VIII. A Dynamical Theory of the Electromagnetic Field

Maxwell 1864

In order to bring these results within the power of symbolical calculation, I then express them in the form of the General Equations of the Electromagnetic Field. These equations express—

(A) The relation between electric displacement, true conduction, and the total current, compounded of both.

(B) The relation between the lines of magnetic force and the inductive coefficients of a circuit, as already deduced from the laws of induction.

(C) The relation between the strength of a current and its magnetic effects, according to the electromagnetic system of measurement.

(D) The value of the electromotive force in a body, as arising from the motion of the body in the field, the alteration of the field itself, and the variation of electric potential from one part of the field to another.

(E) The relation between electric displacement, and the electromotive force which produces it.

(F) The relation between an electric current, and the electromotive force which produces it.

(G) The relation between the amount of free electricity at any point, and the electric displacements in the neighbourhood.

(H) The relation between the increase or diminution of free electricity and the electric currents in the neighbourhood.

There are twenty of these equations in all, involving twenty variable quantities.

(70) In these equations of the electromagnetic field we have assumed twenty variable

quantities, namely,

For Electromagnetic Momentum	F	G	H
,, Magnetic Intensity	α	β	γ
,, Electromotive Force	P	Q	R
,, Current due to true conduction	p	q	r
,, Electric Displacement	f	g	h
,, Total Current (including variation of displacement)	p'	q'	r'
,, Quantity of free Electricity	e		
,, Electric Potential	Ψ		

Between these twenty quantities we have found twenty equations, viz.

Three equations of Magnetic Force	(B)
,, Electric Currents	(C)
,, Electromotive Force	(D)
,, Electric Elasticity	(E)
,, Electric Resistance	(F)
,, Total Currents	(A)
One equation of Free Electricity	(G)
,, Continuity	(H)

These equations are therefore sufficient to determine all the quantities which occur in them, provided we know the conditions of the problem. In many questions, however, only a few of the equations are required.

Maxwell's Equations

The Original Equations

With the knowledge of fluid mechanics MAXWELL[15] has introduced the following eight equations to the electromagnetic fields (the right equations correspond with the original text, the left equations correspond with today's vector notation):

$$\left.\begin{aligned} p' &= p + \frac{d f}{dt} \\ q' &= q + \frac{d g}{dt} \\ r' &= r + \frac{d h}{dt} \end{aligned}\right\} \rightarrow \left.\begin{aligned} J_1 &= j_1 + \frac{\partial D_1}{\partial t} \\ J_2 &= j_2 + \frac{\partial D_2}{\partial t} \\ J_3 &= j_3 + \frac{\partial D_3}{\partial t} \end{aligned}\right\} \Rightarrow \mathbf{J} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.1}$$

$$\left.\begin{aligned} \mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz} \\ \mu\beta &= \frac{dF}{dz} - \frac{dH}{dx} \\ \mu\gamma &= \frac{dG}{dx} - \frac{dF}{dy} \end{aligned}\right\} \rightarrow \left.\begin{aligned} \mu H_1 &= \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \mu H_2 &= \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \mu H_3 &= \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{aligned}\right\} \Rightarrow \mu\mathbf{H} = \nabla \times \mathbf{A} \tag{1.2}$$

$$\left.\begin{aligned} \frac{d\gamma}{dy} - \frac{d\beta}{dz} &= 4\pi p' \\ \frac{d\alpha}{dz} - \frac{d\gamma}{dx} &= 4\pi q' \\ \frac{d\beta}{dx} - \frac{d\alpha}{dy} &= 4\pi r' \end{aligned}\right\} \rightarrow \left.\begin{aligned} \frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} &= 4\pi J_1 \\ \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} &= 4\pi J_2 \\ \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} &= 4\pi J_3 \end{aligned}\right\} \Rightarrow \nabla \times \mathbf{H} = \mathbf{J} \tag{1.3}$$

$$\left.\begin{aligned} P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dx} \\ Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\Psi}{dy} \\ R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\Psi}{dz} \end{aligned}\right\} \rightarrow \left.\begin{aligned} E_1 &= \mu\left(H_3 v_2 - H_2 v_3\right) - \frac{dA_1}{dt} - \frac{d\varphi}{dx} \\ E_2 &= \mu\left(H_1 v_3 - H_3 v_1\right) - \frac{dA_2}{dt} - \frac{d\varphi}{dy} \\ E_3 &= \mu\left(H_2 v_1 - H_1 v_2\right) - \frac{dA_3}{dt} - \frac{d\varphi}{dz} \end{aligned}\right\} \tag{1.4}$$

$$\Rightarrow \mathbf{E} = \mu\left(\mathbf{v} \times \mathbf{H}\right) - \frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi$$

$$\left.\begin{aligned} P &= k f \\ Q &= k g \\ R &= k h \end{aligned}\right\} \rightarrow \left.\begin{aligned} \varepsilon E_1 &= D_1 \\ \varepsilon E_2 &= D_2 \\ \varepsilon E_3 &= D_3 \end{aligned}\right\} \Rightarrow \varepsilon\mathbf{E} = \mathbf{D} \tag{1.5}$$

$$\left.\begin{aligned} P &= -\zeta p \\ Q &= -\zeta q \\ R &= -\zeta r \end{aligned}\right\} \rightarrow \left.\begin{aligned} \sigma E_1 &= j_1 \\ \sigma E_2 &= j_2 \\ \sigma E_3 &= j_3 \end{aligned}\right\} \Rightarrow \sigma\mathbf{E} = \mathbf{j} \tag{1.6}$$

