Relativistic Kinematics

aka

Minkowski Diagrams in Momentum Space

geometry vs algebra

Two-Body Decays

$A \Rightarrow \gamma + \gamma$

$A \Rightarrow B + \gamma$

$A \Rightarrow B + C \quad (m_b = m_c)$

$A \Rightarrow B + C \quad (m_b > m_c)$

Two-Body Reactions

$\gamma + A \Rightarrow \gamma' + A'$ Compton Scattering

$\gamma_a + \gamma_b \Rightarrow \gamma_c + \gamma_d$

$A + B \Rightarrow C + D \quad (m_a = m_b = m_c = m_d)$

$A + B \Rightarrow C + D \quad (m_a > m_b \text{ and } m_c > m_d)$

$A + B \Rightarrow C$

1d elastic

equal masses

One-dimensional relativistic

[edit]

According to Special Relativity,

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where p denotes momentum of any massive particle, v denotes velocity, c denotes the speed of light.

in the center of momentum frame where the total momentum equals zero,

$$p_1 = -p_2$$

$$p_1^2 = p_2^2$$

$$\sqrt{m_1^2c^4 + p_1^2c^2} + \sqrt{m_2^2c^4 + p_2^2c^2} = E$$

$$p_1 = \pm\frac{\sqrt{E^4 - 2E^2m_1^2c^4 - 2E^2m_2^2c^4 + m_1^4c^8 - 2m_1^2m_2^2c^8 + m_2^4c^8}}{cE}$$

$$u_1 = -v_1$$

It is shown that $u_1 = -v_1$ remains true in relativistic calculation despite other differences. One of the postulates in Special Relativity states that the Laws of Physics should be invariant in all inertial frames of reference. That is, if total momentum is conserved in a particular inertial frame of reference, total momentum will also be conserved in any inertial frame of reference, although the amount of total momentum is frame-dependent. Therefore, by transforming from an inertial frame of reference to another, we will be able to get the desired results. In a particular frame of reference where the total momentum could be any,

$$\frac{m_1\, u_1}{\sqrt{1-u_1^2/c^2}} + \frac{m_2\, u_2}{\sqrt{1-u_2^2/c^2}} = \frac{m_1\, v_1}{\sqrt{1-v_1^2/c^2}} + \frac{m_2\, v_2}{\sqrt{1-v_2^2/c^2}} = p_T$$

$$\frac{m_1 c^2}{\sqrt{1-u_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1-u_2^2/c^2}} = \frac{m_1 c^2}{\sqrt{1-v_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1-v_2^2/c^2}} = E$$

We can look at the two moving bodies as one system of which the total momentum is p_T, the total energy is E and its velocity v_c is the velocity of its center of mass. Relative to the center of momentum frame the total momentum equals zero. It can be shown that v_c is given by:

$$v_c = \frac{p_T c^2}{E}$$

Now the velocities before the collision in the center of momentum frame u_1' and u_2' are:

$$u_1' = \frac{u_1 - v_c}{1 - \frac{u_1 v_c}{c^2}}$$

$$u_2' = \frac{u_2 - v_c}{1 - \frac{u_2 v_c}{c^2}}$$

$$v_1' = -u_1'$$

$$v_2' = -u_2'$$

$$v_1 = \frac{v_1' + v_c}{1 + \frac{v_1' v_c}{c^2}}$$

$$v_2 = \frac{v_2' + v_c}{1 + \frac{v_2' v_c}{c^2}}$$

When $u_1 << c$ and $u_2 << c$,

does not "soft land," that is, we assume that $y'(T(\theta)) < 0$, where $T(\theta)$ is the impact time. From (2) and the definitions of f_2 and f_4, we have

$$y'(t)=v \sin \theta f_2'(t)-gf_4'(t)=e^{-f_1(t)}(v \sin \theta-gf_3(t))$$

and hence the impact assumption is

$$f_3(T(\theta))>\frac{v}{g} \sin \theta. \tag{9}$$

In terms of the function

$$\rho(\theta)=\frac{R(\theta)}{v \cos \theta}$$

we have by (1),

$$R(\theta)=x(T(\theta))=v \cos \theta f_2(T(\theta))$$

and hence

$$T(\theta)=f_2^{-1}(\rho(\theta)).$$

The impact assumption (9) is therefore equivalent to

$$f_3(f_2^{-1}(\rho(\theta)))>\frac{v}{g} \sin \theta. \tag{10}$$

Now, $R(\theta)$ is differentiable if and only if $\rho(\theta)$ is differentiable. By (3), $\rho(\theta)$ is defined by $P(\rho(\theta), \theta)=0$, where

$$P(\rho,\theta)=v \sin \theta\rho-gf_4(f_2^{-1}(\rho)).$$

Finally, at $\rho = \rho(\theta)$,

$$\frac{\partial P}{\partial \rho}=v \sin \theta-gf_4'(f_2^{-1}(\rho))f_2^{-1\prime}(\rho)$$
$$=v \sin \theta-gf_3(f_2^{-1}(\rho))<0$$

by (10), and hence $\rho(\theta)$ is differentiable by the Implicit Function Theorem.[7]

[1] G. Galilei, *Two New Sciences* (Elzevirs, Leyden, 1638), translated with a new introduction and notes, by Stillman Drake (Wall and Thompson, Toronto, 1989), 2nd ed., p. 245.
[2] S. Drake and I. Drabkin, *Mechanics in Sixteenth-Century Italy* (University of Wisconsin Press, Madison, 1969), p. 91.
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[6] J. Lekner, "What goes up must come down; will air resistance make it return sooner, or later?," Math. Mag. **55**, 26–28 (1982).
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Minkowski diagrams in momentum space

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I. INTRODUCTION

Minkowski diagrams in configuration space, with points representing events, are often used in undergraduate courses on special relativity. Similar diagrams in momentum space are seldom shown, and the object of this note is to demonstrate their pedagogical usefulness in discussing particle interactions. In configuration space each point has coordinates $(t,\mathbf{x})$; in momentum space the coordinates are $(E,\mathbf{p})$. Two examples should be sufficient to show how such diagrams can be used.

II. EXAMPLES

A. Fission

In this example there is just one space dimension: Minkowski space is two dimensional. A particle of mass m is represented by its mass shell, a hyperbola opening in the positive E direction, given by

$$\left(\frac{E}{c}\right)^2-p^2=(mc)^2.$$

Figure 1 shows two such mass shells belonging to masses m and $M>m$, each labeled by its mass. The scale on the energy axis is chosen as E/c rather than E, so the two mass shells cross the E/c axis at mc and Mc, respectively. Each point on an m mass shell represents a state of a particle of mass m, i.e., possible values of its energy and momentum. A vector from the origin to such a point represents the energy–momentum $(E-p)$ vector of that state.

Consider a particle of mass M at rest, say a uranium nucleus, that undergoes fission to two particles of equal mass m. The vertical arrow in Fig. 1 represents the original uranium $E-p$ vector. $E-p$ conservation implies that the $E-p$ vectors of the two fission fragments add up to the original one, and since the total momentum is zero, the momenta of the two fission fragments must be negatives: their $E-p$ vectors have opposite p components. Symmetry of the m mass shell about the E/c axis then implies that their E/c components are equal, and conservation then implies than each E/c component is equal to $Mc/2$. It is clear from the dia-

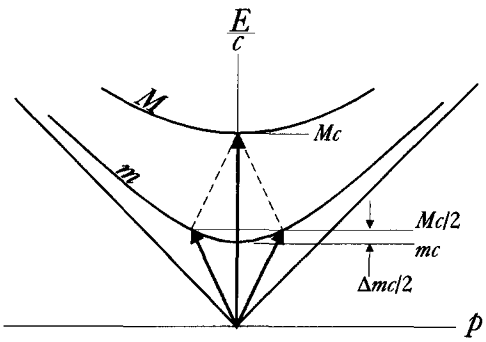

Fig. 1. Fission.

gram that each E/c component is higher than the point at which the m mass shell crosses the E/c axis, i.e., greater than mc, so $m<\frac{1}{2}M$,

$$Mc-2mc\equiv\Delta mc>0.$$

As the fission fragments interact with their surroundings, they slow down and eventually come to rest. Then their total E/c is $2mc$, so the energy they give up to their surroundings is just $\Delta E=\Delta mc^2$. This is the real content of the famous equation $E=mc^2$, involving measurable energy changes rather than absolute values relative to some more or less arbitrarily chosen zero of energy. Note that the mass of the fission fragments is not determined. But because their energies are both $\frac{1}{2}Mc^2$, the mass m and momentum p are related by

$$\left(\tfrac{1}{2}Mc\right)^2-p^2=(mc)^2.$$

The logical order in which to present this in class is first to draw the M mass shell, then the two $E-p$ vectors of the fission fragments, and only then to draw in the m mass shell.

This example is easily generalized to fission fragments of unequal masses. Also, a similar diagram can be used to illustrate fusion or the binding energy of the deuteron. Then M is less than $2m$, and the M mass shell crosses the E/c axis below $2mc$.

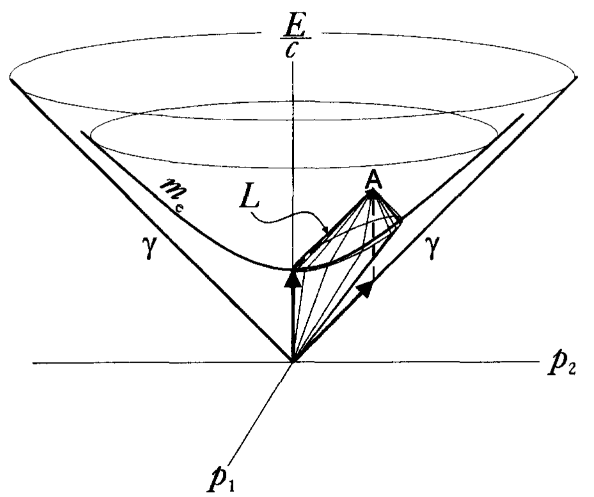

Fig. 2. Compton scattering.

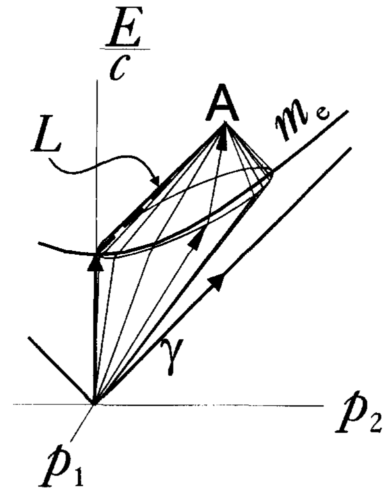

Fig. 3. Compton scattering (detail).

B. Compton scattering

Now take Minkowski space to be three dimensional, as in Fig. 2. The mass shell is now a hyperboloid of revolution. In the figure the intersection of the $(E/c,p_2)$ plane with the electron mass shell is the hyperbola labeled m_e, and the intersection with the light cone consists of the two lines labeled γ. The light cone is the mass shell of the photon, whose equation is

$$\left(\frac{E}{c}\right)^2-|\mathbf{p}|^2=0.$$

The vertical arrow in Fig. 2 is the $E-p$ vector of an electron at rest, and the other arrow represents an incident photon. The system's total $E-p$ vector is represented by the point labeled A (the vector to A is not drawn to avoid confusion). After scattering, the electron $E-p$ vector (again on the electron mass shell) plus the scattered photon $E-p$ vector (again on the light cone) must add up to A. A way to draw this is to construct an inverted light cone L with its vertex at A. The $E-p$ vectors of all possible scattered photons arrive at A from the closed curve, almost a circle, at which L intersects m_e in this three-dimensional space–time (in four dimensions this would be a closed surface, almost a sphere).

Figure 3 is an enlargement of part of Fig. 2. One possible combination of scattered electron and photon $E-p$ vectors is indicated with arrows. The direction of the scattered photon is obtained by projecting its $E-p$ vector onto the (p_1,p_2) plane, so the different lines on the cone represent photons moving in different directions. It is immediately evident that the photon energy E, and hence its frequency ν and wavelength λ, are determined by its direction.

III. CONCLUSION

Other particle interactions can also be visualized on similar Minkowski diagrams. The goal of this note is to show how the dynamics can be visualized, not to perform the calculations. The equations of the mass shells can be used, however, as a starting point for going on to the calculations.

C => A + B   decay

A + B => C   creation

Pa + Pb = Pc

Ma = Mb

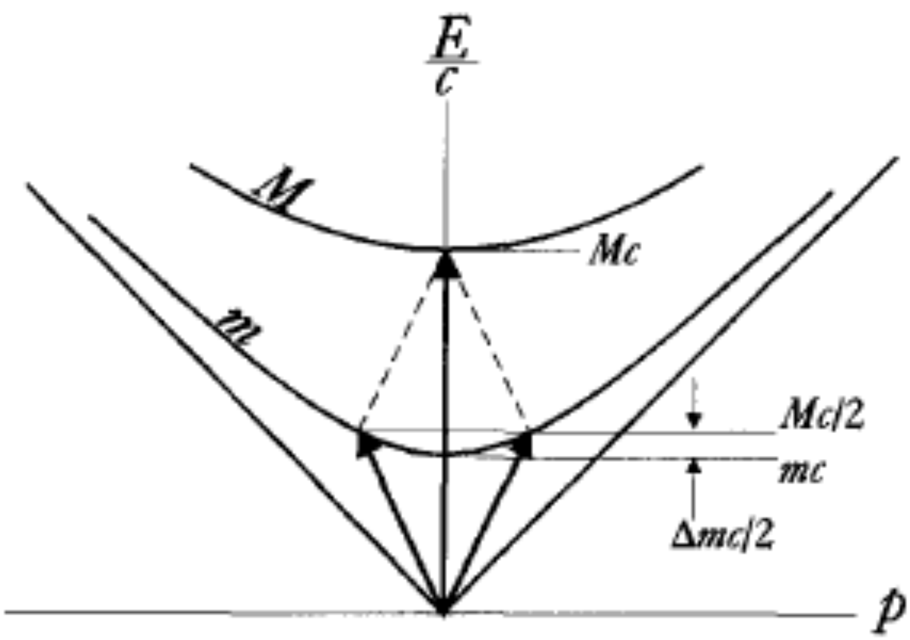

Fig. 1. Fission.

1d inelastic

equal masses

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Analysing collisions using Minkowski diagrams in momentum space

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Abstract

Momentum space and Minkowski diagrams are powerful tools for interpreting and analysing relativistic collisions in one or two spatial dimensions. All relevant quantities that characterize a collision, including the mass, velocity, momentum and energy of the interacting particles, both before and after collision, can be directly seen from a single Minkowski diagram. Such diagrams can also be useful for analysing the differences between Newtonian and relativistic mechanics. As an interesting example, a simple derivation of the Compton wavelength shift formula, based on the geometrical properties of such momentum space diagrams, is also presented.

1. Introduction

A paper by Saletan [1] presented a geometric representation of relativistic interactions, using Minkowski diagrams in momentum space. The method has great intuitive and pedagogical power. In this paper the most important features of such diagrams are discussed in a somewhat more systematic way and some extensions to Saletan's original paper are presented.

For the purposes of this paper the particular units for mass m, energy E and momentum p are unimportant. Multiplying by suitable factors of the speed of light c, all three quantities, i.e. mc^2, E and pc, can be expressed in units of energy. Throughout the paper this convention is adopted and the notation (au) for 'arbitrary unit of energy' is used.

Minkowski diagrams are widely used in configuration space where points represent events, expressed in coordinate notation as (ct, x) (for a 2D diagram) or (ct, x, y) (for a 3D diagram), corresponding to one or two spatial dimensions, respectively. By convention, time ct is along the vertical axis and space is represented along the horizontal axis (or axes) in such diagrams. Drawing a Minkowski diagram with three spatial dimensions is not possible, since it would require a four-dimensional image. Luckily, for many important relativistic phenomena the third spatial dimension can be omitted from the discussion.

As proposed by Saletan [1], Minkowski diagrams can also be constructed in momentum space, with energy E represented along the vertical axis and momentum p represented along

C => A + B decay

A + B => C creation

Pa + Pb = Pc

Ma < Mb

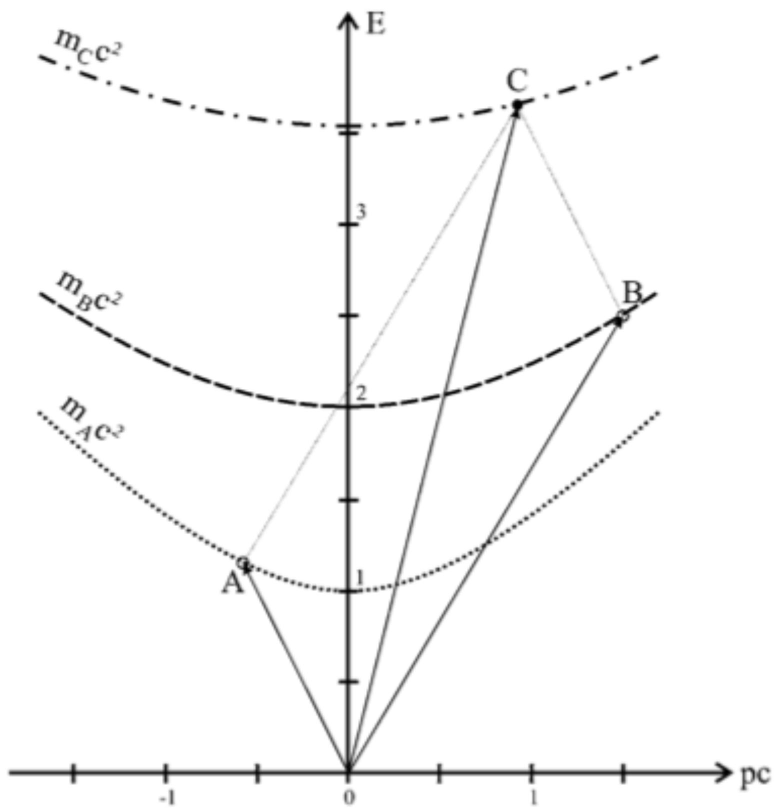


Figure 1. Perfectly inelastic collision in 1D. For all three particles involved in the collision, the *mass* appears as the intersection of the *E*-axis with the hyperbola representing the given particle; the *velocity* appears as the slope of the given energy–momentum vector (relative to the *E*-axis); and the *energy* and *momentum* of each particle appear as the vertical and horizontal components, respectively, of the energy–momentum vector.

1d inelastic

unequal masses

A + B => A' + B'

A' + B' => A + B

Pa + Pb = Pa' + Pb'

Mb > Ma

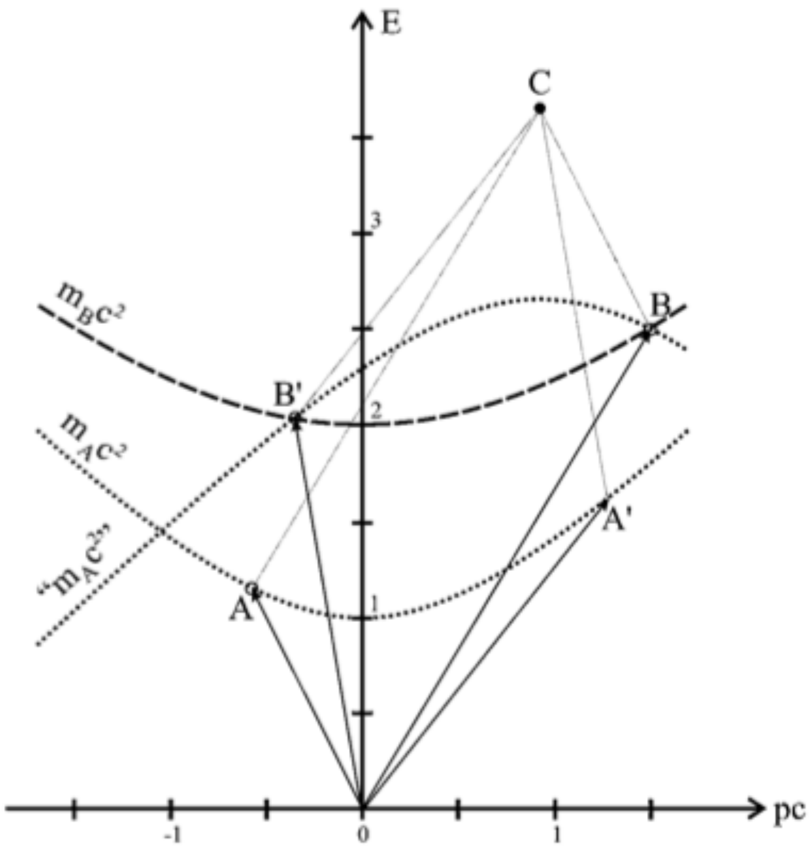
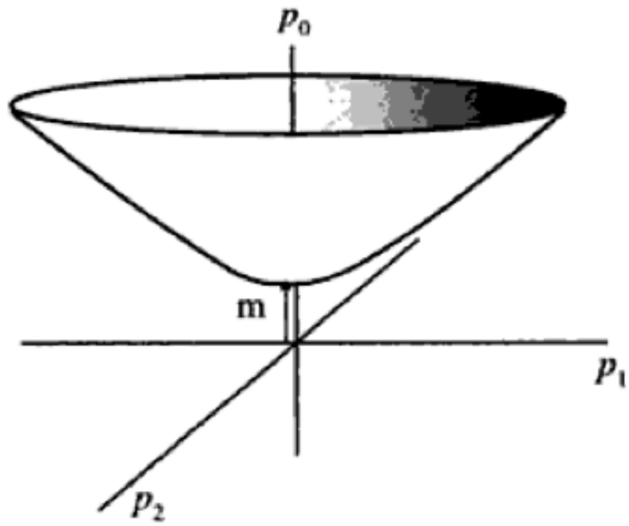


Figure 2. Elastic collision in 1D. From C as the origin, an upside-down version of the hyperbola m_Ac^2 is drawn. This hyperbola (denoted with "m_Ac^2") intersects the hyperbola m_Bc^2 at two points. These are the only two points that satisfy the two requirements listed in the text. B represents particle m_B *before* the collision, B′ represents the same particle *after* the collision. By using the parallelogram rule of vector addition, it is straightforward to draw the final energy–momentum vector for particle m_A (denoted by A′).

1d elastic

unequal masses



Mass Sheels

Figure 3.3. Finite mass hyperbola.

