On the geometry of electromagnetism

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(1): Affine space

INTRODUCTION

What is this series* of articles about? Examine the following figure, which displays Maxwell equations in three different formalisms. Never mind what these equations mean (this is summarized in the caption, but it's a secondary point). My immediate purpose is to call attention on how they *look*. In spite of describing the same physical phenomena, they are as different as three sentences with the same meaning can be in three different languages. The fact that one can discuss the same physics within widely different mathematical formalisms is what will concern us here. In particular, I wish to show that different *geometrical objects* can serve in describing electromagnetism: vector fields, differential forms, even quaternions, as in Maxwell's time, and so forth: "axial" vectors, "polar" vectors .... There is terrible confusion around the latter concepts, which I hope to dispel a little by showing how they relate. This will be, if not the exclusive subject, at the very least the red thread connecting these columns.

The linguistic metaphor should not be stretched too far, but it's apt to some extent. You can say one thing in Japanese and exactly the same thing in English, in most scientific contexts (leaving apart of course, poetry, allusions to political actuality, and jokes). You will not just substitute word for word, however, for both languages use different grammatical categories. Box a of Fig. 1 reproduces something Maxwell told us about the way the world behaves [Ma]. Boxes b and c show how this message translates in two contemporary languages: the "vector fields" formalism of most textbooks and Journals (box b) and the "four-dimensional differential forms" idiom (box c) of many Physics treatises (see, e.g., [Mi]). As one sees, the translation does not consist in a mere change of notation (passing from $\mathcal{H}$ to **H**, for instance), for even when the symbols look alike, they denote different kinds of entities—different "geometrical objects".

$$\begin{aligned} &4\pi C = \mathrm{v}\cdot\nabla \mathrm{H}, && C = J + \dot{D}, \\ &B = \mathrm{v}.\nabla U, && E = -\dot{U} - \nabla\Psi, \\ &B = \mu H, && D = (4\pi)\ \kappa^{-1} E \end{aligned}$$

(a)

$$\begin{aligned} -\partial_t \mathbf{D} + \mathrm{rot}\,\mathbf{H} = \mathbf{J}, \quad & \partial_t \mathbf{B} + \mathrm{rot}\,\mathbf{E} = 0, \\ \mathbf{B} = \mu\mathbf{H}, \quad & \mathbf{D} = \varepsilon\mathbf{E} \end{aligned}$$

(b)

$$\mathrm{dF} = 0, \quad \mathrm{G} = *\mathrm{F}, \quad \mathrm{dG} = \mathrm{J}$$

(c)

Figure 1. Maxwell equations, with given currents: in the style of Maxwell's treatise (box a), in what may be the most widely accepted contemporary formalism (box b), and in modern differential geometric notation (box c). Maxwell used quaternions. (The v means "vector part" of a quaternionic product, and ∇ is the operator $\mathrm{i}d/dx + \mathrm{j}d/dy + \mathrm{k}d/dz$.) Notwithstanding, his formalism is not so far from today's received notation (box b), in which, apart from factors 4π, box a would read $\mathrm{rot}\,\mathbf{H} = \mathcal{C} = \mathbf{J} + \partial_t\mathbf{D}$, $\mathbf{B} = \mathrm{rot}\,\mathbf{A}$, $\mathbf{E} = -\partial_t\mathbf{A} - \nabla\psi$. Boxes b and c say exactly the same thing, but whereas **D**, **B**, etc., denote vector fields, F, G, and J are differential forms (J combines electric current density and electric charge in a single entity, F is "Faraday's tensor" and the star is the so-called "Hodge operator" in Minkowski's metric).

Maxwell's apparently bewildering notation begins to make sense when one remembers he had adopted *quaternions* as basic entities (at least, at the beginning; later, this changed; see e.g., [Cr] or [Sp] for historical accounts

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of the evolution towards the vector formalism of today). Similarly, we shall discover the kinship between (b) and (c) by examining the relations that exist between differential forms and vector fields. Let me hasten to say that quaternions will not be addressed here. Not that they should be confined to the dustbins of history, far from it. (They are quite useful in modern work on robotics, for instance.) But their relevance to electromagnetism was 19th-century illusion. Differential forms, on the other hand, are the right stuff—as I hope to show. But the way they are introduced in classics of differential geometry (which almost all discuss box c, if only in a rather thin chapter) cannot be recommended to Engineers. We shall adopt a different approach, leading to a formalism much closer to the familiar one of box b, by setting aside time and 3D-space, which the four-dimensional equations (c) do not distinguish.

In order so to stay very close to the familiar concepts, we shall have to introduce, and discuss with care, an appropriate *geometrical framework*, consisting of things such as *affine* three-dimensional *space*, the associated vector space, and geometrical objects living therein. (To "live" means that such entities may assume varying values as time goes on.)

That such a critical discussion be necessary at all is not so obvious. We often take for granted 3D space (good old Euclidean space, that is), as the natural framework in which to do physics, and though we shall not depart from this tradition here, I want to stress that this is a *modelling decision*, something that is to a large extent *up to us*, human beings, not something forced on us by the very structure of the Universe. The World *is*, and it certainly has order and structure. But order and structure in our *descriptions* of the world are something else, even if we try our best towards a close match, in the process of *model building.*

This activity—model building—is what distinguishes "pure" mathematics from "applied" ones. Pure mathematicians try to discover, analyse, and classify *all* logically possible abstract structures. People who apply mathematics, including physicists and engineers, use them to construct *specific* abstract structures, which reproduce some of the features of the real world, and thus can help in explaining or predicting the behavior of some definite segment of reality.

So mathematical entities by which we thus describe physics are not a priori frames of our thinking. They are our creation, moulded of course by the structures of the world out there, but still *abstract* things. Therefore, they are more or less adequate as tools with which to deal with the real world, which means one can—and one should—*criticize* the way they are applied, and question their adequacy. This process of critical reevaluation (constantly reinvigorated by new engineering practices, such as programming and computing) is the impetus that forces formalisms to evolve, even in well-understood compartments of physics, like classical electromagnetism, as witnessed by Fig. 1.

The purpose of these articles is thus to critically examine the geometrical concepts which compose the current formalism of electromagnetism. Hence a discussion at two levels: the formal one of mathematics, where one introduces abstractions (such as, for instance, three-dimensional affine space), and the practical one, where one passes judgment on their relevance to model building. This explains the alternation, in these columns, between descriptions of geometrical objects, and discussions of their physical significance.

And now, rather than go on philosophizing, let's do it.

1. AFFINE SPACE

Nothing is built without foundations, so we shall assume some preliminary knowledge: sets, functions and maps, elementary logic, and some familiarity with the basic structures: group, field, vector space ... . Recall that all such structures are sets,[1] but not naked sets: structure is conferred on such sets by specific systems of relations and operations, which tell what can be done with and to the elements of the set.

1.1 Vector spaces

For instance, a *vector space*[2] on the reals is a set of objects called *vectors*, which one can

[1] One can (though this is not the only way) present mathematics in such a light that *all* mathematical objects are sets of some kind.

[2] Defined terms are set in *slanted* style, on first appearance. [Footnote's footnote, Sept. 2002: Fonts have changed, with respect to the original, in what you are

(1) add together (e.g., forming vector $v + w$ from vectors v and w) and (2) multiply by real numbers (e.g., forming vector λv from vector v and real number λ). No need to recall the properties required of these two operations, if the set is to qualify as a vector space. Just be aware that "vector" is a generic name, which may apply to other objects than the familiar two- or three-dimensional vectors of elementary geometry, provided the set of all objects thus considered obeys the vector-space axioms.

For instance, think of the electromagnetic (EM) field, at any instant, in one of these large experience halls in which high-voltage electrical hardware can be tested. At this stage of the discussion, we pretend not to know what the EM field "is", meaning that we are not yet committed to a specific mathematical object by which to *model* this empirical reality, the physical EM field (detectable by its effects on dust, on our hair, etc.). But we know that two EM fields can be superposed, adding up their effects, and that a given field can be scaled up by a factor 2, say, giving conceivable[3] EM fields. So it makes sense[4] to consider the set of all conceivable EM fields, in this hall, as a vector space. One can then envision the evolution of the experiment in the hall as a (continuous) sequence of values of a representative vector in this abstract space, in other words, as a trajectory. And there we are, with the beginning of a *geometrization* of the whole thing.

The main feature which distinguishes such vector spaces of fields (often called "functional spaces") from those of plane or spatial geometry is, of course, dimension. The *dimension* of a vector space is the maximal number of linearly independent vectors, if there is such a maximum (otherwise we have an infinite dimensional space, like the above space of EM fields). A *basis*, or *frame*, in a vector space V of finite dimension n is a family of n linearly independent vectors. Applied mathematicians seem to be especially fond of a particular space of dimension 3, denoted $\mathbb{R}^3$. This is the set of all triples of real numbers $\{x, y, z\}$. Such triples can be added or scaled up the obvious way. Reading such paper titles as "MHD in a subset of $\mathbb{R}^3$", or "Wave propagation in a stratified region of $\mathbb{R}^3$", one might believe that this particular vector space is the natural framework in which to do physics, which I think is silly, and is one of the received ideas I want to challenge here. (But one thing at a time.)

It's an exercise (just pick two bases, and associate their elements two by two) to show that one can always map two vector spaces V and W onto each other, if they have same dimension, by an invertible *linear* map (i.e., a map f such that $f(v + w) = f(v) + f(w)$ and $f(\lambda v) = \lambda f(v)$). So if one is using V to model some physics, W will do just as well. For instance, if the trajectory $t \to v(t)$ in V describes the evolution of a physical system, the trajectory $t \to f(v(t))$ in W provides an equivalent description. This is due to V and W being "of the same form", or as one says, *isomorphic*, via the *isomorphism* f.[5] From this point of view, there is only one *abstract* n-dimensional vector space, and this particular mathematical object we shall label V_n, for future reference.

Isomorphism doesn't mean that V_3 should be *identified* with $\mathbb{R}^3$. Indeed, V_3 and $\mathbb{R}^3$ can be put in one-to-one correspondence by a linear map: just select a basis $\{e_1, e_2, e_3\}$ in V_3, then the generic vector v can be expanded as $v = v^1 e_1 + v^2 e_2 + v^3 e_3$ and thus paired

reading, for better aspect on screens. In the process, "slanted" has become plain old italic. The layout has changed somewhat, too. But apart from typos, no substantial modifications have been done.]

[3] Not the same as *realizable*, of course, due to nonlinear effects: we know only too well that fields of arbitrary magnitude cannot be maintained in the hall. Note here how insidiously the modelling process gives status and credence to mathematical objects that may lack any counterpart in the real world: An electric field 10^{100} volts strong, for instance, is "conceivable", absurd as the very thought of one may be.

[4] It makes sense *from some vantage viewpoint*, of course. The technician in charge of the Van de Graaf may laugh off this "vector space" stuff as pedantic, whereas the person who simulates the experiment on a computer will see it as very natural.

[5] Linear maps are thus, among all possible maps between V and W, those which "preserve the linear structure". Needless to say, mathematicians have devised a language with which to discuss such abstract properties of abstractions. It's the theory of categories [LS], fondly nicknamed "abstract nonsense". A category regroups objects of similar structure, and *morphisms* are structure-preserving associations between pairs of such objects. Linear maps are the morphisms in the category of linear spaces. (A bit later, we'll meet affine maps, which are the morphisms in the category of affine spaces.)

with the triple $\{v^1, v^2, v^3\}$ of its *components* in this frame. But all this depends on the choice of basis, which is arbitrary. So there is no *canonical* way to associate V_3 and $\mathbb{R}^3$, which means, there is no unique, natural way to do it, that would stand out among all others for some good reason.

To say the same thing in different words, the vector v of V_3 can be *represented* by its components, but to say that v *is* the same as its triple of components would be going much too far. Unfortunately, pupils are trained all around the world to consider 3D vectors as triples of numbers, and when they begin to get the idea, one introduces the abstract idea of vector to them. There may be valid reasons for such pedagogy, but the (much sounder, I think) geometrical approach goes exactly the opposite: vectors are geometrical entities, which can be added, stretched, etc., and their Cartesian representation is only a useful computing device, by no means one that should always be used. (We all know examples of problems of geometry which can be solved more easily and more elegantly *without* coordinates than with them.)

