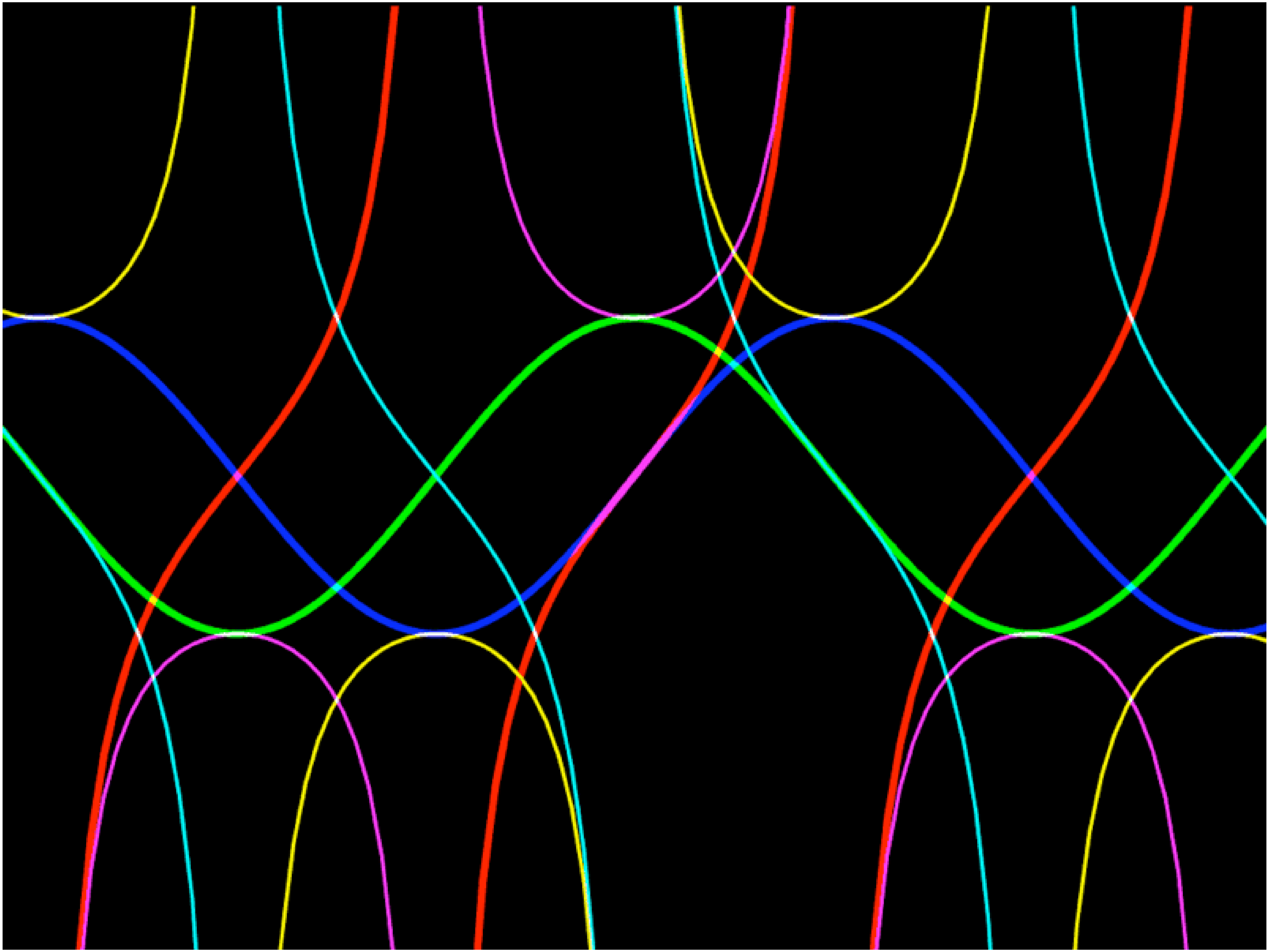
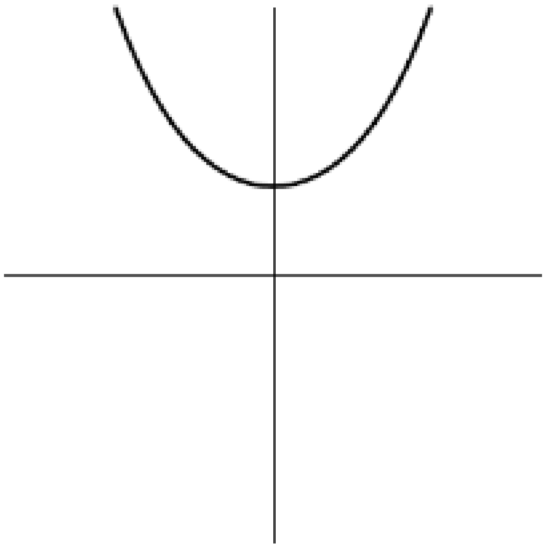


Figure 5.4: More hyperbolic right triangles. The right angle is on the left!

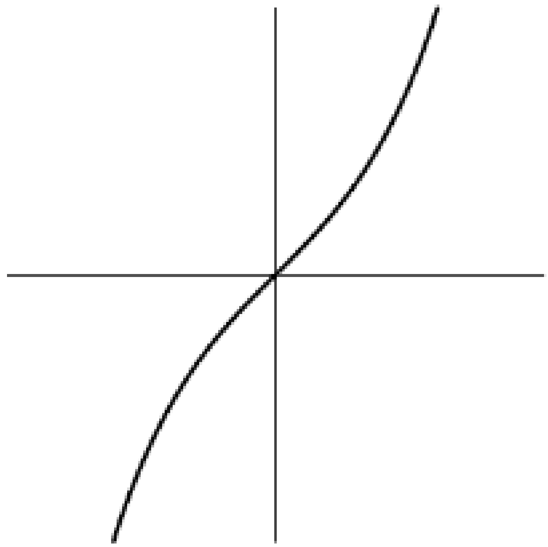
Even More Right Triangles



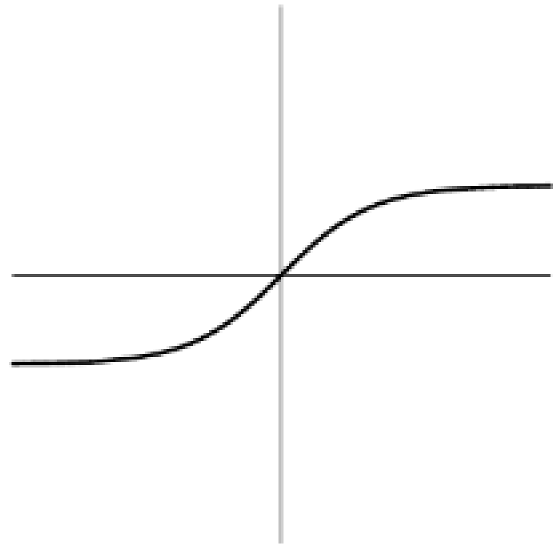




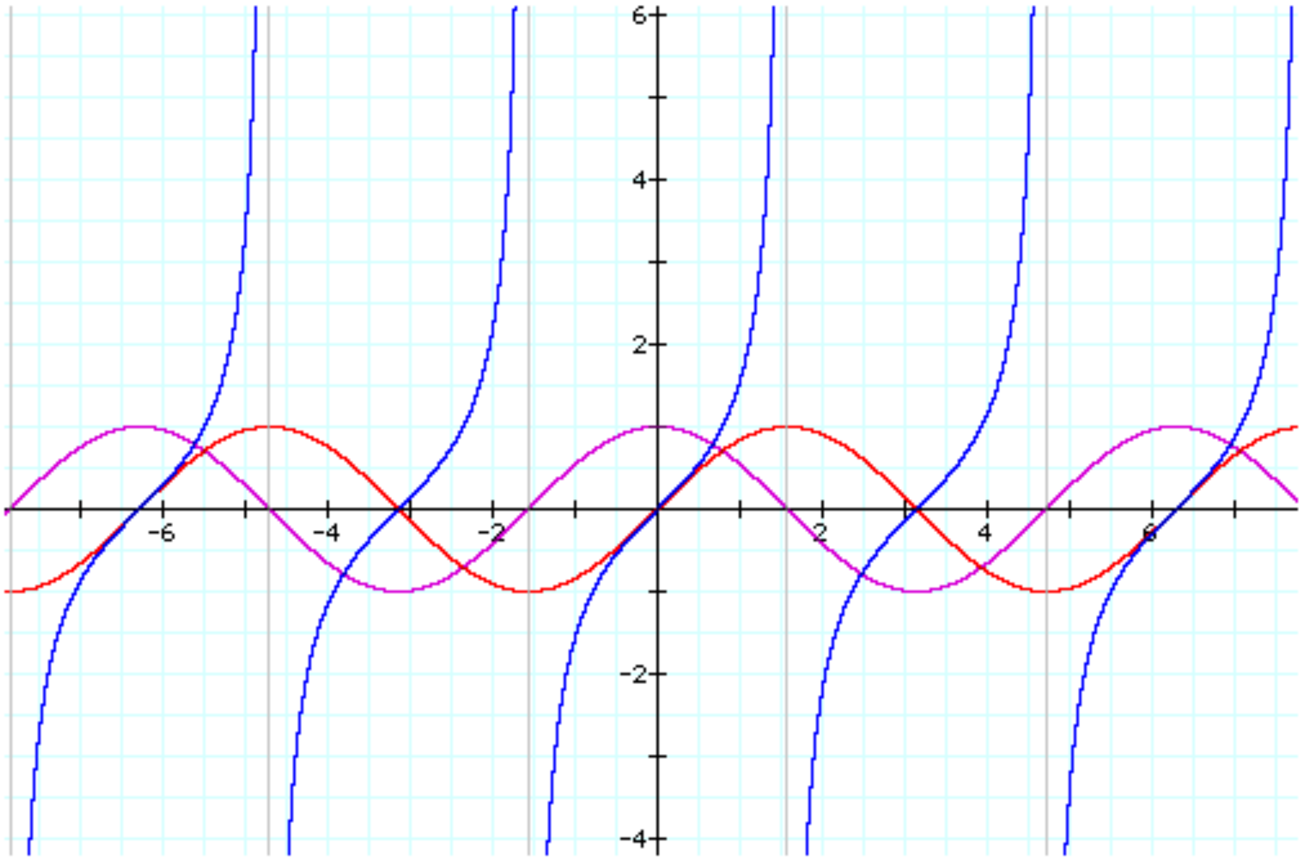
cosh



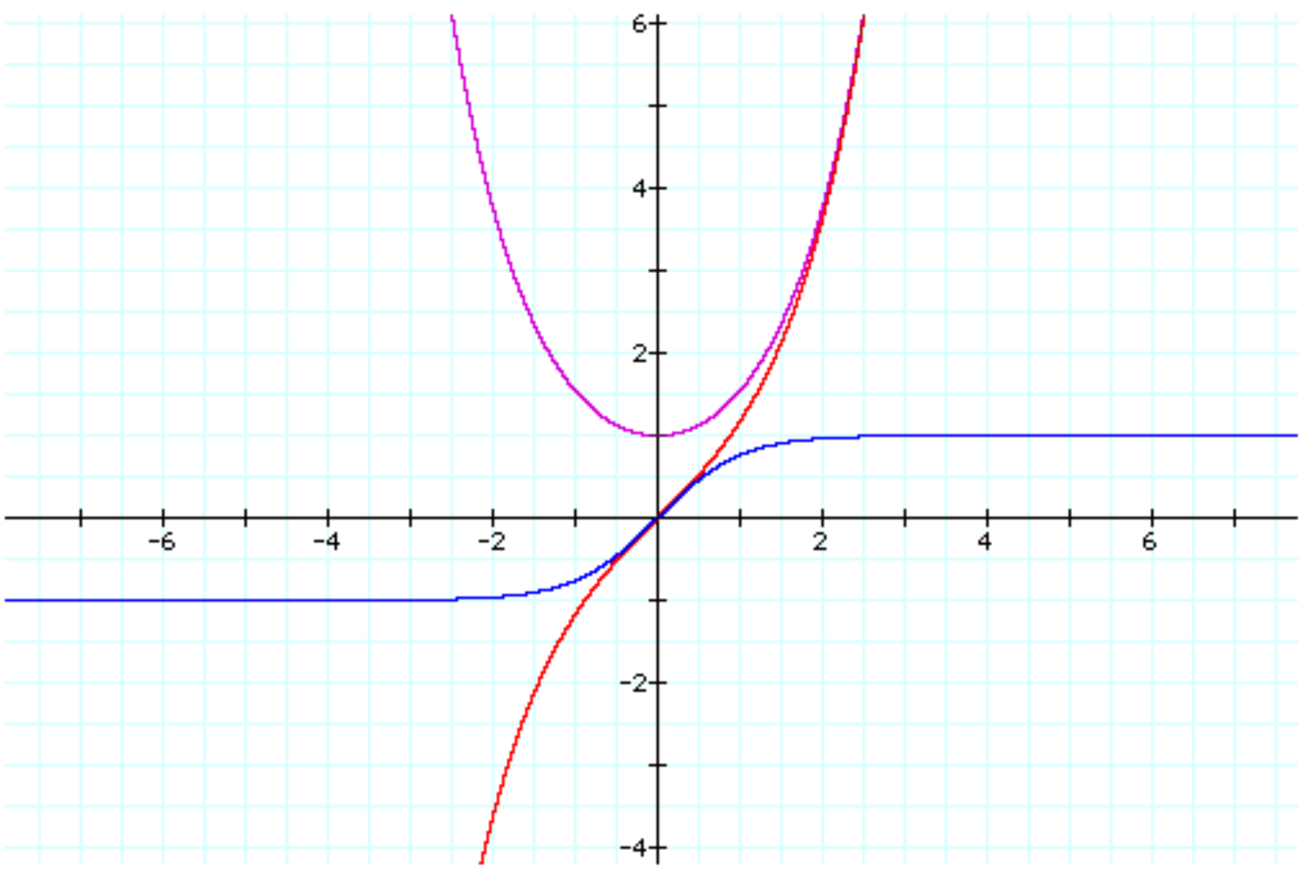
sinh



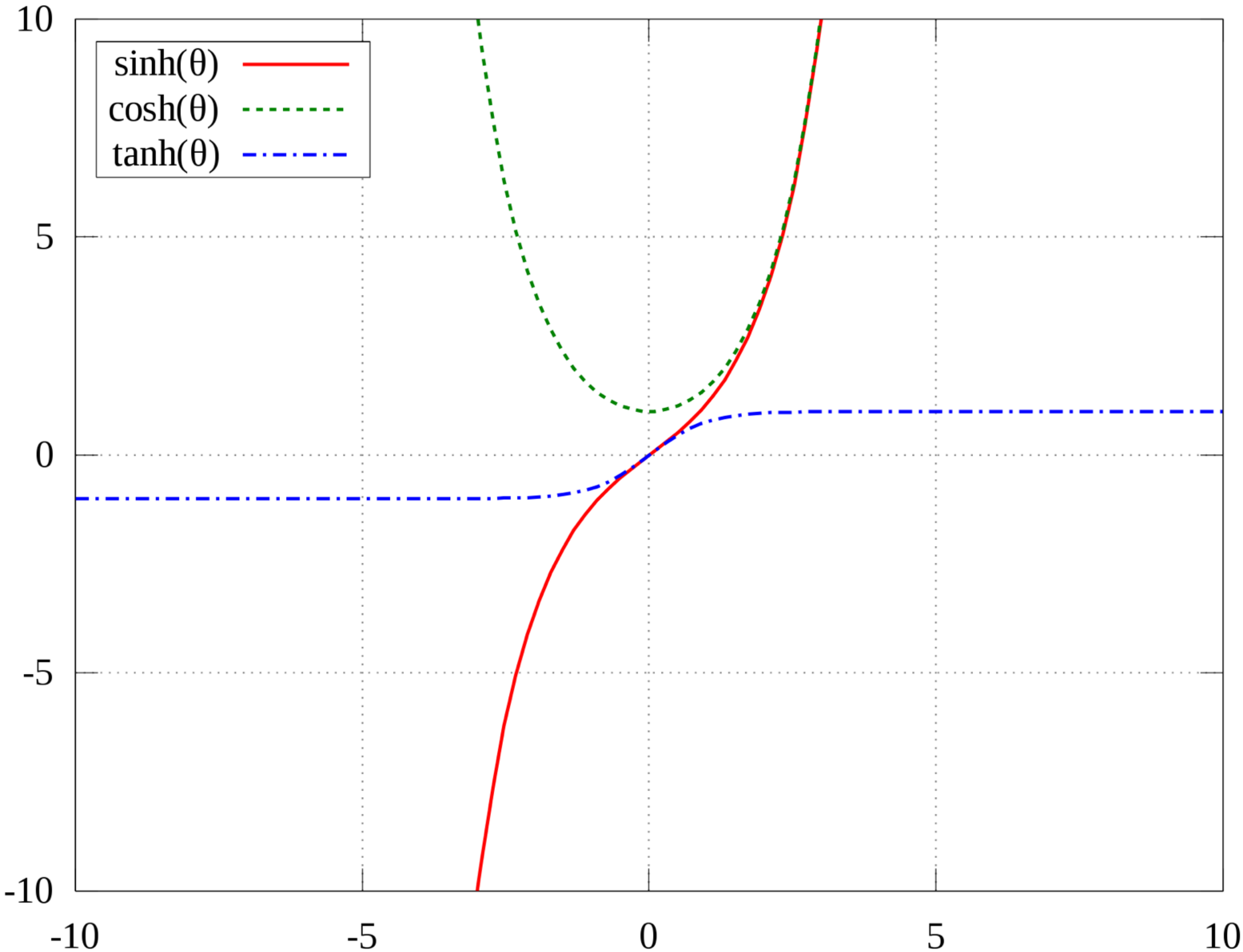
tanh

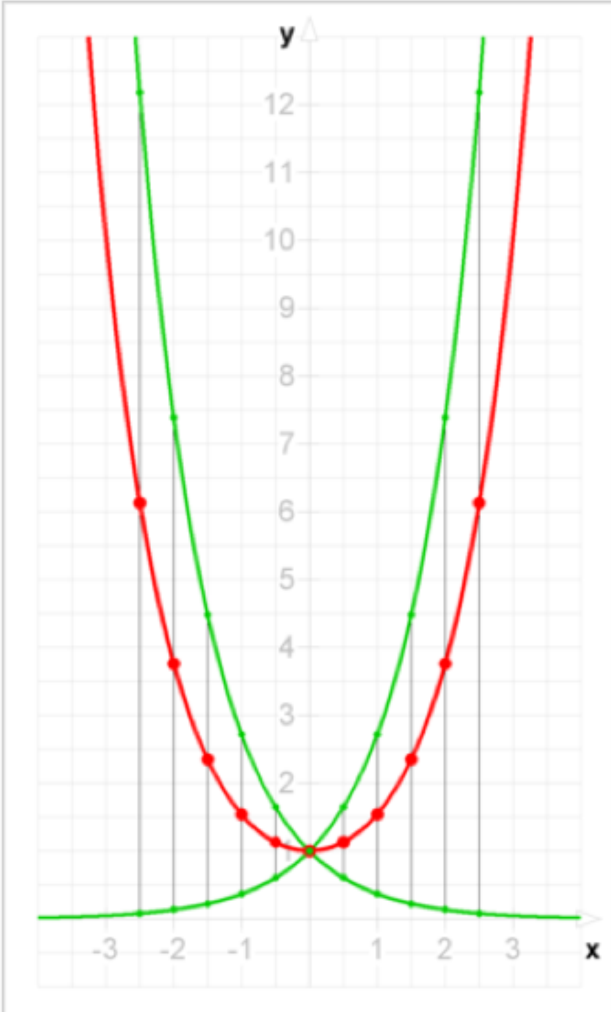


Trigonometric functions on R (cos: purple; sin: red; tan: blue)