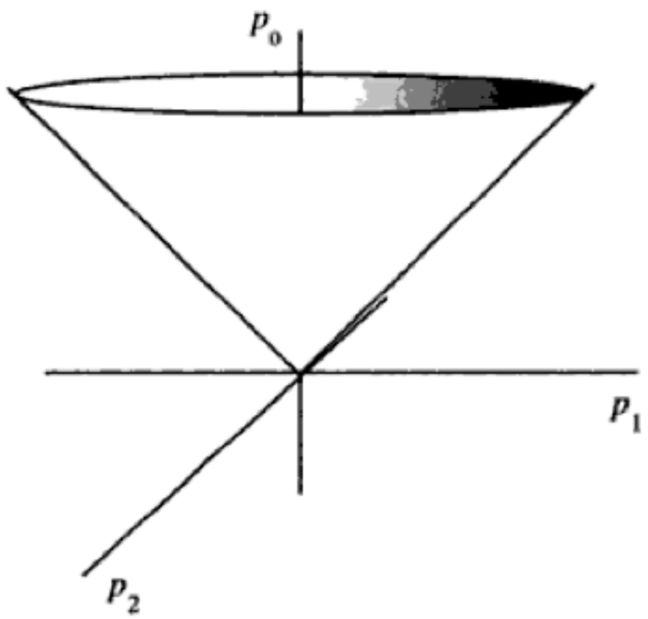


Figure 3.4. Zero mass hyperbola.

mass zero. Their mass hyperbola is the cone depicted in Figure 3.4. Conversely, particles with zero invariant mass travel with the speed of light. One can easily show (see Exercise 1) that, if the four-momenta of two particles are added (i.e., if the corresponding components are added to obtain the components of the sum), the resulting four-momentum is timelike, the invariant being greater than zero, or light like, the invariant being zero. It is lightlike only if the two original four-momentum vectors are themselves lightlike, with their space momenta parallel. It follows that in adding the four-momenta of any number of particles, one always obtains a timelike four-vector (unless, of course, all the particles' four-momenta that are added are lightlike with all three-momentum vectors parallel). This four-momentum has

2d elastic unequal m's (Bokor)

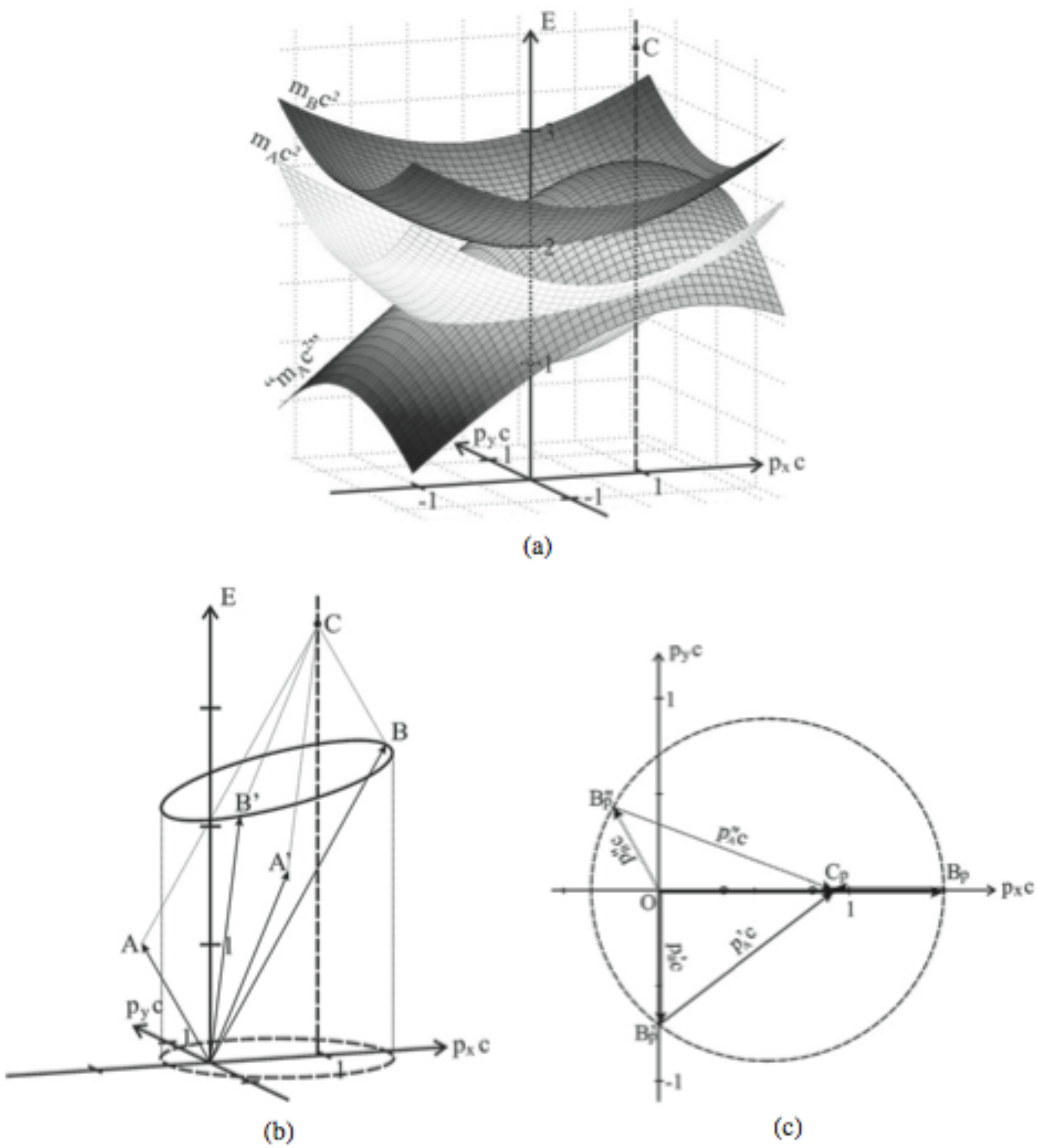
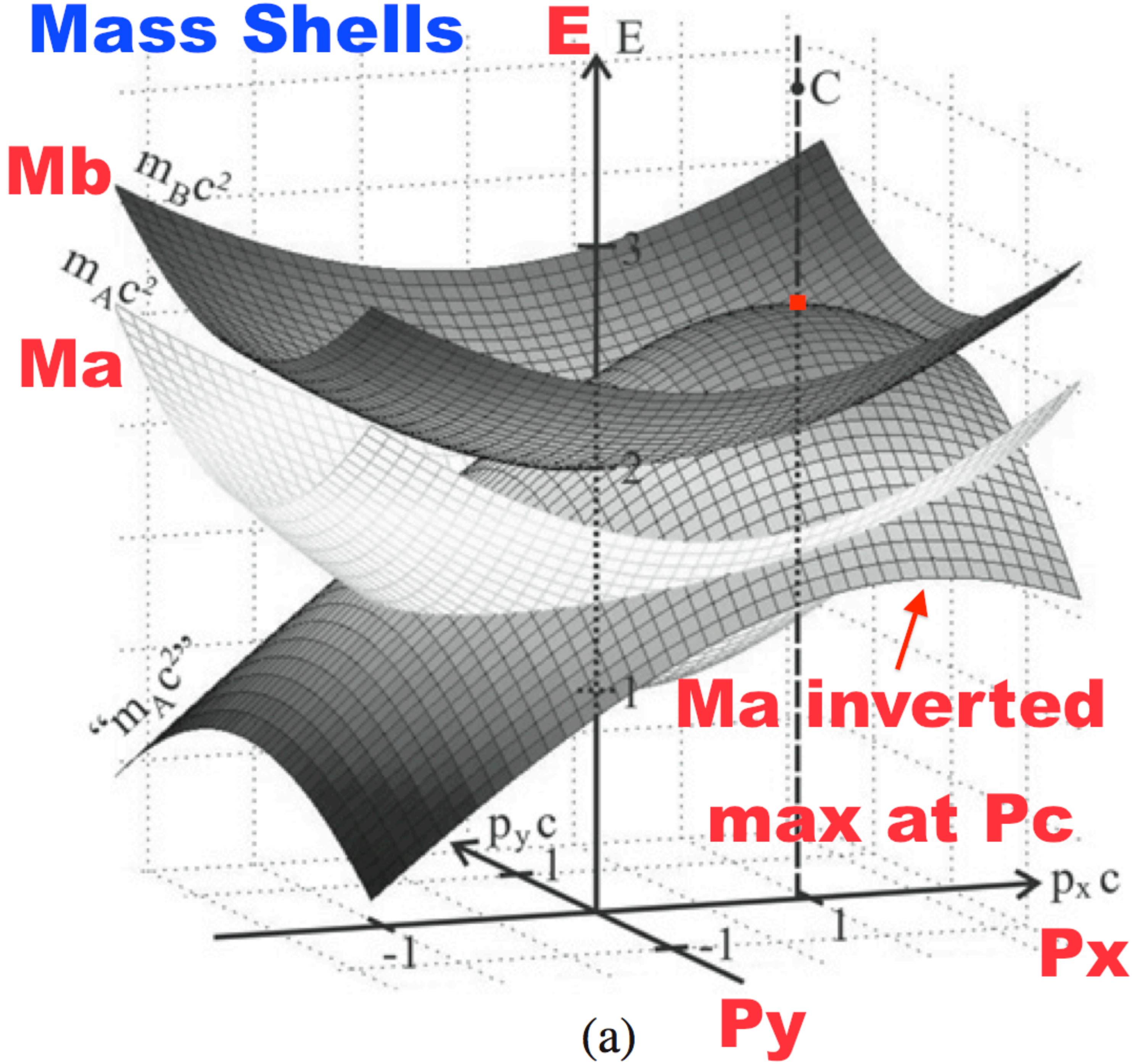


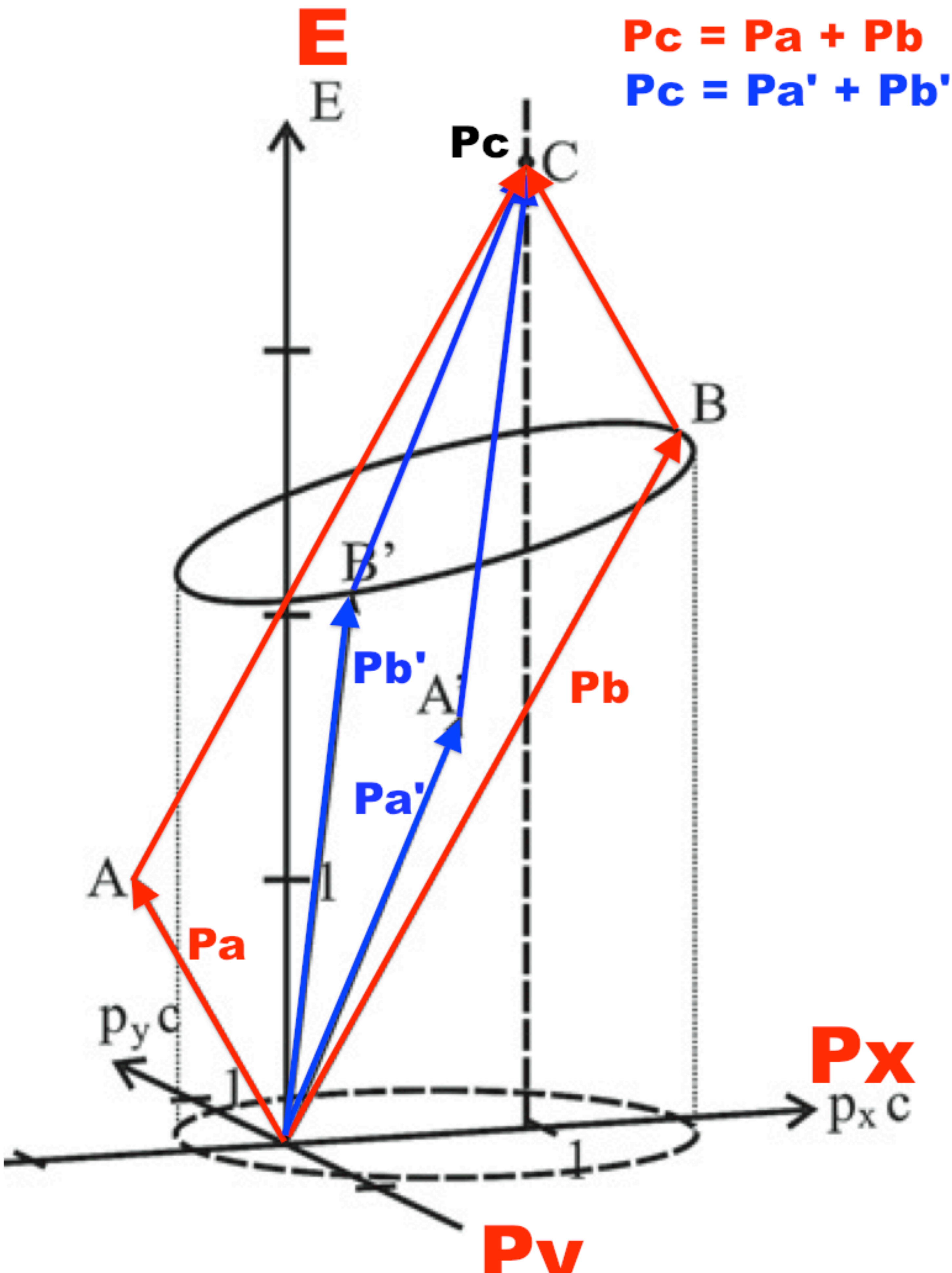
Figure 3. Elastic collision in 2D. (a) As a straightforward generalization of figure 2, an upside-down version of the hyperboloid m_Ac^2 is drawn from C taken as the origin. The intersection curve of this inverted hyperboloid (denoted by 'm_Ac^2') with the hyperboloid m_Bc^2 gives the possible loci for the tip of the energy–momentum vector of particle m_Bc^2 in this interaction. (b) The intersection curve of the two hyperboloids: a tilted ellipse. The energy–momentum vectors A′ and B′ represent *one* of the infinite number of possible final states (corresponding to the case when particle m_B moves along the negative *y*-axis after collision). A and B denote the energy–momentum vectors before collision. The projection of the intersection ellipse on the (p_xc, p_yc) plane is another ellipse, shown as a broken curve. (c) The projected ellipse on the (p_xc, p_yc) plane. All points with a subscript '$_p$' are projections of the corresponding points on (b) onto the (p_xc, p_yc) plane. B_p'' describes another possible final configuration, added here for illustration. (The foci of the ellipse are shown as two small empty circles.)

Mass Shells



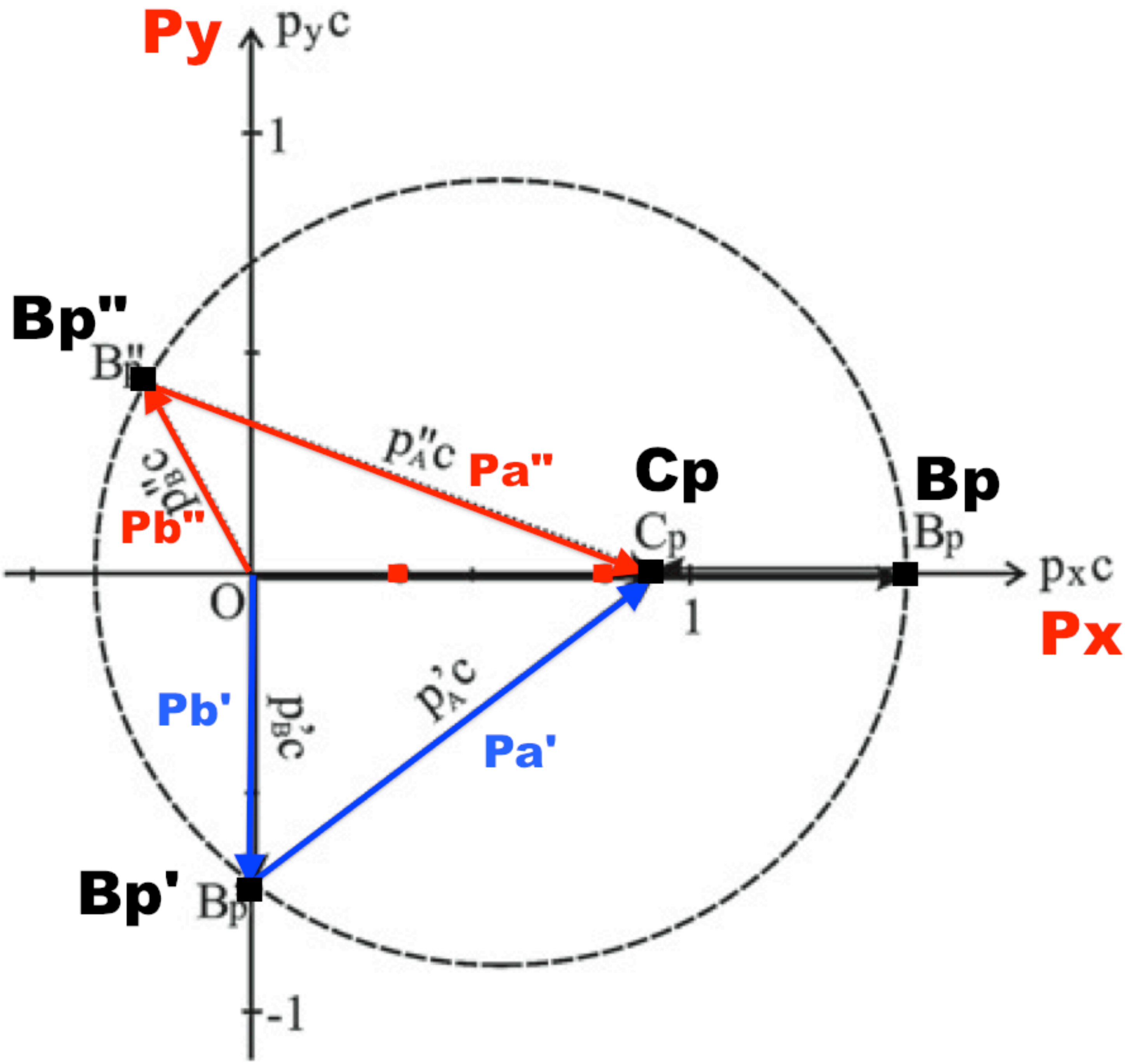
(a)

2d elastic unequal masses



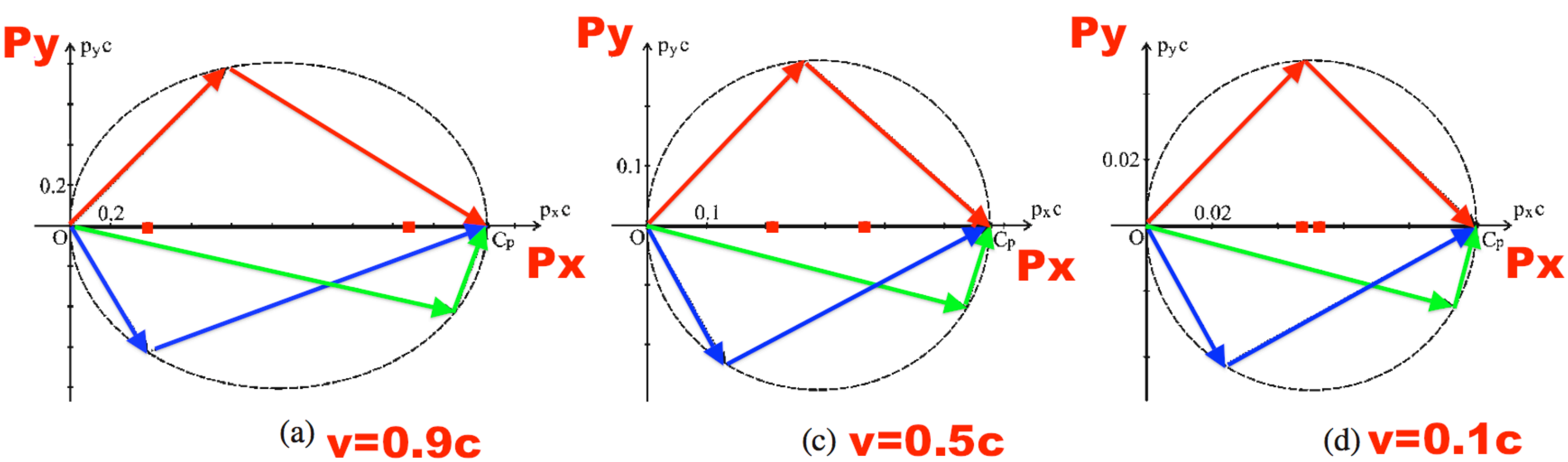
(b)

2d elastic unequal masses



(c)

2d elastic unequal masses



2d elastic unequal masses

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and hence the impact assumption is

$$f_3(T(\theta))>\frac{v}{g} \sin \theta. \tag{9}$$

In terms of the function

$$\rho(\theta)=\frac{R(\theta)}{v \cos \theta}$$

we have by (1),

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and hence

$$T(\theta)=f_2^{-1}(\rho(\theta)).$$

The impact assumption (9) is therefore equivalent to

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Now, $R(\theta)$ is differentiable if and only if $\rho(\theta)$ is differentiable. By (3), $\rho(\theta)$ is defined by $P(\rho(\theta), \theta)=0$, where

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2d Compton Scattering

Compton Mass Shells

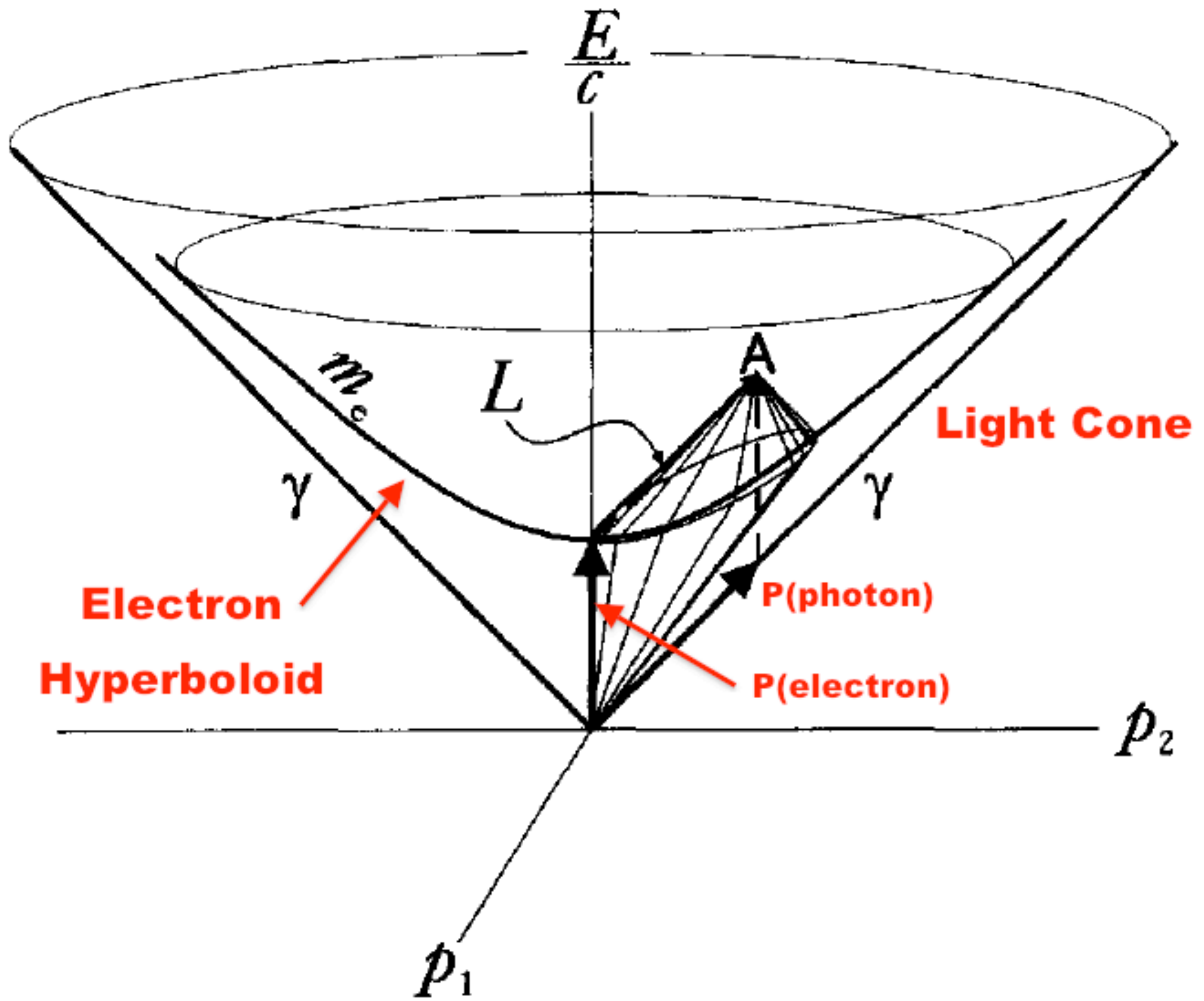

Fig. 2. Compton scattering.