As this theme will recur, I don't wish to hammer in the point right now, but just think about this: when solving a problem in electrotechnics, you don't pick just any system of coordinates (or "reference frame"), you carefully select one which is *adapted* to the device under study. So what makes the interest of coordinates is some peculiarity of the device, which makes some directions in space stand out among others. In the absence of such extra structure, there is no "canonical" way to select a set of coordinate axes, and imposing one would break the symmetry of space in an arbitrary way, devoid of physical justification.

Which prompts the question: What is this allusion to "symmetries of space" supposed to mean? This is where another basic structure, that of *group*, intervenes.

1.2 Affine space

Group is a simpler, more primitive, and hence more general structure than vector space. A vector space is already a group, an additive one: indeed, the operation $+$ admits of a *neutral* element (the vector 0, which is such that $v+0 = v$, for all v), and for each vector v, there is another one, namely $-v$, which yields 0 when added to v. Together with associativity (the fact that $u+(v+w) = (u+v)+w)$), these properties constitute the axioms for the group structure. Moreover a vector space is a *commutative* (or *Abelian)* group, meaning that $v + w = w + v$, something which is not part of the definition of groups in general. So a vector space can be construed as a set with *two* layers of structure: first, the one of Abelian group, conferred on it by vector addition; and a second layer, due to the introduction of scalar multiplication, with the properties required to make it compatible with addition. (This game of *peeling out* layers in a mathematical structure, in order to analyze it, we shall play recurringly.)

So if we forget about the multiplication by reals in V (thus depriving it of one of its structuring features) what remains is an Abelian group, called the associated group of *translations*. Why this name? It's a way to put emphasis on what a given vector v may *do* on other vectors, how it *acts* on them. To vector v, we may associate the map $w \to v+w$, called the "v-translation", and that we shall denote by T_v, so that $v + w$ is the same as $T_v(w)$, the *image* of w under the translation, or "v-translate" of w. Note that T_v is not a *linear* map of V to itself, because 0 does not map to 0. On the other hand, translations are not devoid of "linear" properties, since for instance, it is true that $T_v[(w + w')/2] = [T_v(w) + T_v(w')]/2$ (see Fig. 2—and note how one *must* divide by 2!), and a name will be useful for maps with this property: they are called "affine maps" of V to itself, or "affine transforms" on V.

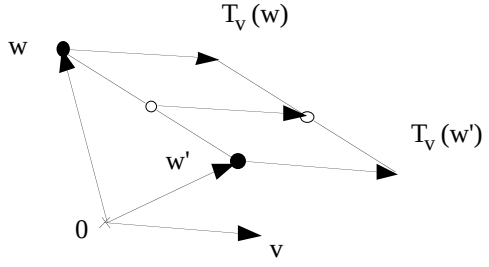

Figure 2. Translation by v, and its affine properties.

Obviously, the vector space structure is not required if one wants to define affine maps. All that is needed is a set, elements of which are now called *points*, not vectors, in which it makes sense to take the point midway between two points, or *barycenter* (and more generally, the barycenter of a given finite set of points to

which weights of nonzero sum are assigned). Affine maps are then defined as maps which preserve barycenters. A set thus equipped with a notion of barycenter is called an *affine space*.

Given a vector space V, there is a way to build from it an associated affine space A, which is the same as V *as a set*, but differently structured. For this, we first define A as a set on which one can perform v-translations: Given a vector v of V and a point a of A, there is another point, denoted $T_v(a)$, which is the v-translate of a, and one postulates the obvious properties ($T_v(T_w(a)) = T_{v+w}(a)$, and so forth). One says that V "*acts* by translations" on A. Next, we assume that any two points of A can be connected by a v-translation, i.e., there is always some v such that $T_v(a) = b$, and that $T_v(a)$ always differs from a when $v \neq 0$. Nothing more natural, now, than writing $v = b - a$, or $T_v(a) = a + v$, so if one selects in A some particular point, denoted 0, one can make the identification between the point $T_v(0)$, alias $0 + v$, and the vector v. Not a *canonical* identification, of course!

Now the barycenter of a and b is the well defined $(b-a)/2$-translate of a, that is, the point $a+(b-a)/2$. *Affine maps*, from one affine space to another, are then defined, as already said, as those which preserve barycenters. (Affine *transforms* are affine maps from the space into itself. Those of them which are one-to-one, and thus invertible, form of course a group, called the *affine group*[6].) They preserve many other properties, as a consequence. For instance, alignment: three points "on the same line" (i.e., one of them is the barycenter of the other two, with adequate weights) are transformed to aligned points. Pairs of parallel lines transform into pairs of parallel lines, and so on. But distances or angles are *not* preserved. In fact, such notions simply don't make sense in affine space: to give them status, we shall have to introduce (but only later) another element of structure, called the *metric* of space. (Note right now, however, that *ratios* of distance between *aligned* points do make sense and are preserved, i.e., are "affine invariants", as one says.)

Everyday examples of affine transformations abound. If for instance, in a museum, you compare a painting with its catalogue reproduction, the two images in your visual field correspond by affine transform (Fig. 3), at least if they are small enough to allow the use of parallel perspective (the one where the eye is supposed to be at infinity).

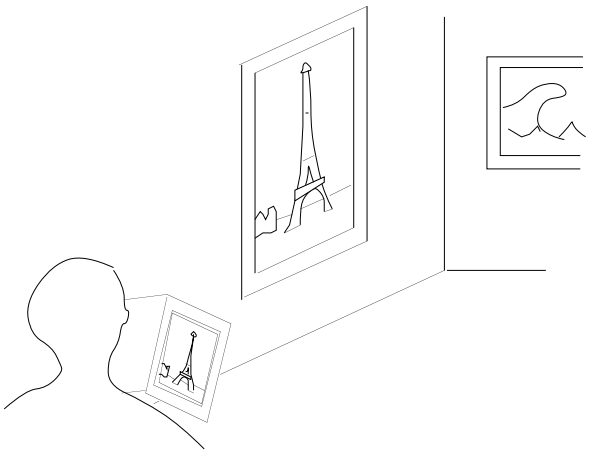

Figure 3. A common case of affine transform. (Note that "vanishing points" are at infinity in this rendering of the situation, so that parallel lines in the painting are seen as parallel lines of the views. More realistic perspective would defeat our purpose!)

To the abstract vector space V_3, it thus corresponds the abstract affine space A_3 (its dimension is, by definition, the dimension of the vector space). Informally, A_3 is what one gets when "forgetting where the origin was" in V_3. Conversely, selecting an origin in A_3 yields V_3, in a non-canonical identification. If one notices that selecting an origin for the space we live in is always an arbitrary move, A_3 emerges as a better model for ambient space[7] than V_3 and—a fortiori—$\mathbb{R}^3$.

In particular, the notion of vector field needs affine space, not only a vector space, to make sense. A *vector field* is a mapping from A_n to V_n. This mathematical entity is very apt to

[6] At this stage, the reader may wish to examine the relation between this affine group, denoted GA_n in dimension n, and the more familiar linear group GL_n of linear transformations in V_n, which is isomorphic (via selection of a basis) to the group of $n \times n$ regular matrices. As a starting point, note that a linear transform on vectors induces an affine transform on points, and that, conversely, any affine transform can be described as the combination of such a special affine transform with a translation. Beware, this is more difficult than one might think.

[7] Meaning, the space we live in. Later we shall use "ambient" in a more technical sense: it will refer to the encompassing space in the modelling, the one in which geometrical objects under consideration all live, a 3D affine space usually, but it may happen to be a manifold of any dimension.

model the (physical) notion of velocity field of, for instance, a mass of fluid: at each point x of A_3, fluid particles have a definite average velocity, represented by a vector $v(x)$ of V_3.

By the way, there is a name, *bound vector*,[8] $\{x, v(x)\}$ consisting of a point in A_n and a vector of V_n which one considers as assigned to this point. (One may also say "a vector at x".) Note that "bound" is *not* a qualifier, there: "bound vector" must be understood as a non-separable aggregate of words. Bound vectors are not vectors, one may even argue, because they do not form, taken together, a vector space: indeed, it makes no sense to add $\{x, v\}$ and $\{y, w\}$, unless $x = y$, or to multiply $\{x, v\}$ by λ. (They do form an affine space, though. Can you see it? First note that a bound vector $\{x, v\}$ can be construed as the pair of points $\{x, x + v\}$.) We are all familiar with the graphic convention according to which a family of bound vectors scattered on the page serves as a picture of a vector field.

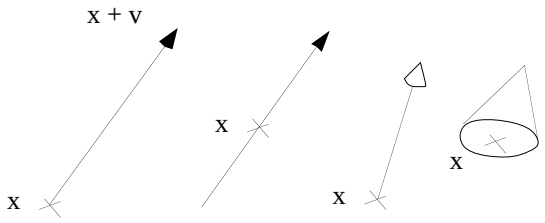

Figure 4. A few standard icons for bound vectors. For other examples, look carefully at plots of vector fields displayed by commercial software packages. As a rule, "three-dimensional" icons, like the two on the right, are to be preferred for 3D fields, and the more compact the icon, the better. (The norm of the vector, in the rightmost one, is rendered by the apparent volume of the cone.) The art of iconology, as applied to the visualization of fields, is still in its infancy. For some serious work in the area, see [C&], and [Tf] for general guidelines. (A reference list of works *not* to be imitated would exceed the size of this Journal.)

This is perhaps the right place for an aside, devoted to the notion of *icon* [Al]. Icons are drawings that stand for an abstract object, be it on our computer's screen or on a piece of paper. The most common icon for the bound vector $\{x, v\}$ is an arrow based at x with its tip at $x + v$ (Fig. 4). It's not *that* good a graphic convention, however, when it comes to visualize fields, because too long arrows tend to clutter in ugly tangles in regions where the field is large, symmetries that may exist are blurred, etc., thus such images often give a wrong idea of the overall field. The pair of points $\{x - v/2, x + v/2\}$ is often a better choice. One may also draw arrows differently, as suggested by Fig. 4.

[8] Caution: Many physicists say "bound" and "free" where I say "free" and "bound", a usage that Burke [Bu] also endorses. But calling "free" a vector whose tail is attached to a point is more than I can swallow.

1.3 Symmetries of physical space and of affine space

Now let's return to these alleged "symmetries" of space. Mathematically speaking, the symmetries of a structure are just its structure-preserving maps, so the symmetries of affine space are affine transforms, by definition. But what is at stake in the present discussion is something else: the ability of such notions to reflect symmetries *of the real world around us*. One of these symmetries is translational invariance: if you move this experiment hall fifty kilometers away, you will observe the same physics inside it. Humankind learned about this long ago, at least as regards *horizontal* translations. With Galileo and Newton, we realized that space was also invariant along the third dimension. Obvious changes in physical phenomena when one climbs up were attributed to the very presence of the Earth and of its gravitational field, but one accepted the idea that the laws of, say, celestial mechanics, would be the same a few light-years away, in any direction. So a prerequisite for all mathematical models of physical space would be translational invariance, and from this point of view, of course, affine space A_3 does qualify.

However, one may object, affine space is *too* symmetrical for the purpose, for one cannot pretend that physics is invariant with respect to scaling and shearing. But *some* aspects of physics are, as there exist experiments which can entirely be described using affine notions only. Figure 5 gives one (the idea comes from [Br], p. 100): The visitor of the science museum is watching a ball rolling along a gutter secured to the wall, and what we see, as in Fig. 3, is a photograph taken from infinity. Our view and his are different, but they correspond via some affine transform, which is enough to agree on our respective predictions: the ball will settle at the point of contact of the gutter with a line *parallel* to the wall's bottom line.

(The floor is supposed to be level, of course, but the wall need not be vertical, as far as it is plane.) Only affine notions are involved.