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \quad \rightarrow \quad \rho + \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} = 0 \quad \Rightarrow \quad -\rho = \nabla \bullet \mathbf{D} \tag{1.7}$$

$$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial j_1}{\partial x} + \frac{\partial j_2}{\partial y} + \frac{\partial j_3}{\partial z} = 0 \quad \Rightarrow \quad -\frac{\partial \rho}{\partial t} = \nabla \bullet \mathbf{j} \tag{1.8}$$

This original equations do not strictly correspond to today's vector equations. The original equations, for example, contains the vector potential **A**, which today usually is eliminated.

Three Maxwell equations can be found quickly in the original set, together with OHM's law (1.6), the FARADAY-force (1.4) and the continuity equation (1.8) for a region containing charges.

The Original Quaternion Form of Maxwell's Equations

In his Treatise[16] of 1873 MAXWELL has already modified his original equations of 1865. In addition Maxwell tried to introduce the quaternion notation by writing down his results also in a quaternion form. However, he has never really calculated with quaternions but only uses either the scalar or the vector part of a quaternion in his equations.

A general quaternion has a scalar (real) and a vector (imaginary) part. In the example below ‚a' is the scalar part and 'ib + jc + kd' is the vector part.

$$\mathbb{Q} = a + ib + jc + kd$$

Here a, b, c and d are real numbers and i, j, k are the so-called HAMILTON'ian[7] unit vectors with the magnitude of √-1. They fulfill the equations

$$i^2 = j^2 = k^2 = ijk = -1$$

and

$$ij = k \qquad jk = i \qquad ki = j$$
$$ij = -ji \qquad jk = -kj \qquad ki = -ik$$

A nice presentation about the rotation capabilities of the HAMILTON'ian unit vectors in a three-dimensional ARGAND diagram was published by GOUGH[6].

Now MAXWELL has defined the field vectors (for example $\mathbf{B} = B_1 i + B_2 j + B_3 k$) as quaternions without scalar part and scalars as quaternions without vector part. In addition he defined a quaternion operator without scalar part

$$\nabla = \frac{d}{dx_1} i + \frac{d}{dx_2} j + \frac{d}{dx_3} k \quad ,$$

which he used in his equations. Maxwell devided a single quaternion with two prefixes into a scalar and vector. This prefixes he defined according to

$$S.\mathbb{Q} = S.(a + ib + jc + kd) = a$$

$$V.\mathbb{Q} = V.(a + ib + jc + kd) = ib + jc + kd$$

The original Maxwell quaternion equations are now for isotrope media (no changes except fonts, normal letter = scalar, capital letter = quaternion without scalar):

Table 15–1

FALSE IN GENERAL (true only for statics)	TRUE ALWAYS
$F = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r^2}$ (Coulomb's law)	$F = q(E + v \times B)$ (Lorentz force) $\rightarrow \nabla \cdot E = \dfrac{\rho}{\epsilon_0}$ (Gauss' law)
$\nabla \times E = 0$ $E = -\nabla\phi$ $E(1) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\rho(2)e_{12}}{r_{12}^2}\, dV_2$ For conductors, $E = 0$, $\phi =$ constant. $Q = CV$	$\rightarrow \nabla \times E = -\dfrac{\partial B}{\partial t}$ (Faraday's law) $E = -\nabla\phi - \dfrac{\partial A}{\partial t}$ In a conductor, E makes currents.
$c^2\nabla \times B = \dfrac{j}{\epsilon_0}$ (Ampere's law) $B(1) = \dfrac{1}{4\pi\epsilon_0 c^2}\displaystyle\int \dfrac{j(2) \times e_{12}}{r_{12}^2}\, dV_2$	$\rightarrow \nabla \cdot B = 0$ (No magnetic charges) $B = \nabla \times A$ $\rightarrow c^2\nabla \times B = \dfrac{j}{\epsilon_0} + \dfrac{\partial E}{\partial t}$
$\nabla^2\phi = -\dfrac{\rho}{\epsilon_0}$ (Poisson's equation) $\nabla^2 A = -\dfrac{j}{\epsilon_0 c^2}$ with $\nabla \cdot A = 0$	$\nabla^2\phi - \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0}$ and $\nabla^2 A - \dfrac{1}{c^2}\dfrac{\partial^2 A}{\partial t^2} = -\dfrac{j}{\epsilon_0 c^2}$ with $c^2\nabla \cdot A + \dfrac{\partial\phi}{\partial t} = 0$
$\phi(1) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(2)}{r_{12}}\, dV_2$ $A(1) = \dfrac{1}{4\pi\epsilon_0 c^2}\displaystyle\int \dfrac{j(2)}{r_{12}}\, dV_2$	$\phi(1, t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(2, t')}{r_{12}}\, dV_2$ and $A(1, t) \pm \dfrac{1}{4\pi\epsilon_0 c^2}\displaystyle\int \dfrac{j(2, t')}{r_{12}}\, dV_2$ with $t' = t - \dfrac{r_{12}}{c}$
$U = \frac{1}{2}\displaystyle\int \rho\phi\, dV + \frac{1}{2}\int j \cdot A\, dV$	$U = \displaystyle\int \left(\dfrac{\epsilon_0}{2} E \cdot E + \dfrac{\epsilon_0 c^2}{2} B \cdot B\right) dV$