2d Compton Scattering

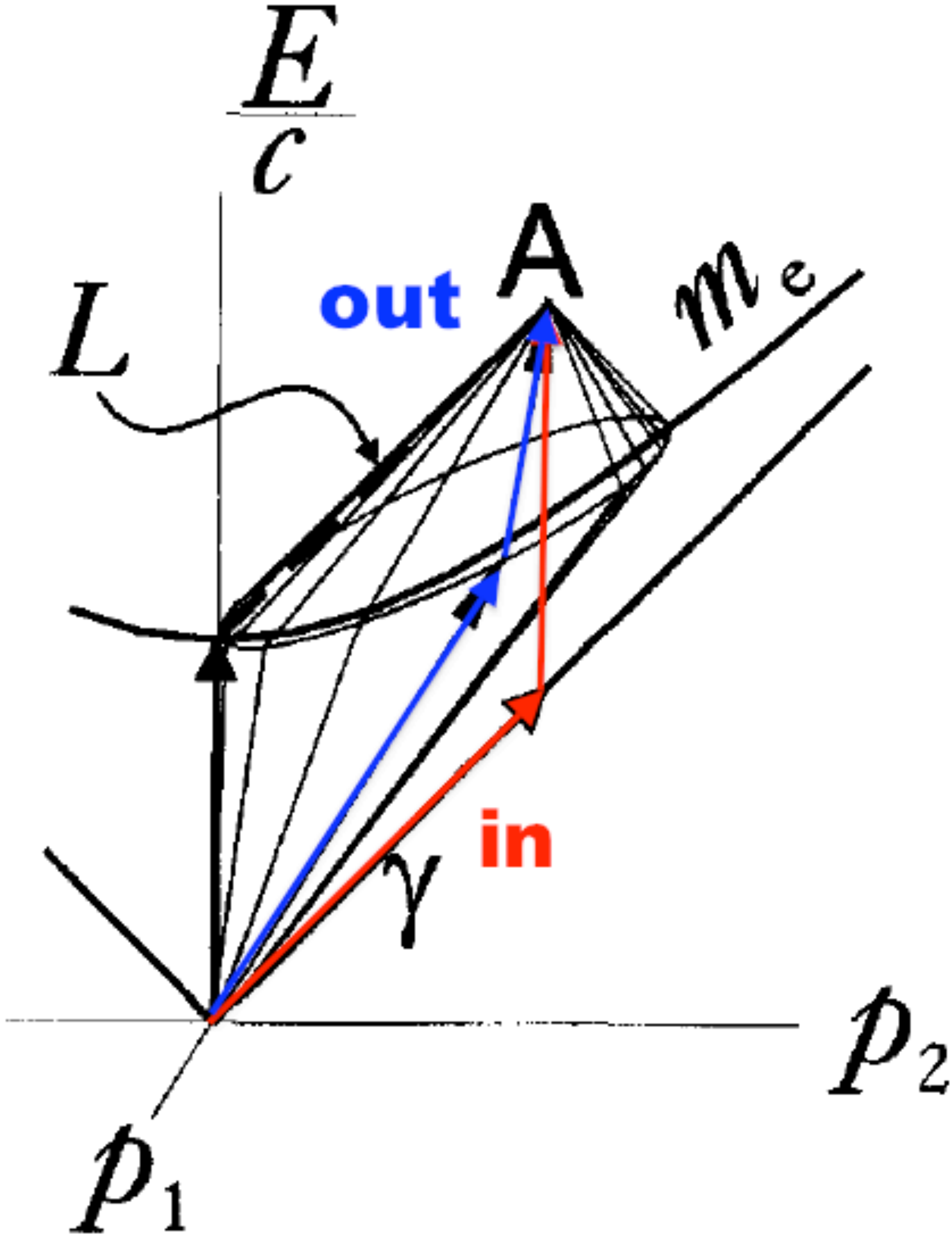

Fig. 3. Compton scattering (detail).

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Minkowski diagrams are widely used in configuration space where points represent events, expressed in coordinate notation as (ct, x) (for a 2D diagram) or (ct, x, y) (for a 3D diagram), corresponding to one or two spatial dimensions, respectively. By convention, time ct is along the vertical axis and space is represented along the horizontal axis (or axes) in such diagrams. Drawing a Minkowski diagram with three spatial dimensions is not possible, since it would require a four-dimensional image. Luckily, for many important relativistic phenomena the third spatial dimension can be omitted from the discussion.

As proposed by Saletan [1], Minkowski diagrams can also be constructed in momentum space, with energy E represented along the vertical axis and momentum p represented along

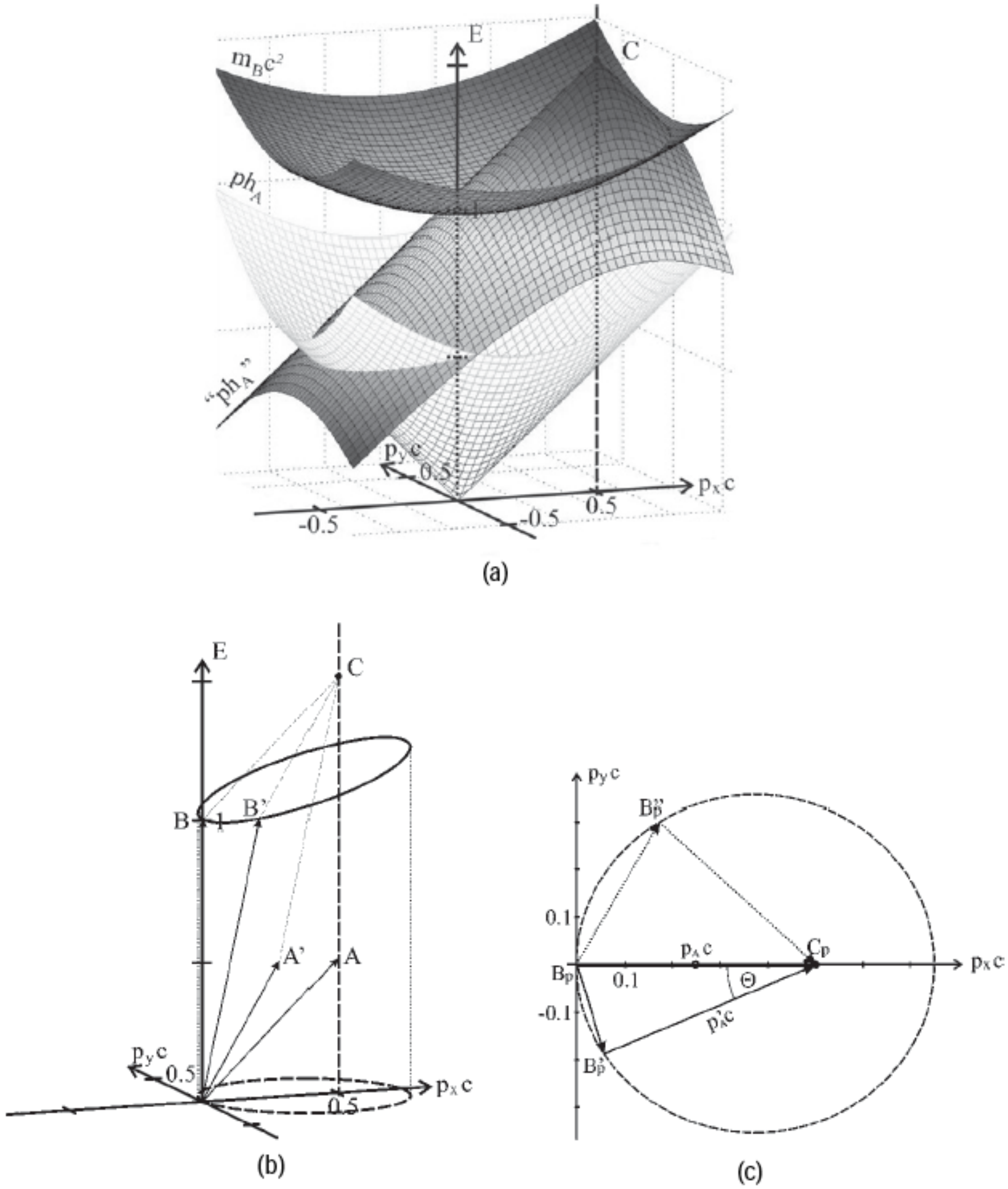
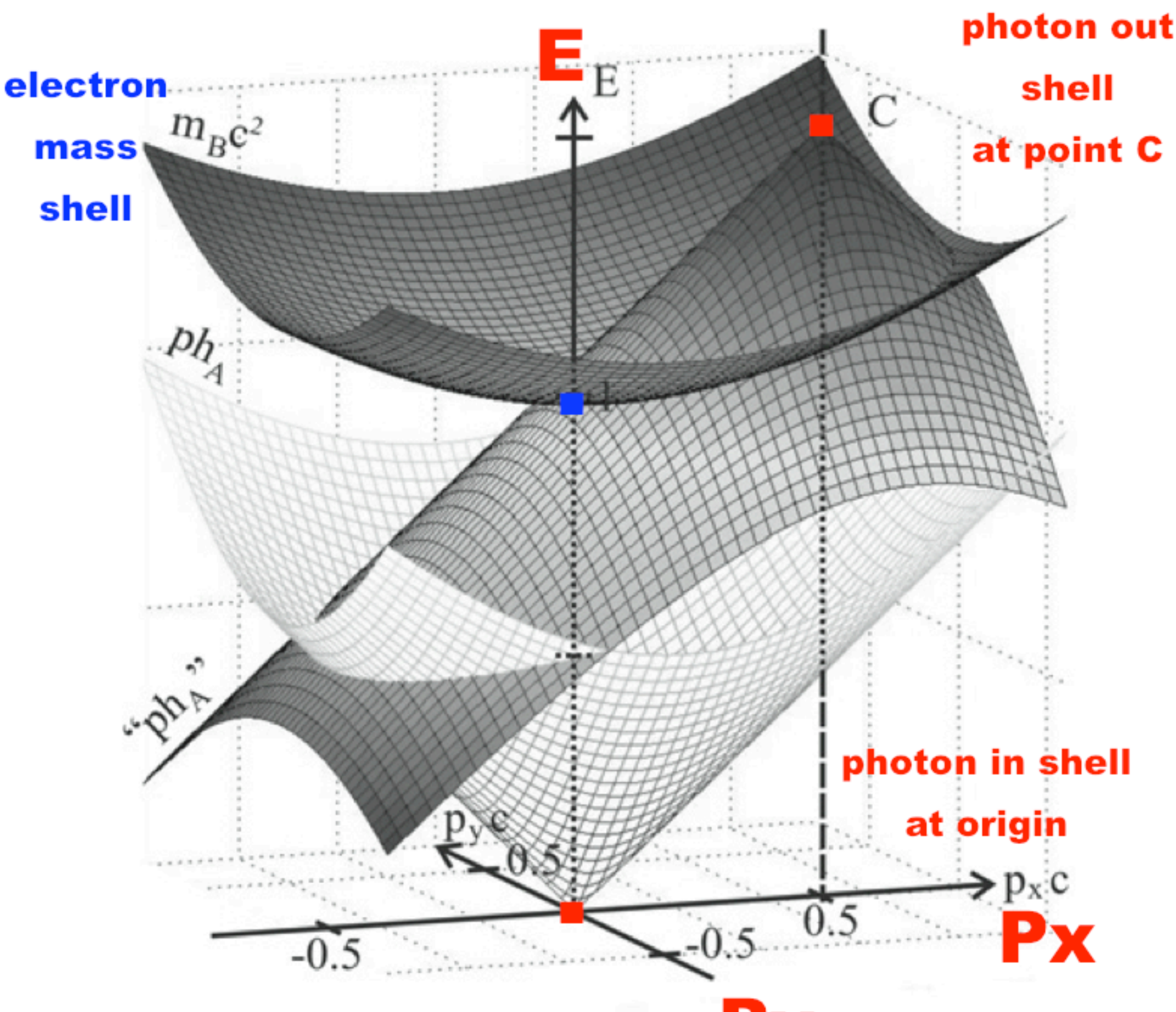


Figure 5. Compton scattering. (a) First the total energy–momentum vector for the system is found (using our initial parameters, the E-component of this vector is 1.5 and the $p_x c$-component is 0.5). The tip of this vector is denoted by C. An upside-down version of the photon cone (denoted by 'ph_A') is then drawn from C; and the intersection curve between the hyperboloid $m_B c^2$ and the inverted cone "ph_A" represents the possible final states for the electron after the interaction. (b) The intersection curve: a tilted ellipse. $B^{\bullet}$ represents one possible final state for the electron and $A^{\bullet}$ denotes the corresponding final state for the photon. A and B denote the energy–momentum vectors before collision. (c) The projected ellipse on the ($p_x c$, $p_y c$) plane. $B_p^{\bullet}$ and $B_p^{\bullet\bullet}$ denote two possible final configurations. The vector pointing from the origin to any point on the ellipse represents a possible momentum vector for the electron and the vector pointing from that point of the ellipse to point C_p gives the corresponding momentum vector for the scattered photon. (The foci of the ellipse are shown as two small empty circles.)

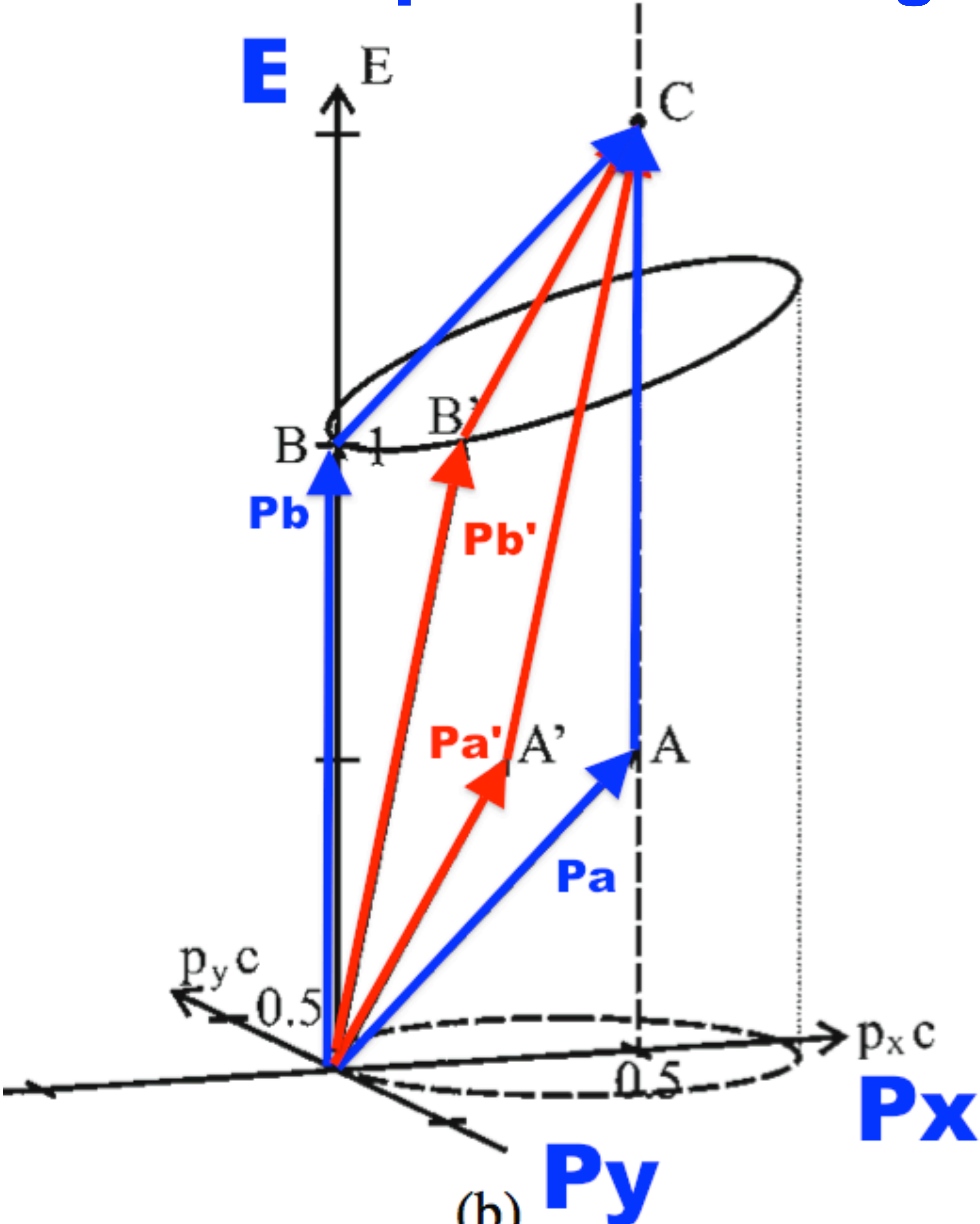
The relative directions of motion for the electron and photon are immediately apparent from the figure. In addition, the length of the momentum vector for the photon in figure 5(c) directly gives the numerical value for its energy too. (Note that the precise shape of the ellipse depends on the energy of the incoming photon, as will be discussed below, but all ellipses, regardless of the initial photon energy, share the two special features listed above.)

2d Compton Scattering



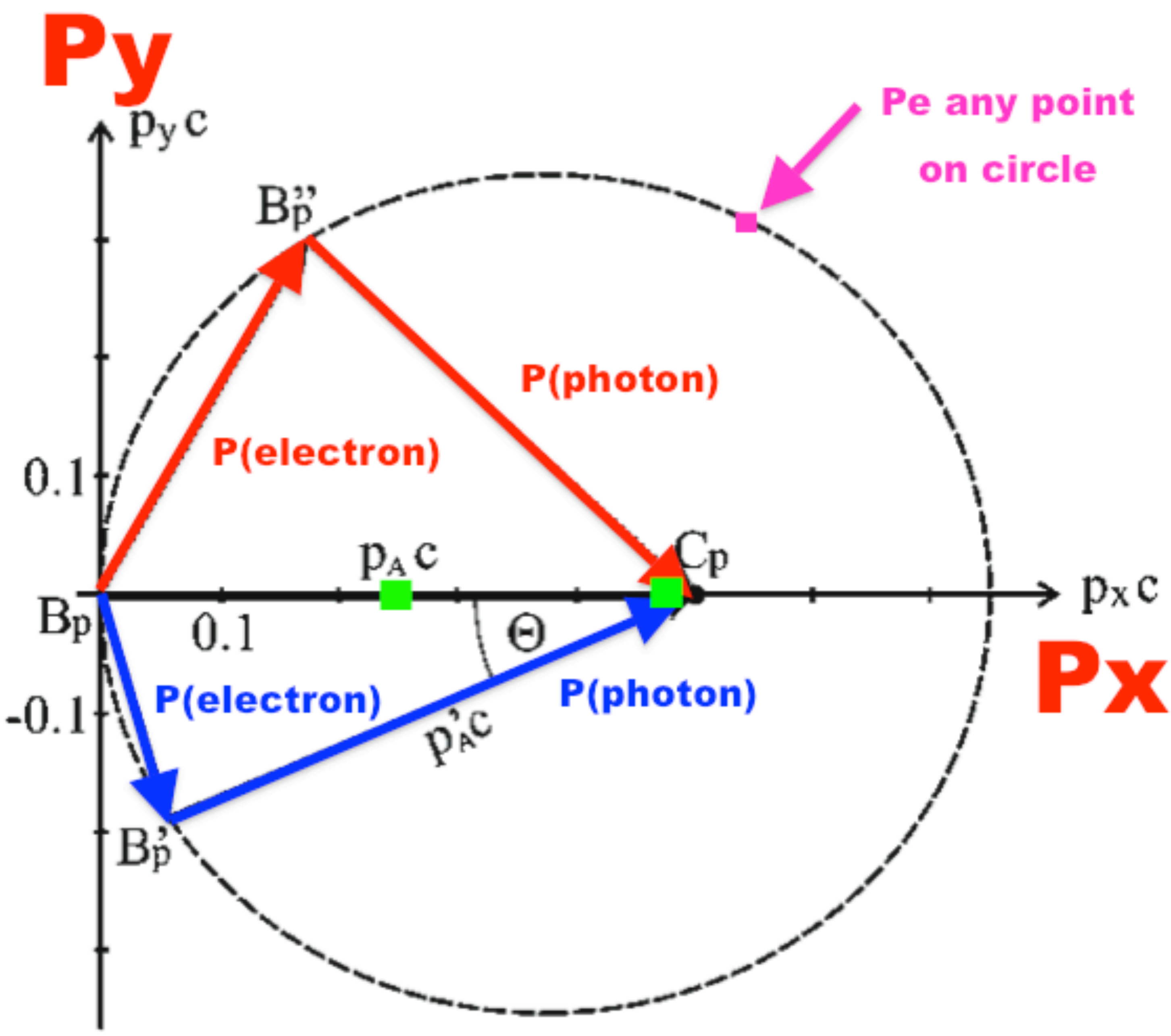
(a)

2d Compton Scattering



(b)

2d Compton Scattering



(c)

Compton Scattering Algebra

Solution

Week 69 (1/5/04)

Compton scattering

We will solve this problem by making use of 4-momenta. The *4-momentum* of a particle is given by

$$P \equiv (P_0, P_1, P_2, P_3) \equiv (E, p_x c, p_y c, p_z c) \equiv (E, \mathbf{p}c). \tag{1}$$

In general, the inner-product of two *4-vectors* is given by

$$A \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3. \tag{2}$$

The square of a 4-momentum (that is, the inner product of a 4-momentum with itself) is therefore

$$P^2 \equiv P \cdot P = E^2 - |\mathbf{p}|^2 c^2 = m^2 c^4. \tag{3}$$

Let's now apply these idea to the problem at hand. We will actually be doing nothing here other than applying conservation of energy and momentum. It's just that the language of 4-vectors makes the whole procedure surprisingly simple. Note that conservation of E and $\mathbf{p}$ during the collision can be succinctly written as

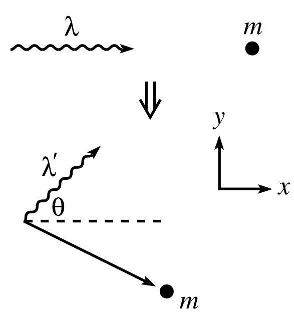
$$P_{\text{before}} = P_{\text{after}}. \tag{4}$$

Referring to the figure below, the 4-momenta before the collision are

$$P_\gamma = \left(\frac{hc}{\lambda}, \frac{hc}{\lambda}, 0, 0\right), \qquad P_m = (mc^2, 0, 0, 0). \tag{5}$$

And the 4-momenta after the collision are

$$P'_\gamma = \left(\frac{hc}{\lambda'}, \frac{hc}{\lambda'}\cos\theta, \frac{hc}{\lambda'}\sin\theta, 0\right), \qquad P'_m = (\text{we won't need this}). \tag{6}$$



If we wanted to, we could write P'_m in terms of its momentum and scattering angle. But the nice thing about this 4-momentum method is that we don't need to introduce any quantities that we're not interested in.