Admittedly, this is a very special case, and there are many physical events in which the symmetry of affine space is broken. Solid dynamics, for instance: rigid bodies you can translate and rotate, but not stretch or deform without altering their inner structure. Rigid bodies are so important in our existence that we need to be able to distinguish, among affine transforms, those which are "deformation-free". Mathematically, what is required for that is a *metric* structure.

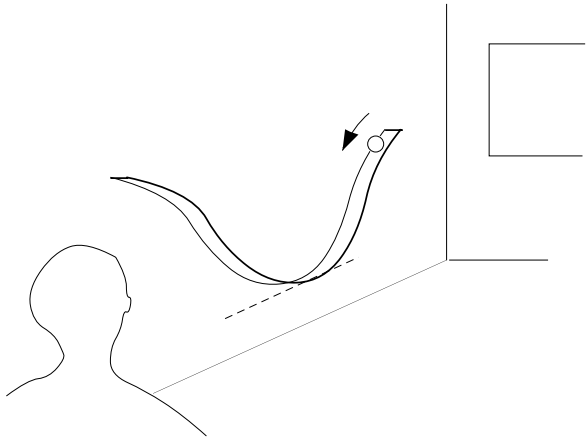

Figure 5. Where will the rolling ball eventually stop? In spite of our different perspectives, we agree with the visitor on that.

1.4 Metric

Metric is conferred onto a vector space V by endowing it with a *dot product*. The dot product $v \cdot w$ of two vectors v and w is a real number, and the correspondence $\{v, w\} \to v \cdot w$ is supposed to be linear with respect to both arguments, symmetrical (i.e., $v \cdot w = w \cdot v$) and most importantly, $v \cdot v > 0$ unless $v = 0$. Then, the square root $|v|$ of $v \cdot v$ is called the *norm* of v (more precisely, the *Euclidean* norm, as there are other kinds of norm—but we won't have to deal with them).

There are a lot of possible dot products on V_n. A way to get them all is to select some frame $\{e_1, e_2, \ldots, e_n\}$ and to set $v \cdot w = \sum_{i,j} g_{ij} v^i w^j$, where the *metric coefficients* g_{ij} are the entries of a square *strictly positive definite* symmetric matrix. As soon as we have adopted a dot product, notions of orthogonality and angle begin to make sense. We also know what "distance" means, speaking of two points x and y of the associated affine space: it's the number $d(x, y) = |x - y|$, of which one immediately sees it satisfies all properties required of a distance ($d(x, y) = d(y, x) > 0$, unless $x = y$, and the triangle inequality). This turns A_n into a *metric space* (any space equipped with such a distance function), with a bonus: the metric is *compatible with the affine structure*, which means that it's invariant by translations, $d(x + v, y + v) = d(x, y)$. Translations are thus *isometries*, i.e., transforms that preserve distance. It's relatively easy to show that isometries in A_n must be affine transforms (let's not feel forced to do it here). But the converse is not true. Those affine transforms that do preserve distances are called *displacements*. They form of course a group, smaller than GA_n. Among them, those that fix at least one point are called *orthogonal transforms.* They include *rotations* (but rotations have another property: they "preserve orientation", a notion we shall soon discuss and criticize), and *mirror reflections* (transforms $v \to v - 2\, u \cdot v\, u$, where u is a *unit vector* ($|u| = 1$). Orthogonal transforms which fix a given point form what crystallographers call a *point group*.[9]

Granted that a metric structure is necessary to correctly model solid dynamics, should we commit ourselves to a metric when only electromagnetism is involved? Definitely, yes. As Fig. 6 should suggest, there are optical experiments which cannot be described in exclusively affine terms, the way a ray of light bounces off a mirror, for example.

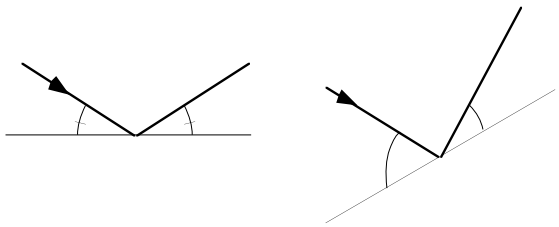

Figure 6. Equality of angles in light-ray reflection is not an affine notion. (Symmetry of the ray and its reflection with respect to the normal is an affine notion, but it's the concept of *normal*, now, which is not an affine one!)

What is very exciting, however, is that all aspects of Maxwell's theory are not alike in this

[9] The reader may wish to check that such a group is isomorphic to the group of orthogonal matrices of order n. A *space* group, in contrast to *point* group, is a subgroup of displacements which contains translations in all three directions.

respect. Some are affine invariant, some require a metric. As we shall little by little discover, both Faraday's law and Ampère's theorem *can* be edicted using only affine notions. On seeing box b of Fig. 1, this statement seems utterly unlikely, doesn't it? For the very definition of curl *does* involve a metric (just try to change the scale of length, for instance). Yet it's true, as we shall see. But this truth is hidden by having represented the physical entities forming the (physical) EM field by these mathematical entities, the vector fields **E**, **H**, **D**, and **B**.

On the other hand, the constitutive laws $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$ in box b are metric-dependent. This is where the metric structure of space intervenes[10] in the laws of electromagnetism. The modern notation of box c neatly makes the distinction, as metric is concentrated, as we'll see later, in the "star-operator" of the middle equation.

In fact, this differential geometric notation is even more general, for the equations $dF = 0$ and $dG = J$ do not depend on all aspects of the affine structure. (Let's say rapidly, though we don't have the technical equipment for such issues at this stage, that "their invariance goup is (much) larger" than the affine group GA_4.) Indeed, Maxwell equations continue to make sense, and to be physically relevant, in situations where the underlying space does *not* possess the symmetries we are used to, as for instance when investigating the magneto-hydrodynamics of a dense star, where space is "warped", as one knows, according to General Relativity. Although physicists, obviously, need to deal with such situations, engineers don't (well, not yet . . .), so the choice of metricized affine space as the framework in which to do our modelling is a reasonable one.

1.5 Orientation

Still, a last element of structure is lacking, which one cannot do without in electrodynamics: orientation of space. Among elements of the framework we are building, it certainly is the most difficult one to discuss, and the source of endless difficulties experienced by students, and not only them, when dealing with fields.

Orientation, like metric, is an element of structure that one may lay over a vector space. These are independent structures. One may have metric without orientation, and the other way around. So here, we assume a given vector space of dimension n, but no dot product.

Consider two frames in V_n , say $\{e_i : i = 1, \ldots, n\}$ and $\{f_j : j = 1, \ldots, n\}$. One may express the e_is as linear combinations of the f_js, hence a "transition matrix" T such that $e_i = \sum_j T_i^j f_j$. As T is regular, its determinant has a definite sign, $+$ or $-$. We say that $\{e_i\}$ and $\{f_j\}$ have *the same orientation* if the sign is $+$, *opposite* orientations if the sign is $-$. (Obviously, the sign is the same if one expresses the f_js in the $\{e_i\}$-frame.) This defines two classes of frames, two of them belonging to the same class if they have same orientation. An *oriented vector space* is a composite mathematical object, a pair, which consists of (1) a (finite dimensional) vector space, (2) *one* of its two orientation classes. So for each vector space, there are *two* oriented vector spaces, with opposite orientations, which can be associated with it. To *orient* V_n consists in making a choice between these two possibilities, that is, designating a distinguished class of frames. It's convenient to name this class Or, and the other one $-Or$. Frames of Or will then be called *direct* frames with respect to this orientation), and those of $-Or$, *skew* frames. (One also says "even" and "odd", hence the notion of "parity" of a frame, which is just the class it belongs to.) The two possible oriented spaces are thus $\{V_n, Or\}$ and $\{V_n, -Or\}$, which we shall abbreviate as ${}^+V_n$ and ${}^-V_n$. (Of course, if one selects ${}^-V_n$ as the oriented space in which to work, then the skew frames are those of Or.)

Take good note that, once a vector space has been oriented, there are direct frames and skew frames, but there is no such thing as direct or skew vectors, except, one may concede, if $n = 1$. A vector is a vector is a vector, and does not become a new object just because the space it belongs to has been oriented! This remark will be important later in our discussion of polar and axial vectors.

Remark. It's all right to consider an oriented vector space as a pair consisting of (1) a space,

[10] And though it's much too early, one can't resist the urge to confirm what the reader may already be suspecting: *things go the other way.* It's the constitutive laws of electromagnetism that give space its metric. After all, don't we make our geodetic surveys with light rays?

(2) *one* of its frames, provided the frame thus privileged serves no other purpose than fixing the orientation class. Pupils asked to "orient the figure" are, unfortunately, often confused by that, for they tend to believe that this is the same as selecting coordinate axes. Not so. Orienting the paper sheet ($n = 2$) means deciding on a "direct" sense of rotation (anticlockwise, most often), but one is not committed to definite axes by that. Orienting space ($n = 3$) means deciding which helices are direct or skew. As one knows, the usual convention for orienting 3D space is the "corkscrew rule", which makes most helices of the real world (shells, staircases, ...) direct, or as one also says, right-handed. $\diamondsuit$

An affine space, now, is oriented by orienting its associate vector space: a "bound frame" at x in A_n , i.e., a set of n independent vectors at x, is direct or skew if the n vectors form a direct or a skew frame in V_n. Hence two new structures: ${}^{+}A_n$ and ${}^{-}A_n$.

Vector subspaces of a given vector space (or affine subspaces of an affine space) can have their own orientation. Orienting a line, in particular, means selecting a vector parallel to it, called the *director* (vector) of the line, which points in what is then, conventionally, the "forward" direction along this line. Note that such orientations of different subspaces are a priori unrelated. Orienting 3D space by the corkscrew rule, for instance, does not imply any orientation in a given plane. Still, the standard orientation of space and of a horizontal plane do match, obviously. How come? Because the vertical direction also is oriented, bottom-up. So, if space is oriented, and if some privileged direction in space is oriented, planes that are "transverse" to this direction (meaning, the intersection reduces to a single point) inherit an orientation, as follows: to know whether a frame in the plane is direct or skew, just append it to the director of the line (i.e., place the latter ahead of the list of frame vectors), and check whether the spatial frame thus obtained is direct or skew.

The recipe can be generalized to all dimensions, so let's introduce a convenient terminology. We say that two subspaces U and W of V are *complementary* if their *span* is all V (i.e., if any v in V can be decomposed as $v = u + w$, with u in U and w in W) and if they are *transverse* ($U \cap W = \{0\}$, which makes the decomposition unique). Now (Fig. 7), we say that U has an *external* or *outer* orientation if an orientation is provided for one of its complements, W say. (For contrast and clarity, we shall call *inner* orientation what was simply "orientation" up to this point.) These notions (which one can trace back to [VW], cf. [VD] and [Sc]) pass to affine subspaces of an affine space the obvious way.

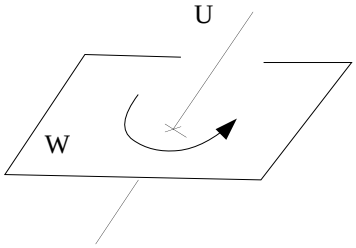

Figure 7. Externally orienting a line U by orienting a plane W transverse to it.

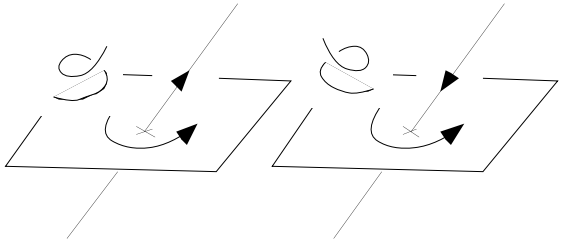

Figure 8. How an externally oriented line acquires inner orientation, depending on the orientation of ambient space. Alternative interpretation: if one knows both orientations, inner and outer, for a line, one knows the ambient orientation. The drawing on the left, then, can be understood as an explanation of *Ampère's rule.*

It's clear that if the encompassing space V itself is oriented, then an outer orientation of U gives it an inner orientation: to know the orientation class of a frame in U, append it to a direct frame of W, thus obtaining a frame in V, and look to which class the latter belongs. But two possible orientations for V make *two* ways to do that, so outer orientation and inner orientation are different, as are their intuitive meanings. For instance, inner orienting a line means distinguishing "forward" and "backward" directions along it. But outer orienting the line, that is to say, inner orienting a transverse plane, amounts to make a choice between the two ways to "turn around" the line. If ambient space is oriented, the "direct" way to turn around a line implies a way to go "forward" along it (see Fig. 8, and note how a play on icons advantageously substitutes for all

this stilted prose!). Similarly, outer orienting a plane means specifying a "crossing direction" through it.