The equations marked by an arrow ($\rightarrow$) are Maxwell's equations.

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\[2ex] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2ex] \nabla \cdot \mathbf{B} = 0 \\[2ex] \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\[2ex] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2ex] \nabla \cdot \mathbf{B} = 0 \\[2ex] \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\[2ex] \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\[2ex] \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Feynman: These Eight Equations Contain All of Classical Physics

(1-4) Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \, \partial \mathbf{E}/\partial t$$

(5) The Conservation of Charge

$$\nabla \cdot \mathbf{J} = -\partial \rho/\partial t$$

(6) The Lorentz Force Law

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

(7) The Law of Motion

$$d\mathbf{p}/dt = \mathbf{F} \text{ where } \mathbf{p} = \gamma m \mathbf{v}$$

(8) Newton's Law of Gravitation

$$\mathbf{F} = -G m_1 m_2 \, \hat{\mathbf{r}}/r^2$$

Maxwell's Equations

Two vector fields, **E** and **B**

Two Divergence Equations

Gauss' Law for Electricity

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

Gauss' Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0$$

Two Curl Equations

Faraday's Law of Induction

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$

Ampere's Law with Maxwell's Extension

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \mathbf{J_d}$$

Maxwell called $\mathbf{J_d}$ the displacement current

For static fields, the four Maxwell Equations are the best formulation

$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$

$\nabla \cdot \mathbf{B} = 0$

$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$

Maxwell's equations are four coupled first-order differential equations
The sources are ρ, $\mathbf{J}$, and each other

For dynamic fields, the wave equations are the best formulation
Decoupling the four Maxwell Equations produces two second-order differential equations

$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \, \partial^2 \mathbf{E}/\partial^2 t = (1/c^2) \, \partial^2 \mathbf{E}/\partial^2 t$

$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \, \partial^2 \mathbf{B}/\partial^2 t = (1/c^2) \, \partial^2 \mathbf{B}/\partial^2 t$

These two wave equations have six degrees-of-freedom, three for E and three for B. They apply in regions where there are no sources.

The vector potential and the scalar potential obey the same wave equation

$$\nabla^2 \mathbf{A} = \partial^2 \mathbf{A} / \partial^2 t$$
$$\nabla^2 \phi = \partial^2 \phi / \partial^2 t$$

These two wave equations have four degrees-of-freedom, one for ϕ and three for A

As we will see later, the vector potential and the scalar potential are the components of the four-potential A_μ

$$A_\mu = (\phi, A_x, A_y, A_z)$$

In the radiation gauge,

$$\partial \phi / \partial t + \nabla \cdot \mathbf{A} = 0$$

the four-potential A_μ obeys the same wave equation

$$[\ \nabla^2 - \partial^2 / \partial^2 t\ ]\ \phi = 0$$

$$[\ \nabla^2 - \partial^2 / \partial^2 t\ ]\ Ax = 0$$

$$[\ \nabla^2 - \partial^2 / \partial^2 t\ ]\ Ay = 0$$

$$[\ \nabla^2 - \partial^2 / \partial^2 t\ ]\ Az = 0$$

Using Feynman's four-vector notation, Maxwell's equations, in the radiation gauge, are given by

$$\nabla_\mu \nabla_\mu\ A_\mu = j_\mu / \epsilon_0$$

$$\nabla_\mu\ j_\mu = 0$$

Note that $\nabla_\mu \nabla_\mu$ becomes $[\ \partial^2 / \partial^2 t, -\nabla^2\ ]$ but ∇_μ becomes $[\ \partial / \partial t, \nabla\ ]$