Compton Scattering Algebra

Conservation of energy and momentum give $P_\gamma + P_m = P'_\gamma + P'_m$. Therefore,

$$\begin{aligned}
(P_\gamma + P_m - P'_\gamma)^2 &= P'^2_m \\
\Longrightarrow P^2_\gamma + P^2_m + P'^2_\gamma + 2P_m(P_\gamma - P'_\gamma) - 2P_\gamma P'_\gamma &= P'^2_m \\
\Longrightarrow 0 + m^2c^4 + 0 + 2mc^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}(1-\cos\theta) &= m^2c^4. \qquad (7)
\end{aligned}$$

Multiplying through by $\lambda\lambda'/(2hmc^3)$ gives the desired result,

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta). \qquad (8)$$

The ease of this solution arose from the fact that all the unknown garbage in P'_m disappeared when we squared it.

Remarks:

1. If $\theta \approx 0$ (that is, not much scattering), then $\lambda' \approx \lambda$, as expected.
2. If $\theta = \pi$ (that is, backward scattering) and additionally $\lambda \ll h/mc$ (that is, $mc^2 \ll hc/\lambda = E_\gamma$), then $\lambda' \approx 2h/mc$, so

$$E'_\gamma = \frac{hc}{\lambda'} \approx \frac{hc}{\frac{2h}{mc}} = \frac{1}{2}mc^2. \qquad (9)$$

   Therefore, the photon bounces back with an essentially fixed E'_γ, independent of the initial E_γ (as long as E_γ is large enough). This isn't all that obvious.

inelastic relativistic collision

A particle of mass m, moving at speed $v = 4c/5$, collides inelastically with a similar particle at rest.

(a) What is the speed v_C of the composite particle?

(b) What is its mass m_C?

Solution by Rudy Arthur:

Call the moving particle 'M', and the particle at rest 'R' (the composite particle is defined to be 'C').

The momentum of the moving particle is

$$p_M = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{4}{3}mc\,. \tag{1}$$

And, the square of its energy is

$$E_M^2 = (mc^2)^2 + (p_M c)^2\,. \tag{2}$$

The energy of the particle at rest is

$$E_R = (mc^2)\,. \tag{3}$$

The square of the energy of the composite particle is

$$E_C{}^2 = (m_C c^2)^2 + (p_C c)^2\,. \tag{4}$$

By conservation of energy: $E_M + E_R = E_C$, or squaring and rearranging,

$$2E_M E_R = E_C^2 - E_M^2 - E_R^2 \tag{5}$$

Substituting (2) and (4) into (5):

$$2E_M E_R = \left((m_C c^2)^2 + (p_C c)^2\right) - \left(2(mc^2)^2 + (p_M c)^2\right)$$

By conservation of momentum, $p_C = p_M$, so this reduces to

$$2E_M E_R = (m_C c^2)^2 - 2(mc^2)^2$$

Squaring again:

$$4E_M^2 E_R^2 = \left((m_C c^2)^2 - 2(mc^2)^2\right)^2 \tag{6}$$

Substituting from (2) and (3) into (6) and expanding on the right,

$$4(mc^2)^2\left((p_M c)^2 + (mc^2)^2\right) = \left((m_C c^2)^4 - 4(mc^2)^2(m_C c^2)^2 + 4(mc^2)^4\right)$$

Rearranging,

$$(m_C c^2)^4 - 4(mc^2)^2(m_C c^2)^2 - 4(mc^2)^2(p_M c)^2 = 0$$

Using (1) this reduces to

$$m_C^4 - 4m_C^2 m^2 - \frac{64}{9}m^4 = 0 \tag{7}$$

Solving for m_C^2 (which must be positive) gives $m_C^2 = \frac{16}{3} m$, so the answer to (b) is

$$m_C = \frac{4}{\sqrt{3}} m . \tag{8}$$

The momentum of the composite particle is

$$p_C = \frac{m_C v_C}{\sqrt{1 - \frac{v_C^2}{c^2}}} . \tag{9}$$

By conservation of momentum $p_M = p_C$, and so, substituting from (1) and (8) into (9)

$$\frac{4}{3} mc = \frac{\frac{4}{\sqrt{3}} m v_C}{\sqrt{1 - \frac{v_C^2}{c^2}}} \tag{10}$$

Solving for v_C gives the answer to (a), $v_C = \frac{c}{2}$.

Inelastic Relativistic Collision

A particle of mass *m*, moving at speed $v = 4c/5$, collides inelastically with a similar particle at rest.

(a) What is the speed v_C of the composite particle?

(b) What is its mass m_C?

Solution by Michael Gottlieb:

(*I choose units for which* $c = 1$.)

Call the moving particle 'M', and the particle at rest 'R'. (The composite particle is defined to be 'C'.)

The momentum and energy of the particle at rest are

$$p_R = 0 \quad E_R = m.$$

The momentum of the moving particle is

$$p_M = \frac{mv}{\sqrt{1-v^2}} = \frac{4/5}{\sqrt{1-\left(\frac{4}{5}\right)^2}}m = \frac{4}{3}m,$$

and its energy is

$$E_M = \frac{p_M}{v} = \frac{(4/3)m}{4/5} = \frac{5}{3}m,$$

For the composite particle, the conservation of energy implies that

$$E_C = E_M + E_R = \frac{8}{3}m,$$

while the conservation of momentum implies that

$$p_C = p_M = \frac{4}{3}m.$$

The speed of the composite particle is

$$v_c = \frac{p_c}{E_c} = \frac{(4/3)m}{(8/3)m} = \frac{1}{2}. \quad \text{(For } c \neq 1,\ v_C = \frac{c}{2}.\text{)}$$

The mass of the composite particle is given by the (positive) solution to

$$m_C^2 = E_C^2 - p_C^2 = \left(\frac{8}{3}m\right)^2 - \left(\frac{4}{3}m\right)^2,$$

$$m_C = \frac{4}{\sqrt{3}}m.$$

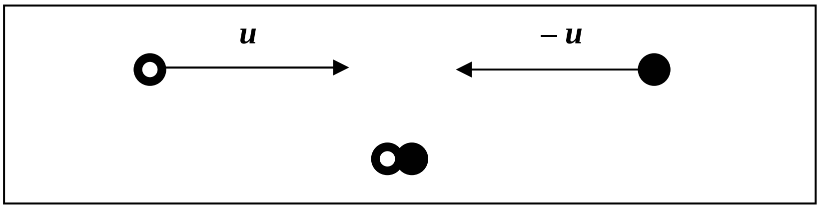
Inelastic Relativistic Collision

A particle of mass *m*, moving at speed $v = 4c/5$, collides inelastically with a similar particle at rest.

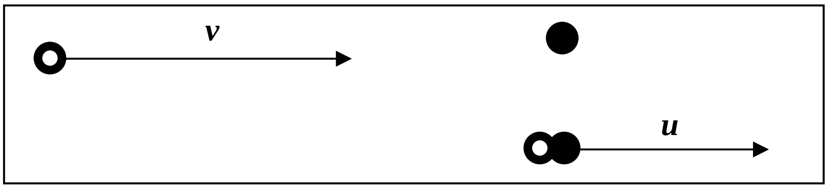
(a) What is the speed v_C of the composite particle?
(b) What is its mass m_C?

Solution by Ilkka Mäkinen:

Call the frame of the particle at rest "the lab frame" and consider the center-of-mass (CM) frame.



CM frame



lab frame

In order for momentum to be conserved the center-of-mass of the system must maintain a constant velocity ***u***; this will be the velocity of the composite particle in the lab frame.

The particle moving at speed v in the lab frame moves at speed u in the CM frame, while $-u$ is the speed of the lab frame relative to the CM frame. We can thus use the relativistic transformation of velocities to find u:

$$\frac{v-u}{1-vu} = u \quad \rightarrow \quad vu^2 - 2u + v = 0$$

$$u = \frac{1}{v} - \sqrt{\frac{1}{v^2} - 1} = \frac{5}{4} - \sqrt{\frac{9}{16}} = \frac{1}{2}$$

Then we can find the composite particle's mass m_c from the conservation of momentum:

$$\frac{mv}{\sqrt{1-v^2}} = \frac{m_C u}{\sqrt{1-u^2}}$$

$$\therefore m_C = \frac{v}{u}\frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}m = \frac{8}{5}\frac{\sqrt{3}}{2}\frac{5}{3}m = \frac{4}{\sqrt{3}}m\,.$$

inelastic relativistic collision

A particle of mass m, moving at speed $v = 4c/5$, collides inelastically with a similar particle at rest.
(a) What is the speed v_c of the composite particle?
(b) What is its mass m_c?

Solution by Ted Jacobson

Choose units where $c = 1$, $v = 4/5$. Let the incoming particle move in the x-direction of the rest frame, and let $\gamma = (1 - v^2)^{-1/2} = 5/3$.

The 4-momentum of the incoming particle (in the rest frame) is (γm, γmv, 0, 0), while the 4-momentum of the particle at rest is (m, 0, 0, 0), so the total 4-momentum is

$$p_\mu = (m(\gamma+1), m\gamma v, 0, 0),$$

which must be conserved, and is therefore the 4-momentum of the composite particle. The mass of the composite particle is the magnitude of its 4-momentum:

$$m_c^{\,2} = p_\mu^{\,2} = m^2(\gamma+1)^2 - m^2\gamma^2v^2 = (\gamma^2(1-v^2) + 2\gamma+1)m^2 = 2(\gamma+1)m^2 = (16/3)m^2.$$

Hence $m_c = (4/\sqrt{3})m$.

The velocity of the composite particle is its 3-momentum divided by its energy:

$$v_c = p_x / p_t = m\gamma v / m(\gamma+1) = (\gamma/(\gamma+1))v = (5/8)v = (5/8)(4/5) = 1/2.$$

Hence $v_c = c/2$.

INELASTIC

BEFORE AFTER

$v = 4/5 \Rightarrow \gamma = 5/3$

$$p_a = (m\gamma,\ m\gamma v,\ 0,\ 0)$$

$$p_b = (m,\ 0,\ 0,\ 0)$$

$$p_c = (m(\gamma+1),\ m\gamma v,\ 0,\ 0)$$

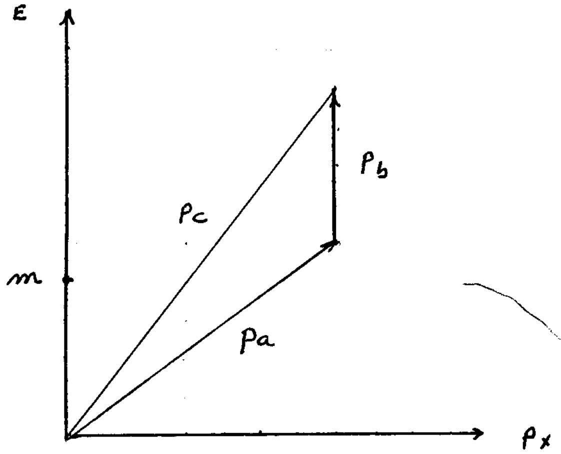
$$m_c = p_c \cdot p_c = (m(\gamma+1))^2 - (m\gamma v)^2$$

$$m_c^2 = \left(m\left(5/3 + 1\right)\right)^2 - \left(m\left(\tfrac{5}{3}\right)\left(\tfrac{4}{5}\right)\right)^2$$

$$m_c^2 = \frac{16}{3} m^2$$

$$m_c = \frac{4}{\sqrt{3}} m$$

$$v_c = \frac{p_x}{p_t} = \frac{m\gamma v}{m(\gamma+1)} = \frac{\left(\frac{5}{3}\right)\left(\frac{4}{5}\right)}{\left(\frac{8}{3}\right)} = \frac{1}{2}$$



$$(\gamma m,\ \gamma m v)$$

$$\left(\frac{5}{3}m,\ \frac{5}{3}m\,\frac{4}{5}\right)$$

$$\left(\frac{5}{3}m,\ \frac{4}{3}m\right)$$

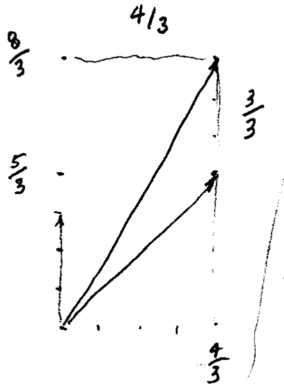
$$(m,\ 0)$$

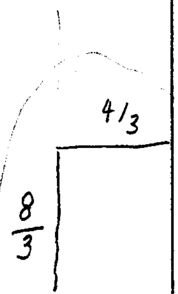
$$\left(\frac{8}{3}m,\ \frac{4}{3}m\right)$$

$$\frac{48}{9} \quad \frac{16}{3}$$

$$\frac{64}{9} - \frac{16}{9} = 5\,\tfrac{1}{3} = \frac{16}{3}$$

$$m_c = \frac{4}{\sqrt{3}}\,m$$





$$v_c = \frac{4/3}{8/3} = \frac{1}{2}$$

$$m_c^2 = \left(\frac{8}{3}\right)^2 - \left(\frac{4}{3}\right)^2$$

$$= \frac{64}{9} - \frac{16}{9}$$

$$= \frac{16}{3}$$

$$m_c = \frac{4}{\sqrt{3}}\,m$$

pion, muon, neutrino

A pion ($m_\pi = 273\ m_e$) at rest decays into a muon ($m_\mu = 207\ m_e$) and a neutrino ($m_\nu = 0$). Find the kinetic energy and momentum of the muon and the neutrino in MeV.

Solution by Michael A. Gottlieb:

(I choose units such that c = 1, and assume that $m_e = 0.511 MeV$.)

Since the pion is at rest conservation of momentum dictates that the momenta of the muon and the neutrino be equal in magnitude (and opposite in direction),

$$p_\mu = p_\nu \,. \tag{A}$$

Since the pion is at rest its energy equals its mass, $E_\pi = m_\pi$. Since the neutrino is massless its energy equals its momentum, $E_\nu = p_\nu$. By conservation of energy, $E_\pi = E_\mu + E_\nu$, so

$$E_\mu = m_\pi - p_\nu \,. \tag{B}$$

Substituting the right sides of (A) and (B) into the left side of the fundamental kinematic equation for the muon $E_\mu^2 - p_\mu^2 = m_\mu^2$ yields

$$\left(m_\pi - p_\nu\right)^2 - p_\nu^2 = m_\mu^2 \,.$$

Solving for p_ν gives (the magnitudes of) the momenta of the decay particles and the kinetic energy (equal to the total energy) of the massless neutrino,

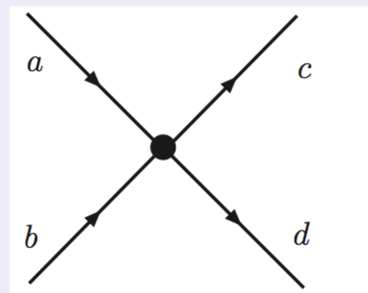
$$p_\nu \left(= p_\mu = E_\nu\right) = \frac{\left(m_\pi^2 - m_\mu^2\right)}{2m_\pi} = 29.65 \text{ MeV} \,.$$

The kinetic energy of the muon equals its total energy minus its mass which, using (B), is $\left(m_\pi - p_\nu\right) - m_\mu = 4.08 \text{ MeV}$.

Example: Relativistic two-body decay

- Consider the decay of a massive particle into two lighter ones, such that **rest masses** satisfy $M > m_1 + m_2$.
- To calculate energies and momenta of decay products use:
  Rest frame of decaying particle: $E = M$, $\underline{P} = 0$;
  Energy conservation: $E = E_1 + E_2$;
  Momentum conservation: $\underline{P} = \underline{p}_1 + \underline{p}_2 \Longrightarrow p_1 = p_2$;
  Energy-momentum relation: $E^2 = m^2 + p^2$.
- Case 1: $m_1 = m_2 = 0 \Longrightarrow E_1 = p_1 = p_2 = E_2 = M/2$
- Case 2: Arbitrary masses, $m_1 \neq 0$, $m_2 \neq 0$:
  $E_{1,2} = \frac{M^2 \pm (m_1^2 - m_2^2)}{2M}$ and $p_1 = p_2 = \frac{\sqrt{(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}}{2M}$.

Example: Relativistic two-body reactions



- Central reaction type in particle physics:
  2-body scattering: $a + b \to c + d$
- Convenient frame of inertia for description:
  centre-of momentum frame,
  characterised by $\underline{p}_a + \underline{p}_b = \underline{p}_c + \underline{p}_d = 0$
- Calculate Lorentz-invariant mass (energy) from:

$$
\begin{aligned}
s = M_{\rm inv}^2 &= (E_a + E_b)^2 - (\underline{p}_a + \underline{p}_b)^2 \\
&= (E_c + E_d)^2 - (\underline{p}_c + \underline{p}_d)^2 .
\end{aligned}
$$

- This is the energy squared in the c.m.-frame: $s = E_{\rm c.m.}^2$.

Example: Relativistic two-body reactions (cont'd)

- Can also calculate the (Lorentz-invariant) momentum transfer from a to c, called t and from a to d, called u:

$$
\begin{aligned}
t &= (E_a - E_c)^2 - (\underline{p}_a - \underline{p}_c)^2 = (E_b - E_d)^2 - (\underline{p}_b - \underline{p}_d)^2 \\
u &= (E_a - E_d)^2 - (\underline{p}_a - \underline{p}_d)^2 = (E_b - E_c)^2 - (\underline{p}_b - \underline{p}_c)^2 .
\end{aligned}
$$

- Properties:
  - $s > 0$, and t, $u \leq 0$
  - $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$.

  Therefore, for massless particles $s + t + u = 0$.
- In the c.m.-frame, and for massless particles:

$$
t = -\frac{E_{\mathrm{c.m.}}^2}{2}\left(1 - \cos\theta_{ac}\right) \quad \text{and} \quad u = -\frac{E_{\mathrm{c.m.}}^2}{2}\left(1 + \cos\theta_{ac}\right).
$$

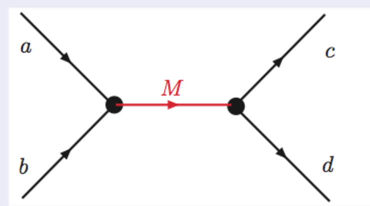
  θ_{ac} is called the "scattering angle".

Particle creation and decay

- Consider a special case of $2 \to 2$-scattering:

  Production of intermediate particle:

  $$a + b \to M \to c + d$$

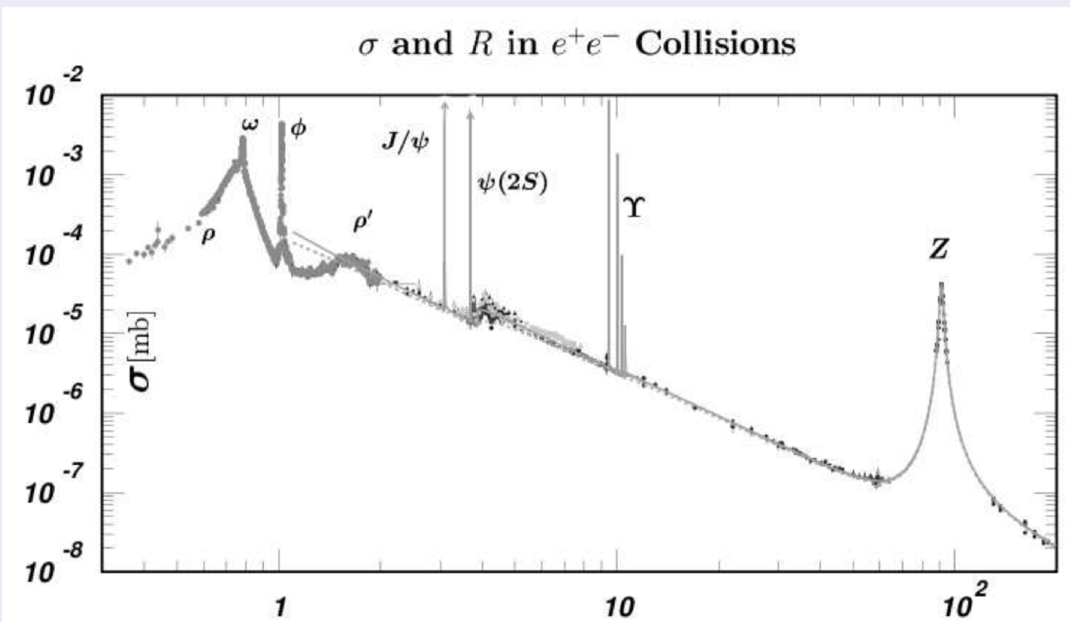


- Energy and momentum of M in c.m.-frame:

  $$E = E_a + E_b,\ \underline{P} = 0$$

- We will see that the probability for this process "resonates", if $s = E^2_{\mathrm{c.m.}} = M^2$ (resonance production).
  The production cross section will yield a peak.
- Note: Cross section is a way to quantify the probability for a process to happen, more on this in Lecture 3.