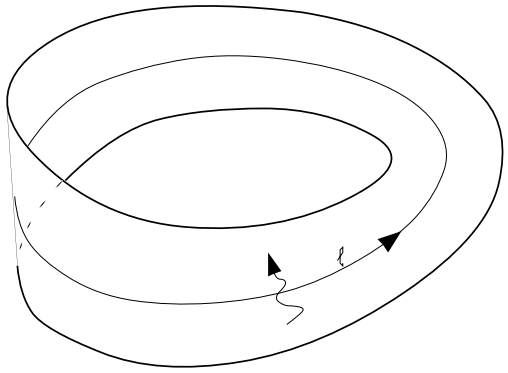

Figure 9. Möbius band, not orientable. (To prove this "experimentally", make such a band, lay on it the drawing of a frame, that will be moved around the band, thus returning at the starting point in inverted position. Be careful to use *transparent* material for both the ribbon and the frame, otherwise, your parlour-trick will fail on its nose.) As the middle line l does not separate two regions, no global crossing direction can be defined for it, so it has no outer orientation with respect to the band.

Curved lines also can be internally and externally oriented. The case of surfaces is a bit more complex. Orienting a surface means orienting all of its tangent planes at all points, in a "consistent" way. For neighboring points, tangent spaces are different, but close enough to have a common transversal. Orientations at these points are consistent if they give the same outer orientation to this transversal, which can always be achieved. But though this ensures consistent orientation locally, it may not be possible to maintain such consistency all over, as this all-time star of mathematical popularizations, the Möbius band, testifies (Fig. 9).

On the other hand, surfaces which enclose a volume can be oriented: "going inside out" defines a consistent crossing direction. This is outer orientation, from which inner orientation stems, if the ambient space is oriented.

Remark. While inner orientability is an intrinsic property, outer orientability always refers to some ambient space. It makes sense, for instance, to speak of outer-orienting a line traced on a surface: this means, as above, defining a consistent crossing direction, from surface points on one side of the line to points on the other side. As Fig. 9 shows, this may not be possible for some lines when the encompassing surface is non-orientable. This demonstrates how different the two notions of orientation can be. ♢

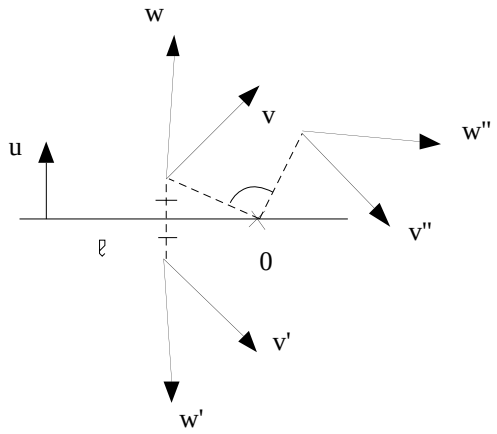

Figure 10. A skew transform (mirror reflection about line l, with unit vector u), and a direct one (rotation around 0).

Before leaving orientation, we need to broach this dangerously vague notion that some geometrical transforms could either "preserve" or "reverse orientation". What is meant by that is their effect on *frames*. Apply an invertible affine transform to n bound vectors forming a frame, you get another bound frame (at another point, in general, cf. Fig. 10). The two frames belong to the same class or they don't. Hence two classes of transforms: the *direct* ones (like rotations), for which a frame and its image belong to the same orientation class, and the *skew* ones (the other way round), like mirror reflections. *(Central symmetry,* i.e., the affine map $x \to -x$, can be direct or skew, depending on the dimension.) One also says "parity preserving" and "parity reversing" transforms.[11] Note that the orientation class Or, as a whole, is mapped to $-Or$ by an odd transform, but this cannot by any means "change the orientation of space", that is to say, our earlier commitment to Or as the class of direct frames!

1.6 Oriented Euclidean space

We are now in possession of a framework in which to model electrical phenomena: *oriented Euclidean three-dimensional affine space,* that will be denoted E_3 (and ${}^+E_3$ when it will be

[11] The notion applies to more general point-to-point transforms than those of GA_n; but the parity of such a transform, then, is only locally defined, and may not be the same at all points.

felt necessary to remind about orientation). It's A_3 coated with two layers of structure: a dot product and an orientation.

Dimension 3 has this in particular[12] that one can define a new operation, the cross product: Given two vectors u and v, the cross product $u \times v$ is a vector orthogonal to both of them, of squared length $|u|^2|v|^2 - (u \cdot v)^2$, and such that the frame $\{u, v, u \times v\}$ be direct. The very notion, therefore, does not make sense without a metric *and* an orientation. To keep oneself aware of that, one might "decorate" the symbol $\times$, like this: $\dot{\times}^+$ or $\dot{\times}^-$. Notice that $u \,\dot{\times}^+ \, v = -u \,\dot{\times}^- \, v$, which clearly explains what is meant when one says that "$\times$ is sensitive to orientation". This would be, however, the only advantage of such heavy notation, which one can't seriously propose.

Remark. If $\bullet$ is a new dot product, one has $u \bullet v = Lu \cdot Lv$, where L is some linear map. One should then be able to express $u \,\dot{\bullet}\times^+ v$, the cross product associated with this new metric, in terms of u, v, L, and $\dot{\times}^+$ (or perhaps, $\dot{\times}^-$). Would you care to try it? It's not so easy an exercise. $\diamondsuit$

Another kind of "sensitivity to orientation" is demonstrated by the following fact: if T is an affine transform, $T(u \times v) = \pm Tu \times Tv$, the sign depending on the parity of T (it's $-$ for a mirror reflection). This contrasts to what happens with respect to vector addition, since one has $T(u + v) = Tu + Tv$, unconditionally.

We can briefly summarize all that by saying that the cross product operation belongs to the structure of *oriented* 3D Euclidean space.

This is true of other operations, notoriously the curl operator. Let's not try to define the curl the "elegant" way, without coordinates, because it's one of these cases where one is better off using them. Having adopted a direct orthonormal Cartesian system of basis vectors and axes, start from a smooth vector field u, take the curl the usual way, and just check that the field rot$u$ thus obtained would have been the same with another system of such axes, which is easy by invoking the Stokes theorem. It's then clear (Fig. 11) that rot$(Mu) = -M(\text{rot}\, u)$, if M is the mirror reflection with respect to a plane m.

[12] One might generalize to $n-1$ vectors in E_n , but this is not usual.

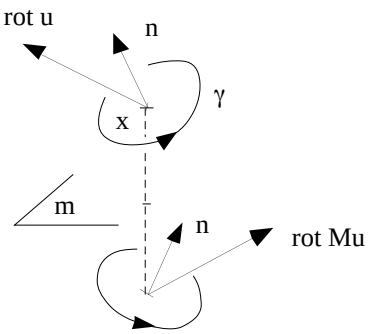

Figure 11. Applying Stokes' theorem to a small patch of surface around x, rimmed by γ, and to its mirror image, to see that rot$(Mu) = -M(\text{rot}\, u)$.

The other differential operations, grad and div, are less capricious. We'll discuss grad at some length next time, but it's obviously immune to orientation diseases. As for div, it's even simpler, as the metric structure is irrelevant in this case: the operator div belongs to the affine structure. (Think of v as the velocity field of some compressible fluid. Then div$v$ expresses the *rate* of change of the volume along the flow, which is an affine concept.)

This concludes our survey of "space", as a framework for modelling: We shall work in E_3, "oriented Euclidean 3D space", while being well aware of the "multilayered" character of this structure. Logically, we should discuss "time" as well, but having no ambition to address Relativity here, we shall be content to consider time as a parameter, which is all right if all phenomena are referred to the previous space E_3. The next part will introduce new geometrical objects: axial vectors, covectors (instead of vectors), and differential forms (instead of vector fields), whose introduction will be motivated by an analysis of the Lorentz force exerted on a moving charge.

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(2): Geometrical objects

We have introduced oriented three-dimensional Euclidean space, denoted E_3, and understood as a three-layer structure: 3D affine space (a set of points on which translations can be performed), plus a dot product, plus an orientation. We now look at denizens of this universe. We'll pay attention to which of these structures they really depend on, and review their use as descriptors of physical entities, with emphasis on the notion of *force*, which itself ushers the electric field and the magnetic field, conceived as differential forms.

2.1 Vectors

Vectors we know well. They belong to the "vector space" structure (V_3) and represent translations: for any two points x and y, there is a unique vector v such that $y = x + v$. (It's indeed the "free" vector v, an element of V_3, not the bound vector $\{x, v\}$ we are speaking about: For it's the same v-translation that will send x', say, to $y' = x' + (y - x)$.) For vectors so to stand for translations is totally in line with their abstract definition. But precisely, what we want to discuss this time is the vector as representing something else than itself, the vector *as a proxy* for some entity of physical interest.

For instance, vectors are often used to represent *position*. Vector r (as it's often denoted) is then assimilated with the point $0 + r$, which involves some arbitrariness, since such a representation depends on a choice of origin. Granted, in many questions of physics, it does make sense to specialize a point to play origin: in atom dynamics, celestial mechanics, etc., there *is* such a privileged point. But otherwise, it's not such a good idea.

A more legitimate use is for *displacements from* a reference position, as one does in continuum mechanics (and now it's *bound* vectors we have in mind). Consider a moving mass of fluid, or a deformable body, represented by a set[1] B, elements of which are called material particles. The material particle sitting at point x at a reference instant ($t = 0$, usually) will be found at time t at a different point $x + \xi_t(x)$. The vector field $x \to \xi_t(x)$ is called *displacement* at time t, and its evolution in time describes what happens to the whole body, provided one knows[2] where each particle stands at $t = 0$. *Virtual* displacements, as one knows, may have to be taken in consideration, and are represented by (time-independent, of course) vector fields, the same way. Metric and orientation of space are irrelevant to such descriptions.

Vectors can also stand for *velocities*. Let's first consider a single particle, which passes at point x at time t. It will be at $x + \delta t\, v$, up to higher order terms, δt seconds later. The bound vector $\{x, v\}$ thus fully represents the particle's motion at time t. To describe its fate over some span of time $[t_1, t_2]$, one will resort to the notions of *trajectory* (a smooth map from the real interval $[t_1, t_2]$ into E_3) and of *field of tangent vectors*,

$$(1) \qquad v(t) = \lim_{\delta t \to 0} [x(t + \delta t) - x(t)]/\delta t,$$

one at each point $x(t)$. (Pause a moment to check that such a limit is well defined, without need for a metric on V_3.)[3] Now if instead of a particle we have the above extended body, a vector *field* will be able to describe its instant motion, while a time-dependent vector field will account for its evolution.

Although metric and orientation of space are there again irrelevant, this time one may object, "But is there not some metric element here, as betrayed by your reference to the *second*, the unit of time? No metric on E_3, all right; but you need a metric (and an orientation, to

[1] B is not a naked set: in order to account for the notion of "material continuity", it must be endowed with a topology (and a bit more: the right structure is that of smooth manifold, actually).

[2] The mathematical device by which such information can be encoded is, of course, a *map*, u_0 say, from B to E_3: the material particle $b \in B$ sits at point $x = u_0(b)$. Such a map is called a *placement.* Note how the initial placement u_0 and the *dis*-placement ξ_t combine to give the placement at time t.