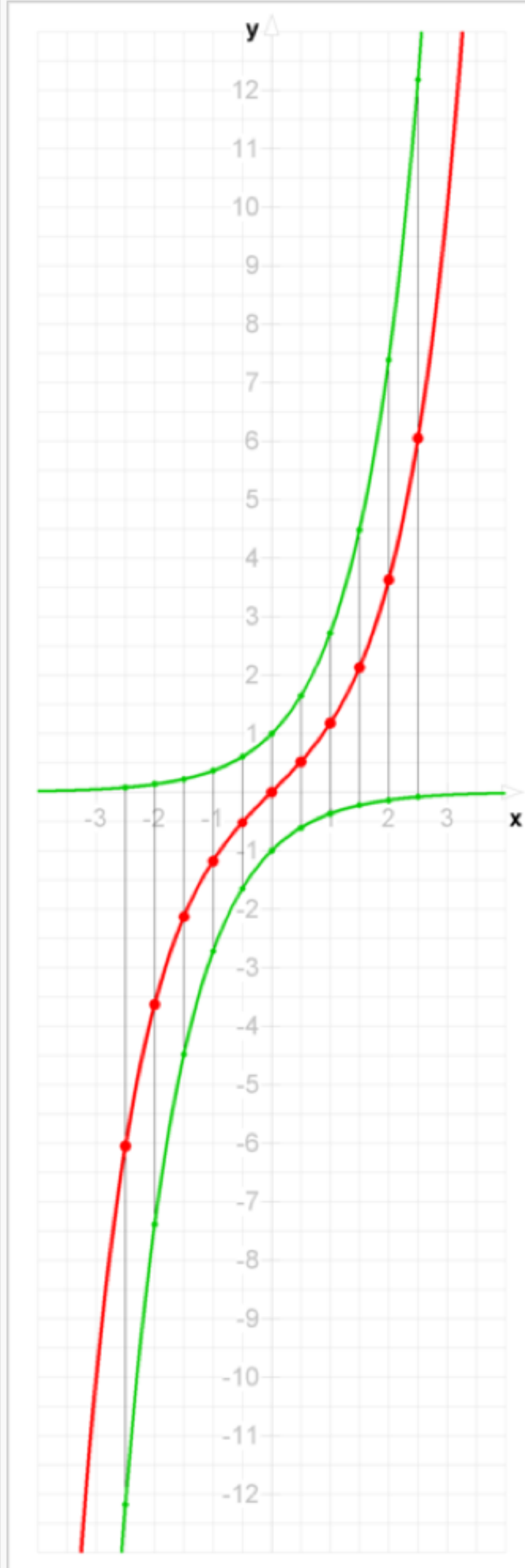
Hyperbolic functions on r (cosh: purple; sinh: red; tanh: blue)


sinh(θ)
cosh(θ)
tanh(θ)
10
5
0
-5
-10
-10
-5
0
5
10



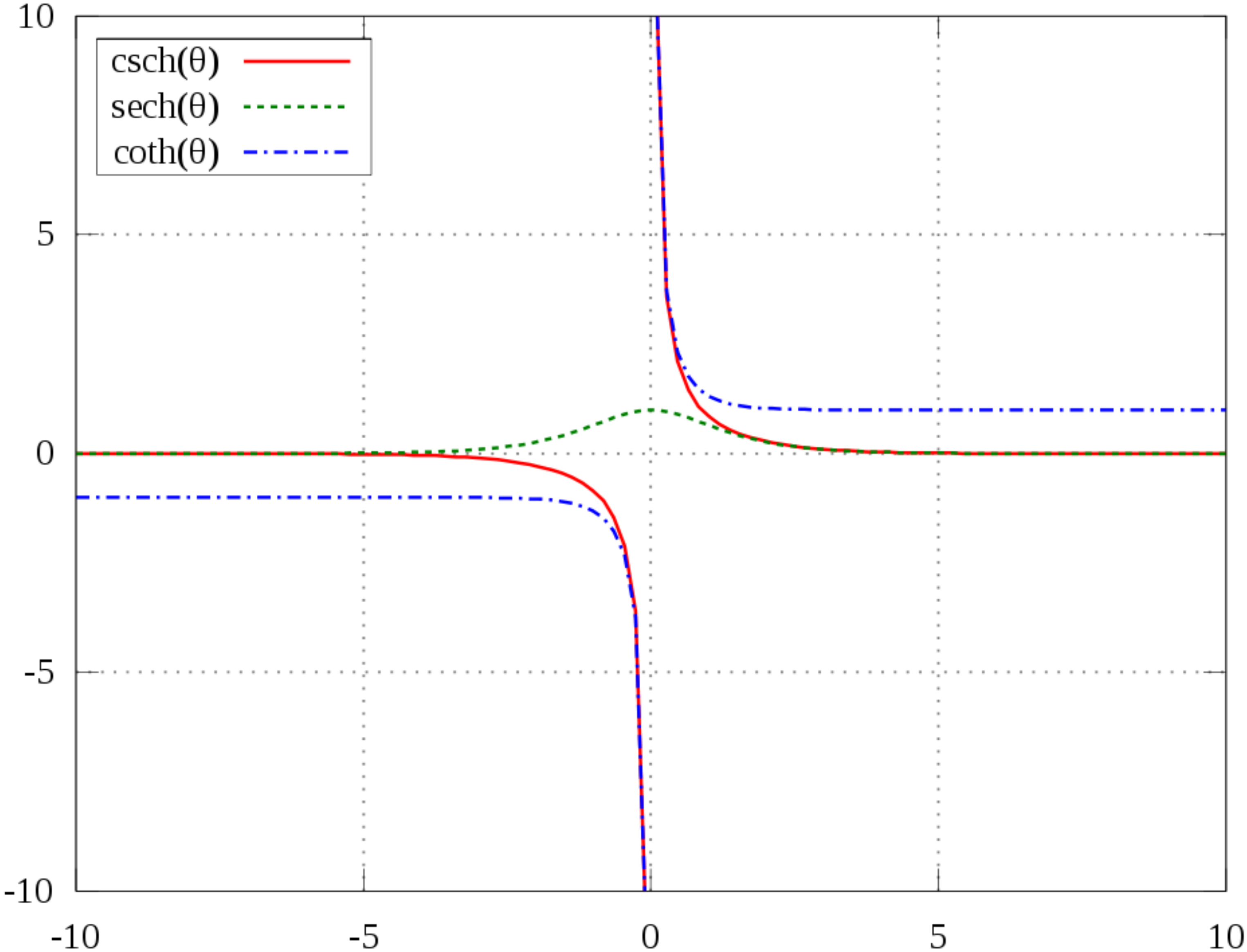
cosh(x) is the average of e^x and e^{-x}

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



cosh(x) is the average of e^x and $-e^{-x}$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



csch(θ)
sech(θ)
coth(θ)
10
5
0
-5
-10
-10
-5
0
5
10

Warning!

The following article is from The Great Soviet Encyclopedia (1979).

It might be outdated or ideologically biased.

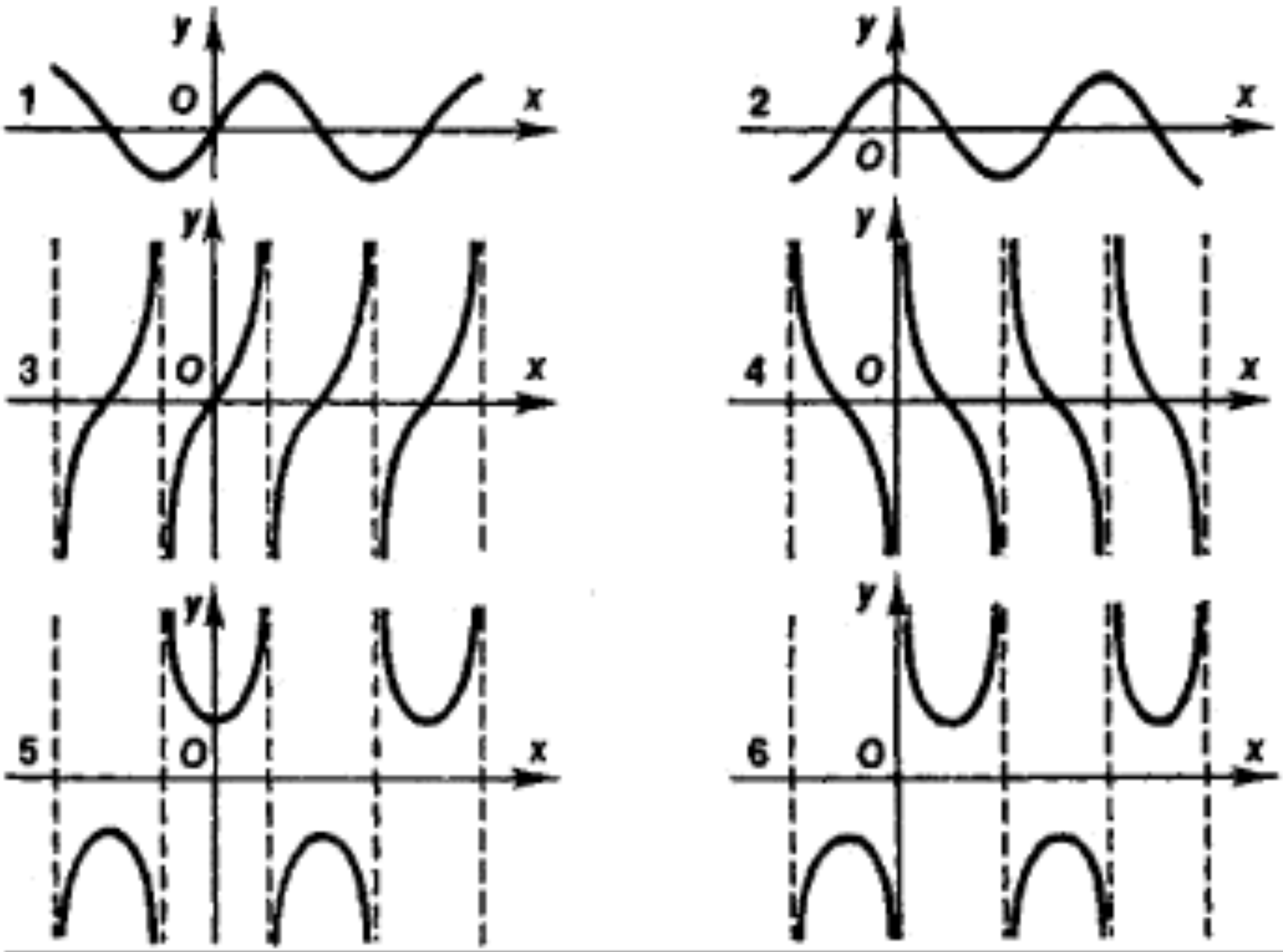


Figure 2. Graphs of trigonometric functions: (1) sine, (2) cosine, (3) tangent, (4) cotangent, (5) secant, (6) cosecant

http://en.wikipedia.org/wiki/File:HyperbolicAnimation.gif

Euclidian Calculus

$$d\cos(\theta)/d\theta = -\sin(\theta)$$

$$d\sin(\theta)/d\theta = \cos(\theta)$$

$$d\tan(\theta)/d\theta = \sec^2(\theta)$$

Hyperbolic Calculus

$$d\cosh(\theta)/d\theta = \sinh(\theta)$$

$$d\sinh(\theta)/d\theta = \cosh(\theta)$$

$$d\tanh(\theta)/d\theta = 1\text{-}\tan^2(\theta)$$

Euclidian to Hyperbolic

$$\cos(\theta) = \cosh(i\theta)$$

$$\sin(\theta) = i \sinh(i\theta)$$

$$\tan(\theta) = i \tanh(i\theta)$$

$$\cot(\theta) = -i \coth(i\theta)$$

$$\sec(\theta) = \operatorname{sech}(i\theta)$$

$$\csc(\theta) = -i \operatorname{csch}(i\theta)$$

Hyperbolic to Euclidian

$$\cosh(\theta) = \cos(i\theta)$$

$$\sinh(\theta) = -i \sin(i\theta)$$

$$\tanh(\theta) = -i \tan(i\theta)$$

$$\coth(\theta) = i \tan(i\theta)$$

$$\operatorname{sech}(\theta) = \sec(i\theta)$$

$$\operatorname{csch}(\theta) = i \csc(i\theta)$$

Rapidity

The dimensionless velocities

$\beta_1 = v_1 / c$

$\beta_2 = v_2 / c$

Rapidity

$\beta = \tanh(r)$

Rapidity adds

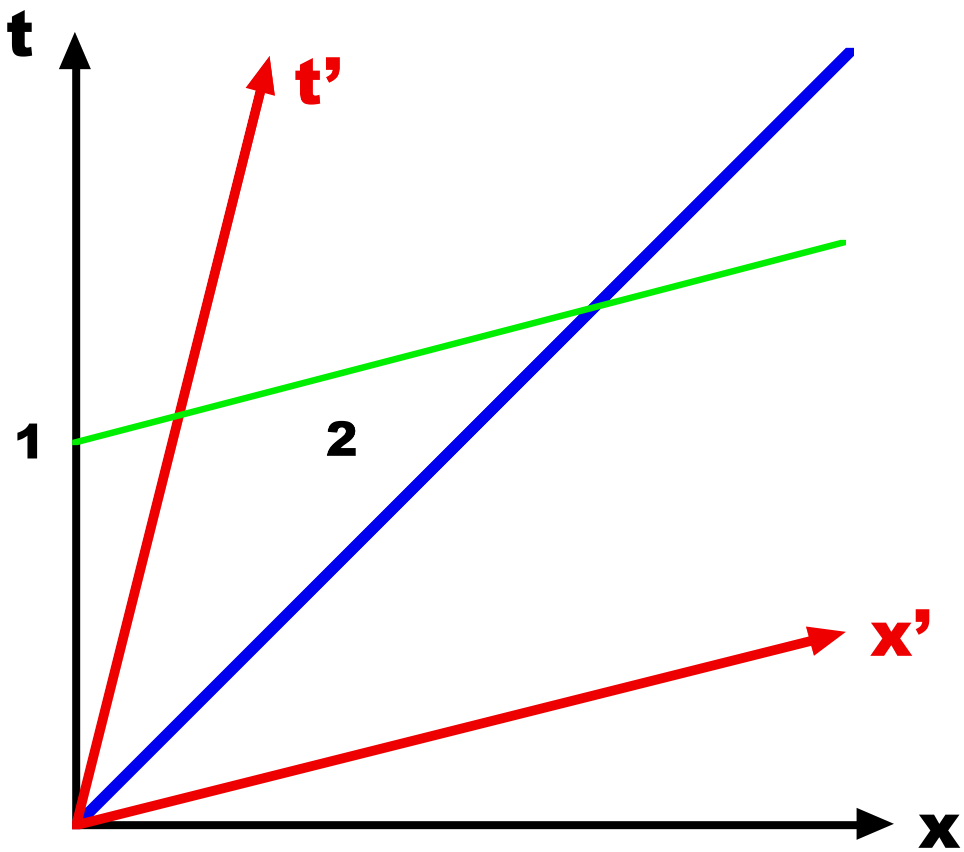
$r_3 = r_1 + r_2$

$\beta = \tanh(r_1 + r_2)$

$\beta = (\tanh(r_1)+\tanh(r_2))/(1+\tanh(r_1) \tanh(r_2))$

Einstein velocity addition

$$\beta = (\beta_1 + \beta_2) / (1 + \beta_1 \beta_2)$$



Two events simultaneous in one frame are not simultaneous in any other frame

Now that we have plotted (a region of the) inertial reference frame S' moving at $v_{rel} = 0.5$ relative to S, we can test the statement regarding the relativity of simultaneity by graphical means.