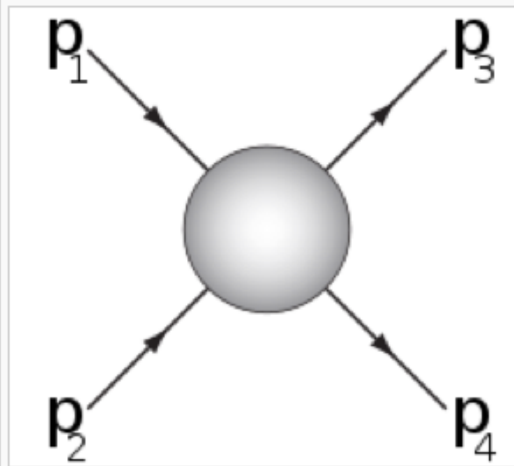
Example for resonance production: $e^+e^- \to$ hadrons



Mandelstam variables

From Wikipedia, the free encyclopedia

In theoretical physics, the **Mandelstam variables** are numerical quantities that encode the energy, momentum, and angles of particles in a scattering process in a Lorentz-invariant fashion. They are used for scattering processes of two particles to two particles.



In this diagram, two particles come in with momenta p_1 and p_2, they interact in some fashion, and then two particles with different momentum (p_3 and p_4) leave.

If the Minkowski Metric is chosen to be $diag(1, -1, -1, -1)$, the Mandelstam variables s,t,u are then defined by

- $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$
- $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$
- $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$

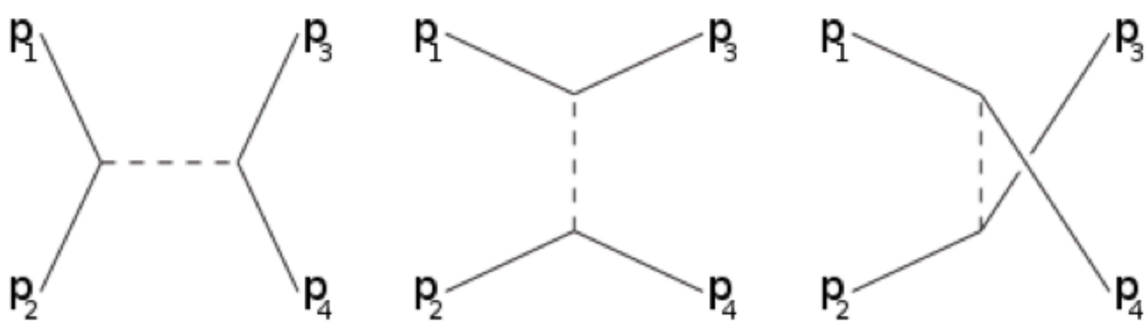
Where p_1 and p_2 are the four-momenta of the incoming particles and p_3 and p_4 are the four-momenta of the outgoing particles, and we are using Planck units (c=1).

s is also known as the square of the center-of-mass energy (invariant mass) and t is also known as the square of the momentum transfer.

Feynman diagrams

[edit]

The letters s,t,u are also used in the terms **s-channel**, **t-channel**, **u-channel**. These channels represent different Feynman diagrams or different possible scattering events where the interaction involves the exchange of an intermediate particle whose squared four-momentum equals s,t,u, respectively.



s-channel **t-channel** **u-channel**

For example the s-channel corresponds to the particles 1,2 joining into an intermediate particle that eventually splits into 3,4: the s-channel is the only way that resonances and new unstable particles may be discovered provided their lifetimes are long enough that they are directly detectable. The t-channel represents the process in which the particle 1 emits the intermediate particle and becomes the final particle 3, while the particle 2 absorbs the intermediate particle and becomes 4. The u-channel is the t-channel with the role of the particles 3,4 interchanged.

The Mandelstam variables were first introduced by physicist Stanley Mandelstam in 1958.

Addition of

[edit]

Note that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

where m_i is the mass of particle i.

Proof

[edit]

To prove this, we need to use two facts:

- The square of a particle's four momentum is the square of its mass,

$$p_i^2 = m_i^2 \qquad (1)$$

- And conservation of four-momentum,

$$\begin{aligned} p_1 + p_2 &= p_3 + p_4 \\ p_1 &= -p_2 + p_3 + p_4 \end{aligned} \qquad (2)$$

So, to begin,

$$\begin{aligned} s &= (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ t &= (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 \\ u &= (p_1 - p_4)^2 = p_1^2 + p_4^2 - 2p_1 \cdot p_4 \end{aligned}$$

First, use (1) to re-write these,

$$\begin{aligned} s &= m_1^2 + m_2^2 + 2p_1 \cdot p_2 \\ t &= m_1^2 + m_3^2 - 2p_1 \cdot p_3 \\ u &= m_1^2 + m_4^2 - 2p_1 \cdot p_4 \end{aligned}$$

Then add them

$$\begin{aligned} s + t + u &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_1 \cdot p_4 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\left(m_1^2 + p_1 \cdot p_2 - p_1 \cdot p_3 - p_1 \cdot p_4\right) \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\left(m_1^2 + p_1 \cdot (p_2 - p_3 - p_4)\right) \end{aligned}$$

Then use eq (2) to simplify further,

$$\begin{aligned} s + t + u &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\left(m_1^2 - p_1 \cdot p_1\right) \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\left(m_1^2 - m_1^2\right) \end{aligned}$$

So finally,

$$\boxed{s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2}$$

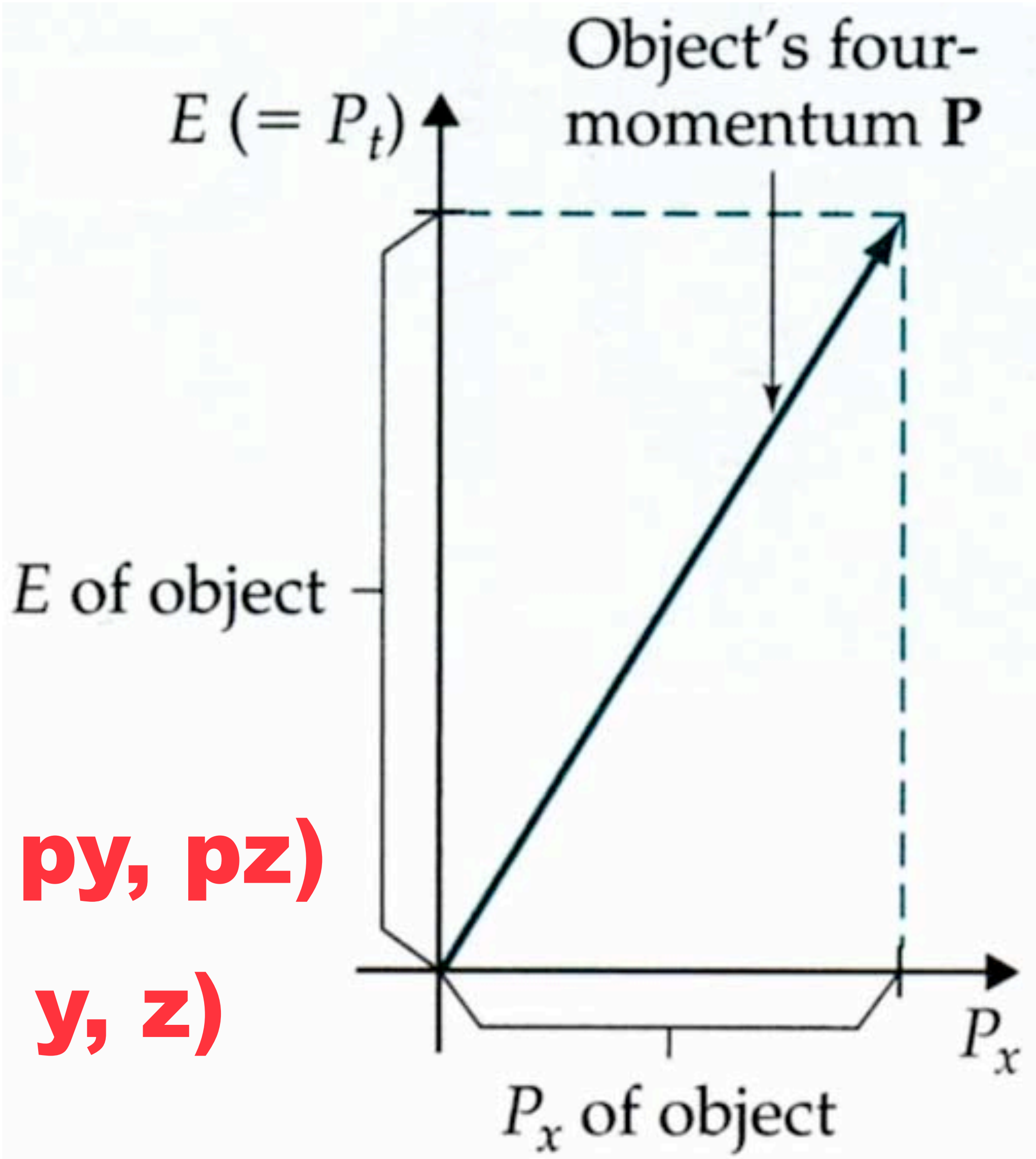
Relativistic Dynamics: Conservation of Four-Momentum

Energy-Momentum Diagrams:

Minkowski diagrams in four-momentum space

Definition:

A particle's energy-momentum diagram plots E versus P_x



Typically we assume

$$P_y = P_z = 0$$

So that

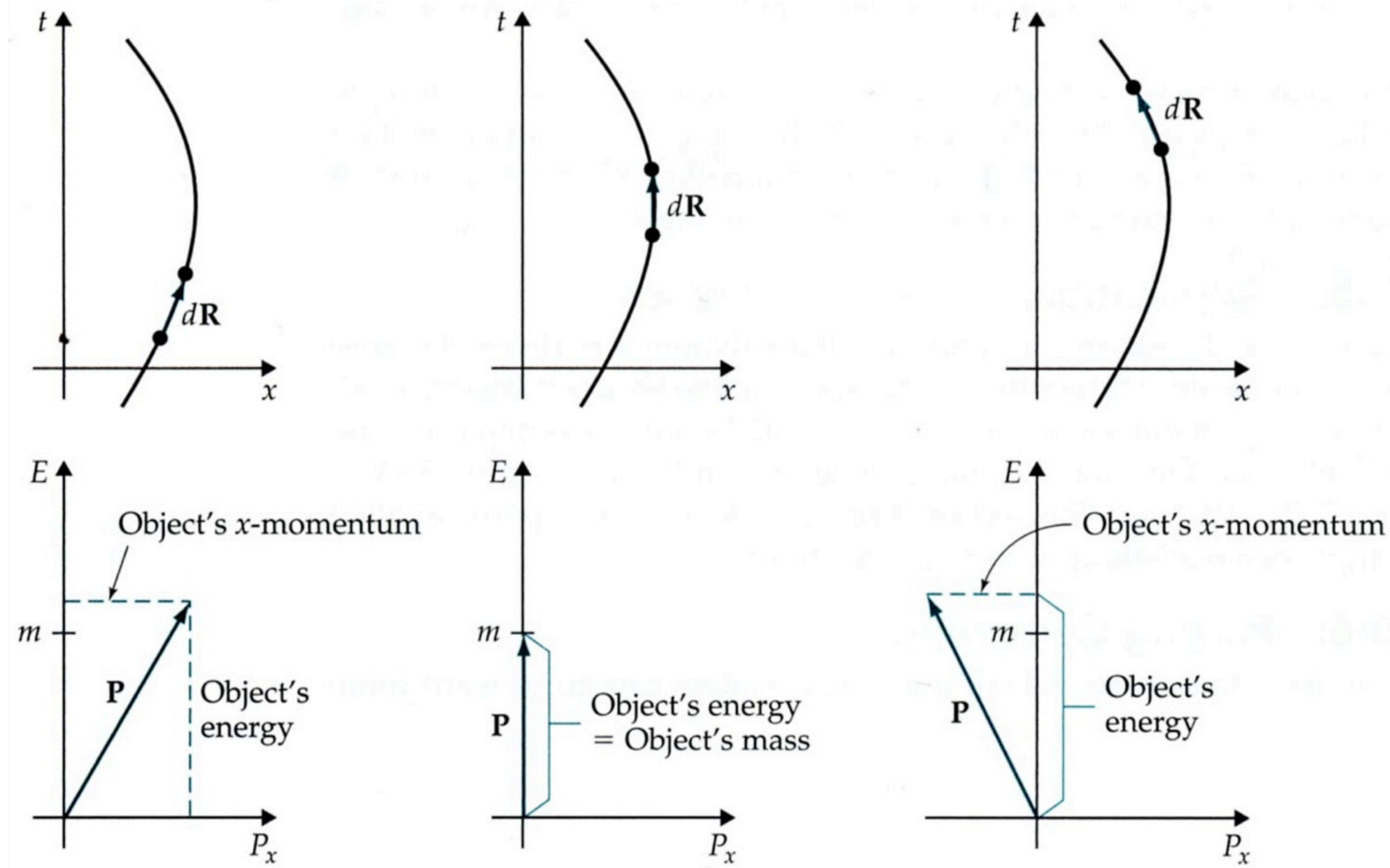
$$p = |P_x|$$

p = (E, px, py, pz)

x = (t, x, y, z)

Relativistic Dynamics: Conservation of Four-Momentum

Energy-Momentum Diagrams ...



Properties:

$d\mathbf{R}$ *tangent* to worldline

Slope at event: $1/v_x$

At a given event:

$\mathbf{P}$ is parallel to $d\mathbf{R}$

Slope of $\mathbf{P} = m\frac{d\mathbf{R}}{d\tau}$:

$1/v_x$

So $\frac{P_x}{E} = \frac{mv_x}{\sqrt{1-v^2}} \frac{\sqrt{1-v^2}}{m} = v_x$.

Mass is magnitude of $\mathbf{P}$: $m = \sqrt{E^2 - p^2}$

m is the rest mass

E^2 = p^2 + m^2

vx = px / E

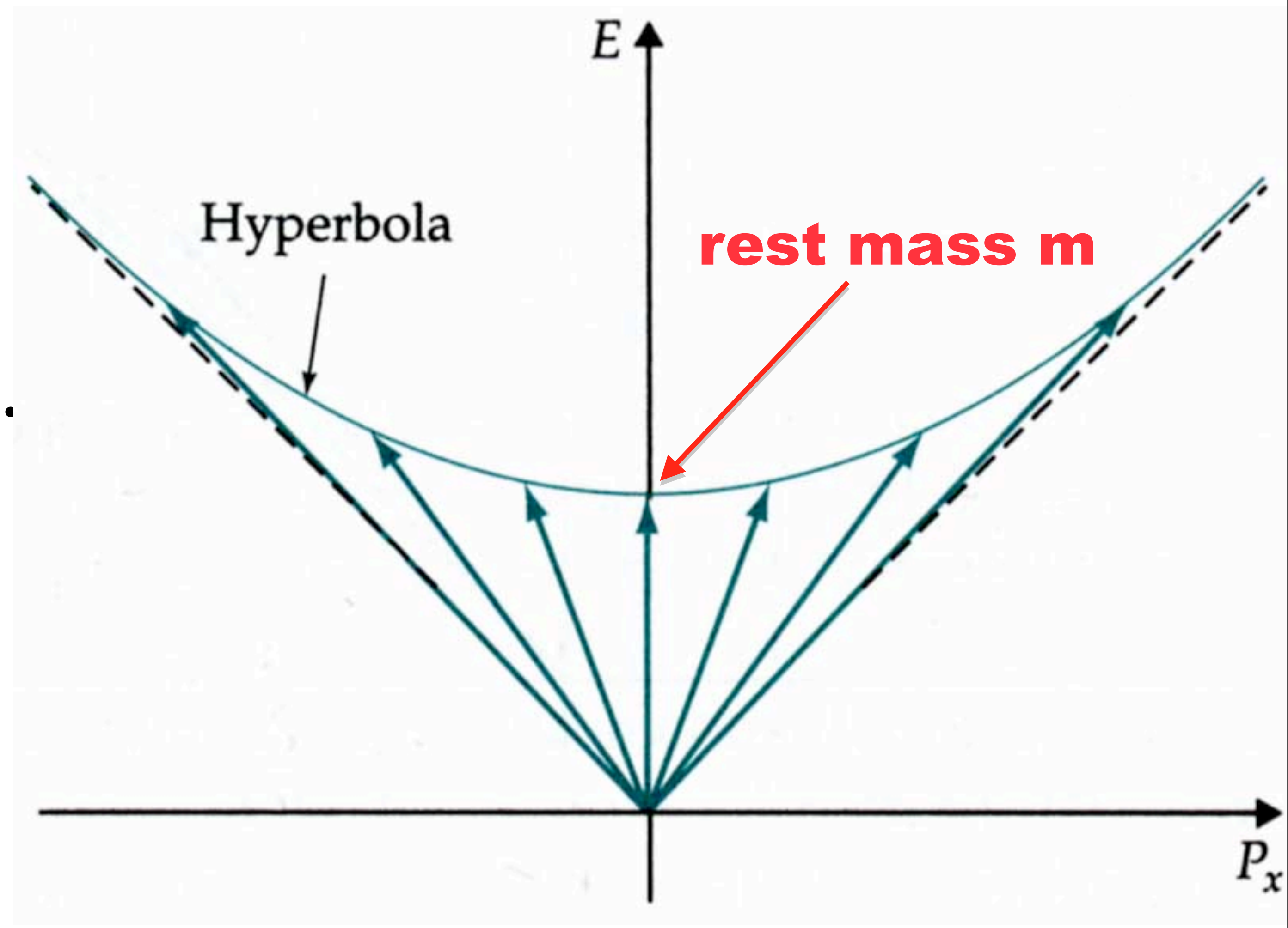
Relativistic Dynamics: Conservation of Four-Momentum

Energy-Momentum Diagrams ...

Mass is magnitude of $\mathbf{P}$:

$$m^2 = E^2 - p^2$$

Tip of vector lies on a hyperbola.



Arrow on E-P diagram is *frame dependent.*

Relativistic Dynamics: Conservation of Four-Momentum

Energy-Momentum Diagrams ...

Example:

$$v_x = 3/5$$

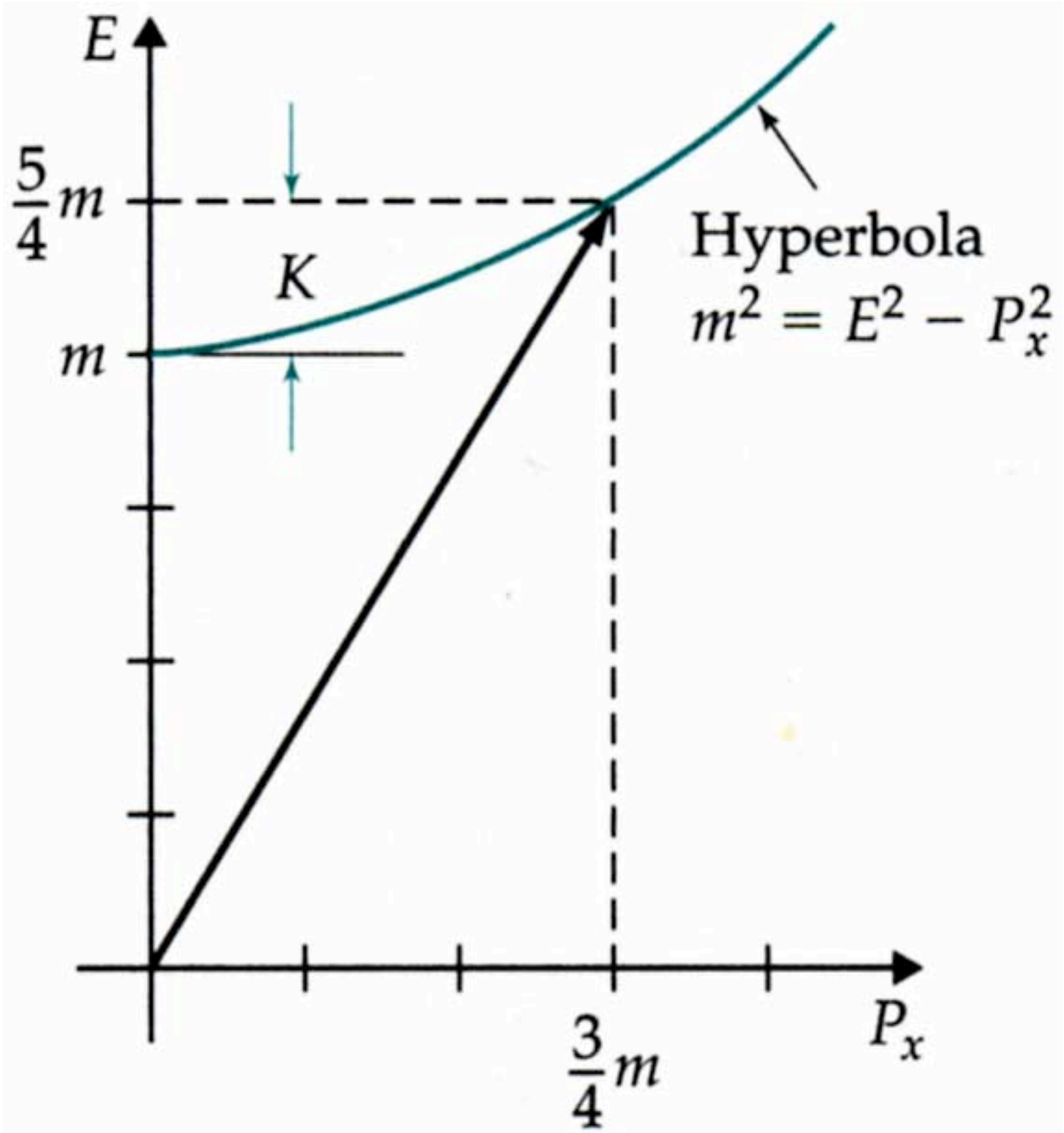
So $\mathbf{P}$ has slope $5/3$.

$$E = \tfrac{5}{4}m$$

$$P_x = \tfrac{3}{4}m$$

$$K = E - m = \tfrac{1}{4}m$$

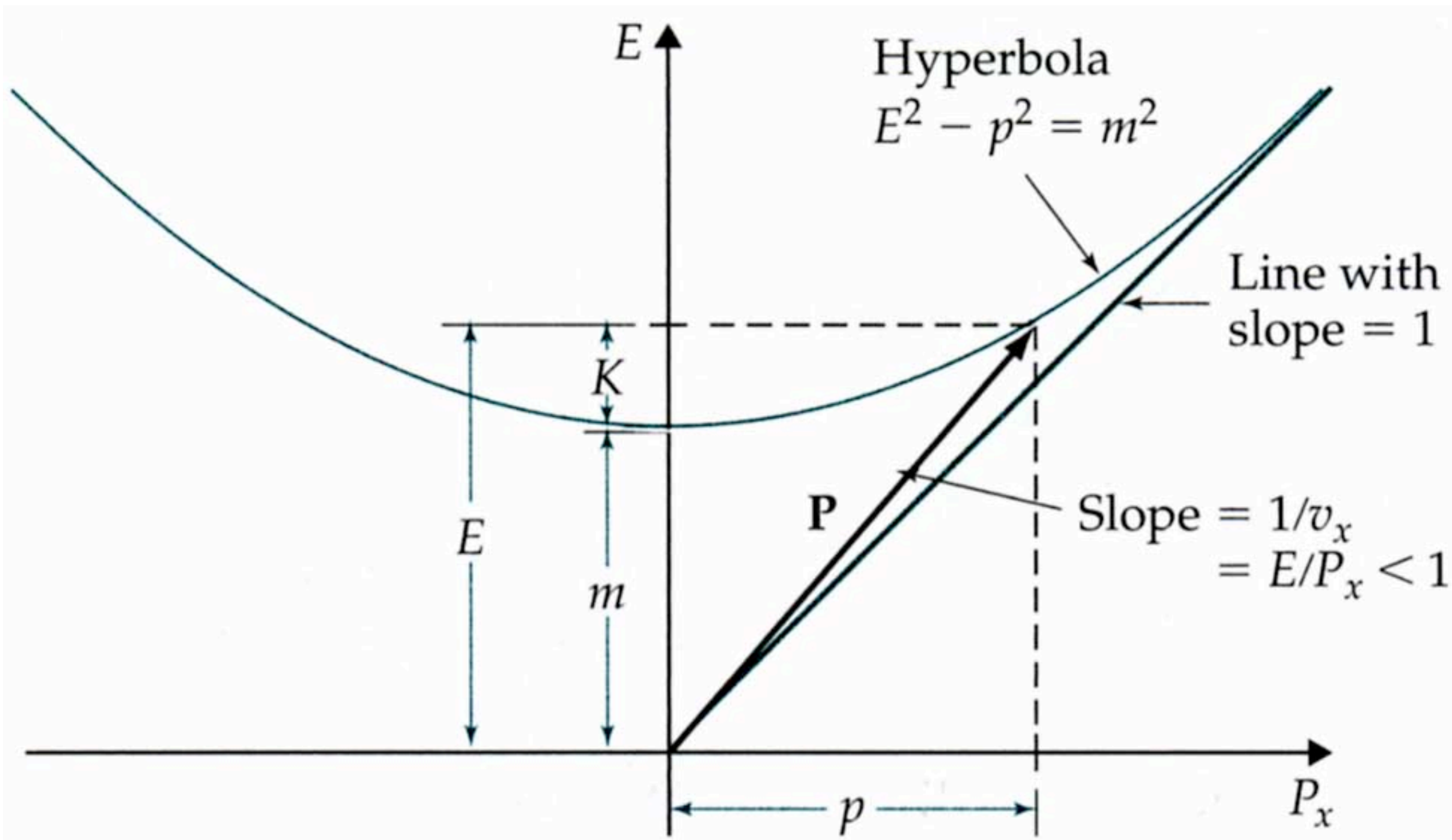
$$K > \tfrac{1}{2}mv^2 = \tfrac{9}{50}m$$



Relativistic Dynamics: Conservation of Four-Momentum

Energy-Momentum Diagrams ...