[3] The trajectory is more than its supporting curve: it's this curve plus a specific way to run along it (just as a graded ruler is more than a plain ruler). Notice how the supporting curve is oriented (it's *inner* orientation) by the law of motion.

boot) on the time-axis." Right on! But please make allowance for the necessity to proceed step by step in this deconstruction process we have initiated. We focus on the structure of *space*, for the moment, and other fundamental categories such as time, energy, electric and magnetic charge, etc., will have to wait in line. So we take seconds, joules, coulombs, and webers[4] for granted. (It's meters and inches that are under attack!) Yet, let's acknowledge that when it comes to velocities, vector fields cannot do the job *alone*, some extra structure (here, a chronometry) is needed.

Vectors are also frequently cast in the role of *rotations*. We shall dwell for a while on this example, which will lead to the construction of a new geometric object.

2.2 "Axial vectors"

You know the trick: When a solid has one of its points anchored at a fixed position a, its velocity field v is given by

$$v(x) = s \times (x - a), \tag{2}$$

where the *spin vector*[5] s may depend on time. The instantaneous axis of rotation is then the line supporting s, and the norm $|s|$ is a measure of the rate of spin. As we know, defining the cross product requires a metric and an orientation of space. One cannot object about metric, since the very notion of rotation depends on it. But orientation? Look at a spinning baseball; its velocity field (the v of (2)) is what it is, and exists independently of any orientation convention. Yet s, in (2), *does* depend on orientation, since $\times$ itself does! Change the orientation and you need to change s into $-s$ in (2) in order to obtain the same velocity field.

So here again, the spin vector cannot do its job alone. It needs extra structure in background. But whereas one part of this structure, the metric, is clearly relevant (as was the chronometry in the velocity example), another part, orientation, seems artificially introduced here, since it cancels out in (2). To say the same thing in a different way, what truly represents the rotatory motion is *not* the spin vector s alone, but a composite object, the *pair* $\{s, Or\}$ where Or is the chosen orientation for ambient space. Moreover, since the opposite pair $\{-s, -Or\}$ represents the same instantaneous velocity field, we have two *equivalent* descriptors for the same instant motion (and only two: no other pair will do). Formally, we may denote this as

$$\{s, Or\} \equiv \{-s, -Or\}, \tag{3}$$

an equivalence between the two pairs.

In mathematics, when objects are equivalent in some respect, we often bundle them together, putting like with like, and start considering each of the "equivalence classes" thus obtained as a new object in its own right. This is how, to recall only one well-known example, rational numbers are defined as equivalence classes of pairs of nonzero integers: We consider two ordered pairs of signed integers $\{m, n\}$ and $\{m', n'\}$ as equivalent if $mn' = m'n$, and we dump all equivalent pairs in the same class q. (Then we justify the abuse of notation $q = m/n = m'/n'$, and happily go ahead.)

In the spirit of such tradition, what follows, which Voigt[6] first did around 1910, appears as a rather natural move. Let's consider an equivalence class for relation (3) as a geometrical object in its own right, that we shall call (very provisionally) a *rotator*, and denote by $\tilde{s}$. So, formally,

$$\tilde{s} = \{\{s, Or\}, \{-s, -Or\}\}. \tag{4}$$

Thus defined as an equivalence class, the rotator $\tilde{s}$ is *not* the same kind of object as s or $-s$. But it can be *represented* by one of these, in a very definite way: If space is oriented once and for all, this establishes a one-to-one association between vectors and rotators by which, as one sees, s stands for $\tilde{s}$ if the chosen orientation is Or.[7] But let's insist again on the fact that $\tilde{s}$

[4] Grams, no. Can you see why?

[5] It's a vector at point a, thus a *bound* vector. So the "axial" objects we are about to define will also be point-bound. But I stop insisting on this distinction from now on. Whether bound or free objects are meant should be obvious from the context each time.

[6] Cf. [Po], which points to [Vg]. Post credits Voigt for the introduction of the term "tensor", too.

[7] The representative would of course be $-s$ if the chosen orientation was $-Or$. (The notation $\tilde{s}$ is not unimpeachable: it betrays a bias in favor of Or as the symbol for the standard orientation.)

exists in its own right, just as s, whether space be oriented or not.[8]

Of course "rotator" is not a good name, since such geometrical objects can serve for other things than velocity fields in a rotatory motion. (To quote only one, of which the reader will have been aware already, and to which we shall of course return, the magnetic induction field **B** can be represented by a field of such "rotators", one at each point.) But the name chosen by Voigt for objects like $\tilde{s}$, *axial vectors*, appears much worse, in retrospect.

For again (remember, we had the same trouble with "bound vector", last time), "axial vector" must be understood as a single, unbreakable label for this new kind of geometrical object. "Axial" is *not*, definitely not, *an adjective* that would point to some quality possessed by either $\tilde{s}$ or s, some "axiality". Such a thing cannot exist, anyway. It can't be an attribute of "the vector" $\tilde{s}$, if only because $\tilde{s}$ is not a vector. Neither can it be an attribute of s, which just *plays the role* of $\tilde{s}$. So there is here a quite unfortunate choice of terms,[9] aggravated by the habit to call "polar vectors", for contrast, the "ordinary" vectors.

Remark. Alternative denominations exist: e.g., *twisted vector*, which has the same drawbacks (but also some advantages, to be discussed in due time), or *pseudo vector*, which is a bit better (since a pseudo X is not supposed to be an X). But be careful: authors may use such names for again slightly different objects. $\diamondsuit$

Let us check that this geometrical object is indeed able, by itself and without the preexistence of an orientation, to represent rotatory motion. Given a metric, consider an axial vector $\tilde{s}$ at point a. Let's select one of the two elements of the class, say $\{s, Or\}$. Metric and orientation Or define a cross product $\times$. Now, the instantaneous velocity field is $x \to s \times (x - a)$. Had we chosen the other element of the class (4), two signs would change (one in front of s, one in the definition of the cross product), and we would get the same field.

Figure 1 shows how to design a convenient icon for axial vectors. (Figure 2 gives the 2D version.) The idea is simply to replace symbols by icons in (4), and to do some stylizing (since the two arrows do double duty, keep only one, etc.). Hence the icon for the axial vector $\tilde{s}(x)$ at point x: it consists of a plain segment (not a vector), with x at midpoint, and of an outer orientation of its supporting line. This is satisfactorily suggestive of the notion of "turning around" the rotation axis, at a speed proportional to the length of the segment—the very notion we wanted to capture in the first place.

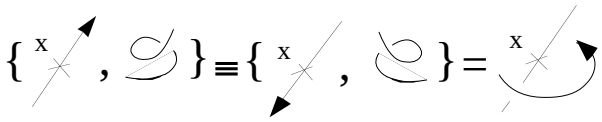

Figure 1. Merging the icons for vector and orientation to produce an icon for the axial vector.

Remark. Notice how an axial vector confers an *outer* orientation on its supporting line, whereas a polar vector gives it *inner* orientation. This suggests an alternative way to define these objects in affine space. A (bound) polar vector is made of (1) a point, (2) a line through this point, (3) a real number (the length of the vector), (4) an *inner* orientation of the line, that is, a sense or "pointing direction". An axial vector is made of the same items, except for (4), which is an *outer* orientation of the line, that is, a sense or "turning direction". $\diamondsuit$

Note that no ambient orientation is suggested by the icon. *None should be*, because the axial vector does *not* depend on orientation. (This was the whole point in defining it the way we did.) Like the bound vector, it's an "affine object", meaning that, of the three layers of structure that make E_3, only the affine structure is necessary to its existence. Yet axial vectors differ a lot from vectors, and I can't resist quoting Burke's elegant argument (Fig. 2) to show to which extent they do so.

[8] Maybe an analogy can help, for what it's worth. Think of E_3 (the world ...) as a stage. Orientation, metric, etc., are elements of the set; $\tilde{s}$ is a character in the play; s and $-s$ are actors, who may alternate in the part, depending on which set (Or or $-Or$) is installed.

[9] Though one tries hard to avoid such things in mathematics, they do happen. A "*signed* measure" is not a *measure*, "*free* Abelian groups" are not Abelian groups that would happen to be "free", and so on. So much for logic!

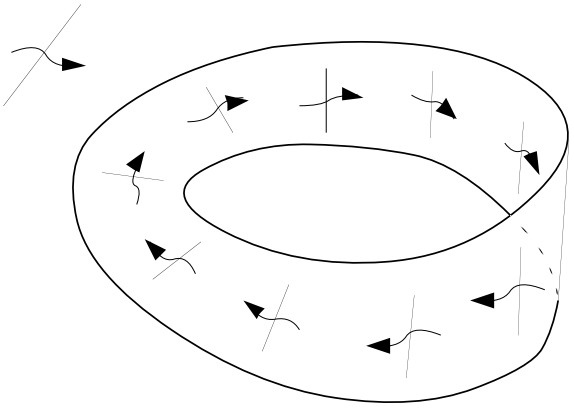

Figure 2. 2D icon for axial vectors, and how it's used in Burke's graphic illustration [Bu]: "To appreciate that a twisted vector is an independent notion, consider the problem of finding a continuous nonzero vector field on the Möbius strip which is everywhere transverse to the edge. No such vector field exists, but a twisted vector field with these properties does."

Axial vectors can be subjected to stretching, turning, mirroring, etc., i.e., to all geometric transforms, just like vectors.[10] Since a skew transform M (such as a mirror reflection) changes the orientation class of frames, it sends the representative $\{s, Or\}$ of $\tilde{s}$ to $\{Ms, -Or\}$, not to $\{Ms, Or\}$. Hence the different behavior of axial and polar vectors under mirror reflection (Fig. 3). This is well seen by using icons, for one need only apply the transformation to all graphical elements that compose the icon[11] in order to visualize the transform of an object.

Exercise. Study Fig. 4 and comment on its meaning.

There is nothing simple in what precedes, so one may wonder whether representing spin by a vector was such a good idea in the first place. Thinking about dimension n instead of 3 may help, there. In all dimensions n, rotations are represented, via an orthonormal basis, by $n \times n$ orthogonal matrices. Instantaneous velocity fields in such rotations (or if one prefers, "infinitesimal rotations") are then represented by $n \times n$ skew-symmetric matrices, which depend on $n(n-1)/2$ parameters, and this happens to equal n when $n = 3$. So it's only when $n = 3$ that vectors can stand for infinitesimal rotations,[12] thanks to the cross-product trick[13] of (2). It's a spurious association. No wonder we had so much trouble!

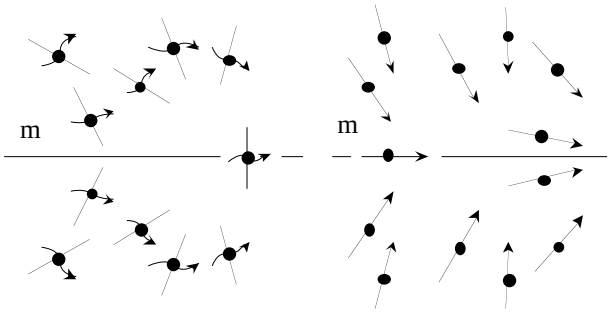

Figure 3. Two fields (suggested by a few scattered icons), each invariant by mirror reflection, but one made of axial vectors, the other one of polar vectors.

We gained something of value, however, by this brush with axial vectors: awareness that vectors and vector fields, these workhorses of calculus, are not always the best tool for the job at hand. What we want to challenge next is their use to represent *forces*.

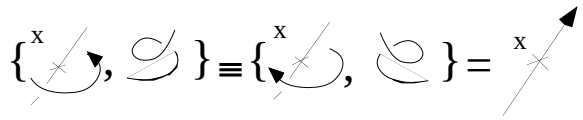

Figure 4. How polar vectors, too, could be defined in terms of axial ones.

2.3 Covectors

Is force a vector? Or, to be precise about the meaning of "is" (we shall feel free to abuse the language from now on, so let it be the last time I fuss like that), "is this physical manifestation we call force properly described by this (well understood) mathematical abstraction, the vector in Euclidean space?"