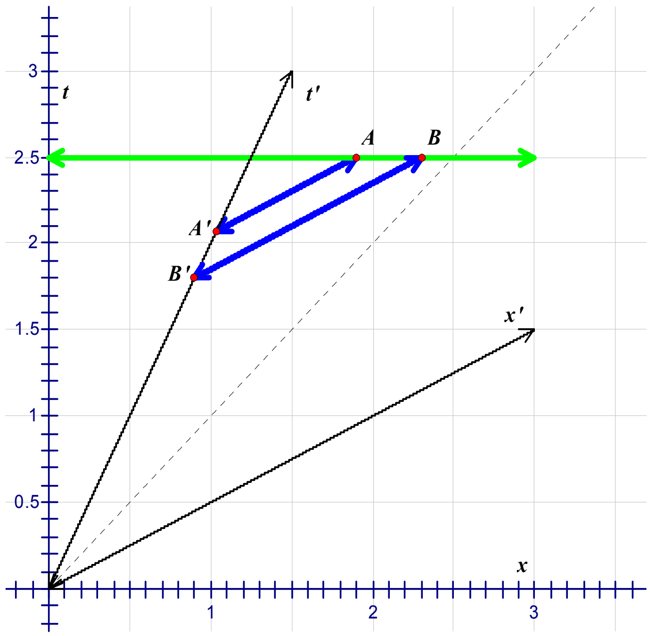


Figure 3-II.2 Relativity of simultaneity.

For an observer in S, we represent simultaneous events A and B occurring at time t, by plotting a line parallel to the x axis through points A and B. For an observer in S', similar lines plotted parallel to the x' axis through point A and B correspond with time measurements A' and B' on the t' axis. Not only are events A and B not simultaneous in S', their chronological order is reversed.

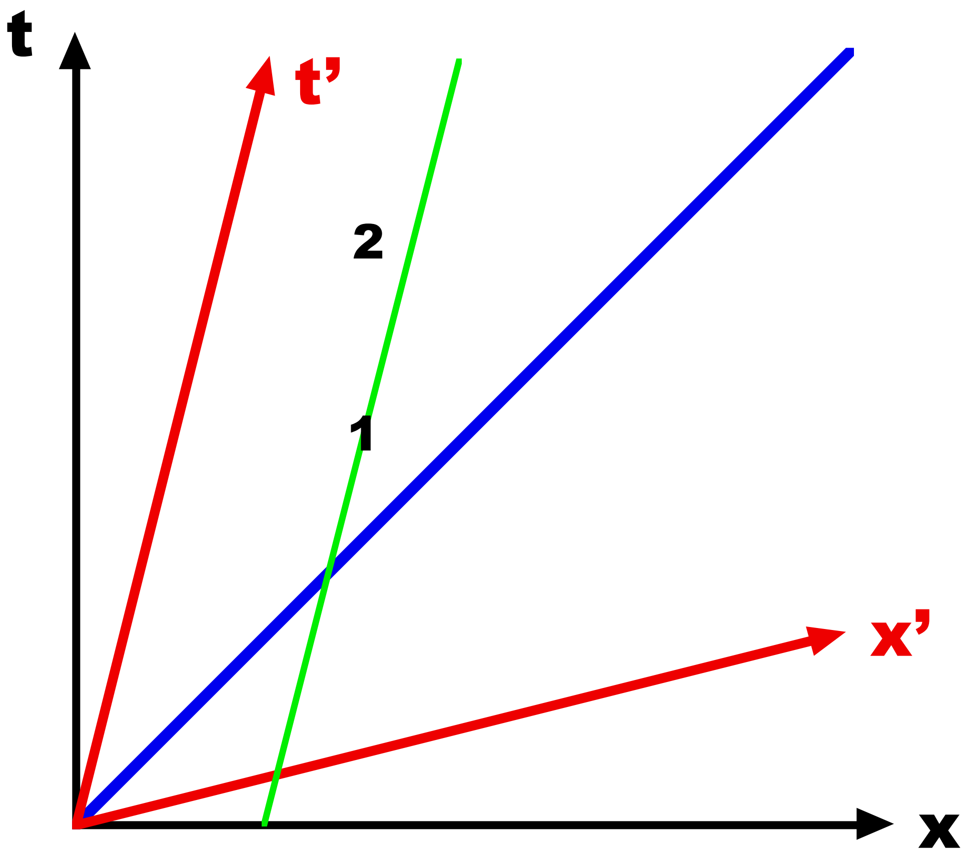
3-1. Relativity and swimming

The idea here is to illustrate how remarkable is the invariance of the speed of light (speed of light is the same in all free-float frames) by contrasting it with the case of a swimmer making her way through water.

Light goes through space at 3×10^{8} meters per second, and the swimmer goes through water at 1 meter per second. "But how can there otherwise be any difference?" one at first asks oneself.

For a light flash to go down the length of a 30-meter spaceship and back again it takes

$$\begin{aligned} time &= (distance)/(speed) \\ &= 2\times(30\,metes)/(3\times10^{8}\,meters/second) \\ &= 2\times10^{-7}\,second \end{aligned}$$



Two events at the same place in one frame are not at the same place in any other frame