Summary:



Methods for Conservation of Four-Momentum Problems:

Two objects, 1 & 2, collide:

Before: $\mathbf{P}_1, \ \mathbf{P}_2$

After: $\mathbf{P}_3, \ \mathbf{P}_4$

Conservation of 4-momentum: **p1+p2=p3+p4**

$$\begin{pmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{pmatrix} + \begin{pmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{pmatrix} = \begin{pmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{pmatrix} + \begin{pmatrix} E_4 \\ P_{4x} \\ P_{4y} \\ P_{4z} \end{pmatrix}$$

Each component must be conserved *separately*.

Methods for Conservation of Four-Momentum Problems ...

Example 1:

Before:

Rock 1: $m_1 = 12 \text{ kg} \quad v_{1x} = 4/5$

Rock 2: $m_2 = 28 \text{ kg} \quad v_{2x} = 0$

After:

Rock 1: $v_{3x} = -5/13$

Question: What is v_{4x}?

Methods for Conservation of Four-Momentum Problems ...

Answer:

Before:

Rock 1:

$$E_1 = \frac{m_1}{\sqrt{1 - v_{1x}^2}} = \frac{m_1}{\sqrt{1 - (4/5)^2}} = 5/3(12 \text{ kg}) = 20 \text{ kg}$$

$$P_{1x} = \frac{m_1 v_{1x}}{\sqrt{1 - v_{1x}^2}} = (5/3)(4/5)(12 \text{ kg}) = 16 \text{ kg}$$

Rock 2:

$$E_2 = \frac{m_2}{\sqrt{1 - v_{2x}^2}} = m_2 = 28 \text{ kg}$$

$$P_{2x} = \frac{m_2 v_{2x}}{\sqrt{1 - v_{2x}^2}} = 0 \text{ kg}$$

Methods for Conservation of Four-Momentum Problems ...

After:

Rock 1:

$$E_3 = \frac{m_1}{\sqrt{1 - v_{3x}^2}} = \frac{m_1}{\sqrt{1 - (-5/13)^2}} = \frac{m_1}{\sqrt{144/169}} = (13/12)(12\ \mathrm{kg}) = 13\ \mathrm{kg}$$

$$P_{3x} = \frac{m_1 v_{3x}}{\sqrt{1 - v_{3x}^2}} = (13/12)(-5/13)(12\ \mathrm{kg}) = -5\ \mathrm{kg}$$

Conservation:

$$\begin{pmatrix} E_4 \\ P_{4x} \\ P_{4y} \\ P_{4z} \end{pmatrix} = \begin{pmatrix} E_1 \\ P_{1x} \\ P_{1y} \\ P_{1z} \end{pmatrix} + \begin{pmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{pmatrix} - \begin{pmatrix} E_3 \\ P_{3x} \\ P_{3y} \\ P_{3z} \end{pmatrix}$$

$$= \begin{pmatrix} 20\ \mathrm{kg} \\ 16\ \mathrm{kg} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 28\ \mathrm{kg} \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 13\ \mathrm{kg} \\ -5\ \mathrm{kg} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 35\ \mathrm{kg} \\ 21\ \mathrm{kg} \\ 0 \\ 0 \end{pmatrix}$$

Relativistic Dynamics: Conservation of Four-Momentum

Methods for Conservation of Four-Momentum Problems ...

After:

Check mass of Rock 2:

$$m = \sqrt{E_{4x}^2 - P_{4x}^2}$$
$$= \sqrt{(35 \text{ kg})^2 - (21 \text{ kg})^2} = 7 \text{ kg} \sqrt{5^2 - 3^2} = 28 \text{ kg}$$

Finally,

$$v_{4x} = \frac{P_{4x}}{E_4} = \frac{21 \text{ kg}}{35 \text{ kg}} = \frac{3}{5}$$

Relativistic Dynamics: Conservation of Four-Momentum

Methods for Conservation of Four-Momentum Problems ...

Example 2:

Problem: Solve previous problem using E-P diagram.

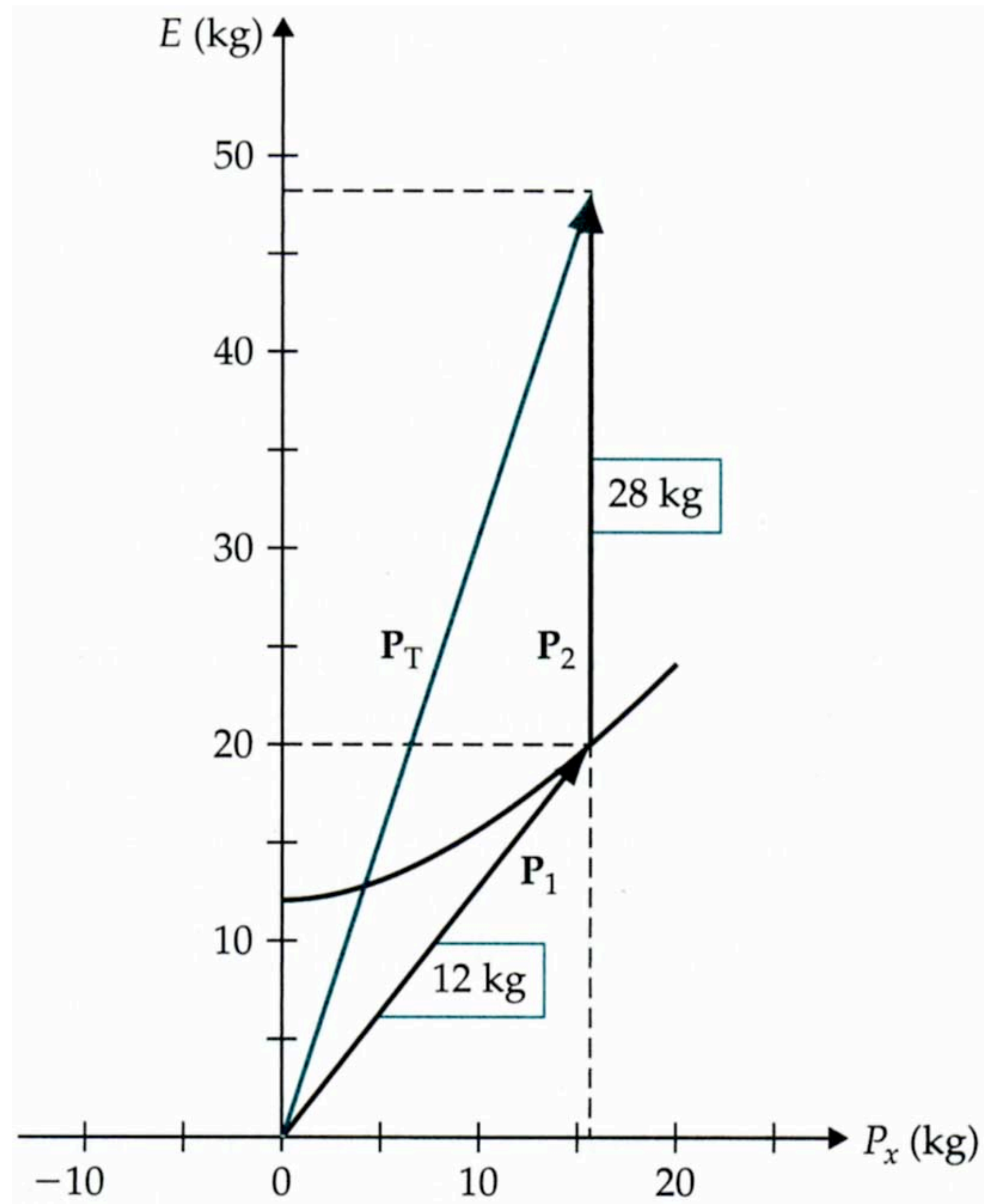
Answer: Add 4-momenta vectors.

Before:

Total 4-momentum $\mathbf{P}_T$:

$$E_T = 48 \text{ kg}$$

$$P_{Tx} = 16 \text{ kg}$$



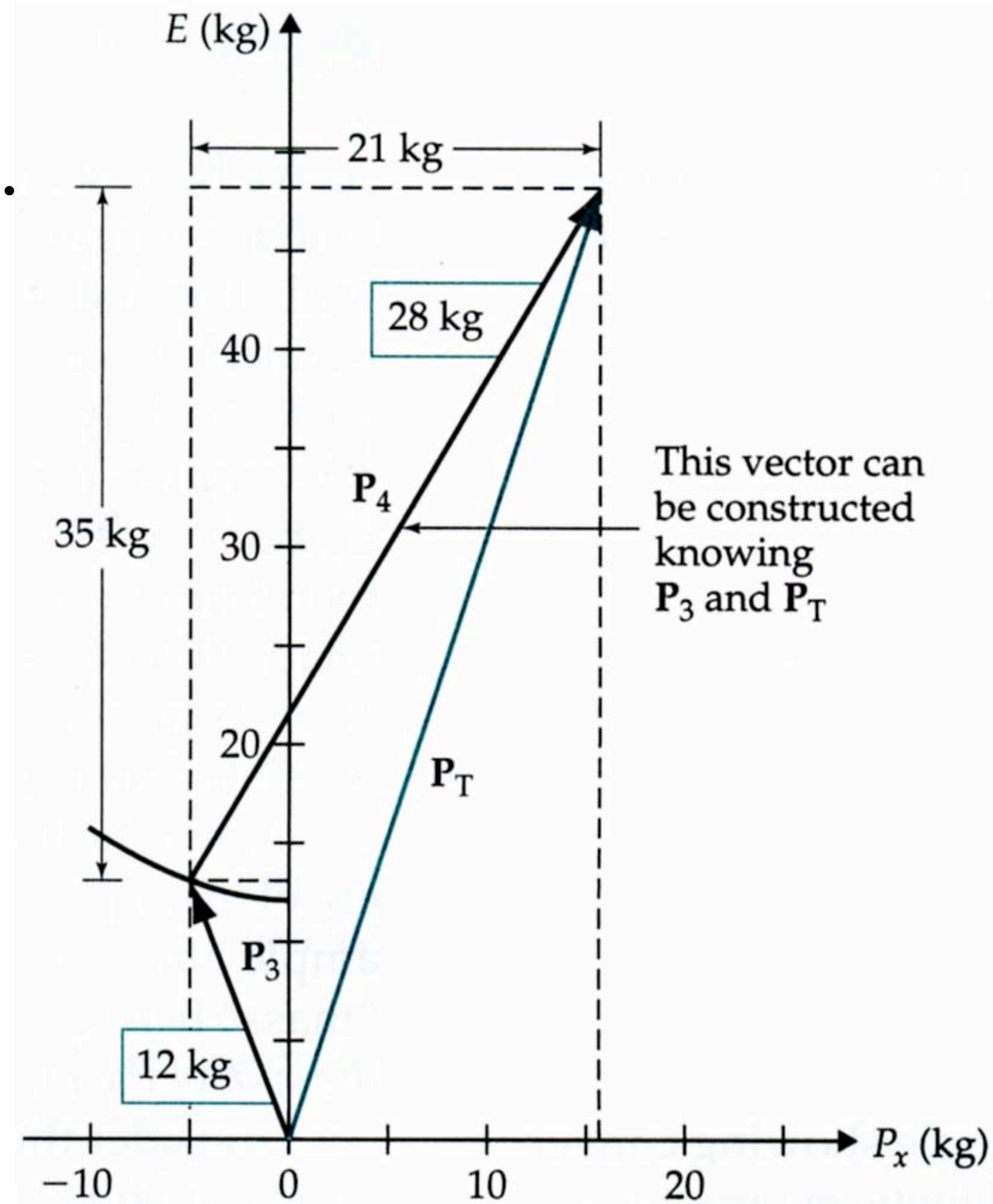
Methods for Conservation of Four-Momentum Problems ...

Example 2 ...

After: Vectors must add to same $\mathbf{P}_T$.

Know: $\mathbf{P}_3 = (E_3, P_{Tx}, 0, 0)$
$$= (13\ \mathrm{kg}, -5\ \mathrm{kg}, 0, 0)$$

Then: $\mathbf{P}_4 = \mathbf{P}_T - \mathbf{P}_3$.



Relativistic Dynamics: Conservation of Four-Momentum

Mass of a Collection of Particles:

Relativistic energy = Rest energy + Relativistic kinetic energy.

This is conserved.

Can mass and energy be inter-converted?

Yes!

One general case:
Mass of a system of particles $\neq$ Sum of individual masses.

mass is not additive

Relativistic Dynamics: Conservation of Four-Momentum

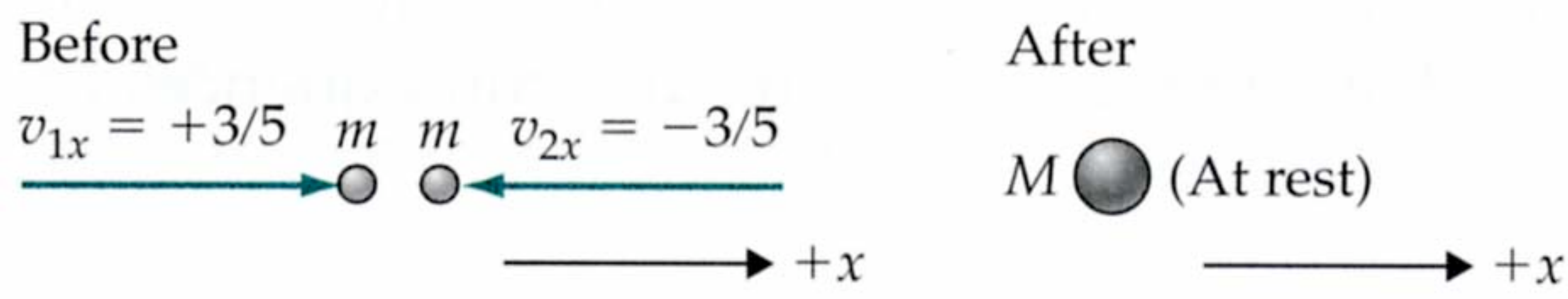
Mass of a Collection of Particles ...

Example: Inelastic collision.

Problem:

Two putty balls: $m = 4\ \mathrm{kg}$ at $v_{1x} = 3/5$ and $v_{2x} = -3/5$.

Collide and stick together into mass M.



Question: What is the total mass M after the collision?

Relativistic Dynamics: Conservation of Four-Momentum

Mass of a Collection of Particles ...

Before: 4-momentum

$$P_{1x} + P_{2x} = \frac{m(3/5)}{\sqrt{1-(3/5)^2}} + \frac{m(-3/5)}{\sqrt{1-(-3/5)^2}} = 0$$

After: 4-momentum

Mass at rest: Conservation implies total $P_x = 0$

Before: Energy?

Newtonian: Kinetic energy converted to heat (?). Also,

$$M = m + m = 2m = 8 \text{ kg}$$

Relativistic energy: Final object is at rest ($E_{\text{after}} = M$).

Conservation: $M = E_1 + E_2 = \frac{m}{\sqrt{1-(3/5)^2}} + \frac{m}{\sqrt{1-(-3/5)^2}}$

$$= \frac{2m}{\sqrt{16/25}} = \frac{10}{4} m = 10 \text{ kg}$$

"Mass" is larger by 2 kg!

Relativistic Dynamics: Conservation of Four-Momentum

Mass of a Collection of Particles ...

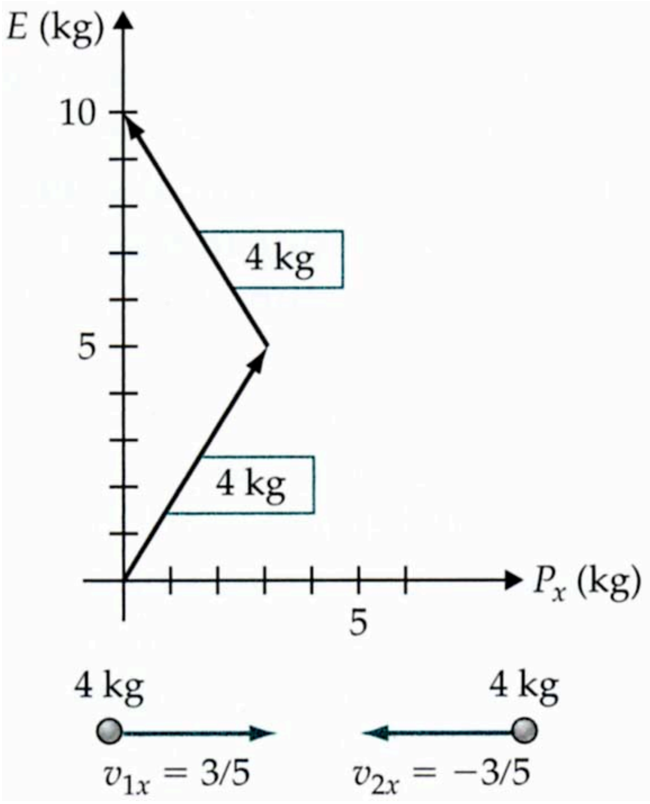
Where did this “mass” come from?

Option: Increase in thermal energy; if so, alot!

Rather: Mass is a property of the system as a whole.

Relativistic Dynamics: Conservation of Four-Momentum

Mass of a Collection of Particles ...

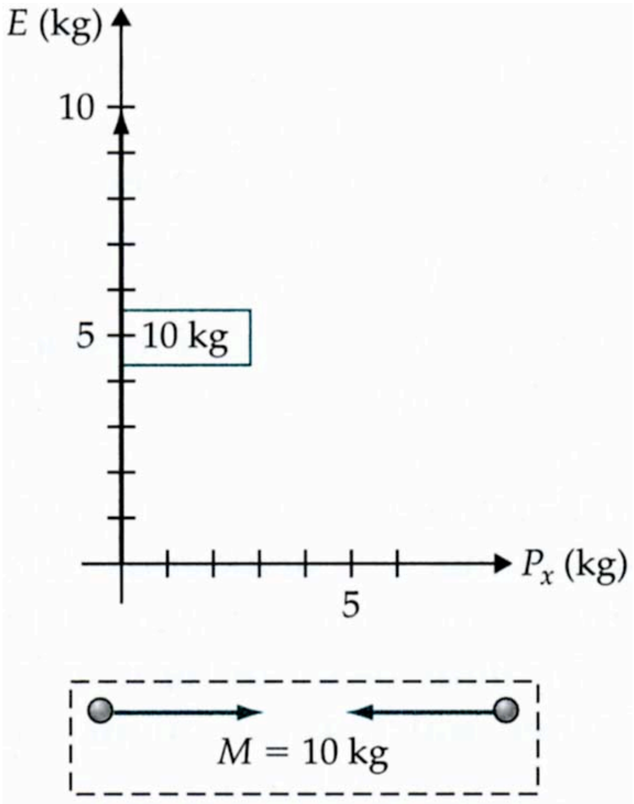


Extra mass already present *before* collision:

Balls moving as one system:

$$M = \sqrt{E_T^2 - P_{Tx}^2}$$

$$= E_1 + E_2 = 5\ \mathrm{kg} + 5\ \mathrm{kg} = 10\ \mathrm{kg}$$



Lessons:

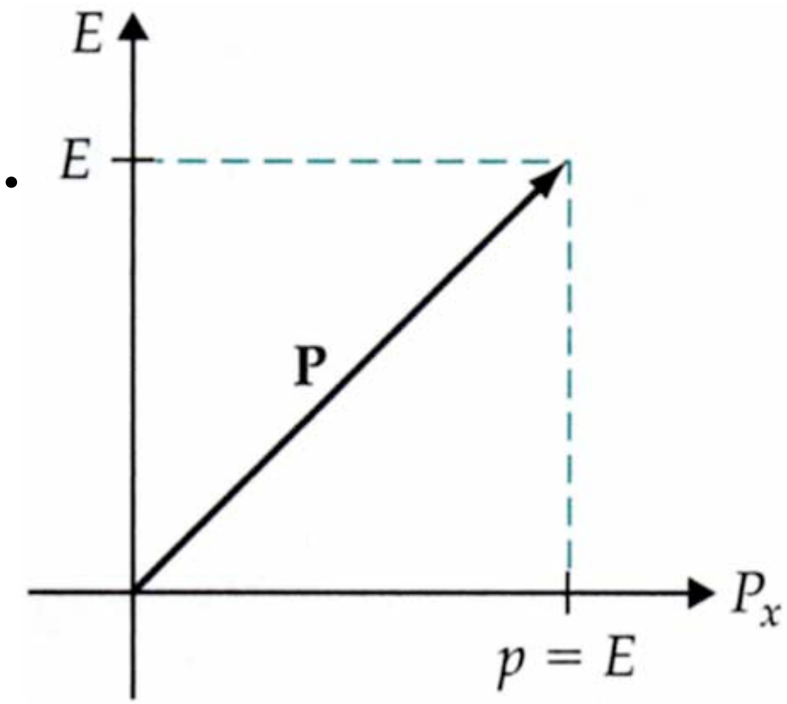
System mass unchanged.