Using modern notation, Maxwell's equations, in the radiation gauge, are given by

$$\partial_\mu \partial^\mu\ A^\nu = 4\pi\ j^\nu$$

$$\partial_\nu j^\nu = 0$$

TABLE 3.1 ANALOGY BETWEEN THE ELECTROMAGNETIC AND GRAVITATIONAL FIELD THEORIES

	Electromagnetism	Gravitation (Linear Approximation)
Source of field	j^ν	$T^{\mu\nu}$
Conservation law	$\partial_\nu j^\nu = 0$	$\partial_\nu T^{\mu\nu} = 0$
Field	A^ν	$h^{\mu\nu}$
Field equation	$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 4\pi j^\nu$	$\partial_\lambda \partial^\lambda h^{\mu\nu} - 2\partial_\lambda \partial^{(\nu} h^{\mu)\lambda} + \partial^\mu \partial^\nu h - \eta^{\mu\nu} \partial_\lambda \partial^\lambda h + \eta^{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} = -\kappa T^{\mu\nu}$
Gauge transformation	$A^\mu \to A^\mu + \partial^\mu \Lambda$	$h^{\mu\nu} \to h^{\mu\nu} + \partial^{(\nu} \Lambda^{\mu)}$
Preferred gauge condition	$\partial_\mu A^\mu = 0$	$\partial_\mu (h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h) = 0$
Field equation in preferred gauge	$\partial_\mu \partial^\mu A^\nu = 4\pi j^\nu$	$\partial_\lambda \partial^\lambda (h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h) = -\kappa T^{\mu\nu}$
Energy-momentum exchange between field and particle	$\partial_\nu T_{(m)\mu}{}^{\nu} = F_{\mu\nu} j^\nu$	$\partial_\nu T_{(m)\mu}{}^{\nu} = \frac{\kappa}{2} m h_{\alpha\beta,\mu} T_{(m)}^{\alpha\beta}$
Equation of motion of particle	$\frac{d}{d\tau} p_\mu = q F_{\mu\nu} u^\nu$	$\frac{d}{d\tau} P_\mu = \frac{\kappa}{2} m h_{\alpha\beta,\mu} u^\alpha u^\beta$
Energy-momentum of particle	$p_\mu = m u_\mu$	$P_\mu = m u_\mu + m\kappa h_{\mu\alpha} u^\alpha$
Proper time interval	$d\tau^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$	$d\tau^2 = (\eta_{\alpha\beta} + \kappa h_{\alpha\beta}) dx^\alpha dx^\beta$

The Principle of Relativity

All laws of physics must be invariant under Lorentz transformations.

Invariant means:

(1) The law has the same mathematical form

(2) All numerical constants have the same values

A potpouri of four-vectors and four-operators

spacetime 4-vector $s = (t, x, y, z)$

energy-momentum 4-vector (E, p_x, p_y, p_z)

4-potential $A_\mu = (\phi, A_x, A_y, A_z)$

4-current $j_\mu = (\rho, j_x, j_y, j_z)$

Feynman's ∇ **in 4-dimensions** $\nabla_\mu = (\partial/\partial t, \nabla)$

Feynman's D'Alembertian $= \nabla_\mu \nabla_\mu = (\partial^2/\partial t^2, -\nabla^2)$ 4**-Laplacian**

In modern notation we use ∂_μ **and** $\partial_\mu \partial^\mu$

All 4-vectors transform in precisely the same way.

The Structure of Spacetime

Virtual Text 2 starts by saying

“Lorentz transformations are just hyperbolic rotations.”

I am going to go one step further by asserting

Special Relativity is hyperbolic geometry.

Newtonian Space And Time

Space and time are different separate entities.

Distances are calculated using the Pythagorean Theorem

$s^2 = x^2 + y^2 + z^2$

Euclidean Spacetime, Cartesian Spacetime, Galilean Spacetime

For 90 proofs of the Pythagorean Theorem see:
http://www.cut-the-knot.org/pythagoras/index.shtml

Spacetime in SR (aka Minkowski Spacetime)

Space and time are one entity.

Distances are calculated differently

$s^2 = t^2 - x^2 - y^2 - z^2$

In two dimensions

$s^2 = t^2 - x^2$

This is how distance is measured in two-dimensional hyperbolic geometry.

Calculating Distances

In Euclidean Spacetime, distances are calculated using the inner product

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{r} \cdot \mathbf{r} = \mid \mathbf{r} \mid^2 = x^2 + y^2$$

In Minkowski Spacetime, distances are calculated using a different inner product

$$\mathbf{s} = \begin{bmatrix} t \\ x \end{bmatrix}$$

$$\mathbf{s} \cdot \mathbf{s} = \mid \mathbf{s} \mid^2 = t^2 - x^2$$

Formally, we can define the inner product using a matrix

For Euclidean Spacetime

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For Minkowski Spacetime

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Distance is given by the inner product $\mathbf{s} \cdot \mathbf{s} = \mathbf{s}^T\ \mathbf{M}\ \mathbf{s}$

The matrix M that defines distance is called the metric.

On the west coast

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad s^2 = t^2 - x^2 - y^2 - z^2$$

On the east coast

$$\mathbf{M} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad s^2 = -t^2 + x^2 + y^2 + z^2$$

The important thing about distances—also called intervals—is that they have the same value independent of the coordinate system used to measure them.