[10] If an object $\tilde{o}$ is defined as a class of equivalent tuples $\{o_1, o_2, \ldots\}$, and if some transform T can act on all components o_i (which in general are objects of different types), then $T\tilde{o}$ is defined as the class of $\{To_1, To_2, \ldots\}$, provided the latter compound be equivalent to $\{To'_1, To'_2, \ldots\}$ when $\{o_1, o_2, \ldots\}$ and $\{o'_1, o'_2, \ldots\}$ are equivalent. Only transforms for which this condition holds are legitimate.

[11] That's the rationale for good icon design [Al]: if $I(o)$ is the icon of object o, and T a geometric transform, the icon $I(T(o))$ of the transformed object $T(o)$ should be $T(I(o))$.

[12] The "right" geometrical objects for infinitesimal rotations in all dimensions are bivectors, which we shall soon encounter. Cf. [He], where Hestenes has recast a large part of classical physics in the language of multivectors. A good summary of his views can be found in [Hs]. It's fascinating and recommended reading, even if he overstates his case at times.

[13] A binary operation with the properties of the cross product can exist only in 3 and 7 dimensions [Ec]. Case $n = 3$ we know about. I don't know what the implications of the case $n = 7$ are.

The answer seems obvious: force has a magnitude, right? A direction, right? Hence, it's a vector, what else?

Well, it's no so obvious that force has direction. If you kick a golf ball, yes, it goes along with your shove. (Doesn't it?) But this is a simple, point-like object. What of a rugby ball? Should we take the direction in which the body moves as the direction of the thrust we exert? Playing a few minutes with a gyroscope is enough to cast this in doubt. It's easy to see that, in all cases where we can assign a definite direction to a force, this is the direction taken by material *objects* to which the force is applied, and since this direction is determined in part by the shape and structure of such objects, it cannot be *attributed to* the force. Force has no *intrinsic* pointing direction.

So let's take the question from another angle. Consider a physical force field, such as the gravitational field, or the electric field. (Assume, for simplicity, a static field, so time is no concern.) We want to describe this empirical reality by some mathematical abstraction. How do we know about the force field? By the deformations it causes on material structures, by the *displacements* it imparts on loose objects placed in the field. And since we can represent displacements by vectors, there lies our handle, in which one will recognize the time-honored *principle of virtual work*: Imagine (to treat the case of the electric field), an electric charge χ coulombs strong, placed at point x, where χ is a scalar factor meant to tend to zero[14], and let's displace it to $x + \lambda v$, where v is a vector (the "virtual displacement") and λ another "vanishing" scalar factor. The work involved in this displacement, or virtual work, is a smooth function of $\lambda\chi$ and v, which one can Taylor-expand in $\lambda\chi$. The leading term of this expansion is of the form $\lambda\chi f_x(v)$, where $v \to f_x(v)$ is some *linear* map (this is the key point). Now *if we know the map* $v \to f_x(v)$, *we know the force* at point x. This is as complete a characterization of force as one may desire.

In this description, the force at point x "is", therefore, a linear map of type[15] *VECTOR* $\to$*REAL*, that is to say—by the very definition of dual space—an element of the *dual* of the vector space V_3. This dual, V_3^*, is a three-dimensional vector space too, so its elements would deserve the generic name of vectors. But they are not of the same type as the vectors of V_3, so we call them *covectors*, instead. Force, as we see, is a covector.

The force *field*, now, is a field of covectors, one at each point. Calling that a *covector field* would make perfect sense. But it happens that fields of covectors have another name: one calls them *differential forms* (DF) for reasons that will little by little become apparent. Anyhow, such a covector field, or DF, appears as the right geometric object by which to represent a force field.

As a corollary, this will give us the right mathematical representation of the electric field. Humankind, by a protracted process of experimentation and theorization, recognizes the existence of a particular substance called "electric charge" and of a physical manifestation, called the "electromagnetic (EM) field", which affects the space around us in ways which are revealed by, precisely, the behavior of electrically charged objects. More specifically, a moving particle of (vanishingly small) charge q appears subject to a force, the *Lorentz force*, which (1) is proportional to q, (2) depends in part on the velocity of the particle. We thus distinguish two parts in this force, the static one (which a nonmoving particle feels) and the dynamic one (due to motion), and hence, we also distinguish two aspects, two facets, of the EM field: the electric one and the magnetic one. The electric field is this part of the EM phenomenon that is revealed by forces on nonmoving charged particles.

So, assuming by convention a unit charge, the electric field is akin to a force field. *The electric field is a covector field.* It "is", more precisely, the mathematical object we may denote as follows

$$(5) \qquad e = x \to (v \to e_x(v)),$$

a compact and spiked (but convenient, as you will see) expression, that should be parsed as follows: e (the differential form that represents the electric field) is the field $x \to e_x$, where e_x is a covector at x, which itself is the linear map $v \to e_x(v)$, where the real number $e_x(v)$

[14] since actually putting a *finite* charge there would alter the field.

[15] When a function f sends all or part of a set X to another set Y, we say that "the type of f is $X \to Y$".

is interpreted as a virtual work. More precisely, the work yielded by the field when a charge χ at x is pushed to $x + \lambda v$ would be $\lambda\chi\, e_x(v)$, up to terms of higher order in $\lambda\chi$.

Remark. It will be convenient to write the value $e_x(v)$ as $\langle e_x, v\rangle$, with the covector on the left and the vector on the right (that's Dirac's "bra–ket" notation), and to informally refer to this number as "the effect of e_x on v", since it's the value of a map, e_x, that acts on vectors. Note that we can say "the effect of v on e_x" as well, since a vector can be seen as a linear map over covectors, by reflexivity of the duality relationship (the dual of V_n^* is V_n). $\diamondsuit$

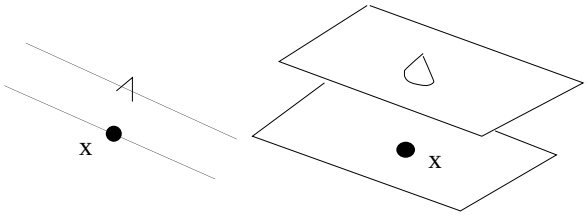

Figure 5. Icons for a covector at x, in 2D and 3D. (Their origin can be traced back to [VW].) The length of the segment or the area of the plane patches are not meaningful.

We said that force should not be construed as pointing towards some direction, the way a vector does. This doesn't imply that force has no directionality at all, for spatial directions are not all alike with respect to a force covector. The map $v \to e_x(v)$ has a *kernel*, made of vectors at x such that $e_x(v) = 0$. They define, in 3D, a plane containing x, along which virtual work is zero, to first order. (So this plane is tangent to the equipotential surface of the force field through x, if there are[16] such surfaces.) Under the virtual work interpretation, the force at x should, actually, be visualized as a pair of parallel planes (Fig. 5). One is the previous null plane, passing through x. And—just as the tip of the velocity vector of a particle was the point reached after one second of movement, assuming no change in velocity—the other plane is the one reached by releasing one joule of virtual work, again assuming uniformity of the force field.

[16] Which is not necessarily the case. One may imagine force fields such that the virtual work involved in pushing a particle form x to y depends on the trajectory followed, not only on the end-points. It means that the above "null planes" can't be quilted together to envelop surfaces. A central result of differential geometry, the *Frobenius theorem* (see, e.g., [Sc], p. 82) tells when they can.

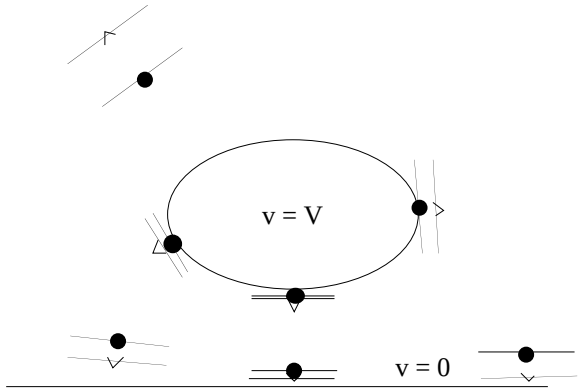

Figure 6. The electric field, between an electrode at potential V, and the ground.

Remember this hall for electrostatic experiments in the first installment ([B1], p. 19)? See on Fig. 6 how such icons nicely visualize the electric field around a charged isolated conductor, and near the ground. This pictorial representation of the force field does not depend on a metric. In particular, the notion of orthogonality of field *lines* (which have no status so far) with respect to the conducting surfaces is irrelevant. Note how Fig. 6 can be looked at from any angle, and retain its meaning. (Same remark about Fig. 5 of last issue [B1].)

It should not be felt as counterintuitive that the larger the covector (which measures the intensity of the electric field), the thinner its icon, i.e., the closer the two planes which compose it. It's because higher intensity means closer equipotentials. Figure 7 should make that clear. (Since a covector ω is a linear map from vectors to reals, it makes sense to ask the question, "given a covector ω and a vector v, what is the value of $\langle\omega, v\rangle$?", which Fig. 7 answers.)

We are now prepared to introduce the notion of gradient of a function. If a smooth function f maps A_n (the affine space) to reals, we may expand it in the vicinity of a point x, thus obtaining $f(y) = f(x) +$ <a linear part in $y - x$> + <higher order terms>. The "linear part" here, considered as a function of $y - x$, is a covector, which we may denote as $\mathrm{d}f(x)$, and call the *differential* of f *at* point x. Hence a covector field $\mathrm{d}f$ (the map $x \to \mathrm{d}f(x)$), i.e. a differential form, which it would be natural to call the differential of f. It's more common to call it its *gradient*, however. No surprise here: we expected the force field to be the gradient of the potential function, when there is one. But a great risk of confusion, because "gradient" may

mean something else when a metric structure is present. Let's address this delicate issue.

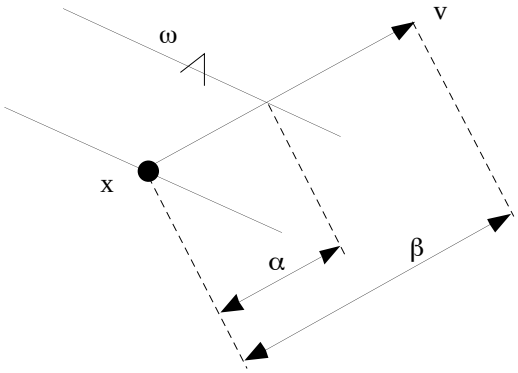

Figure 7. Since $\langle \omega, v \rangle = 1$ when the tip of vector v lies on the "arrowed front" of the covector ω, the value of $\langle \omega, v \rangle$ for the vector v of this figure is the ratio β/α, by linearity. (This ratio is an *affine* notion, since only one direction is concerned. No metric involved.) Large ratios, i.e., big covectors, thus correspond to closely spaced planes in the covector's icon.

2.4 Vectors as proxies for covectors

Suppose that a dot product has been defined. Then, each linear map $v \to \langle \omega, v \rangle$, that is to say, each covector ω, has an associated *Riesz vector*, defined as the unique vector w such that $w \cdot v = \langle \omega, v \rangle$ for all v. Conversely, of course, each vector w generates a linear map, that is a covector. (Hence an isomorphism, non-canonical, between V_n and V_n^*.) A vector field, therefore, generates a differential form. To save on notation, I will denote the DF thus associated with a vector field w as ${}^1w$, where the 1 refers to something called the "degree" of the DF (later to be defined, but of no importance right now). Same notation, of course, for the covector ${}^1w$ (or ${}^1w(x)$ if we need to refer to its location), generated by the vector w at point x.

So if a force field is described by a DF ω, it can as well be described by the vector field w such that ${}^1w = \omega$. By the very definition of the iconic planes in Fig. 5, w is orthogonal to them, and hence the field lines of w are orthogonal to the equipotentials of the field. This restores the sense of pointing directionality of force that we shunned a moment ago, and also gives status to the notion of field lines. These lines support the test-particle trajectories, so there is no doubt that metric is physically relevant here. But what it tells about is the structure of *space*, as revealed by the dynamics of simple particles,[17] not the structure of the over-imposed electric field.