Mass is *not* individually additive.

Mass is a property of *system as a whole*.

Relativistic Dynamics: Conservation of Four-Momentum

Four-Momentum of Light:
"Particle" of light: Burst or flash of short duration.
4-momentum of light flash moving +x with energy E.



$\mathbf{P}$ is parallel to worldline, which has slope = 1.

$$\frac{p}{E} = v = 1$$

So for light: $p = E$.

Light carries momentum: (E&M light "pressure" experiment, 1903)

Light has zero mass:

$$m^2 = E^2 - p^2 = 0$$

Relativistic Dynamics: Conservation of Four-Momentum

Four-Momentum of Light ...

Example 3:

Annihilation: Particle + Antiparticle produces light.

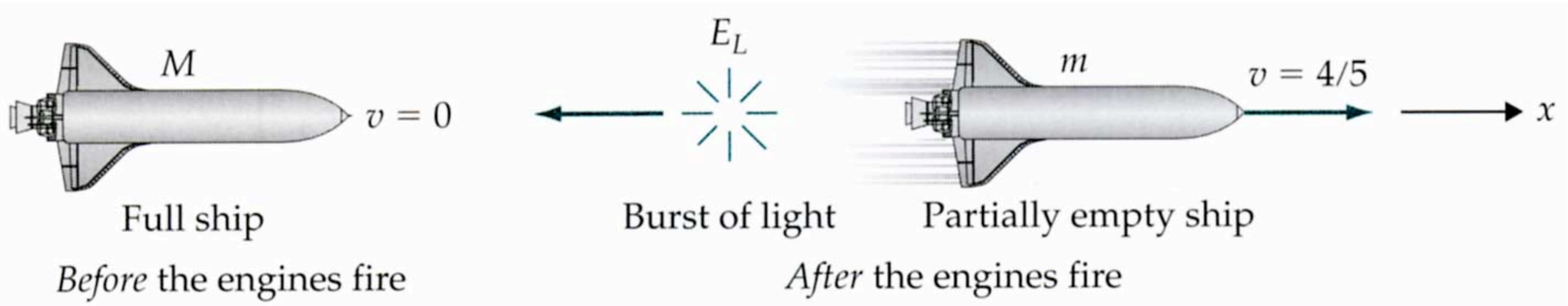
Problem:

Matter-Antimatter rocket engine produces light pulse.

Initially, rocket mass $M = 90,000\ \mathrm{kg}$.

Rocket fires, emitting pulse with energy E_L.

Rocket then moving at $v = 4/5$.



Question: What is the rocket's final mass m?

Four-Momentum of Light ...

Answer:

Before: 4-momentum $(P_t, P_x, P_y, P_z) = (M, 0, 0, 0)$.

After: System = Ship + Light Pulse.

Pulse: $p_L = E_L \ , \quad P_{px} = -E_L$

$$(P_{pt}, P_{px}, P_{py}, P_{pz}) = (E_L, -E_L, 0, 0)$$

Ship now mass m: $P_{st} = \dfrac{m}{\sqrt{1-v^2}} = \dfrac{m}{\sqrt{1-(4/5)^2}} = \dfrac{5}{3}m$

$$P_{sx} = +p = Ev = \frac{4}{3}m$$

$$P_{sy} = P_{sz} = 0$$

Relativistic Dynamics: Conservation of Four-Momentum

Four-Momentum of Light ...

Conservation:

$$\begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_L \\ -E_L \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{5}{3}m \\ \frac{4}{3}m \\ 0 \\ 0 \end{pmatrix}$$

$$M = E_L + \frac{5}{3}m$$

$$E_L = \frac{4}{3}m$$

$$v = \frac{4}{5}$$

$$m = \frac{1}{3}M = 30,000 \text{ kg}$$

Fuel mass = $M - m = 60,000 \text{ kg}$ to accelerate ship to $v = \frac{4}{5}$.

Relativistic Dynamics: Conservation of Four-Momentum

Particle Physics:

Elementary particles small: Can accelerate to v ~ c.

Definition: Electron volt is the energy gained by an electron passing through a 1V battery:

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} = 1.782 \times 10^{-36} \text{ kg}$$

Example:

Electron: $m = 0.511 \text{ MeV}$ at $v = 4/5$.

Relativistic energy: $E = \frac{m}{\sqrt{1 - v^2}} = \frac{5}{3} m = 0.852 \text{ MeV}$

Relativistic kinetic energy: $K = E - m = 0.341 \text{ MeV}$

Relativistic momentum: $p = Ev = 0.682 \text{ MeV}$

Particle Physics ...

Example 4: Kaon decays to 2 pions

Problem:

Kaon: K^0 meson with $m = 498 \text{ MeV}$.

Pion: π^0 meson with $m = 135 \text{ MeV}$.

Decay: $\Delta\tau = 36 \text{ ns}$

$$K^0 \rightarrow \pi^0 + \pi^0$$

Question: After decay what is the speed of the pions?

Relativistic Dynamics: Conservation of Four-Momentum

Particle Physics ...

Answer:

Conservation: $$\begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_1 \\ p_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E_2 \\ P_{2x} \\ P_{2y} \\ P_{2z} \end{pmatrix}$$

So: $P_{2y} = P_{2z} = 0$

$P_{2x} = -p_1$

Pion 2 moves in -x direction, same momentum: $p_2 = |P_{2x}| = p_1$.

They both have same mass and so same relativistic energies:

$$E_2 = \sqrt{m^2 + p_2^2} = \sqrt{m^2 + p_1^2} = E_1$$

Relativistic Dynamics: Conservation of Four-Momentum

Particle Physics ...

$$M = 2E_1$$

Plug in numbers:

$$E_1 = M/2 = 249 \text{ MeV}$$

$$p_1 = \sqrt{E_1^2 - M^2} = \sqrt{(249 \text{ MeV})^2 - (135 \text{ MeV})^2} = (209 \text{ MeV}$$

$$v_1 = \frac{p_1}{E_1} = \frac{209 \text{ MeV}}{249 \text{ MeV}} \approx 0.84$$

$$v_1 = v_2$$

Kaon mass converted to pion kinetic energy.

does not "soft land," that is, we assume that $y'(T(\theta)) < 0$, where $T(\theta)$ is the impact time. From (2) and the definitions of f_2 and f_4, we have

$$y'(t)=v \sin \theta f_2'(t)-gf_4'(t)=e^{-f_1(t)}(v \sin \theta-gf_3(t))$$

and hence the impact assumption is

$$f_3(T(\theta))>\frac{v}{g} \sin \theta. \tag{9}$$

In terms of the function

$$\rho(\theta)=\frac{R(\theta)}{v \cos \theta}$$

we have by (1),

$$R(\theta)=x(T(\theta))=v \cos \theta f_2(T(\theta))$$

and hence

$$T(\theta)=f_2^{-1}(\rho(\theta)).$$

The impact assumption (9) is therefore equivalent to

$$f_3(f_2^{-1}(\rho(\theta)))>\frac{v}{g} \sin \theta. \tag{10}$$

Now, $R(\theta)$ is differentiable if and only if $\rho(\theta)$ is differentiable. By (3), $\rho(\theta)$ is defined by $P(\rho(\theta), \theta)=0$, where

$$P(\rho,\theta)=v \sin \theta\rho-gf_4(f_2^{-1}(\rho)).$$

Finally, at $\rho = \rho(\theta)$,

$$\frac{\partial P}{\partial \rho}=v \sin \theta-gf_4'(f_2^{-1}(\rho))f_2^{-1\prime}(\rho)$$
$$=v \sin \theta-gf_3(f_2^{-1}(\rho))<0$$

by (10), and hence $\rho(\theta)$ is differentiable by the Implicit Function Theorem.[7]

[1]G. Galilei, *Two New Sciences* (Elzevirs, Leyden, 1638), translated with a new introduction and notes, by Stillman Drake (Wall and Thompson, Toronto, 1989), 2nd ed., p. 245.
[2]S. Drake and I. Drabkin, *Mechanics in Sixteenth-Century Italy* (University of Wisconsin Press, Madison, 1969), p. 91.
[3]K. Symon, *Mechanics* (Addison-Wesley, Reading, MA, 1953), p. 38.
[4]H. Erlichson, "Maximum projectile range with drag and lift, with particular application to golf," Am. J. Phys. **51**, 357–361 (1983).
[5]T. de Alwis, "Projectile motion with arbitrary resistance," Coll. Math. J. **26**, 361–366 (1995).
[6]J. Lekner, "What goes up must come down; will air resistance make it return sooner, or later?," Math. Mag. **55**, 26–28 (1982).
[7]R. Courant, *Differential and Integral Calculus* (Interscience, New York, 1961), Vol. II, p. 114.

Minkowski diagrams in momentum space

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I. INTRODUCTION

Minkowski diagrams in configuration space, with points representing events, are often used in undergraduate courses on special relativity. Similar diagrams in momentum space are seldom shown, and the object of this note is to demonstrate their pedagogical usefulness in discussing particle interactions. In configuration space each point has coordinates $(t,\mathbf{x})$; in momentum space the coordinates are $(E,\mathbf{p})$. Two examples should be sufficient to show how such diagrams can be used.

II. EXAMPLES

A. Fission

In this example there is just one space dimension: Minkowski space is two dimensional. A particle of mass m is represented by its mass shell, a hyperbola opening in the positive E direction, given by

$$\left(\frac{E}{c}\right)^2-p^2=(mc)^2.$$

Figure 1 shows two such mass shells belonging to masses m and $M>m$, each labeled by its mass. The scale on the energy axis is chosen as E/c rather than E, so the two mass shells cross the E/c axis at mc and Mc, respectively. Each point on an m mass shell represents a state of a particle of mass m, i.e., possible values of its energy and momentum. A vector from the origin to such a point represents the energy–momentum $(E-p)$ vector of that state.

Consider a particle of mass M at rest, say a uranium nucleus, that undergoes fission to two particles of equal mass m. The vertical arrow in Fig. 1 represents the original uranium $E-p$ vector. $E-p$ conservation implies that the $E-p$ vectors of the two fission fragments add up to the original one, and since the total momentum is zero, the momenta of the two fission fragments must be negatives: their $E-p$ vectors have opposite p components. Symmetry of the m mass shell about the E/c axis then implies that their E/c components are equal, and conservation then implies than each E/c component is equal to $Mc/2$. It is clear from the dia-

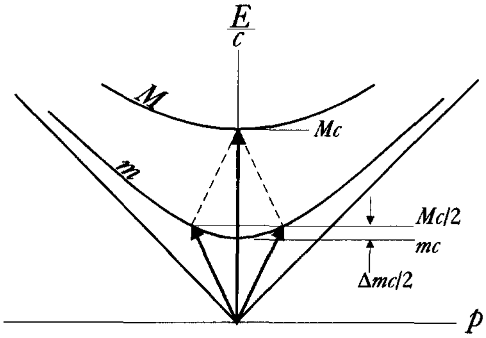

Fig. 1. Fission.

gram that each E/c component is higher than the point at which the m mass shell crosses the E/c axis, i.e., greater than mc, so $m<\frac{1}{2}M$,

$$Mc-2mc\equiv\Delta mc>0.$$

As the fission fragments interact with their surroundings, they slow down and eventually come to rest. Then their total E/c is $2mc$, so the energy they give up to their surroundings is just $\Delta E=\Delta mc^2$. This is the real content of the famous equation $E=mc^2$, involving measurable energy changes rather than absolute values relative to some more or less arbitrarily chosen zero of energy. Note that the mass of the fission fragments is not determined. But because their energies are both $\frac{1}{2}Mc^2$, the mass m and momentum p are related by

$$\left(\tfrac{1}{2}Mc\right)^2-p^2=(mc)^2.$$

The logical order in which to present this in class is first to draw the M mass shell, then the two $E-p$ vectors of the fission fragments, and only then to draw in the m mass shell.

This example is easily generalized to fission fragments of unequal masses. Also, a similar diagram can be used to illustrate fusion or the binding energy of the deuteron. Then M is less than $2m$, and the M mass shell crosses the E/c axis below $2mc$.

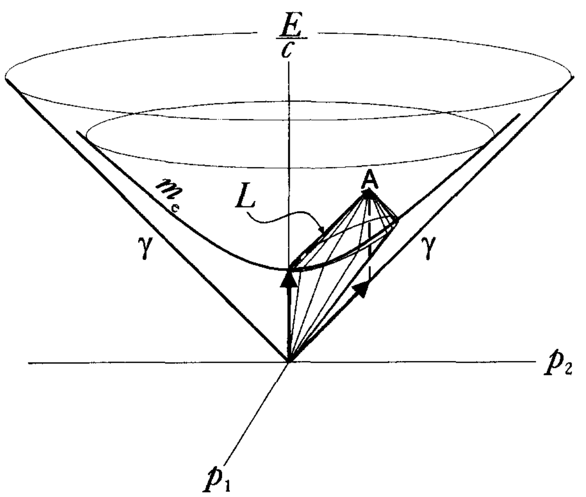

Fig. 2. Compton scattering.

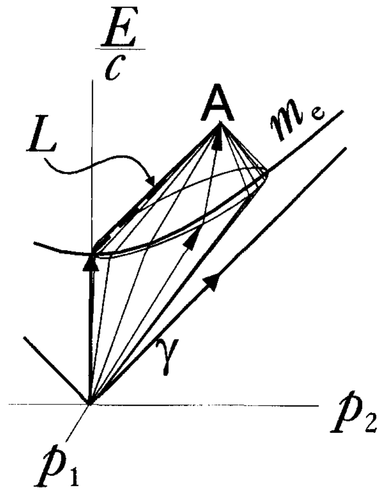

Fig. 3. Compton scattering (detail).

B. Compton scattering

Now take Minkowski space to be three dimensional, as in Fig. 2. The mass shell is now a hyperboloid of revolution. In the figure the intersection of the $(E/c,p_2)$ plane with the electron mass shell is the hyperbola labeled m_e, and the intersection with the light cone consists of the two lines labeled γ. The light cone is the mass shell of the photon, whose equation is

$$\left(\frac{E}{c}\right)^2-|\mathbf{p}|^2=0.$$

The vertical arrow in Fig. 2 is the $E-p$ vector of an electron at rest, and the other arrow represents an incident photon. The system's total $E-p$ vector is represented by the point labeled A (the vector to A is not drawn to avoid confusion). After scattering, the electron $E-p$ vector (again on the electron mass shell) plus the scattered photon $E-p$ vector (again on the light cone) must add up to A. A way to draw this is to construct an inverted light cone L with its vertex at A. The $E-p$ vectors of all possible scattered photons arrive at A from the closed curve, almost a circle, at which L intersects m_e in this three-dimensional space–time (in four dimensions this would be a closed surface, almost a sphere).

Figure 3 is an enlargement of part of Fig. 2. One possible combination of scattered electron and photon $E-p$ vectors is indicated with arrows. The direction of the scattered photon is obtained by projecting its $E-p$ vector onto the (p_1,p_2) plane, so the different lines on the cone represent photons moving in different directions. It is immediately evident that the photon energy E, and hence its frequency ν and wavelength λ, are determined by its direction.

III. CONCLUSION

Other particle interactions can also be visualized on similar Minkowski diagrams. The goal of this note is to show how the dynamics can be visualized, not to perform the calculations. The equations of the mass shells can be used, however, as a starting point for going on to the calculations.

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Analysing collisions using Minkowski diagrams in momentum space

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Abstract

Momentum space and Minkowski diagrams are powerful tools for interpreting and analysing relativistic collisions in one or two spatial dimensions. All relevant quantities that characterize a collision, including the mass, velocity, momentum and energy of the interacting particles, both before and after collision, can be directly seen from a single Minkowski diagram. Such diagrams can also be useful for analysing the differences between Newtonian and relativistic mechanics. As an interesting example, a simple derivation of the Compton wavelength shift formula, based on the geometrical properties of such momentum space diagrams, is also presented.

1. Introduction

A paper by Saletan [1] presented a geometric representation of relativistic interactions, using Minkowski diagrams in momentum space. The method has great intuitive and pedagogical power. In this paper the most important features of such diagrams are discussed in a somewhat more systematic way and some extensions to Saletan's original paper are presented.

For the purposes of this paper the particular units for mass m, energy E and momentum p are unimportant. Multiplying by suitable factors of the speed of light c, all three quantities, i.e. mc^2, E and pc, can be expressed in units of energy. Throughout the paper this convention is adopted and the notation (au) for 'arbitrary unit of energy' is used.

Minkowski diagrams are widely used in configuration space where points represent events, expressed in coordinate notation as (ct, x) (for a 2D diagram) or (ct, x, y) (for a 3D diagram), corresponding to one or two spatial dimensions, respectively. By convention, time ct is along the vertical axis and space is represented along the horizontal axis (or axes) in such diagrams. Drawing a Minkowski diagram with three spatial dimensions is not possible, since it would require a four-dimensional image. Luckily, for many important relativistic phenomena the third spatial dimension can be omitted from the discussion.

As proposed by Saletan [1], Minkowski diagrams can also be constructed in momentum space, with energy E represented along the vertical axis and momentum p represented along

the horizontal axis (or axes). In this diagram a point (E, pc) or (E, p_xc, p_yc) expresses the *state* of a particle as it moves with energy E and momentum p. Some general features of these diagrams can be understood by considering the relativistic expressions for energy and momentum:

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

and

$$p = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} v \quad \text{(in 1D)}, \tag{2}$$

$$p_x = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} v_x \qquad \text{and} \qquad p_y = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} v_y \quad \text{(in 2D)}, \tag{3}$$

yielding the well-known relation between mass, energy and momentum:

$$E^2 - (pc)^2 = (mc^2)^2. \tag{4}$$

Equation (4) expresses the invariance of mass. Different observers will disagree about the energy and momentum of a given moving particle, but the particular algebraic combination of E and p on the left-hand side of (4) gives the same value for all observers: mass is invariant.

It can be seen from (4) that in momentum-space Minkowski diagrams all possible states of a given particle with mass m lie on the same hyperbola (for motion in 1D) or hyperboloid of revolution (for motion in 2D). A vector drawn from the origin to any point on this hyperbola (or hyperboloid) is the energy–momentum vector of the particle.

The energy–momentum vector can be used effectively for the visualization of all relevant properties of the particle; the intersection point of the E-axis with the hyperbola (or hyperboloid) on which the tip of the vector lies represents the particle's mass; and the slope of the vector relative to the E-axis represents its speed (as a fraction of c). An attractive feature of such diagrams is that the total energy and total momentum of a system of particles can be visualized in a single step by adding up the individual energy–momentum vectors and looking at the vertical or horizontal components of the resulting total energy–momentum vector. In what follows, it will be demonstrated how all important features of both inelastic and elastic collisions can be seen at a single glance on such Minkowski diagrams.

2. Collisions

In classroom problems involving collisions, it is typically the masses and initial velocities of the colliding particles that are given; students are then supposed to find

(a) the mass and velocity of the resulting single particle (in *perfectly inelastic collisions*) and
(b) the final velocities of the colliding particles (in *elastic collisions*).

2.1. Perfectly inelastic collision between two particles

Figure 1 presents the momentum-space Minkowski diagram of a *perfectly inelastic* collision between two particles having masses of m_A and m_B, respectively. The masses and initial velocities are the following:

$m_Ac^2 = 1$ (au), $m_Bc^2 = 2$ (au), $v_A = -0.5\,c$, $v_B = 0.6\,c$.

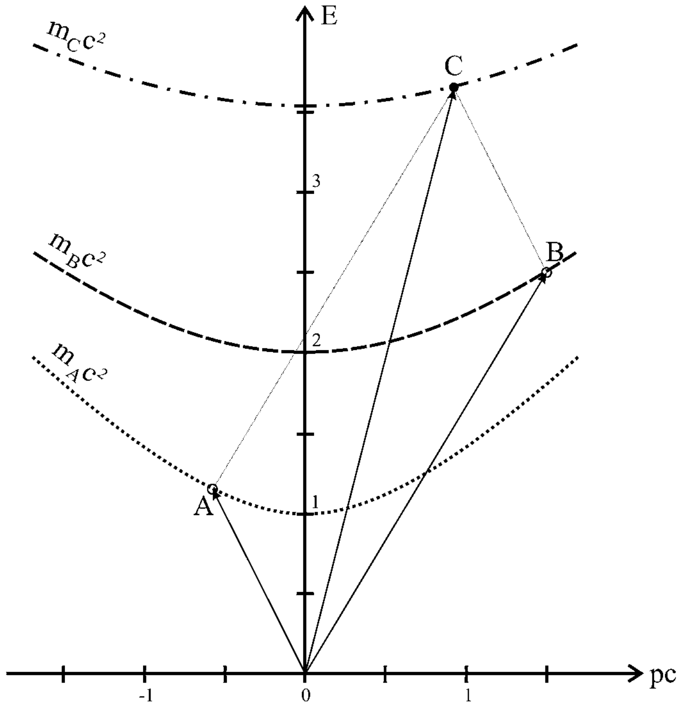

Figure 1. Perfectly inelastic collision in 1D. For all three particles involved in the collision, the *mass* appears as the intersection of the *E*-axis with the hyperbola representing the given particle; the *velocity* appears as the slope of the given energy–momentum vector (relative to the *E*-axis); and the *energy* and *momentum* of each particle appear as the vertical and horizontal components, respectively, of the energy–momentum vector.

In the collision the two particles are combined into a single particle m_C which moves with a velocity v_C. The combined particle is represented by an energy–momentum vector which is the vectorial sum of the energy–momentum vectors of the two colliding particles.

Writing the equations for conservation of energy and conservation of momentum and solving them algebraically yields

$m_C c^2 = 3.536$ (au) and $v_C = 0.252\, c$.

Even though these numbers cannot be deduced to such precision from figure 1, the figure does 'tell the entire story' of the collision and provides *quantitative* answers to all relevant questions for all three particles involved in the collision (as explained in the figure caption).

The solution for m_C and v_C is uniquely determined. When treating inelastic collisions algebraically, this fact is usually explained by noting that we have *two equations* (conservation of energy and conservation of momentum) for *two unknowns*. Using a Minkowski diagram such as figure 1, the uniqueness of the solution is trivially apparent: the combined particle is represented as the sum of two vectors, so both the 'length' of this sum vector (i.e. $m_C c^2$) and its orientation (i.e. $v_C\, c^{-1}$) are uniquely determined.