The Lorentz Transformations

Coordinate system changes in Minkowski Spacetime are given by the Lorentz transformations

The Lorentz transformations are the only linear coordinate transformations that produce invariant distances. The transformations must be linear so that the worldlines of free particles are straight lines in all coordinate systems.

$$t' = At + Bx$$

$$x' = Ct + Dx$$

The Lorentz transformations are given by

$$t' = \frac{t - vx/c}{\sqrt{1 - (v^2/c^2)}}$$

$$x' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}}$$

Written more compactly

$$t' = \gamma t - \beta\gamma x$$

$$x' = \gamma x - \beta\gamma t$$

Where

$$\beta = v/c$$

$$\gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$$

As the author of Virtual Text 2 asserted, we will see later that the Lorentz transformations are hyperbolic rotations.

http://www.univie.ac.at/future.media/moe/galerie/struct/struct.html

http://webphysics.davidson.edu/applets/Minkowski/Minkowski_FEL.html

Matrix form

This Lorentz transformation is called a "boost" in the *x*-direction and is often expressed in matrix form as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix}.$$

This transformation matrix is universal for all four-vectors.

More generally for a boost in any arbitrary direction $(\beta_x, \beta_y, \beta_z)$,

$$\begin{bmatrix} c\,t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta_x\,\gamma & -\beta_y\,\gamma & -\beta_z\,\gamma \\ -\beta_x\,\gamma & 1+(\gamma-1)\dfrac{\beta_x^2}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} \\ -\beta_y\,\gamma & (\gamma-1)\dfrac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_y^2}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} \\ -\beta_z\,\gamma & (\gamma-1)\dfrac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\dfrac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_z^2}{\beta^2} \end{bmatrix} \begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix}$$

where $\beta = \frac{v}{c} = \frac{|\vec{v}|}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Hyperbolic Geometry

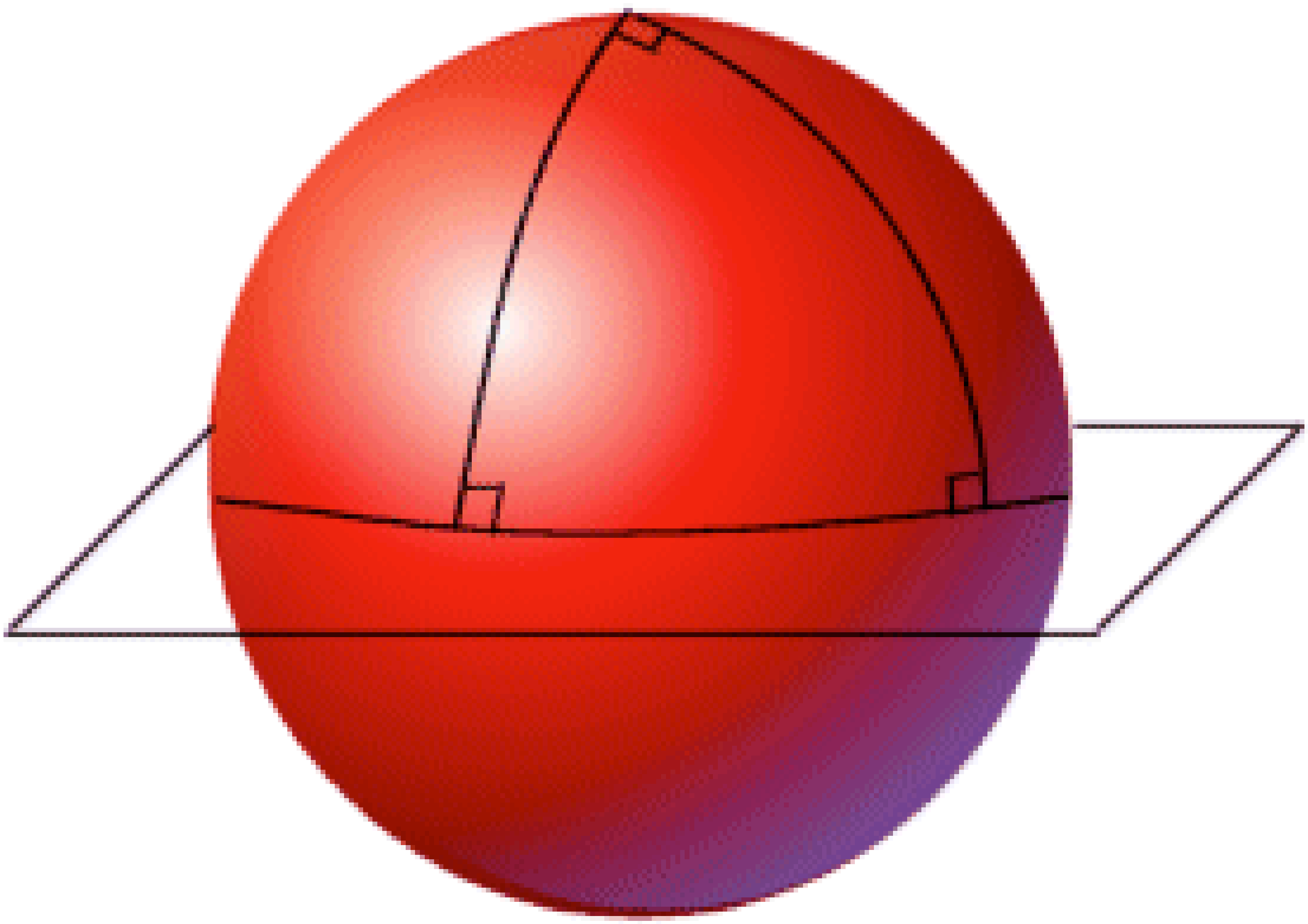
Three forms of geometry
Euclid's Fifth Postulate