Minkowski diagrams

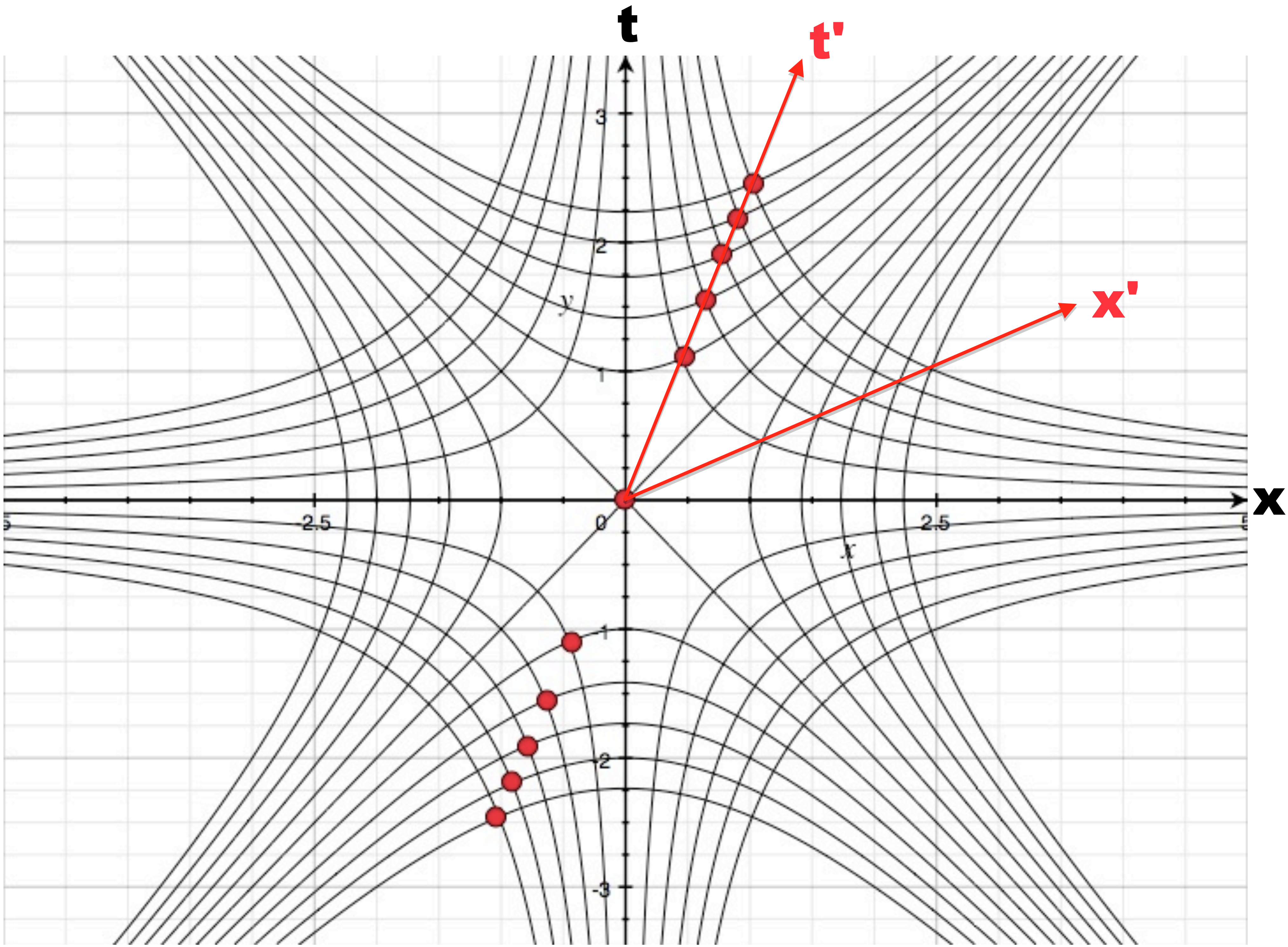
Two great things

(1) Arbitrarily many reference frames

(2) Constantly reminded that space is hyperbolic

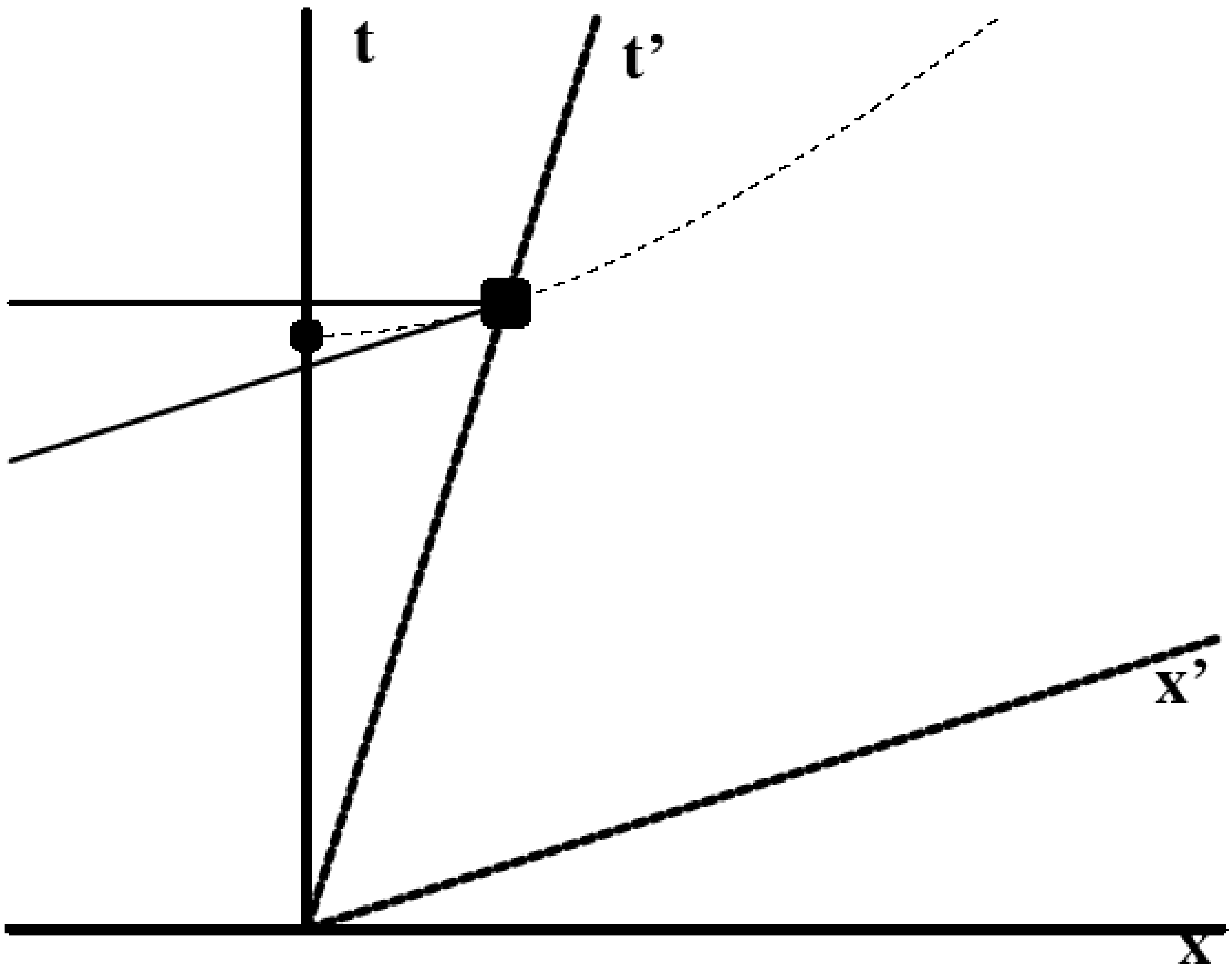
One not so great thing

(1) Hyperbolic distortion

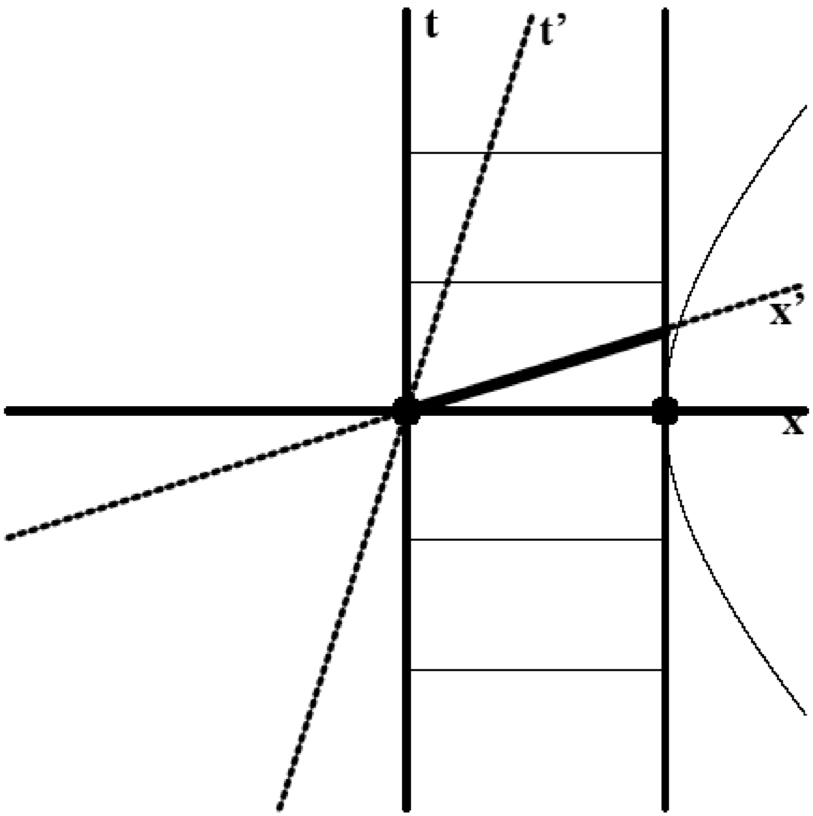


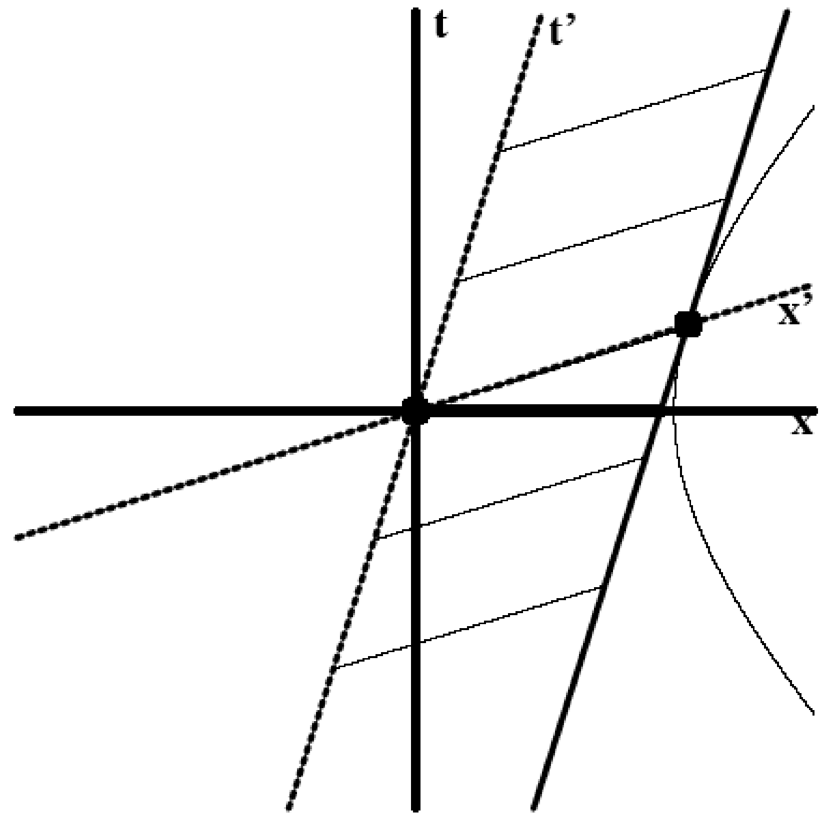
calibration hyperbolas

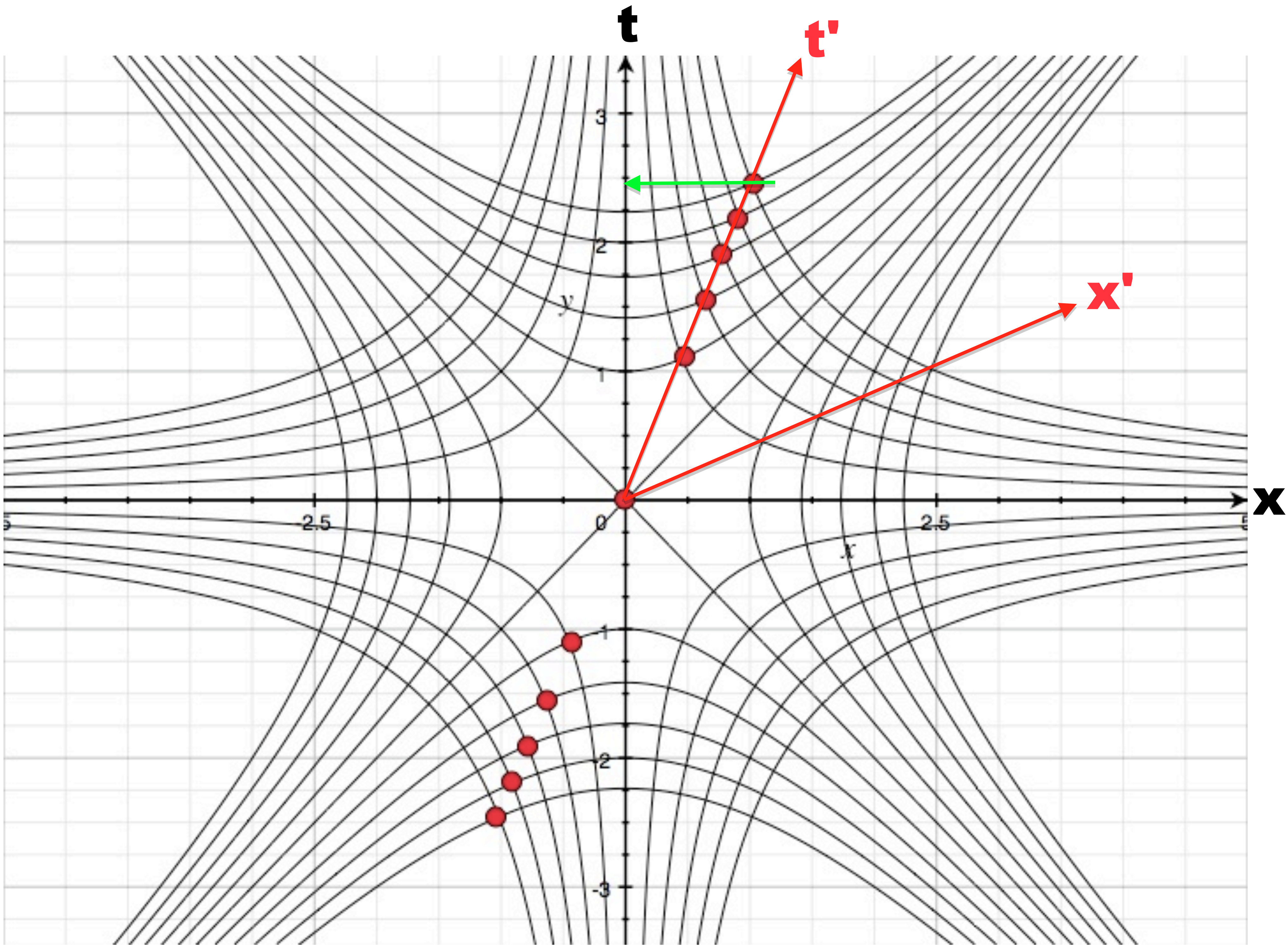
Time Dilation



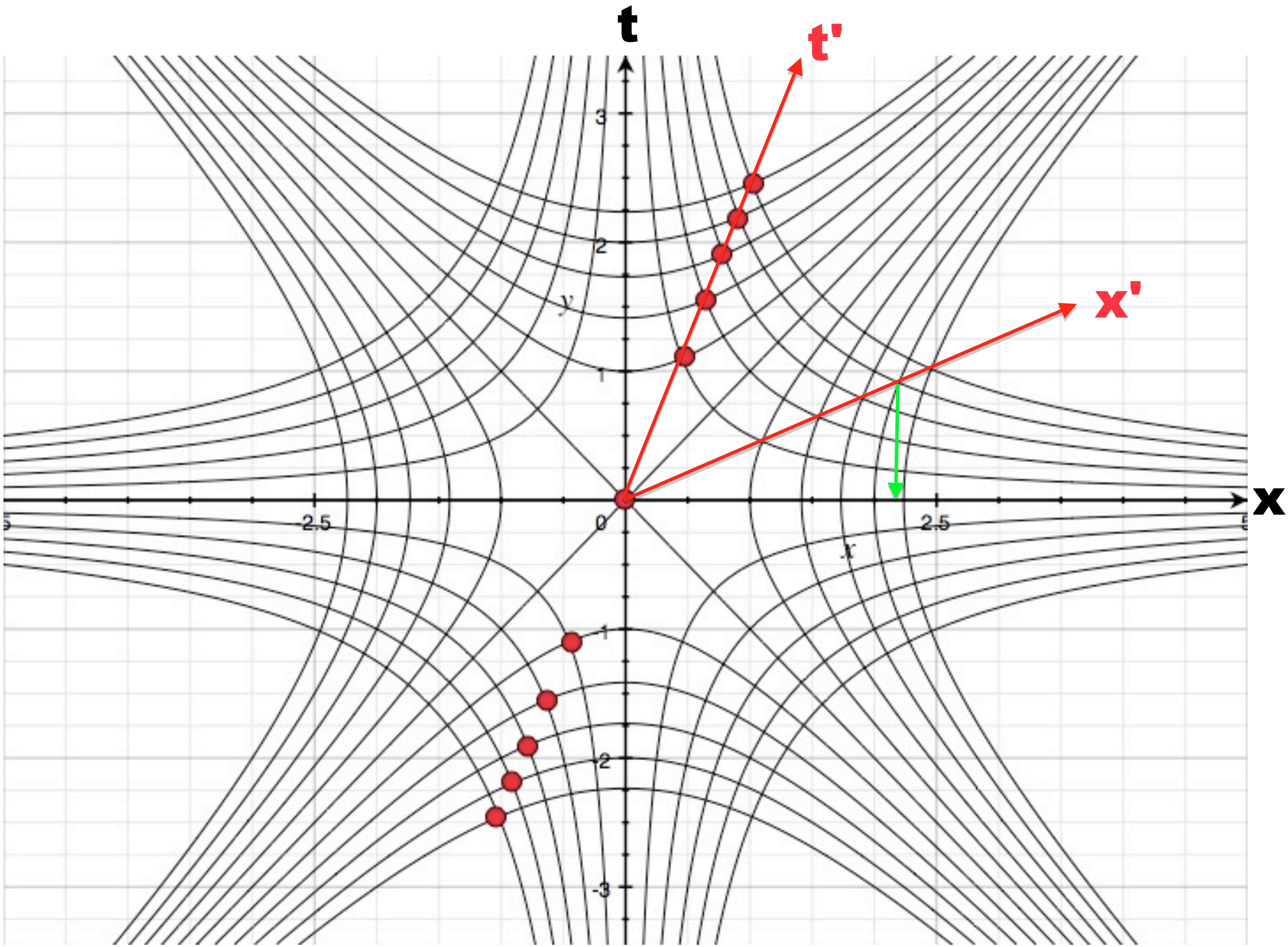
Length Contraction







time dilation



length contraction

X’ is Moving Right

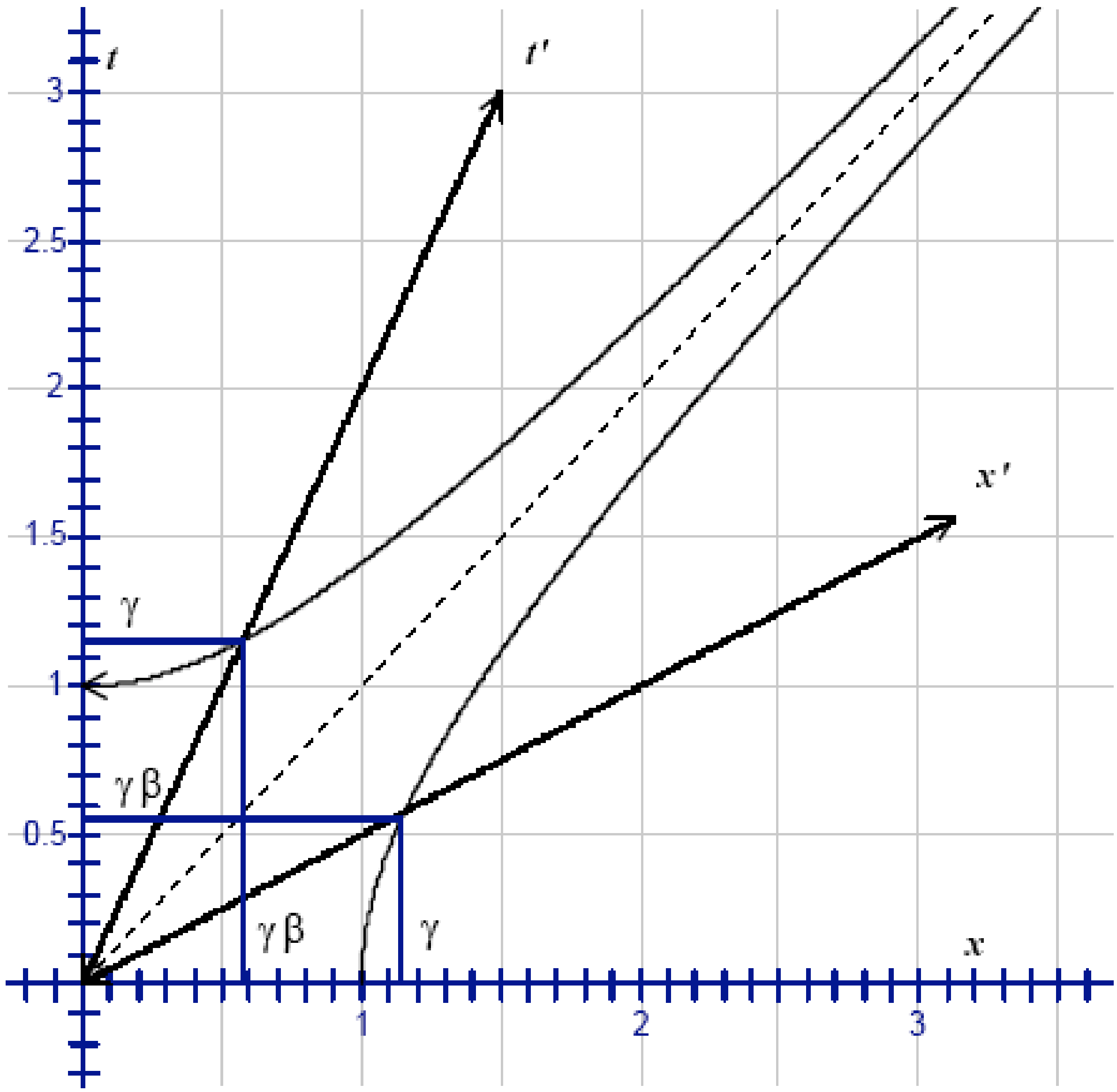


Figure L-VI.1 x' moving to the right of x.

X is Moving Left

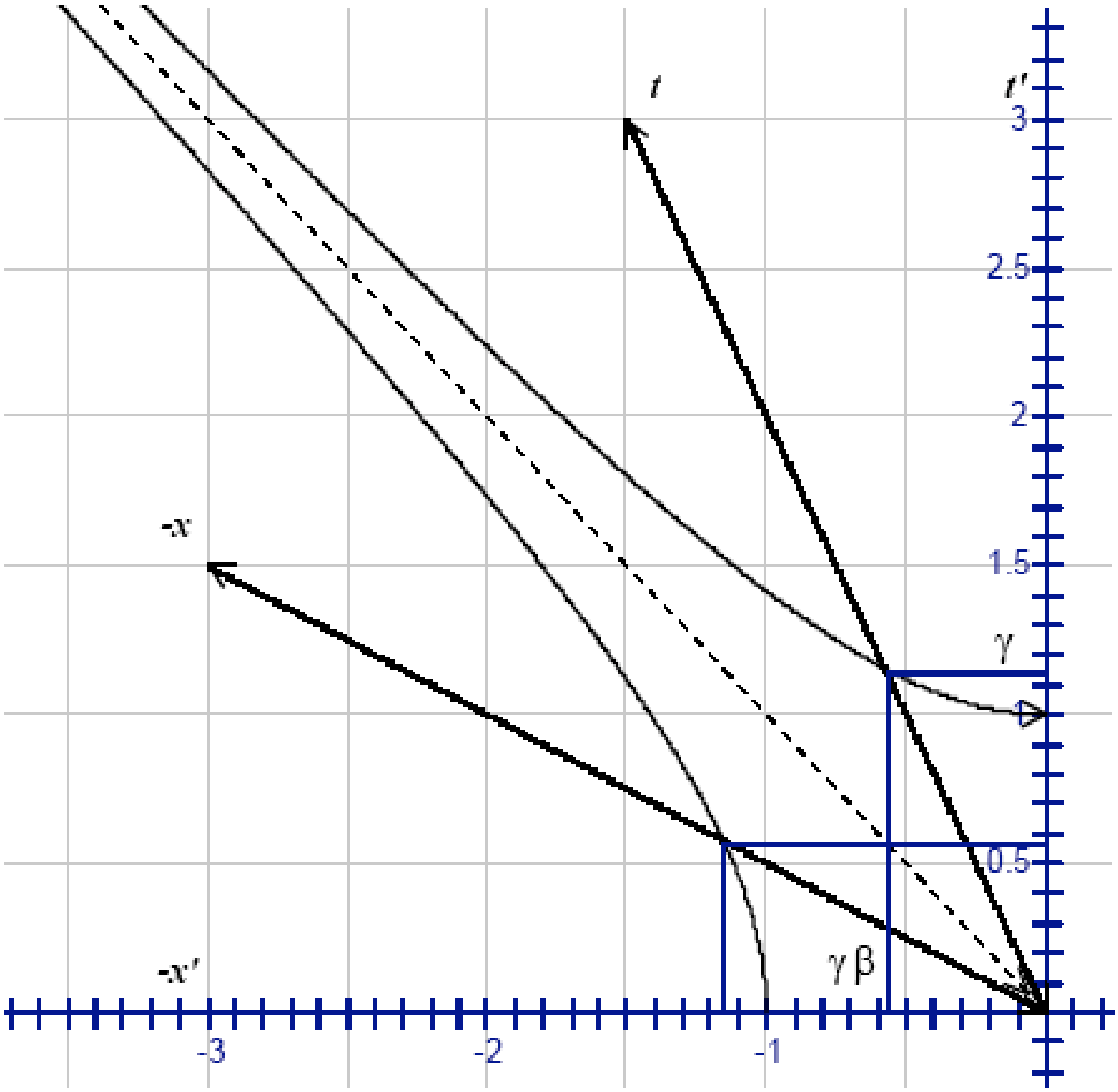


Figure L-VI.2 x moving to the left of x'.

Lorentz transformation matrix Λ

$$\mathbf{\Lambda} = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Hyperbolic