The best way to make that obvious is to imagine two metrics that would only differ by the chosen unit of length, say $\cdot_i$ for inches and $\cdot_m$ for meters. The *same* electric field e is then represented by two *different* vector fields $\mathbf{E}_i$ and $\mathbf{E}_m$, linked[18] by the equality

$$(6) \qquad \mathbf{E}_i(x) \cdot_i v = \mathbf{E}_m(x) \cdot_m v$$

for all test vectors v. This is reflected in the choice of units: volts per inch for $\mathbf{E}_i$, volts per meter for $\mathbf{E}_m$, whereas the DF itself is, so to speak, in "volts per vector". One should imagine, at each point of space, a machine where one can insert a vector, to then see a dial give the number of joules (recall that joule = volt $\times$ coulomb) available by letting a unit charge (one coulomb) drift from x to $x + v$ (all that, of course, to first order and virtually).

The proxy fields $\mathbf{E}_i$ and $\mathbf{E}_m$, therefore, cannot do their job alone. Both require a specific metric, irrelevant to the virtual work available at each point, which reflects the real nature of the electric field: a differential form.

Now, let's go back to the notion of gradient. A pressure gradient, for instance, is routinely given in millibars per kilometer, in technical meteo reports, showing that what people have in mind there is the vector field representative, not the differential form. We must commit ourselves to some nonvarying use, so we shall denote the differential form by $\mathrm{d}f$ and reserve the notation $\mathrm{grad}\, f$ to the proxy vector field with respect to a background metric which, hopefully, will be fixed once and for all. This way, therefore, $\mathrm{d}f = {}^1(\mathrm{grad}\, f)$.

Remark. If one applies the DF $\mathrm{d}f$ to a specific vector field v, one gets a scalar field

$$(7) \quad x \to \langle \mathrm{d}f(x), v(x) \rangle \equiv (\mathrm{grad}\, f)(x) \cdot v(x),$$

which is the one often denoted by $\partial f/\partial v$, or better $\partial_v f$, that is, the *derivative of f along*

[17] Note that a particle with complex inner structure, like a spinning top charged off-center, would behave differently.

[18] The differential form e is the same in both cases, so $e = {}^{1_i}\mathbf{E}_i = {}^{1_m}\mathbf{E}_m$, as we would be forced to write if we insisted on having the metric explicitly appear in the notation. Just keep in mind that this left upperscript "1" implicitly points to the metric.

v.[19] (The notion of *normal derivative,* $\partial_n f$, is a case in point.) An *affine* notion, as one sees, despite the appearance of a dot product on the right of (7). By a reversal of viewpoint, a vector field can thus be seen as an *operator,* acting on functions, which has all the formal properties of a *derivation* (linearity, $\partial_v(f+g) = \partial_v f + \partial_v g$, and Leibniz rule, $\partial_v(fg) = f\partial_v g + g\partial_v f$). *Vector fields are derivations.* We won't make much of this important observation for the time being, except justify the use of an otherwise bizarre notation, ∂_i, for the fields of basis vectors in what follows. $\diamondsuit$

2.5 Components

We avoided using frames and components up to now, and indeed, one can do much mileage without them. They can't be ignored, however, if only to make contact with other work using them.

So let's assume a smoothly varying field of frames over A_n : at each point, n independent bound vectors $\{\partial_1(x), \partial_2(x), \ldots, \partial_n(x)\}$, forming a frame at x, that we may denote $\partial(x)$, and also forming n vector fields ∂_1, ∂_2, ..., ∂_n. The field $x \to \partial(x)$ is called a *reference frame.*[20] Any vector field v can then be written

$$v = \sum_{i=1,\ldots,n} v^i \partial_i, \tag{8}$$

with of course $v(x) = \sum_i v^i(x)\partial_i(x)$ at each point. The v^is are the *components* of v in this reference frame. We define an associated *dual* frame, or *coframe,* by introducing the covectors $d^i(x)$ such that

$$\langle d^i(x), \partial_j(x)\rangle = 0 \text{ if } i \neq j, 1 \text{ if } i = j. \tag{9}$$

A covector field ω can then be written $\omega(x) = \sum_i \omega_i(x) d^i(x)$ at each point, that is[21]

$$\omega = \sum_{i=1,\ldots,n} \omega_i d^i. \tag{9}$$

As a result, the scalar field $\langle \omega, v\rangle$ is equal to

$$x \to \sum_{i=1,\ldots,n} \omega_i(x) v^i(x), \tag{10}$$

which is of course generally abbreviated as $\sum_i \omega_i v^i$ or even $\omega_i v^i$ (the Einstein convention). Be careful, *this is not a scalar product,* but a so-called "duality product".

Now suppose there is a metric, defined by the dot products $g_{ij} = \partial_i \cdot \partial_j$ (there, position dependent). Then, the Riesz vector of ω is w such that $\langle \omega, v\rangle = \sum_{i,j} g_{ij} w^j v^i = \sum_i \omega_i v^i$, hence $\omega_i = \sum_j g_{ij} w^j$. The components ω_i of ω are then called the *covariant components* of $\ldots w$! These components do not depend on w and the basis only (as do its *contravariant* components, i.e., the w^is in the expansion $w = \sum_i w^i \partial_i$), but also on the metric, to which the w^is owe nothing, so it's a misleading symmetry that is suggested by this unfortunate terminology.

Where does it come from, by the way? If the basis is changed to $\bar\partial_i$, with the new basis vectors given as $\bar\partial_i = \sum_j A_i^j \partial_j$ in terms of the old ones, the new components $\bar v^i$ must satisfy $\sum_i \bar v^i \bar\partial_i = \sum_i v^i \partial_i$, hence $v^i = \sum_j A_j^i \bar v^j$: they transform "the other way" with respect to the basis vectors. A similar calculation, based on (9)(10), shows that "covariant components" of w transform just as basis vectors do: $\bar\omega_i = \sum_j A_i^j \omega_j$.

This is why vectors and covectors are sometimes called "contravariant vectors" and "covariant vectors". One may find this debatable. (After all, defining a change of frames by the way the ∂_is change is an arbitrary choice. One might as well give the new basis covectors in terms of the old ones. Then, it's the ω_is that would be "contra"!) But much worse, covectors (and all other sorts of tensors) are sometimes *defined* by their behavior under frame changes, and that is really old-fashioned, a remembrance of the time when vectors were not conceived as

[19] So one has $\langle \mathrm{d}f, v\rangle = \partial_v f$, exhibiting a duality that would better be seen by writing $\langle \mathrm{d}f, v\rangle = \langle f, \partial v\rangle$. But no notational system can satisfy all needs with equal success.

[20] Be well aware—this is a nasty little trap, that very few authors warn about, [Sc] being one of the exceptions—that whether a *coordinate system,* that is, a set of n functions $\xi^i : A_n \to \boldsymbol{R}$ that map points to n-tuples of coordinates, induces basis vectors (∂_i is just the one corresponding to the i-th partial derivative), *the converse is not true,* even if one restricts attention to the neighborhood of a point. This is why we talk of *components,* not of *coordinates.*

[21] The symbol d^i thus stands with advantage for the "dx^i" of the physics literature, a badly thought-of notation in many respects.

autonomous objects, but as frame-related sets of numbers.[22]

Remark. Components of $\mathrm{d}f$ are the partial derivatives $\partial_i f$. So $\partial_v f = \sum_i \partial_i f v^i$. Components $(\mathrm{grad}\, f)^i$ of $\mathrm{grad}\, f$, such that $\mathrm{grad}\, f = \sum_i (\mathrm{grad}\, f)^i \partial_i$, verify $\partial_i f = \sum_j g_{ij} (\mathrm{grad}\, f)^j$.$\diamondsuit$

2.6 Gyroscopic forces, bi-covectors

Now, let's deal with the "magnetic part" of the Lorentz force, the one due to motion. It has the experimentally demonstrable property of being *gyroscopic*, that is, to depend on the *actual* velocity vector V in such a way that the *virtual* work for a v parallel to V is null. (Electrons in a steady magnetic field twirl around field lines in complex motion, but neither lose nor acquire energy.[23])

A *gyroscopic force field,* therefore, can be characterized as a *covector-valued map,* which at each point sends the actual velocity vector V of the particle passing there to a covector $\omega(V)$, with the essential property that

$$\langle \omega(V), V \rangle = 0 \text{ for all } V. \tag{11}$$

By linearity,

$$0 = \langle \omega(v+V), v+V \rangle = \langle \omega(v), V \rangle + \langle \omega(V), v \rangle$$

for any pair of vectors $\{v, V\}$. This suggests to define a new entity, denoted b, acting on such pairs to yield a real number, which we shall denote $\langle b; v, V \rangle$, and define as

$$\langle b; v, V \rangle = \langle \omega(V), v \rangle$$

where $\omega(V)$ is the force covector. So we have here (at each point) a mapping of type *VECTOR* $\times$ *VECTOR* $\rightarrow$ *REAL*, which is linear with respect to both arguments, and *alternating*, meaning that $\langle b; v, V \rangle = -\langle b; V, v \rangle$ whatever v and V. Such an object is called a *bicovector*, or 2-covector. (They form a vector space, which has a dual, elements of which are *bivectors*.) A field of 2-covectors is a *differential form of degree* 2, or 2-form for short, in reference to the number of vectors acted upon. So *the magnetic part of the EM field is a* 2*-form*, called "magnetic induction". It's like having a distribution of machines with two slots (to be filled in this precise order), one for the virtual displacement, one for the actual velocity, the dial then giving the virtual work. Figure 8 displays a suitable icon [Bu].

What of vector proxies? There's an obvious alternating bilinear map that one can associate with a vector u: it's

$$\{v, V\} \rightarrow u \cdot (v \times V), \tag{12}$$

but this time *both* metric and orientation are necessary to build this associate bicovector, that I shall denote ${}^2u$. (Thus, by definition, $\langle {}^2u; v, V \rangle = u \cdot (v \times V)$, or $(V \times u) \cdot v$.) In the stage analogy, therefore, we can have a vector field playing the part of the magnetic induction field, but only if metric *and orientation* have previously been set, say g and Or. Then there is a vector field **B** which describes the magnetic part of the EM field, in the precise sense that the Lorentz force *vector* (mind that!) on a unit charge of velocity V is $V \times \mathbf{B}$. Of course, if one substitutes $-Or$ to Or, it's $-\mathbf{B}$ which acts.[24]

One can do without metric and orientation, thanks to the concept of 2-form: **B** is just a proxy for the 2-form $b = {}^2\mathbf{B}$, which is a purely affine object. One can keep the metric and do without the orientation by introducing an axial vector $\tilde{\mathbf{B}}$, the one represented by the pair $\{\mathbf{B}, Or\}$. Hence the oft-repeated assertion that "magnetic field is an axial vector". But is there any wisdom in thus disposing of the orientation while keeping the metric? Especially when it calls for as difficult a concept as "axial vector"?

[22] It goes this way: Build composite objects $\omega = \{\partial_1, \ldots, \partial_n;\ \omega_1, \ldots, \omega_n\}$, made of n vectors and n real numbers. Say ω and $\bar{\omega}$ are equivalent if there is a regular matrix A such that $\bar{\partial}_i = \sum_j A_i^j \partial_j$ and $\bar{\omega}_i = \sum_j A_i^j \omega_j$, and call the equivalence class a "covariant" vector. (Note that, in order to define vectors in the first place by such a method, you would have to deal with classes of $n(n+1)$-tuples of numbers. No wonder "old tensor" calculus is so dreaded.) With this approach, an axial vector is a (class of) similar sets of numbers, but with the last n components a^i transforming as $a^i = \mathrm{sign}(\det(A)) \sum_j A_j^i \bar{a}^j$.

[23] The energy build-up in synchrotrons is due to an *electric* field, which changes direction at each half-turn.

[24] And if one substitutes inches for meters... But no need to repeat this.