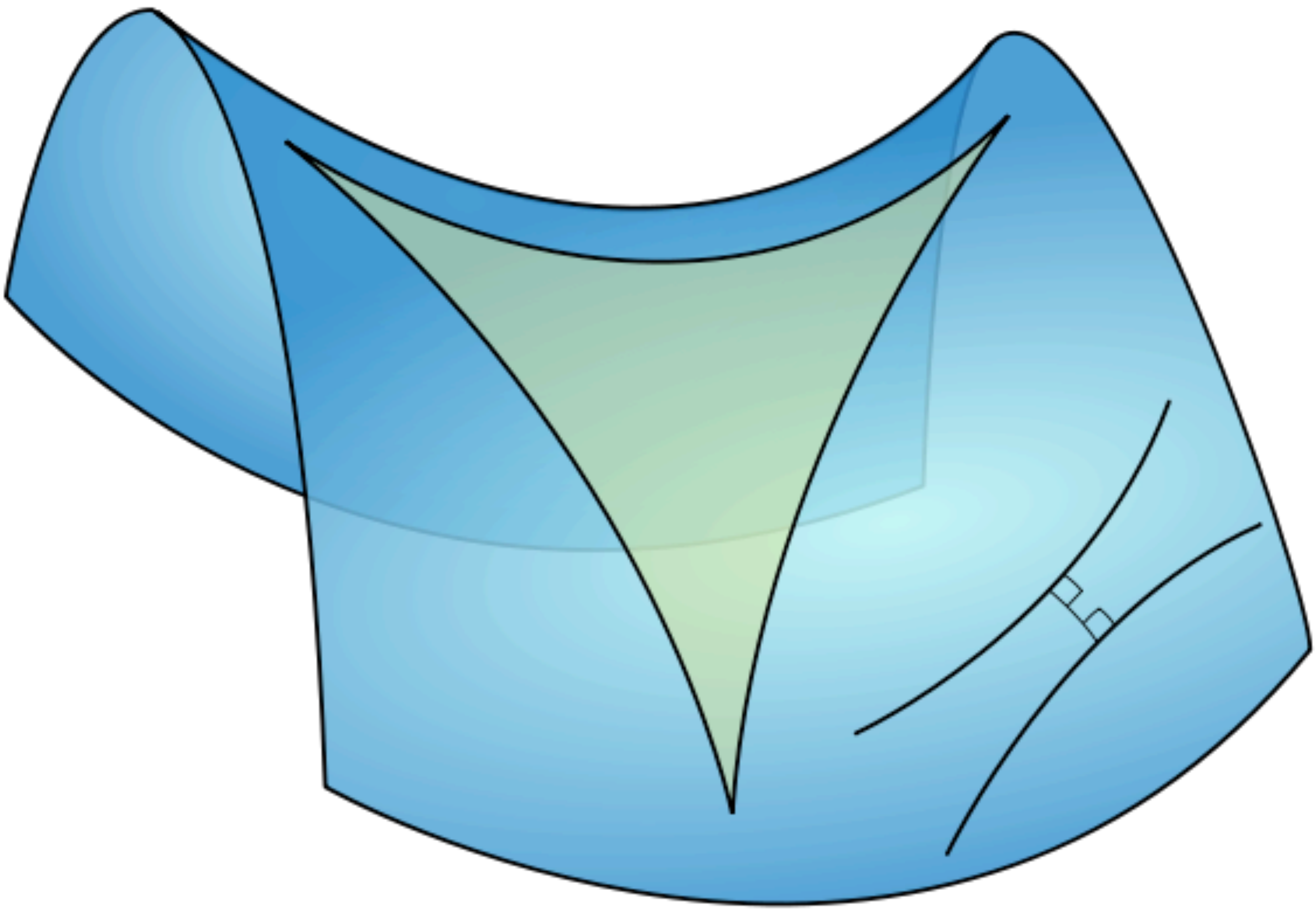
1 parallel line =>	Flat (zero curvature) Euclidean Geometry
no parallel lines =>	positive curvature Spherical Geometry
infinite number =>	negative curvature Hyperbolic Geometry

The three geometries

	Elliptical Spherical Riemannian	Parabolic Planar Euclidean	Hyperbolic Saddle Lobachevskian
curvature	positive	0	negative
triangles	$>180^o$	$=180^o$	$<180^o$
parallel lines	zero	1	infinite
projections	Mercator Gall-Peters Dymaxion Cuboctahedron		Poincare disk Poincare half plane Klein disk Minkowski