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

Making Photons using Relativistic Electrons

Synchrotrons
Wigglers
Undulators
and
X-Ray Lasers

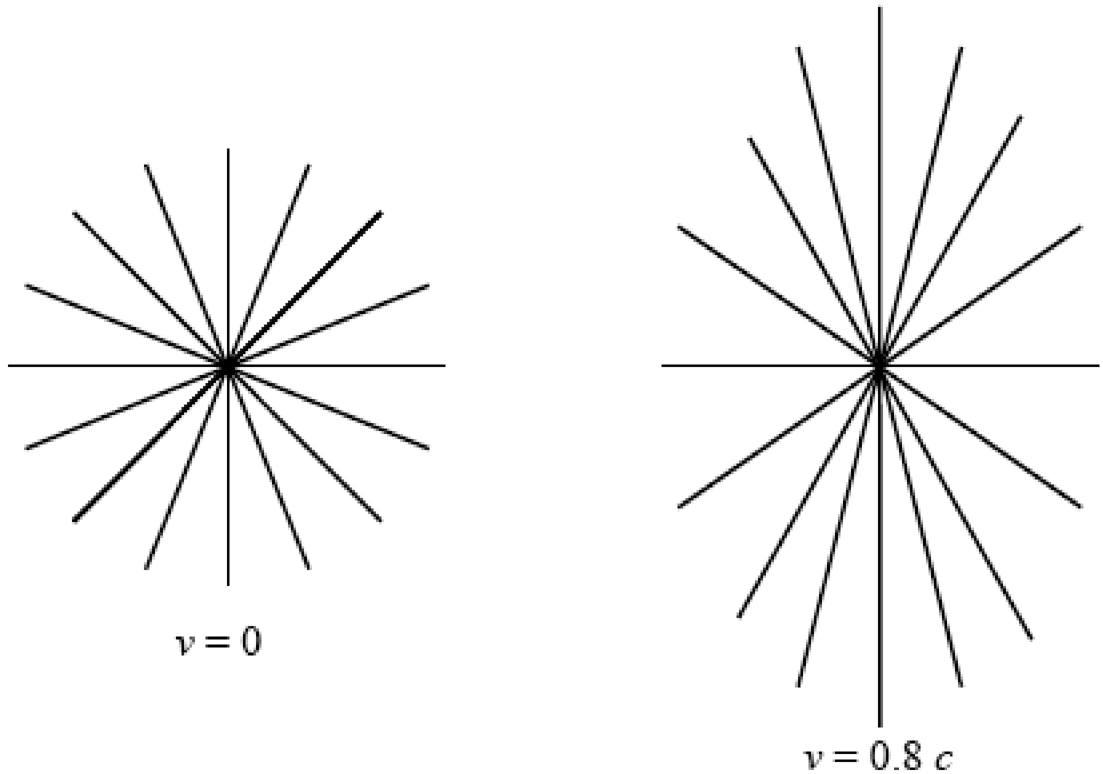
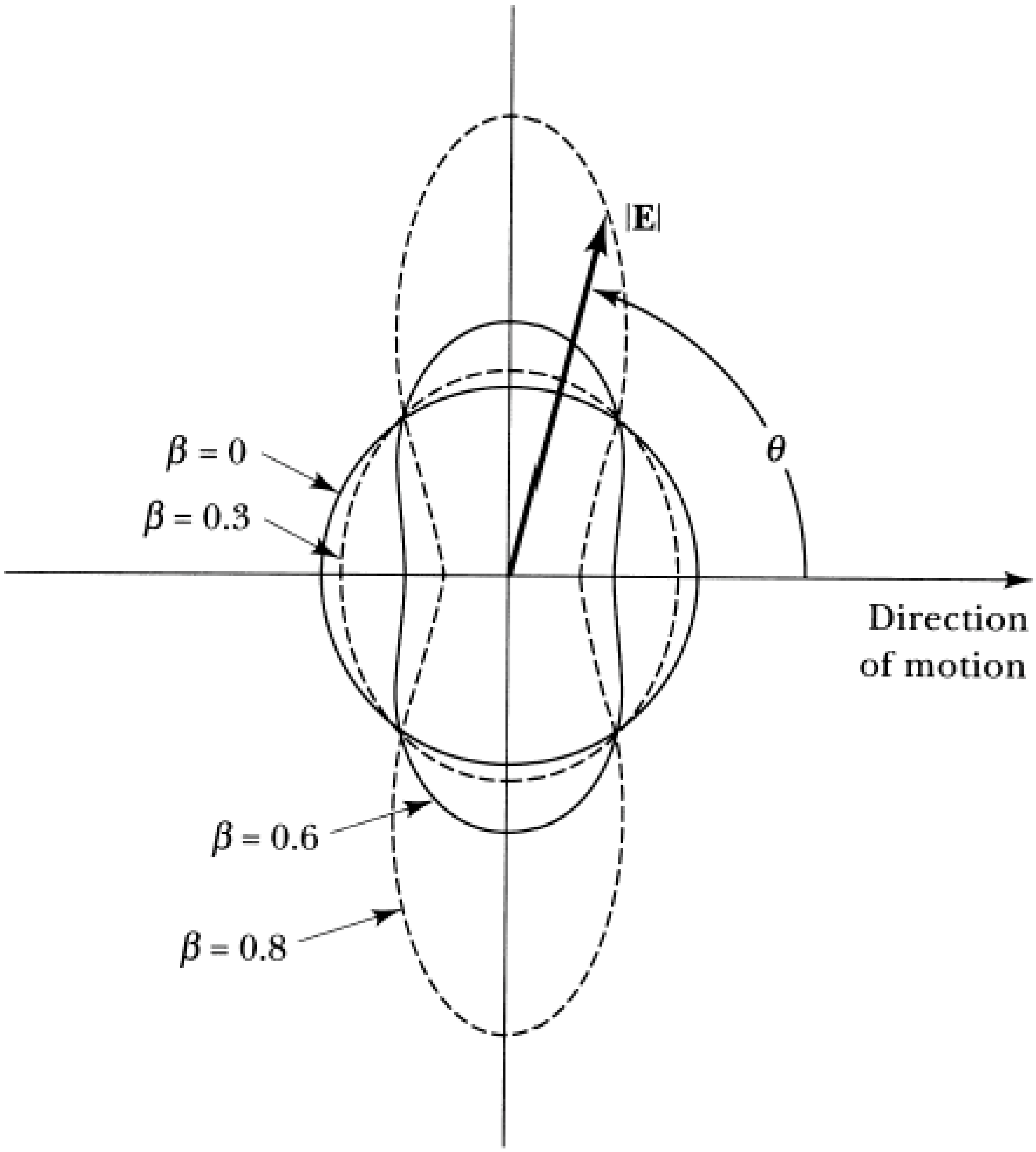


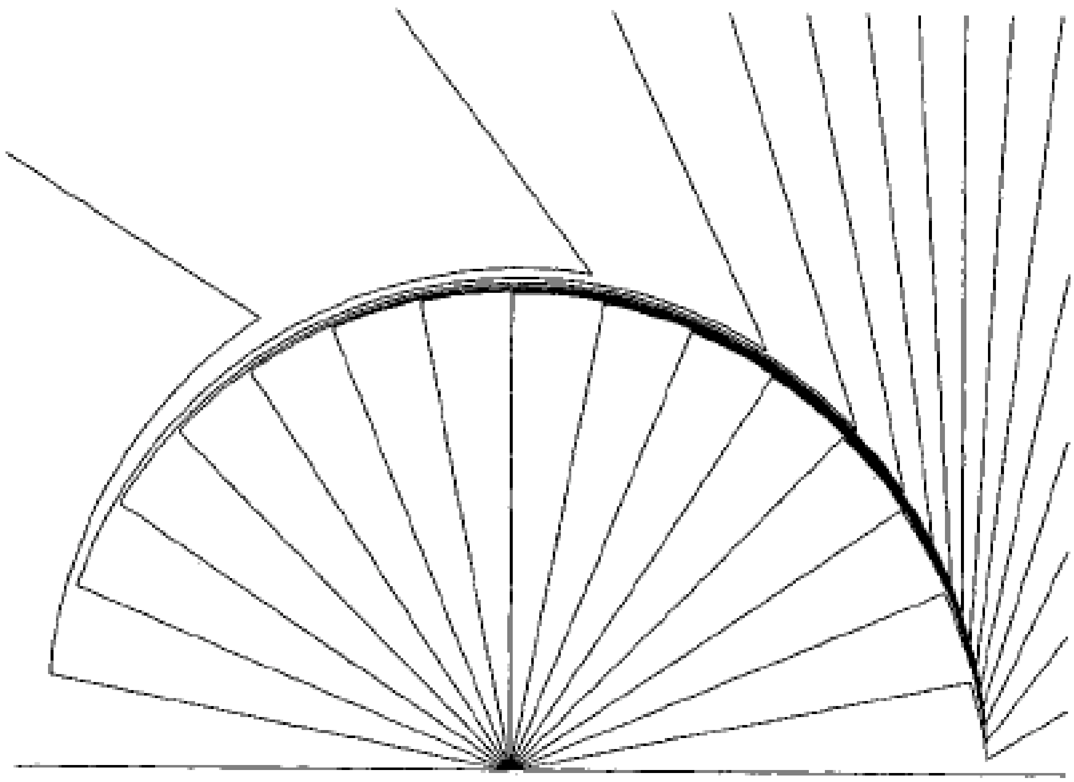
Figure 10: The electric field lines of a point charge.

Compression of the Electric Field Lines



Before $\beta = 0.95$

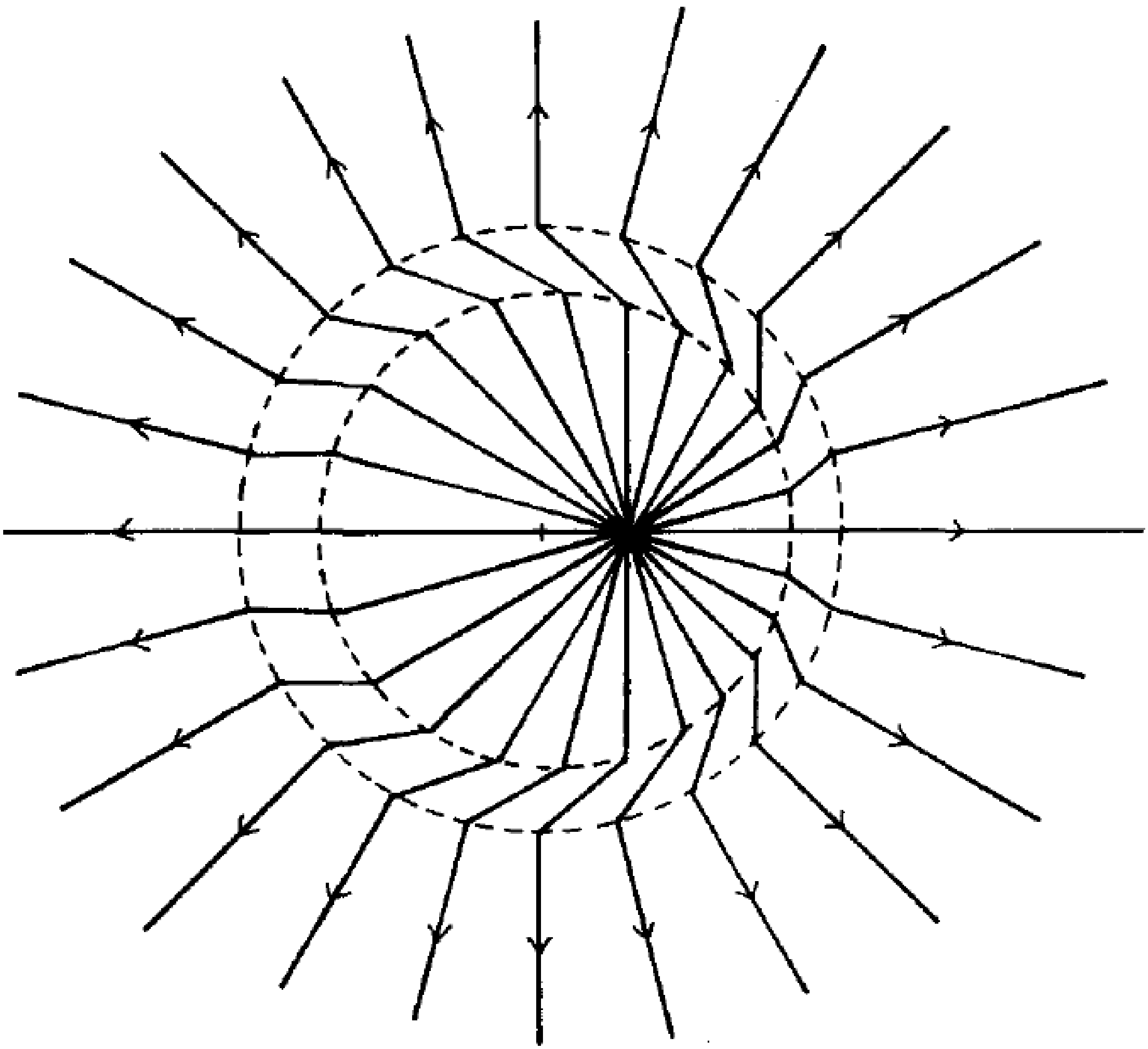
After $\beta = 0$



radial lines ~ r^{-2}

tangential lines ~ r^{-1}

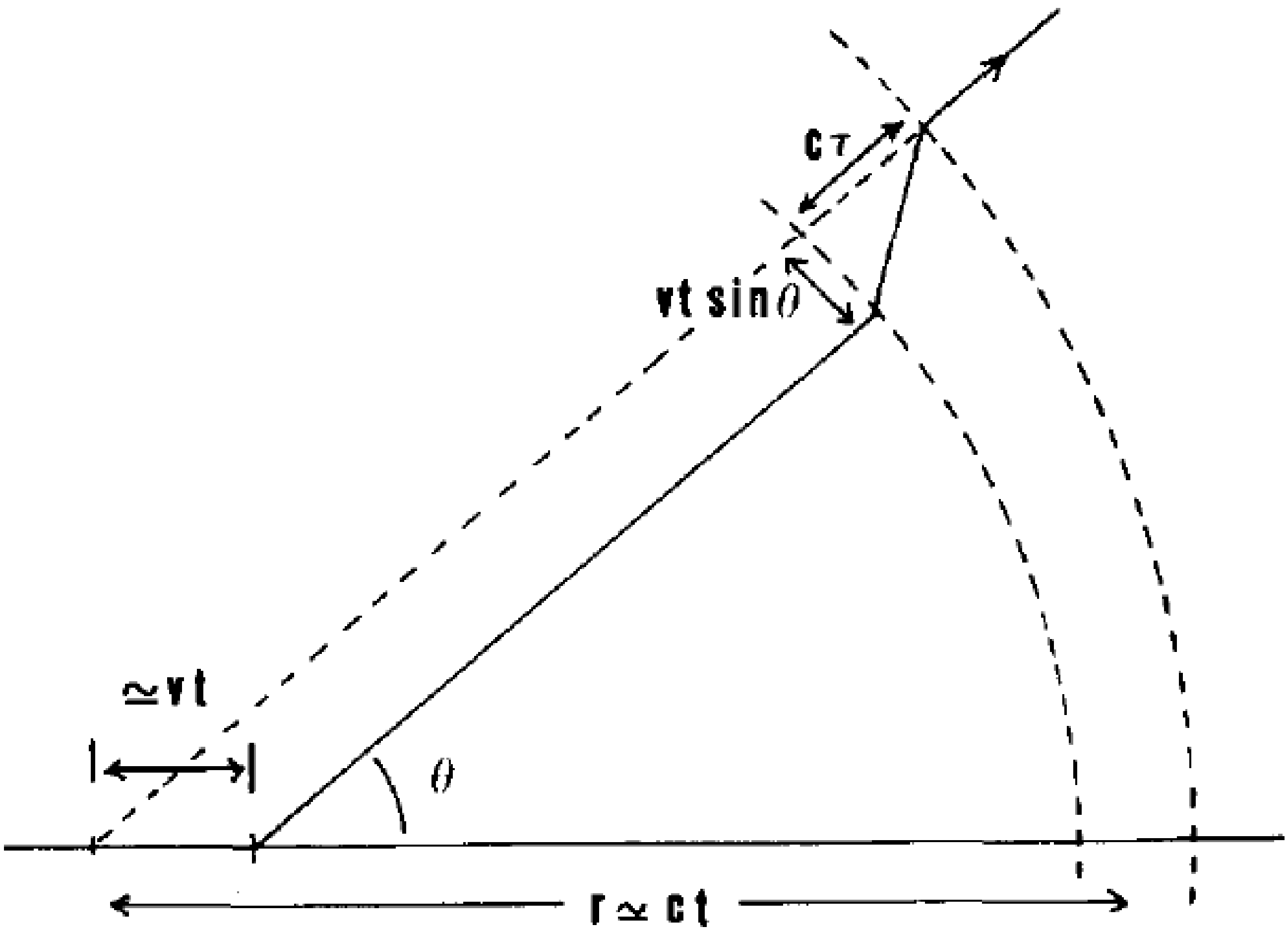
At rest, then acceleration a for time τ, finally at fixed velocity v