Perfectly inelastic collisions yield a unique value for the mass m_C of the created particle and its velocity vector v_C, *regardless of the number of spatial dimensions* involved. In the algebraic treatment this is explained by noting that as we increase the number of dimensions—and hence the number of *unknowns* in the components of v_C—the number of available *equations* increases at the same rate, hence the total number of independent equations (and the total number of unknowns) becomes 3 and 4, in 2D and 3D collisions, respectively. In the Minkowski diagram treatment, again, the reasoning is simpler: the energy–momentum vector of the created particle is the sum of two energy–momentum vectors; hence its 'length' (i.e. m_C) and its orientation (which tells us the components of v_C) are uniquely determined.

One of the most important features of perfectly inelastic collisions is that $m_C > m_A + m_B$, i.e. the final particle has larger mass than the sum of the two original masses. This point is immediately apparent in the figure. Even the numerical value of the mass excess $m_C - (m_A + m_B)$, i.e. the part of the kinetic energy which was converted to mass during the collision, can be read off from the diagram in a straightforward way.

Fission is an inelastic collision such as the one presented in figure 1, but 'played backward in time'. In that case a diagram quite similar to figure 1 can be used for the analysis. Fission was discussed in detail in Saletan's paper [1], including the explanation of mass defect, so it will not be discussed here.

2.2. Elastic collision between two particles in 1D

Let us consider an *elastic* collision with the same initial conditions as in figure 1. The task is to find the final velocities $v_{A'}$ and $v_{B'}$ of the two particles after they bounced off each other. In the momentum-space Minkowski diagram we should thus find two energy–momentum vectors which satisfy the following requirements:

(1) they must add up to produce the same combined energy–momentum vector as the two initial energy–momentum vectors did (i.e. their sum must point at C);
(2) the tips of the two vectors must lie on the hyperbolae representing $m_A c^2$ and $m_B c^2$, respectively.

A simple geometrical method to solve the problem is presented in figure 2. Again, *a single diagram* tells the whole story: all particle speeds, energies and momenta, both before and after collision, are shown quantitatively.

A unique solution exists for $v_{A'}$ and $v_{B'}$ in a one-dimensional elastic collision. Just like for inelastic collisions, this fact is usually explained from algebra by noting that we have the same number of equations as unknowns: two equations (conservation of energy and conservation of momentum) for two unknowns. However, a Minkowski diagram such as figure 2 even tells 'this part of the story' at a single glance: if two hyperbolae intersect, they intersect at exactly *two* points. One of these two points corresponds to the initial configuration of the particles. The other uniquely determines the state of the particles after collision.

2.3. Elastic collision between particles in 2D

It is straightforward to generalize the previous discussion to collisions in two spatial dimensions. The Minkowski diagram becomes a three-dimensional figure where each colliding particle is represented by a hyperboloid of revolution, rather than a hyperbola, as equations (3) and (4) suggest. Figure 3(a) presents such a diagram for the same initial parameters as figures 1 and 2. (The axis along which the two particles were moving before collision is the x-axis.)

In the usual algebraic discussion of elastic collisions in 2D it is noted that there is an *infinite number* of possible final states for the two particles: we have four unknowns (the x- and y-components of the final velocities $\boldsymbol{v}_{A'}$ and $\boldsymbol{v}_{B'}$), but only three independent equations (energy conservation plus the two components of momentum conservation). Again, however, a single glance at the Minkowski diagram in figure 3(a) is sufficient to provide a simple alternative explanation for the infinite number of possible final states: the two hyperboloids

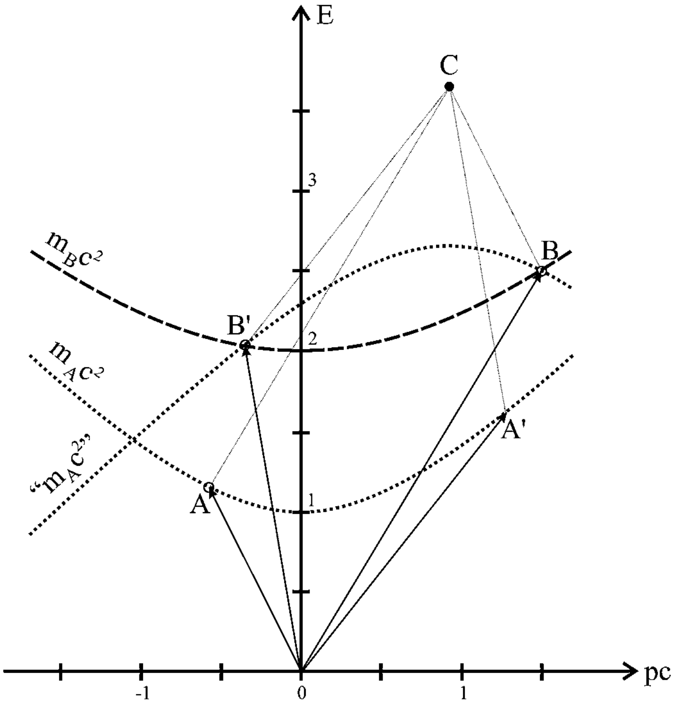

Figure 2. Elastic collision in 1D. From C as the origin, an upside-down version of the hyperbola m_Ac^2 is drawn. This hyperbola (denoted with "m_Ac^2") intersects the hyperbola m_Bc^2 at two points. These are the only two points that satisfy the two requirements listed in the text. B represents particle m_B *before* the collision, B′ represents the same particle *after* the collision. By using the parallelogram rule of vector addition, it is straightforward to draw the final energy–momentum vector for particle m_A (denoted by A′).

intersect in a *continuous curve*, not just two discrete points as in the 1D case (see figure 2), resulting in an infinite number of possible solutions.

Figure 3(b) depicts the intersection curve of the two hyperboloids: a tilted ellipse. Figure 3(c) shows this projected ellipse on the (p_xc, p_yc) plane. The kind of partial, 'momentum-only' Minkowski diagram depicted in figure 3(c) may also have its pedagogical uses. Any vector pointing from origin O to an arbitrary point on the ellipse gives a possible momentum vector for particle m_B, and the vector pointing from that point of the ellipse to point C_p gives the corresponding momentum vector for particle m_A. The total momentum vector of the system is represented by $\mathbf{OC_p}$.

3. Special cases

3.1. Special case #1: 2D elastic collision between a particle and an identical particle at rest

A frequently discussed special case of 2D elastic collisions is when two identical particles collide (i.e. $m_A = m_B$), and one of them is initially at rest. A well-known result of the Newtonian treatment of this problem is that after collision the two particles move *perpendicularly* to each other. That this result is correct only for small initial speeds can be readily illustrated using momentum-only Minkowski diagrams.

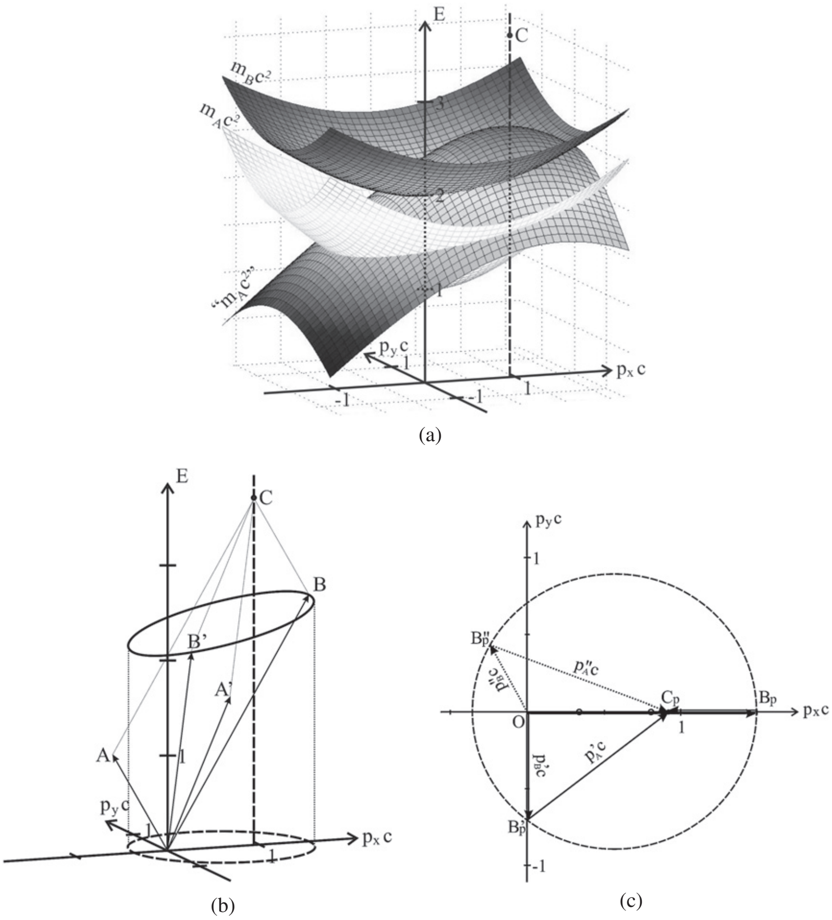

Figure 3. Elastic collision in 2D. (a) As a straightforward generalization of figure 2, an upside-down version of the hyperboloid m_Ac^2 is drawn from C taken as the origin. The intersection curve of this inverted hyperboloid (denoted by 'm_Ac^2') with the hyperboloid m_Bc^2 gives the possible loci for the tip of the energy–momentum vector of particle m_Bc^2 in this interaction. (b) The intersection curve of the two hyperboloids: a tilted ellipse. The energy–momentum vectors A′ and B′ represent *one* of the infinite number of possible final states (corresponding to the case when particle m_B moves along the negative y-axis after collision). A and B denote the energy–momentum vectors before collision. The projection of the intersection ellipse on the (p_xc, p_yc) plane is another ellipse, shown as a broken curve. (c) The projected ellipse on the (p_xc, p_yc) plane. All points with a subscript 'p' are projections of the corresponding points on (b) onto the (p_xc, p_yc) plane. B_p'' describes another possible final configuration, added here for illustration. (The foci of the ellipse are shown as two small empty circles.)

Figure 4 presents momentum-only Minkowski diagrams for this type of collision, with $m_Ac^2 = m_Bc^2 = 1$ (au). As seen in figure 4(a) (corresponding to a highly relativistic case) the relative orientation of the final momentum vectors can deviate significantly from 90°. As we approach the classical case, the relative orientation of the final momentum vectors becomes closer and closer to perpendicular: for decreasing v_A the eccentricity of the ellipse decreases and the diagram begins to resemble a circle. In the Newtonian approximation the diagram *is* a circle with OC_p as its diameter; hence each possible momentum configuration corresponds

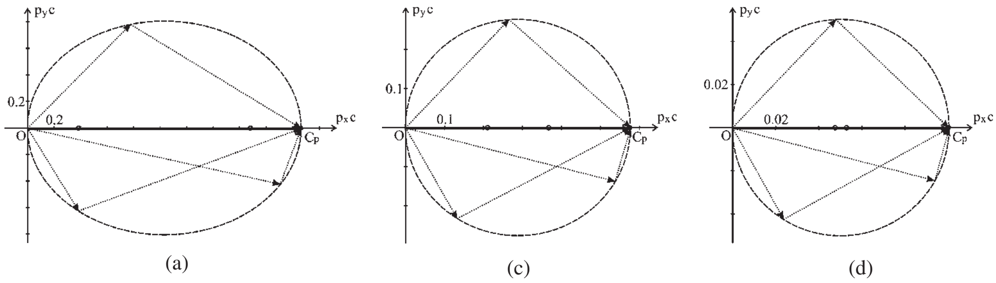

Figure 4. Elastic collision in 2D between identical particles, one of them at rest. The initial speed of particle m_A is (a) $v_A = 0.9\ c$, (b) $v_A = 0.5\ c$, and (c) $v_A = 0.1\ c$. Just like for figure 3(c), the relativistic treatment always results in an *ellipse* for the curve describing the possible configurations after collision. In this special case, however, O and C_p (the starting point and the tip of the total momentum vector, respectively) are the *two end points of the major axis* of the ellipse. Some possible configurations for the final momenta of the two particles are depicted as vectors with dotted lines. (Again, the foci of the ellipses are shown as two small empty circles.) The figure also illustrates quantitatively how the Newtonian formula for total momentum $m_A \cdot v_A$ deviates from the correct relativistic one, but approaches it at small velocities. The total momentum is (in (au)) 2.06, 0.58 and 0.10, for figures 4(a), (b) and (c), respectively. The corresponding Newtonian values would be 0.90, 0.50 and 0.10, respectively.

to drawing two adjacent chords between the end points of a diameter, yielding, according to Thales' theorem, perpendicularly oriented final momenta.

3.2. Special case #2: Compton scattering

Another special case for 2D elastic collision is Compton scattering: here a photon collides with an electron originally at rest. As a result, part of the photon's energy is transferred to the electron, causing it to move in some direction, while the photon itself will be scattered in some other direction. The energy loss for the photon results in a wavelength shift. The main points of the momentum-space Minkowski diagram analysis for this interaction were discussed in Saletan's paper [1], so I just give a short summary here and make some additional remarks.

For any photon $E^2 - (pc)^2 = 0$, so photons are represented in a momentum-space Minkowski diagram with a *cone*, denoted with ph_A in figure 5(a). (Unlike the 'light cone' in configuration space Minkowski diagrams, the cone in figure 5(a) represents *all* photons, travelling with *any* energy at *any* time on the (x, y) plane.) That photons are particles with zero mass is immediately apparent from the Minkowski diagram, since the cone in figure 5(a) can be thought of as a degenerate hyperboloid which intersects the E-axis at $E = 0$.

The electron is represented by the hyperboloid denoted by m_B. For simplicity, the energy unit in this example is chosen so that the electron's mass is unity: $m_B c^2 = 1$. The incoming photon is assumed to move along the positive x-axis with energy $E_A = p_A c = 0.5$. The geometrical method to find the final states for the photon and the electron is similar to figure 3(a) (see captions to figure 5). Figure 5(b) shows the intersection curve (again, a tilted ellipse). As in figure 3(b), the interaction ellipse can next be projected onto the $(p_x c, p_y c)$ plane. The result is shown in figure 5(c). Two special features are apparent on this ellipse, compared to figure 3(c) (which represented an arbitrary collision between two material particles):

(1) the left end-point of the major axis is *at the origin*;

(2) point C_p, the projection of C, coincides with the *right-hand-side focus*.

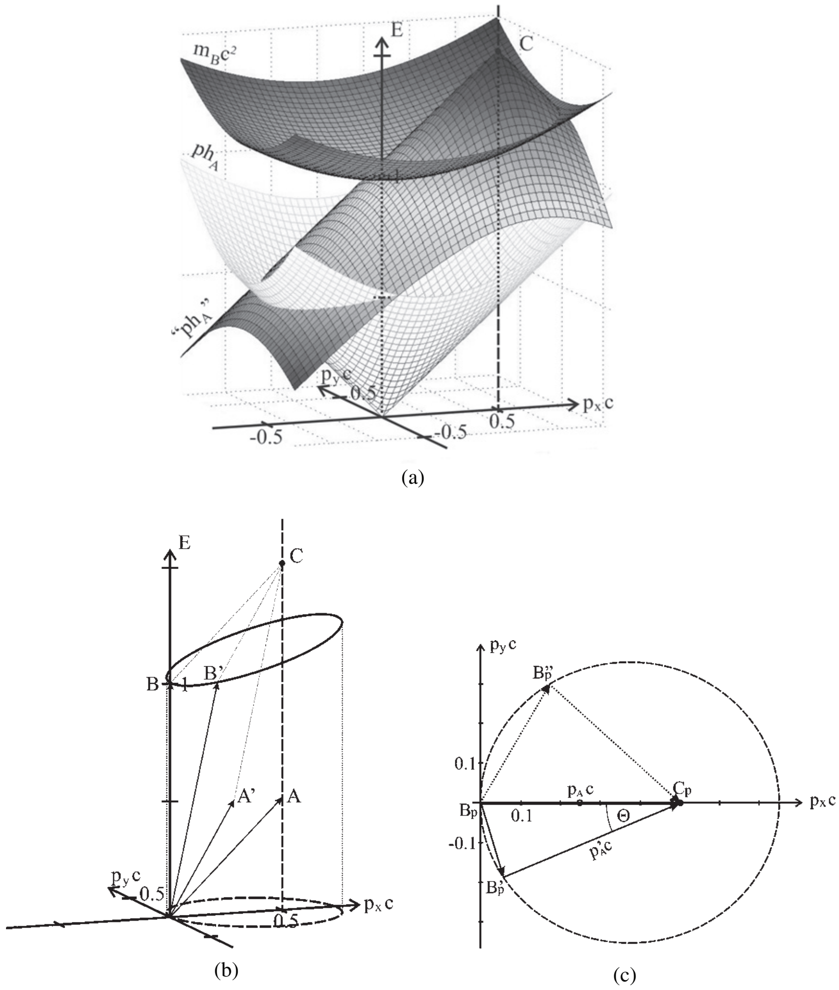

Figure 5. Compton scattering. (a) First the total energy–momentum vector for the system is found (using our initial parameters, the E-component of this vector is 1.5 and the p_xc-component is 0.5). The tip of this vector is denoted by C. An upside-down version of the photon cone (denoted by 'ph$_A$') is then drawn from C; and the intersection curve between the hyperboloid m_Bc^2 and the inverted cone "ph$_A$" represents the possible final states for the electron after the interaction. (b) The intersection curve: a tilted ellipse. B′ represents *one* possible final state for the electron and A′ denotes the corresponding final state for the photon. A and B denote the energy–momentum vectors before collision. (c) The projected ellipse on the (p_xc, p_yc) plane. B_p' and B_p'' denote two possible final configurations. The vector pointing from the origin to any point on the ellipse represents a possible momentum vector for the electron and the vector pointing from that point of the ellipse to point C_p gives the corresponding momentum vector for the scattered photon. (The foci of the ellipse are shown as two small empty circles.)

The relative directions of motion for the electron and photon are immediately apparent from the figure. In addition, the length of the momentum vector for the photon in figure 5(c) directly gives the numerical value for its energy too. (Note that the precise shape of the ellipse depends on the energy of the incoming photon, as will be discussed below, but all ellipses, regardless of the initial photon energy, share the two special features listed above.)

Some qualitative features of Compton scattering are immediately obvious from figure 5(c): (1) all scattered photons have smaller energies (i.e. larger wavelength) than the initial photon; (2) as the scattering angle increases, the energy of the scattered photon decreases, and the backward scattered photon has the smallest energy (i.e. the largest wavelength).

The equation of the 'Compton ellipse' of figure 5(c) can be found in a straightforward way. The equation of the hyperboloid for $m_B c^2$ is

$$E = \sqrt{m_B^2 c^2 + p_x^2 + p_y^2} \cdot c \tag{5}$$

and the equation of the upside-down cone 'ph_A' is

$$E = \left(p_A + m_B c - \sqrt{(p_x - p_A)^2 + p_y^2}\right) \cdot c. \tag{6}$$

Equating the right-hand sides of equations (5) and (6) yields the equation of the projected intersection ellipse in the (p_x, p_y) plane:

$$\frac{\left(p_x c - \frac{a}{2}\right)^2}{a^2} + \frac{(p_y c)^2}{b^2} = 1, \tag{7}$$

where

$$a = \frac{p_A c (p_A + m_B c)}{2p_A + m_B c} \tag{8}$$

and

$$b = p_A c \sqrt{\frac{m_B c}{2p_A + m_B c}} \tag{9}$$

are the semi-major axis and the semi-minor axis, respectively. The focal length is

$$f = \sqrt{a^2 - b^2} = \frac{p_A^2 c}{2p_A + m_B c}. \tag{10}$$

Using equations (8)–(10) we find particularly simple expressions for the eccentricity e and the so-called parameter Π of the ellipse:

$$e \equiv \frac{f}{a} = \frac{p_A}{p_A + m_B c} \tag{11}$$

$$\Pi \equiv \frac{b^2}{a} = \frac{p_A m_B c^2}{p_A + m_B c}. \tag{12}$$

The geometrical meaning of eccentricity e is how much the ellipse deviates from a circle (as $e \to 0$, the ellipse becomes a circle and the two foci coincide with the centre of the circle). As seen from equation (11), the eccentricity of the Compton ellipse has a physical meaning too: if the initial photon energy is much smaller than the electron's mass, i.e. $(p_A c)/(m_B c^2) \ll 1$, the ellipse becomes a circle, and the possible momentum vectors of the photon are along radii of that circle (see figure 5(c)). This implies that the scattered photon only changes its direction, while losing only a negligible fraction of its energy.

The equation of an ellipse has its most elegant algebraic form in the polar coordinate system having one of the focal points as its origin. The length of the radius vector r as a function of the polar angle θ is given as

$$r = \frac{\Pi}{1 - e\cos\theta}. \tag{13}$$

As seen in figure 5(c), in the Compton ellipse the length of the 'radius vector' that belongs to a given photon scattering angle Θ represents the final photon momentum $p'_A c$ corresponding to that scattering angle. Adapting equation (13) for figure 5(c) and substituting expressions (11) and (12) for Π and e, we obtain

$$p'_A c = \frac{m_B c^2}{1 + \frac{m_B c}{p_A} - \cos\Theta}. \tag{14}$$

Substituting the de Broglie expressions for the initial and final photon momenta, $p_A = \frac{h}{\lambda}$ and $p'_A = \frac{h}{\lambda'}$ (where h is Planck's constant) and multiplying through both sides by the denominator in equation (14) we get

$$\frac{h}{\lambda'} c\,(1 - \cos\Theta) = m_B c^2 \frac{\lambda' - \lambda}{\lambda'}, \tag{15}$$

hence

$$\Delta\lambda = \frac{h}{m_B c}\,(1 - \cos\Theta)\,. \tag{16}$$

The well-known formula for the Compton wavelength shift of the photon is thus obtained by exploiting the geometric properties of the Compton ellipse rather than the purely algebraic method of writing the equations for the conservation of momentum and conservation of energy, and then eliminating the electron's velocity and scattering angle from the equations. Whereas the mathematics employed here in deriving equation (16) is not necessarily simpler than the purely algebraic method, the geometric approach presented here might be more elegant and attractive in the eyes of many students, especially since neither the electron's velocity nor its scattering angle appear anywhere in the derivation.

4. Concluding remarks

Momentum-space Minkowski diagrams may serve as a powerful tool in classroom discussions of particle–particle and particle–photon interactions. In many cases a *single diagram* can give students not only a quick qualitative description of all relevant features of the interaction at a single glance, but it can also provide them with a rough numerical value for all the essential physical quantities.

References

[1] Saletan E J 1997 Minkowski diagrams in momentum space *Am. J. Phys.* **65** 799–800