For this spherical triangle the sum of the internal angles = 270 degrees





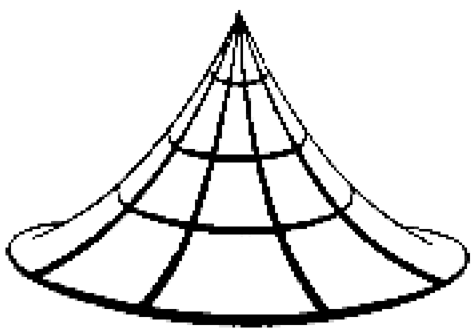


FIG. 234a

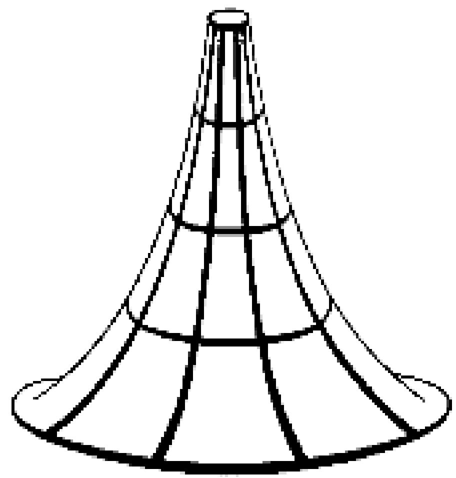


FIG. 234b

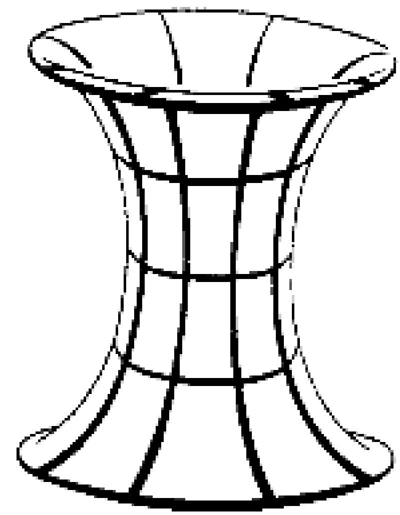
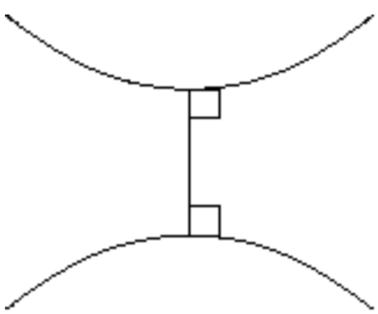
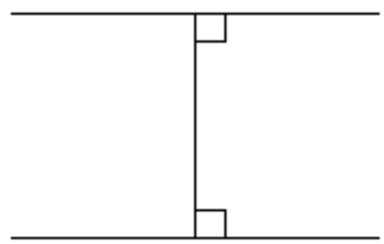