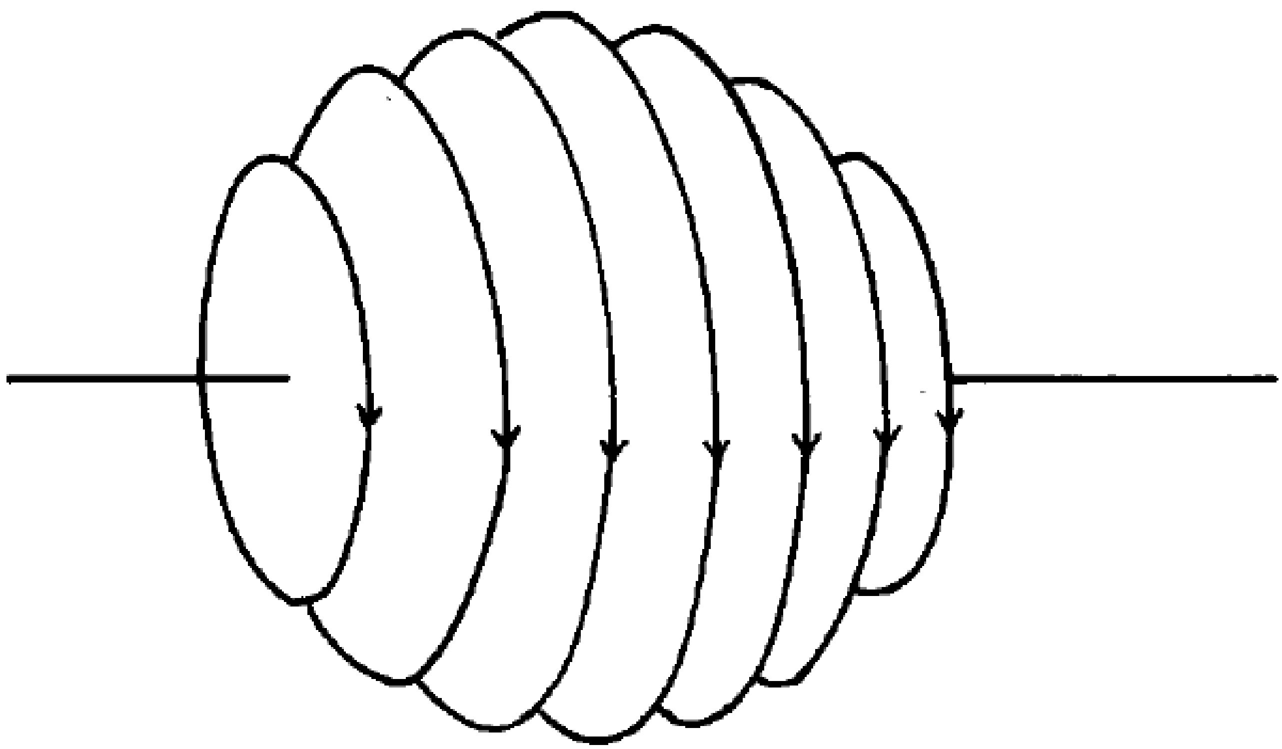
From the Geometry

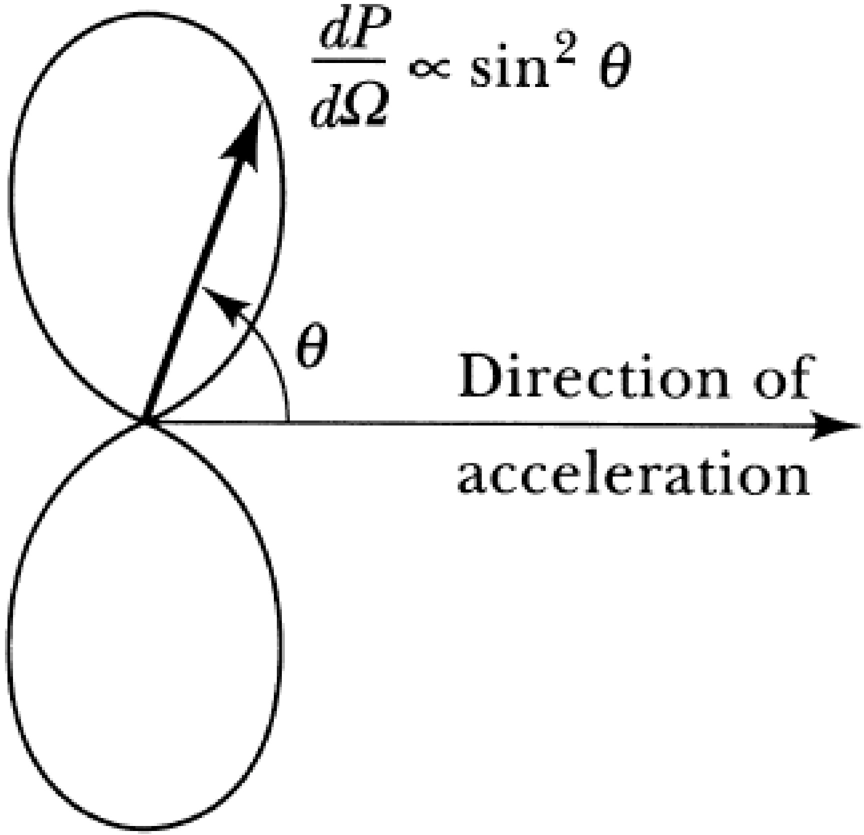
Transverse component = vt sin θ

Radial component = cτ



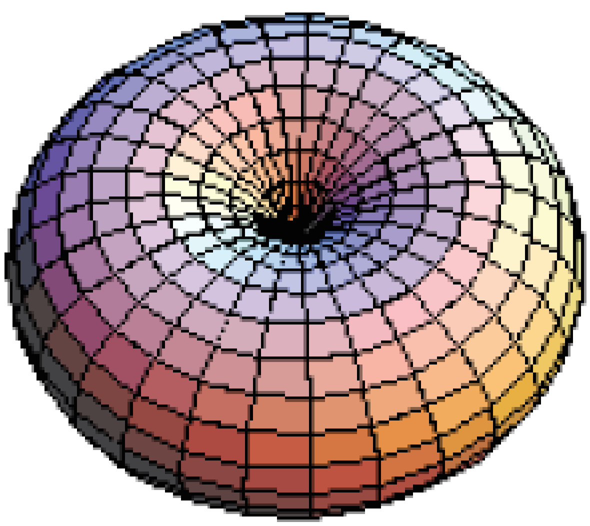
Transverse Magnetic Field





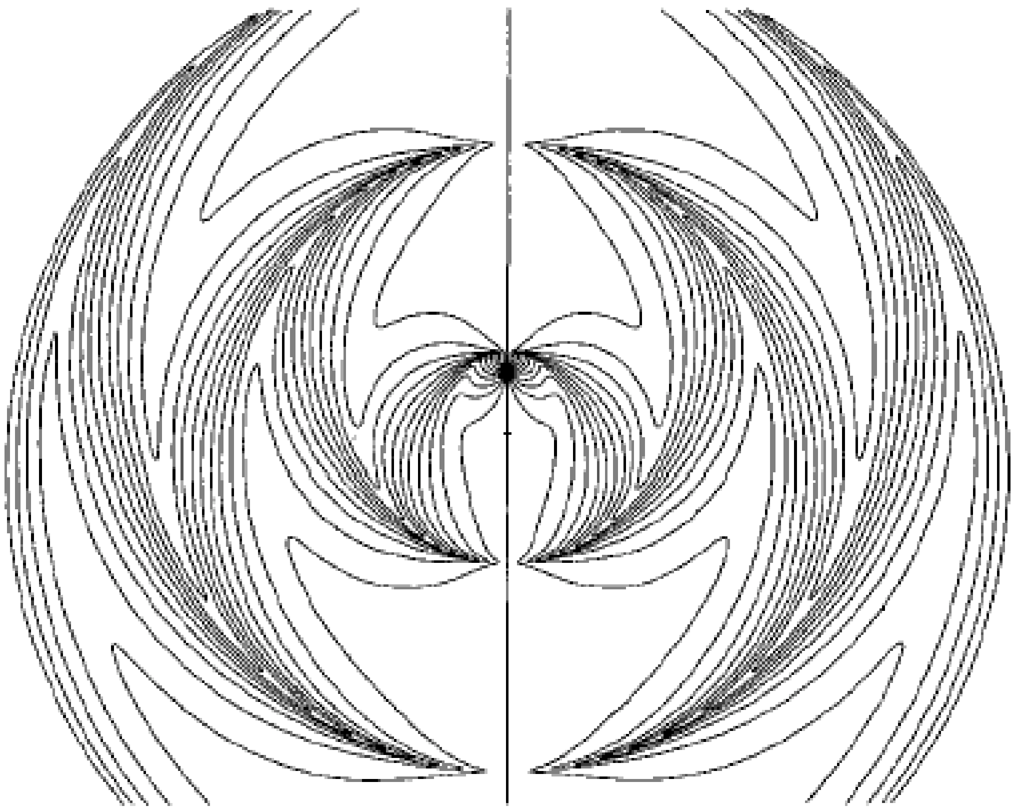
Low Velocity Limit

Non-Relativistic Limit

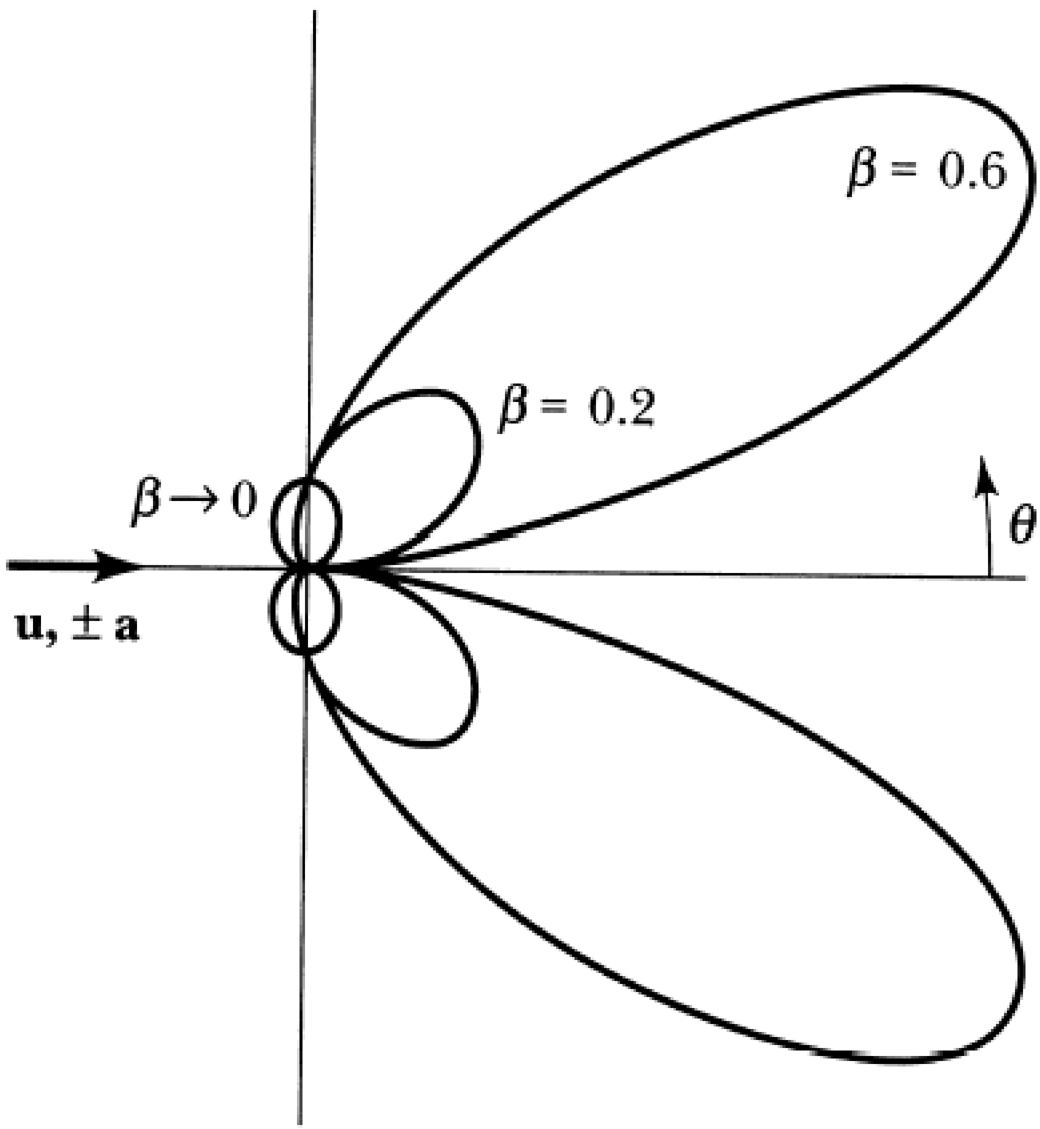


Dipole Pattern

β = 0.90

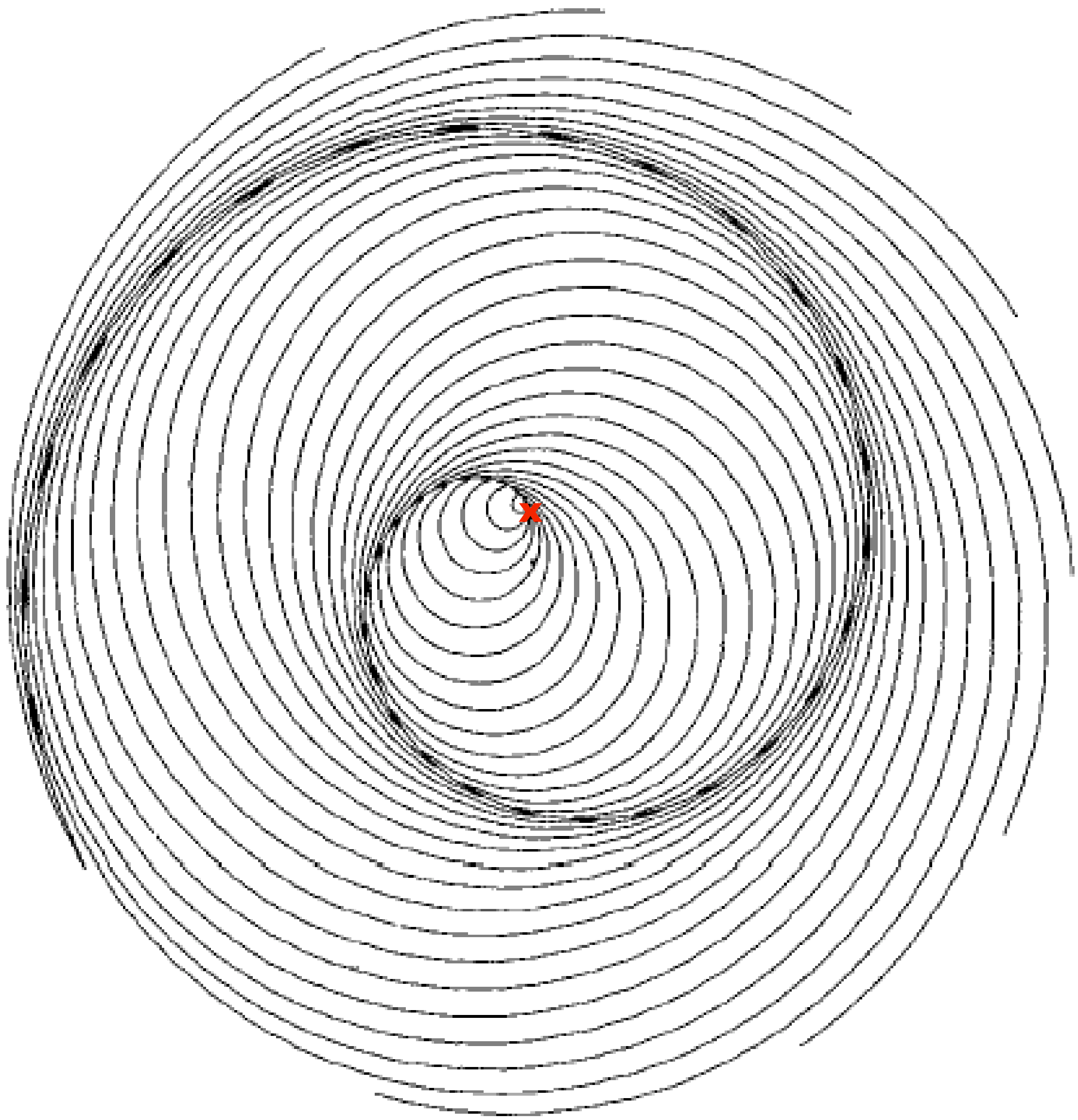


Relativistic with Acceleration Parallel to the Velocity

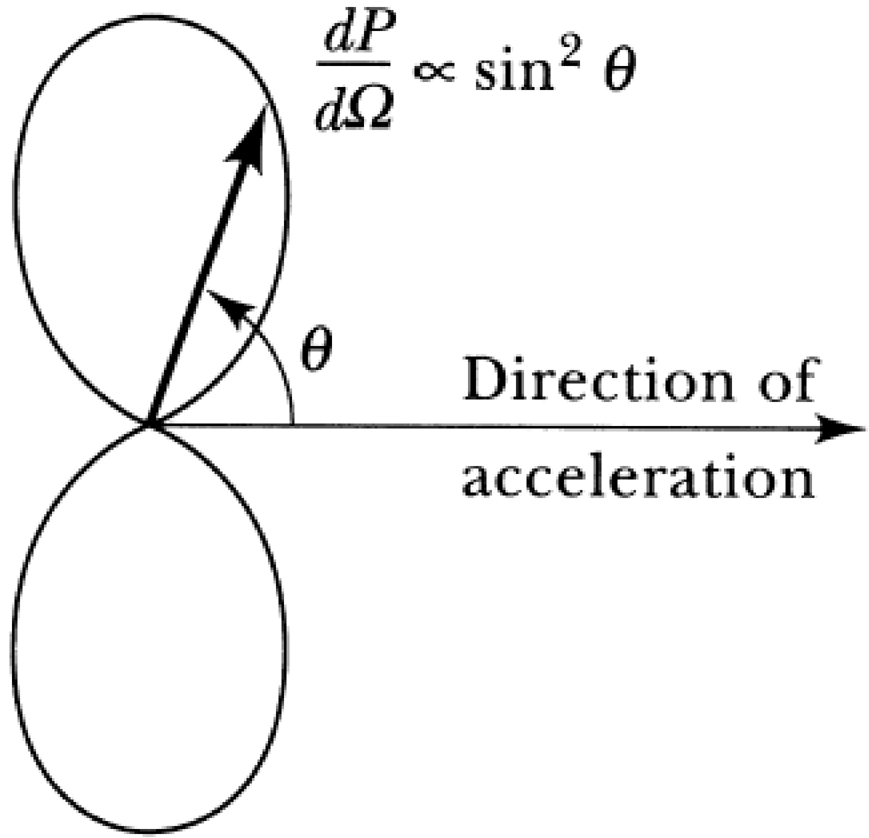


Synchrotron Radiation

β = 0.95

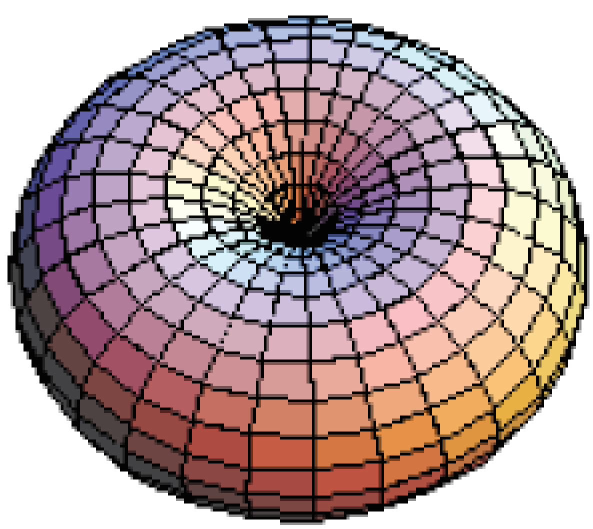


Synchrotrons are Ultra-Relativistic with Acceleration Perpendicular to the Velocity