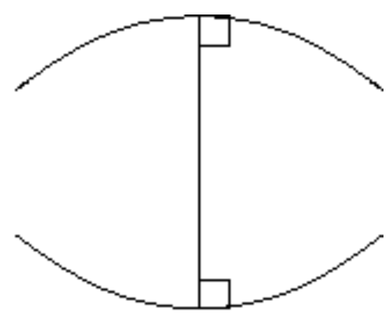
FIG. 234c



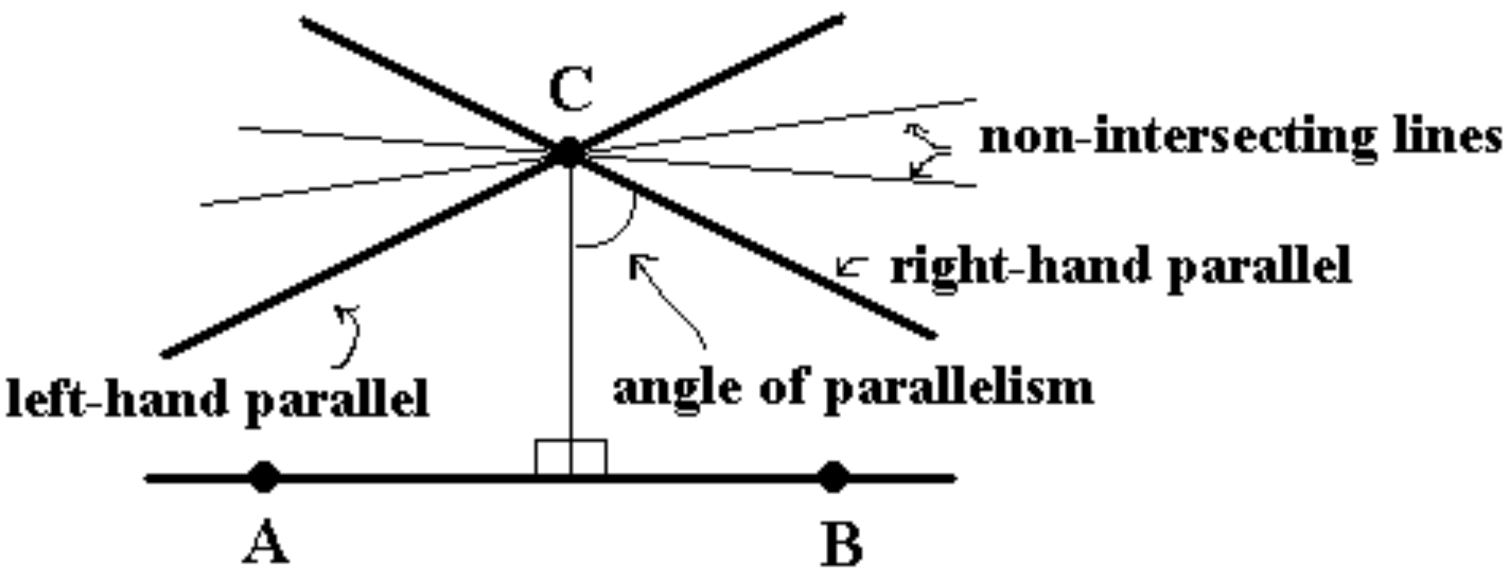
Hyperbolic

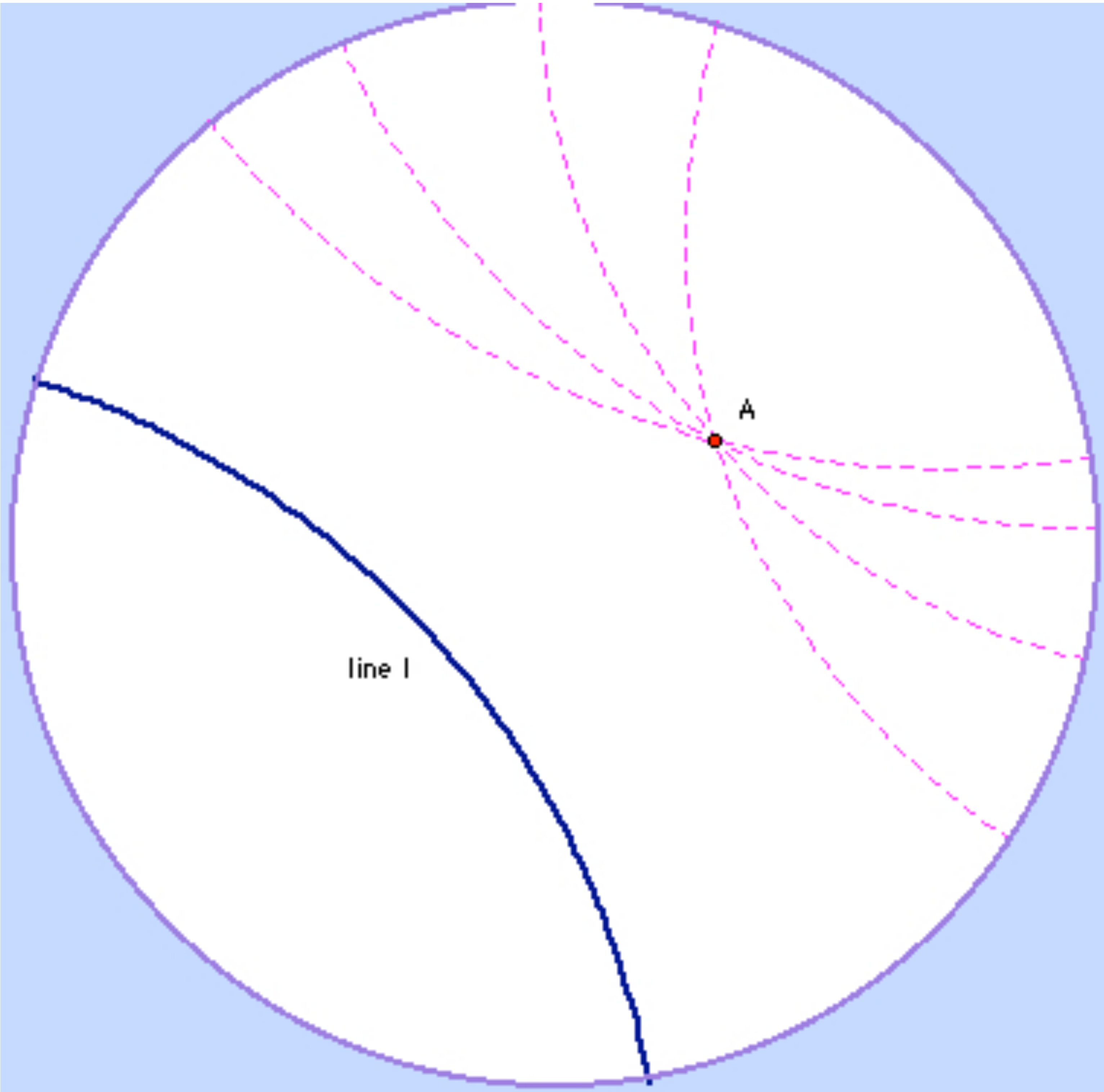


Euclidean



Elliptic


C
non-intersecting lines
right-hand parallel
left-hand parallel
angle of parallelism
A
B


A
line l

How can we project the hyperbolic plane onto the Euclidian plane?