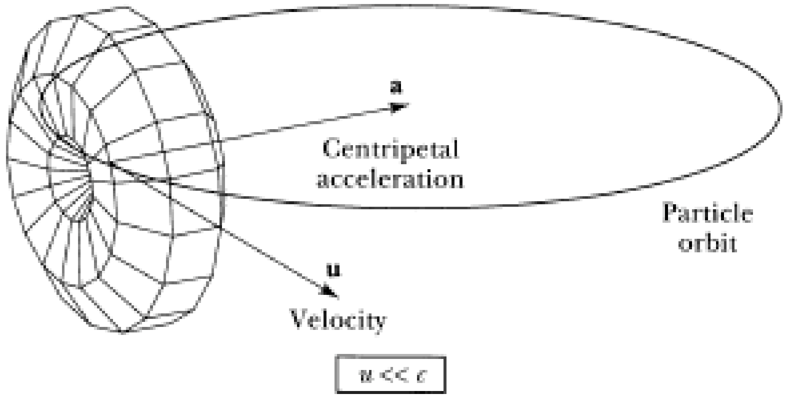


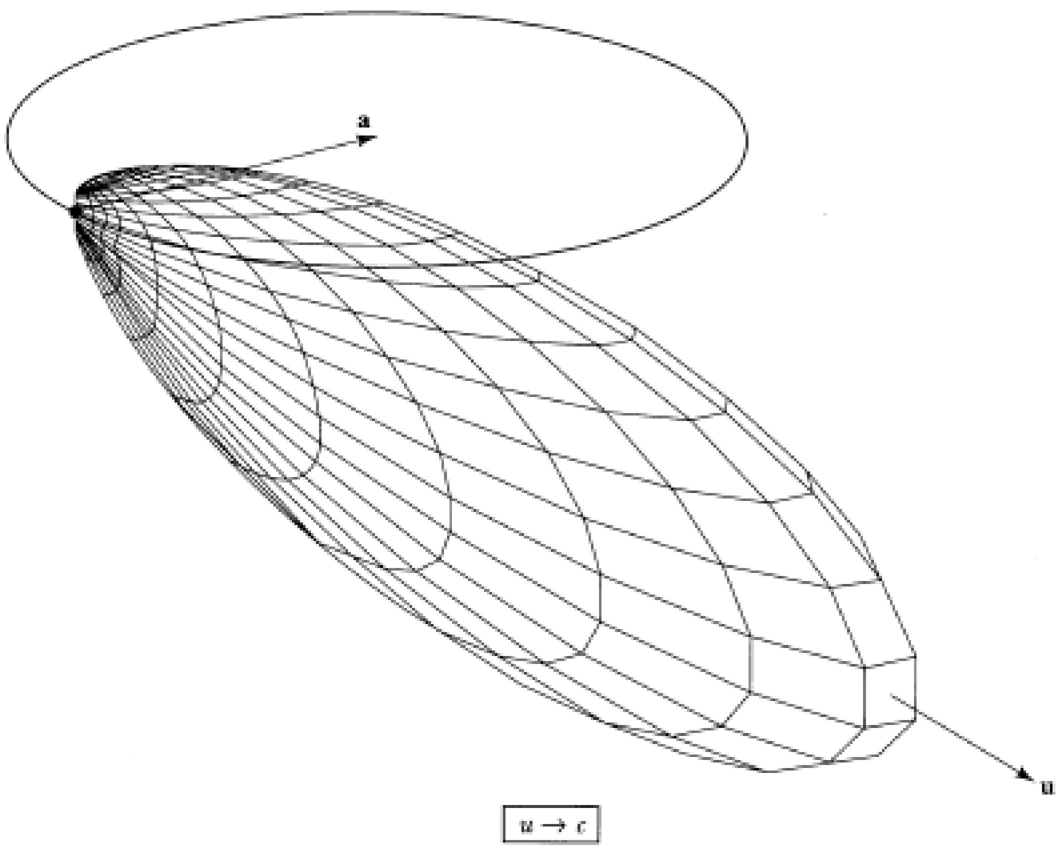
Low Velocity Limit

Non-Relativistic Limit

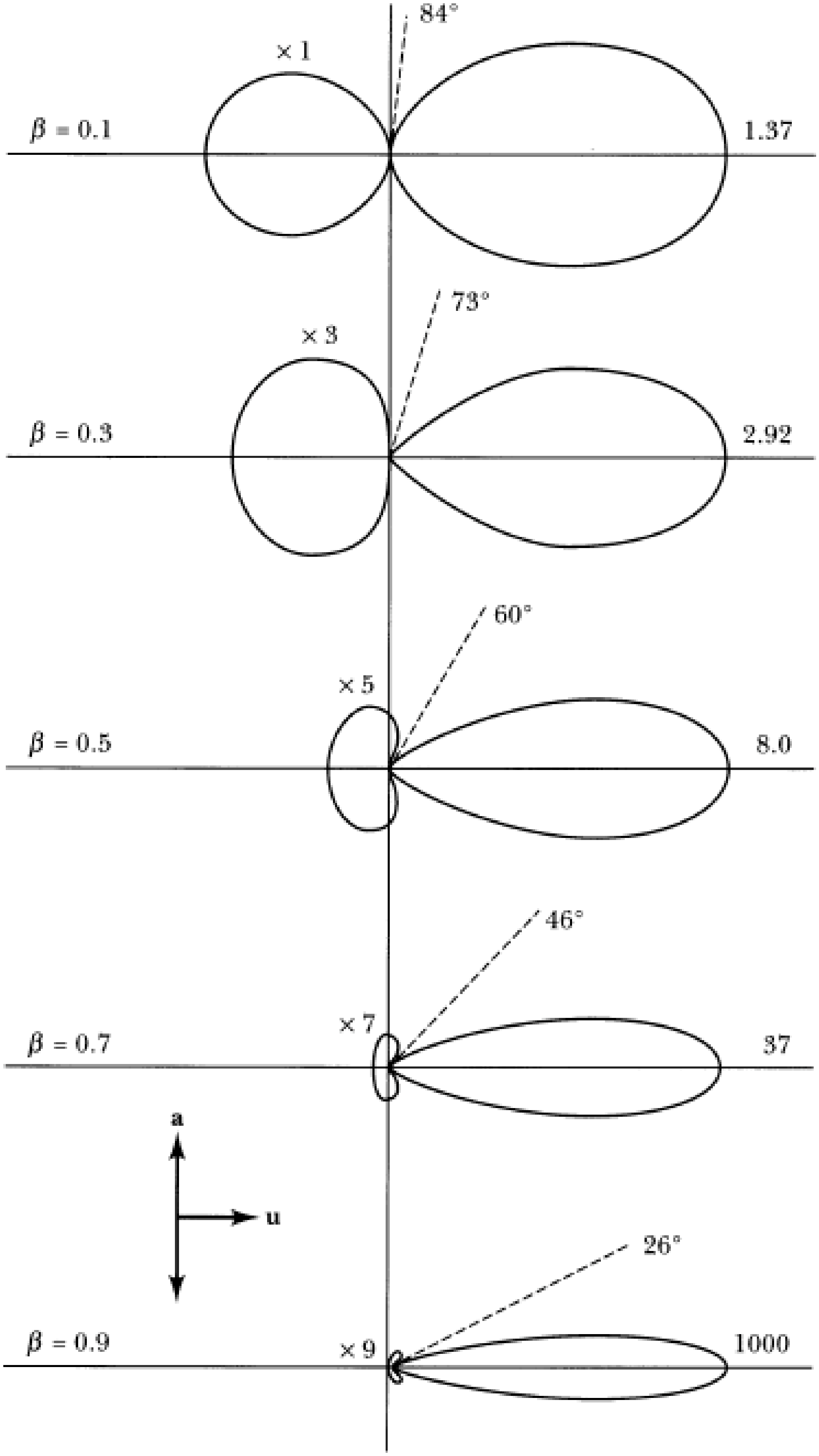


Electron Frame

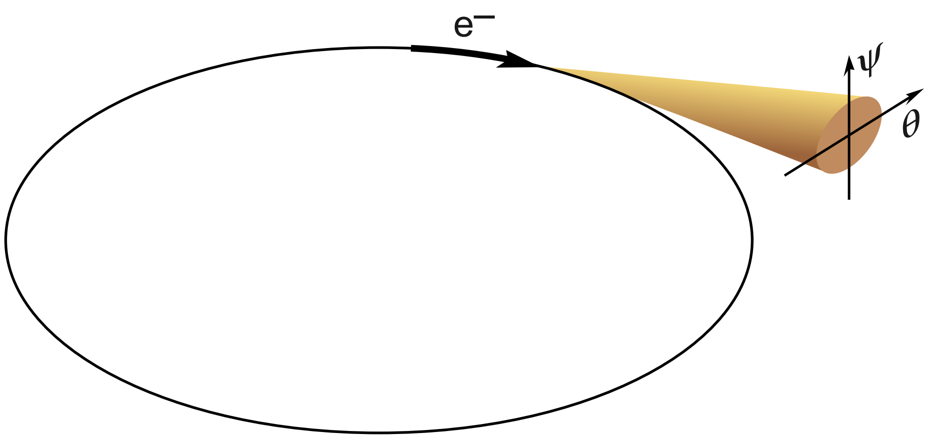


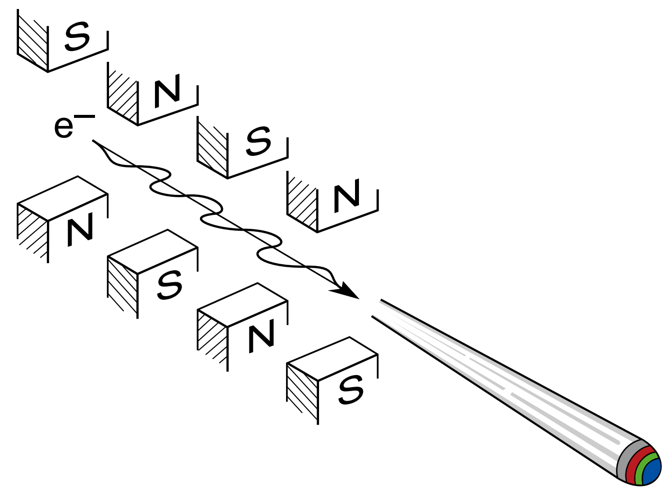


Lab Frame


84°
× 1
β = 0.1
1.37
73°
× 3
β = 0.3
2.92
60°
× 5
β = 0.5
8.0
46°
× 7
β = 0.7
37
a
u
26°
β = 0.9
× 9
1000

Bright and Powerful X-Rays from Relativistic Electrons





Synchrotron radiation

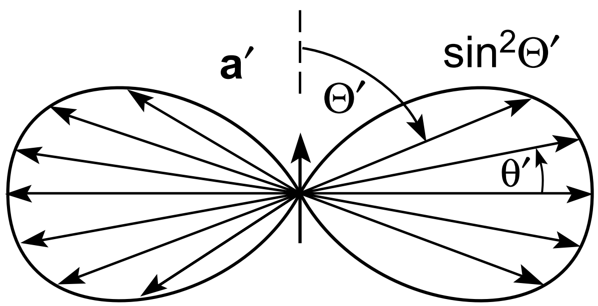
- 10^{10} brighter than the most powerful (compact) laboratory source
- An x-ray "light bulb" in that it radiates all "colors" (wavelengths, photons energies)

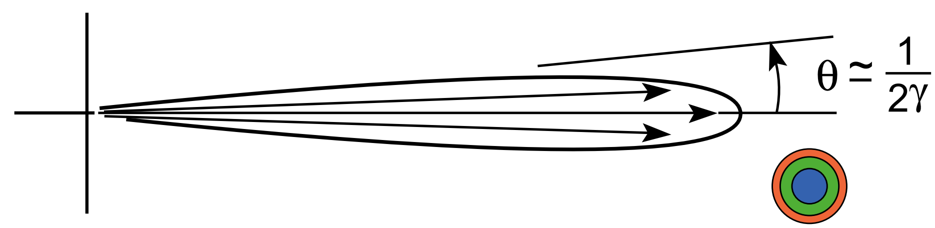
Undulator radiation

- Lasers exist for the IR, visible, UV, VUV, and EUV
- Undulator radiation is quasi-monochromatic and highly directional, approximating many of the desired properties of an x-ray laser

Synchrotron Radiation in a Narrow Forward Cone

Frame moving with electron | **Laboratory frame of reference**



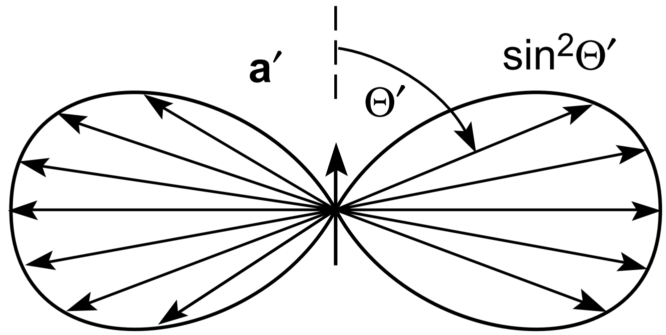


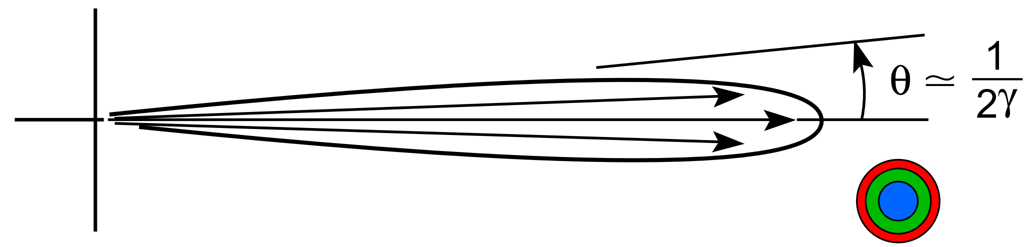
$$\tan\theta = \frac{\sin\theta'}{\gamma(\beta + \cos\theta')} \tag{5.1}$$

$$\theta \simeq \frac{1}{2\gamma} \tag{5.2}$$

Relativistic Electrons Radiate in a Narrow Forward Cone

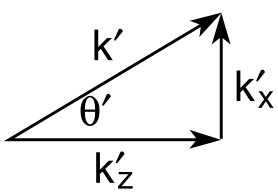
Dipole radiation



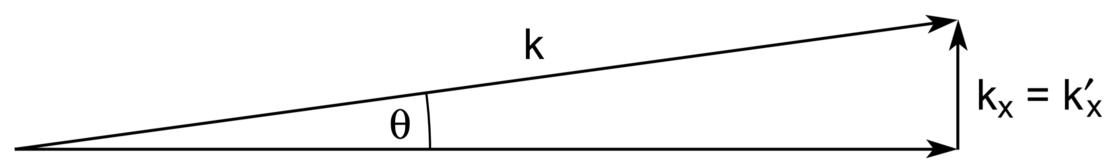


Frame of reference moving with electrons

Laboratory frame of reference



Lorentz transformation

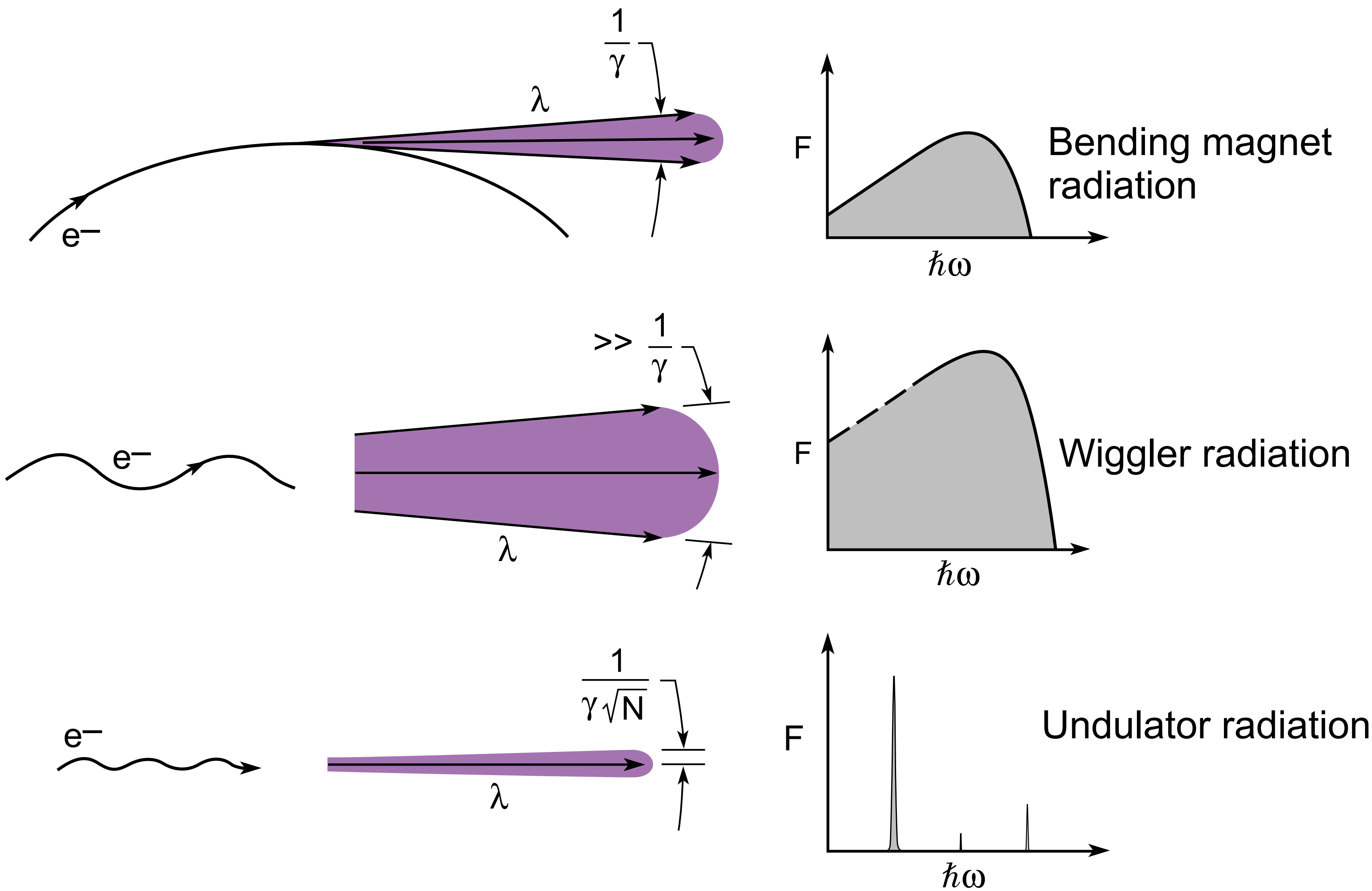


$k' = 2\pi/\lambda'$

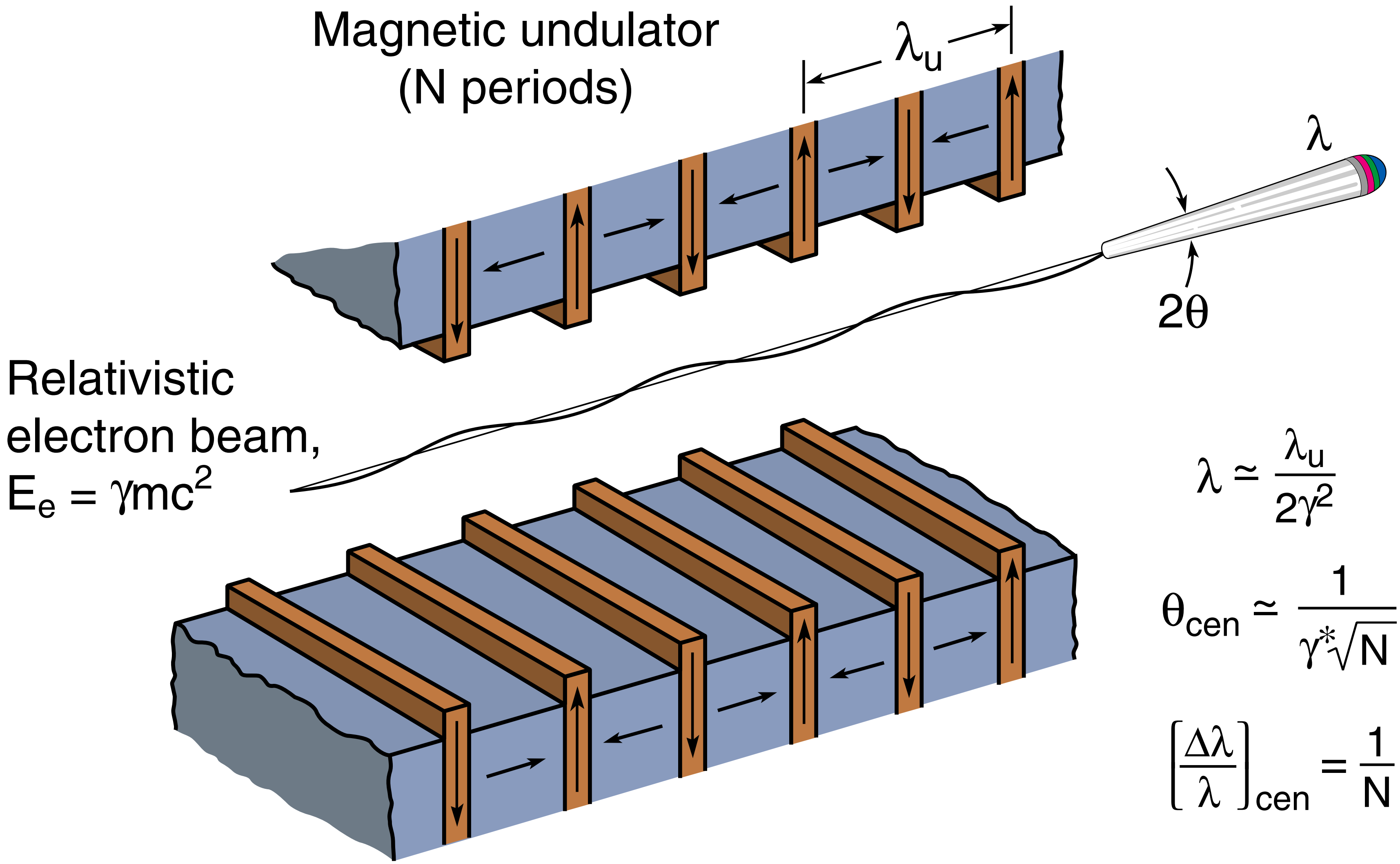
$k_z = 2\gamma k'_z$(Relativistic Doppler shift)

$$\theta \simeq \frac{k_x}{k_z} \simeq \frac{k'_x}{2\gamma k'_z} = \frac{\tan\theta'}{2\gamma} \simeq \frac{1}{2\gamma}$$

Three Forms of Synchrotron Radiation



Narrow Cone Undulator Radiation, Generated by Relativistic Electrons Traversing a Periodic Magnet Structure

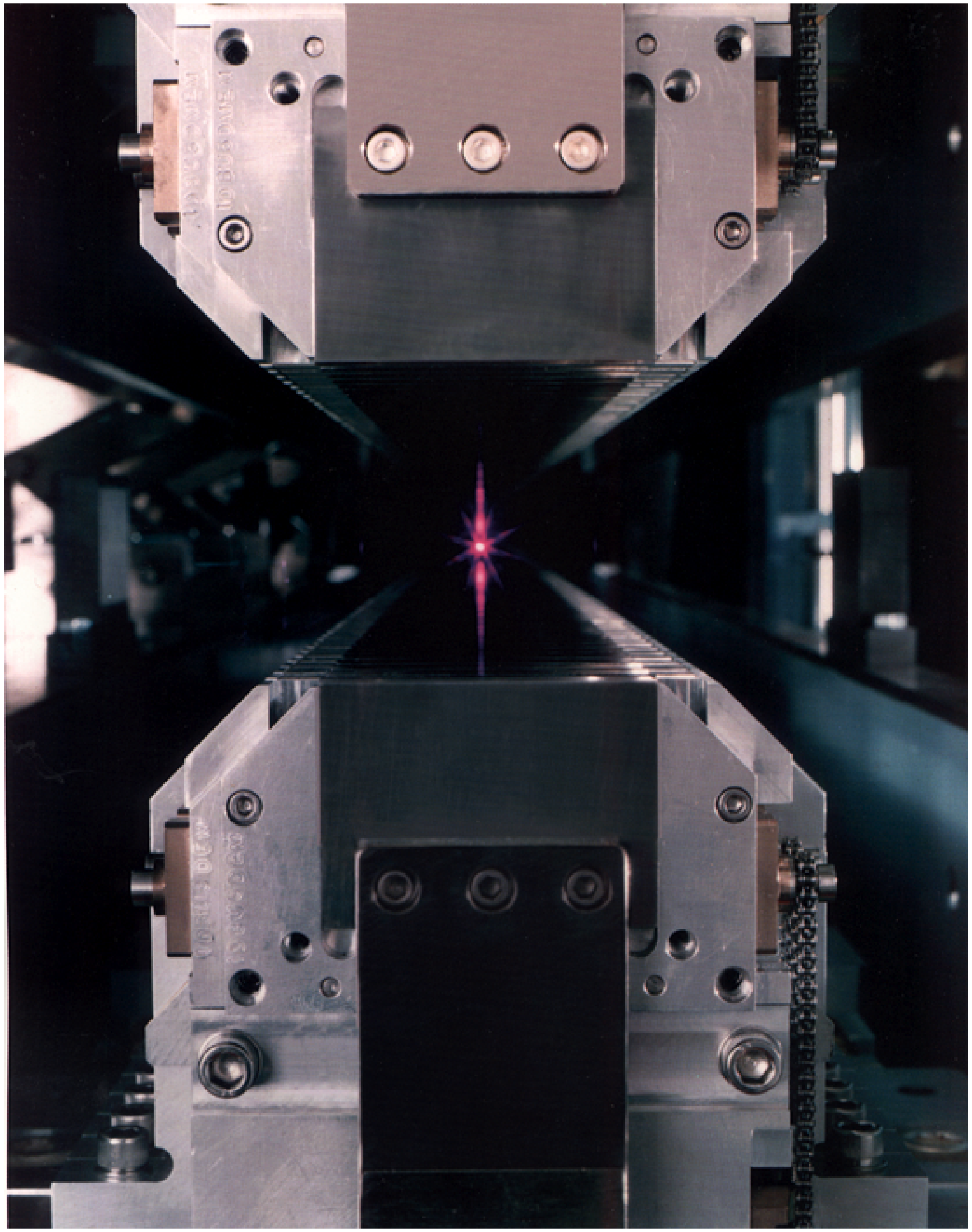


$$\lambda \simeq \frac{\lambda_u}{2\gamma^2}$$

$$\theta_{cen} \simeq \frac{1}{\gamma^* \sqrt{N}}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_{cen} = \frac{1}{N}$$

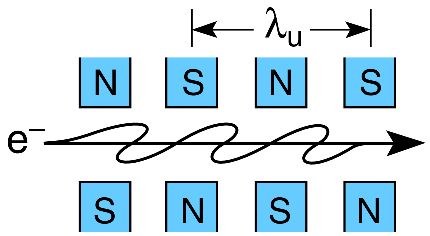
An Undulator Up Close



ALS U5 undulator, beamline 7.0, N = 89, λ_u = 50 mm

Undulator Radiation

Laboratory Frame of Reference

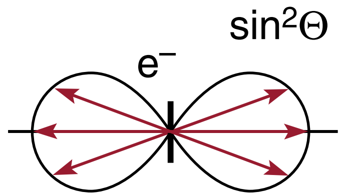


$E = \gamma mc^2$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

N = # periods

Frame of Moving e−



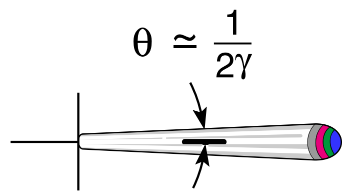
e^- radiates at the Lorentz contracted wavelength:

$$\lambda' = \frac{\lambda_u}{\gamma}$$

Bandwidth:

$$\frac{\lambda'}{\Delta\lambda'} \simeq N$$

Frame of Observer



Doppler shortened wavelength on axis:

$$\lambda = \lambda'\gamma(1 - \beta\cos\theta)$$

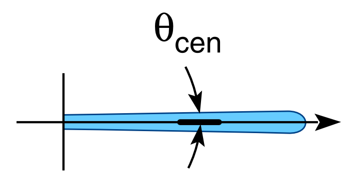
$$\lambda = \frac{\lambda_u}{2\gamma^2}(1 + \gamma^2\theta^2)$$

Accounting for transverse motion due to the periodic magnetic field:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

where $K = eB_0\lambda_u/2\pi mc$

Following Monochromator



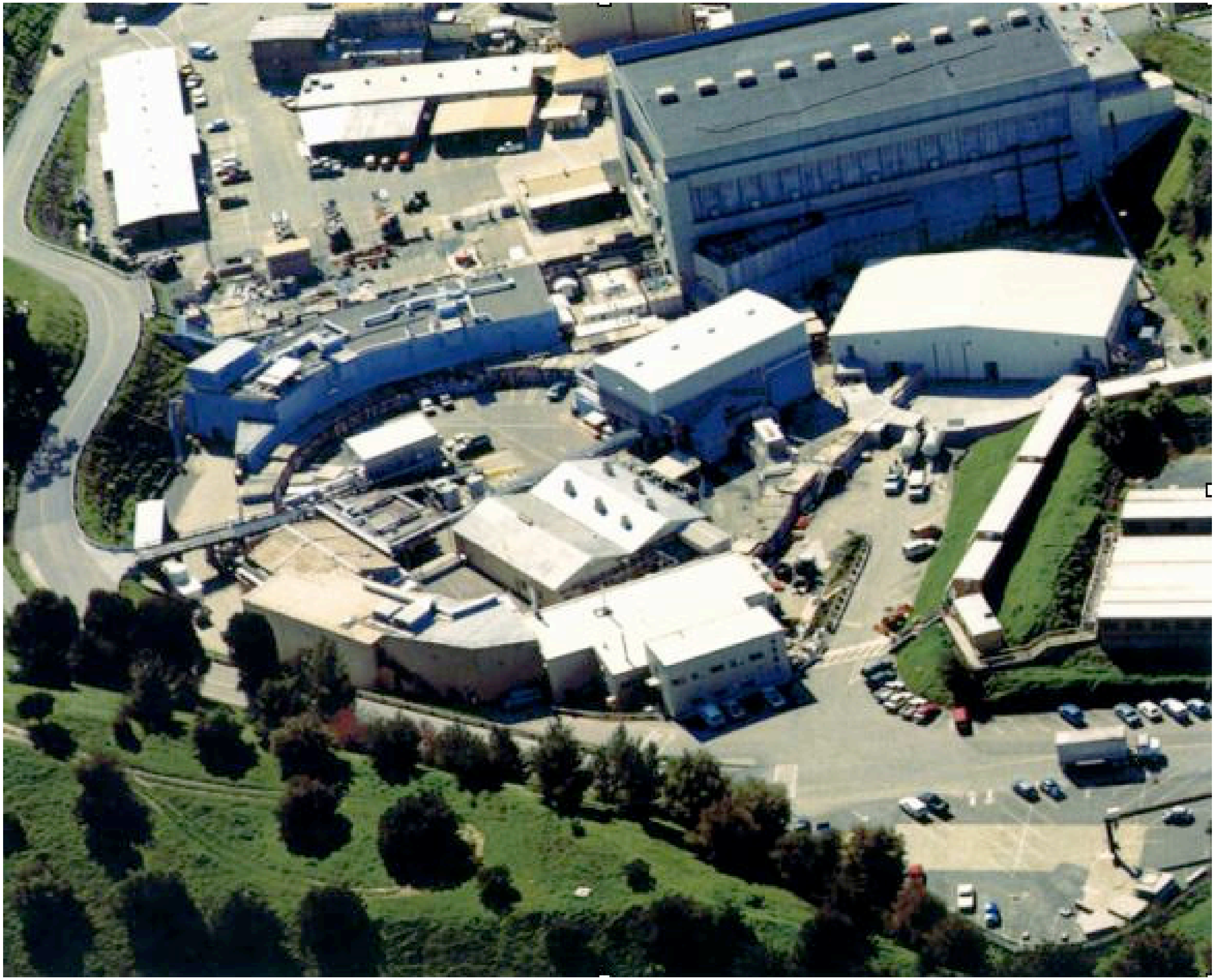
For $\frac{\Delta\lambda}{\lambda} \simeq \frac{1}{N}$

$$\theta_{cen} \simeq \frac{1}{\gamma\sqrt{N}}$$

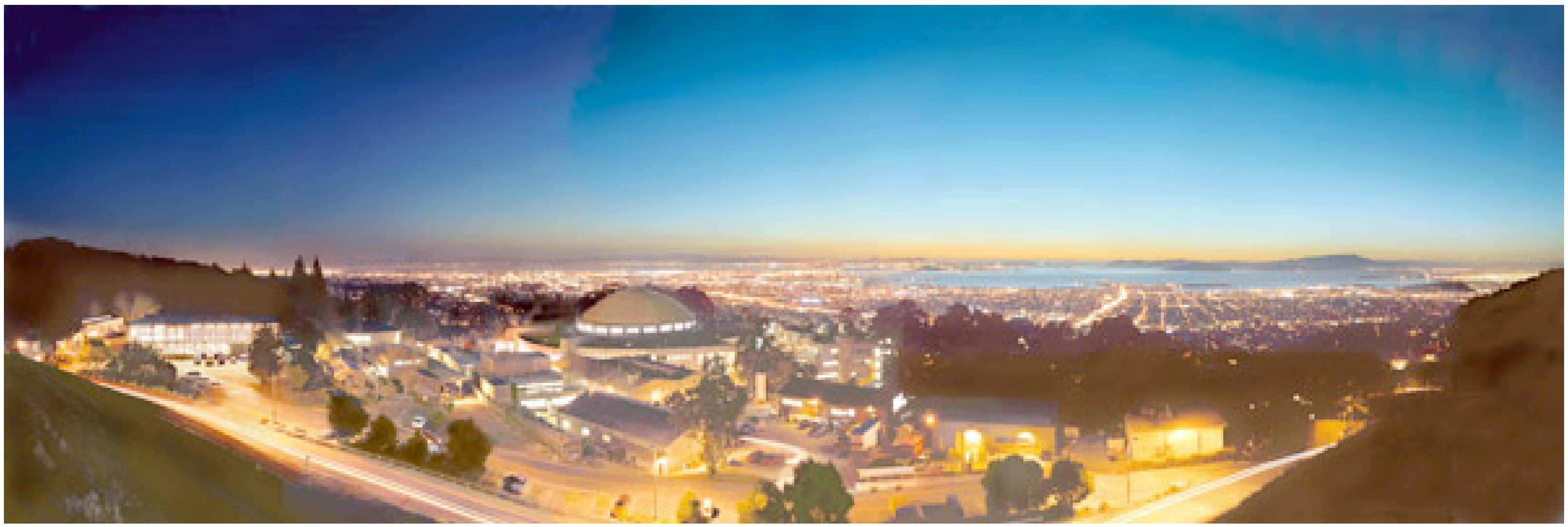
typically

$$\theta_{cen} \simeq 40 \text{ rad}$$

The Synchrotrons that I Have Loved




Argonne Central Campus
Central Lab/Office Bldg.
Argonne Guest House
APS Conference Center
Center for
Nanoscale Materials
Booster/Injector
RF/Extraction Bldg.
Linac
Experiment Hall
Storage Ring
Lab/Office
Module




Damping
Rings
Main Linac
LCLS
Injector
PEP-II
End
Station B
SABER
NLCTA
End
Station A
SPEAR3
Guest
House
LCLS
LCLS
Near Hall
LCLS
Far Hall
CEH
BABAR