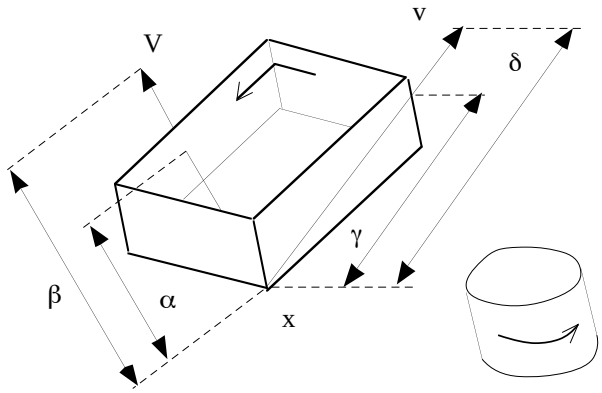

Figure 8. A 2-covector at x needs two pairs of parallel planes (with a way to tell which is first, here the caret in front of x). So it has a definite associated spatial direction (its kernel, to which its vector proxy will be parallel). The length of the icon along this direction is not meaningful. Note the suggestion of a flux tube. (Strong fields correspond to narrow tubes.) The effect on the pair of vectors is the ratio $\beta\delta/\alpha\gamma$ (again an affine notion), but you need to shear the cross-section first, to let one plane absorb one of the vectors. Rules of this kind make such icons too cumbersome to be useful as such. On the right, a simplified one.

2.7 Twisted covectors

The same way we defined axial vectors, we may introduce new affine objects,

$$\tilde{\omega} = \{\{\omega, Or\}, \{-\omega, -Or\}, \tag{13}$$

where ω is a covector. But "axial covectors" would be a poor name for them, wrongly suggesting the existence of some axis which is nowhere in sight on the icon (Fig. 9). We shall call them *twisted* covectors, and unify the convention by thus referring to all objects which "carry orientation in their bag", including axial vectors, from now on *twisted* vectors. Of course vector proxies for twisted covectors are twisted vectors. Having no immediate use for such objects, however, we shall not deal with them right now, but return to the analysis of Maxwell's equations, beginning with *Faraday's law*, in search for motivation.

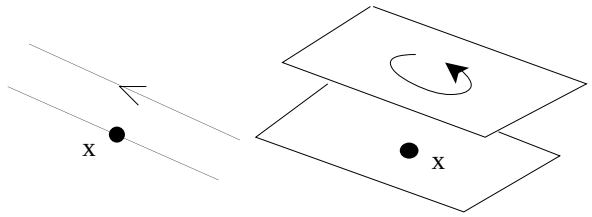

Figure 9. Icons for a *twisted* covector [Bu], in 2 and 3 dimensions.

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(3): Integration, Stokes, Faraday's law

Where do we stand? We have identified *three* systems of mathematical entities, closely related but distinct, by which the physical electromagnetic field can be represented: (1) The pair of vector fields **E** and **B**, (2) The pair of differential forms e and b, (3) The pair consisting of **E** plus the *axial* vector field $\tilde{\mathbf{B}}$—the definition of which will briefly be recalled. Now, we show how Faraday's law is expressed in these systems.

3.1 Three systems of representation

In the previous chapters, we have found a minimalist description of the EM field. With the barest equipment in background (affine 3D space), it was possible to display geometric objects which tell about forces felt by electric charges: a field $x \to e_x$ of covectors, denoted e, which gives the Coulomb force, and a field $x \to b_x$ of 2-covectors, denoted b, which gives the part of the Lorentz force proportional to the particle's velocity. (We'll say the "dynamic part" of the Lorentz force, for shortness, the "static part" being of course Coulomb force.)

Since the EM field was defined as "what pushes loose electric charges", the concept of electric charge being taken as a given, the above two geometric objects (known as "differential forms", the 1-form e and the 2-form b, both time-dependent) contain all the relevant information about the field. Neither the orientation of space nor its metric played any role in it. Of course, one may have to introduce them later, in order to deal with other aspects of physics. But up to now, and to say it briefly, we have been able to describe the electromagnetic field via *affine* objects.[1]

As suggested last time, it's like having at each point of space a machine with two slots and two dials. Insert a vector v in the first slot, and the first dial displays the (virtual) work yielded by the field in the virtual displacement v of a particle of charge unity standing at this point, hence the "static" force that a passing charged particle will feel. This is the electric side of the machine, so to speak. It has a magnetic side, too: To exploit it, insert the actual speed V of such a particle in the second slot, and read off the corresponding virtual work (still with respect to v), on the second dial. If we agree to denote by $i_u b$ a covector[2] of the form $v \to b(u, v)$, the "dynamic" force is therefore the covector $-i_V b$, and the Lorentz force is the covector $e - i_V b$.

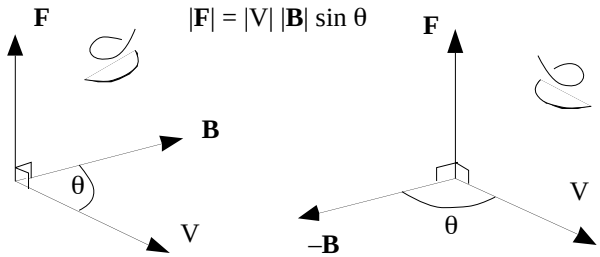

Figure 1. Construction of the vector proxy for the "dynamic" force, here denoted **F**, from the velocity vector V and the magnetic-induction vector proxy **B**. Metric information (lengths of V and **B**, and the angle they form) is enough to know the length of **F** and its supporting line, but an orientation is needed to know which way **F** points. If one decides to change the orientation of space (right side of the figure), one must represent the same magnetic induction by $-\mathbf{B}$ instead of **B**, since **F** doesn't change. So, for a given metric, the same 2-form b has *two* vector proxies, **B** and $-\mathbf{B}$, one for each orientation of ambient space.

The vector formalism replaces these pretty complex machines by simple pairs of vectors: **E** and **B**, one pair at each point. The relation with e and b is given by $e = {}^1\mathbf{E}$ and $b = {}^2\mathbf{B}$, in last installment's notation.[3] Now the above virtual work is $(\mathbf{E} + V \times \mathbf{B}) \cdot v$, which

[1] One can do with even *less* than an affine structure: Since p-covectors make sense at each point of a differentiable manifold (they act on tangent vectors), differential forms make sense too. But the generality thus gained would not compensate for the increased conceptual difficulty. Differential manifolds are apt to model our intuitive notion of "three-dimensional continuum" that ambient space seems to have, at least at our scale. Some physicists currently speculate about the possibility of not even assuming such continuity [Bw].

[2] This is called the *inner product* or *contraction* of the 2-covector b by the vector u. We'll use it later.

[3] For the reader's convenience, let's recall this notation: given a vector u, ${}^1u$ denotes the covector $v \to u \cdot v$ and ${}^2u$ the 2-covector $\{v, w\} \to u \cdot (v \times w)$. The metric is involved in both mappings. Moreover, since the cross product $\times$ is orientation-dependent, the correspondence

is another way of saying that the Lorentz force is the covector ^{1}(**E** + $V \times$ **B**), the same covector as $e - i_V b$. Owing to long familiarity with the expression **E** + $V \times$ **B**, one may consider the {**E**–**B**}-machines as simpler than the $\{e$–$b\}$-ones, but this is an illusion. The {**E**–**B**}-machines cannot work without a whole infrastructure in background: the dot product and the orientation, both necessary to give sense to the cross product operation (see Fig. 1). The $\{e$–$b\}$-machines, more basic, can work in autonomy. It's in that sense that our description deserved to be called "minimal".

From this point of view, the "{**E**–$\tilde{\mathbf{B}}$}-machines" stand in between, for this mode of representation necessitates only a metric in background, no orientation (cf. Fig. 2). At each point, there is a regular (or "polar") vector **E** and a "twisted" (or "axial") vector $\tilde{\mathbf{B}}$. The force vector **F** is of course the same, but is obtained by a slightly different rule, a kind of "orientation-free" variant of the cross product, which Fig. 2 displays.

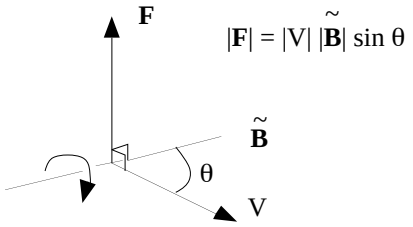

Figure 2. Construction of the dynamic-force vector **F** from the velocity vector V and the magnetic-induction *twisted*-vector proxy $\tilde{\mathbf{B}}$. This time, orientation of ambient space is irrelevant: the outer orientation of the support of $\tilde{\mathbf{B}}$, which $\tilde{\mathbf{B}}$, as a twisted vector, brings with it, suffices to show which way **F** must point: so that turning it 90° around $\tilde{\mathbf{B}}$ will bring it in the same plane as V and $\tilde{\mathbf{B}}$.

Is the simplification thus brought in by the use of an axial vector worth the trouble? The issue will be more clearcut after we have compared the expressions of Faraday's law within each of the three systems. For this, we need to know how to integrate differential forms first.

Exercise. Both figures 1 and 2 can be read as geometric constructions of the icon of the Lorentz-force vector proxy, assuming a metric in background. Similarly, it should be possible to build the Lorentz-force *co*-vector's icon from those of V and of the 2-covector b (see last issue for iconic conventions). Can you propose such a construction? (You may find it a rather involved and cumbersome recipe. Decide by yourself, however, to which extent this judgment is biased by long-time familiarity with vectors' manipulation.)

between u and ^2u also depends on orientation. (Exercise: Check that $i_v{}^2u = -^1(v \times u)$.)

3.2 Integration of forms

Integrating differential forms is a simple matter: they are, as we presently see, geometrical objects which are *meant* to be integrated. More precisely, p-forms have canonically defined integrals over p-dimensional *inner* oriented manifolds. Cases p = 1 and 2, that is, lines and surfaces, are the ones of interest for us.

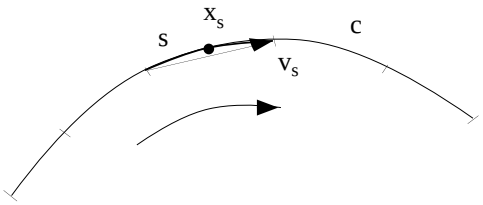

Figure 3. Subdividing an oriented curve c in order to set up a Riemann sum such as (1); shown, a generic "curved segment" s, and the associated point and vector.

Suppose we are given the electric field as a smooth 1-form e and are being asked to find the corresponding electromotive force along an oriented smooth curve c (Fig. 3). Let's chop the curve into a finite family $\mathcal{S}$ of adjacent curve-segments and pick a point x_s in each of them. To each such short curve s, we may associate a vector v_s which joins its endpoints; we make this vector point in the direction indicated by the curve's orientation. Let the covector $e(x_s)$ at point x_s act on v_s, yielding the real number $\langle e(x_s), v_s \rangle$. Summing up these numbers, we obtain the following "Riemann sum"

$$(1) \qquad I(c; \mathcal{S}, e) = \sum_{s \in \mathcal{S}} \langle e(x_s), v_s \rangle .$$

Now, by definition, the integral of e over c is the limit

$$\int_c e = \lim_{\mathcal{S} \to \infty} I(c; \mathcal{S}, e),$$

where "$\mathcal{S} \to \infty$" means that the number of curved segments tends to infinity as the subdivision is repeatedly refined.

We shall not attempt to make this "$\mathcal{S} \to \infty$" notion more precise. This would raise a series of technical points (for instance, make sure that no finite part of the curve escapes subdivision),

and thus hide the main ideas. Moreover, there is no big difference, in how one would tackle the passage to the limit, with respect to the Riemann integration theory for functions, to which the reader has probably been exposed once (and once is enough ...).

What does make an important difference, on the other hand, is the absence here of any reference to the *lengths* of the vectors v_s. Such lengths are not defined anyway, since we did not assume a metric on the ambient space. Had we done that, there would be a notion of length of lines, of area of surfaces, of volume of 3D regions: In one word, a *measure*[4] (in the sense of classical measure theory [Ha]), induced by the metric, on manifolds of any dimension. Here, there is no need for such a measure: the differential form e carries with itself, so to speak, all it needs to be integrated over an oriented line.

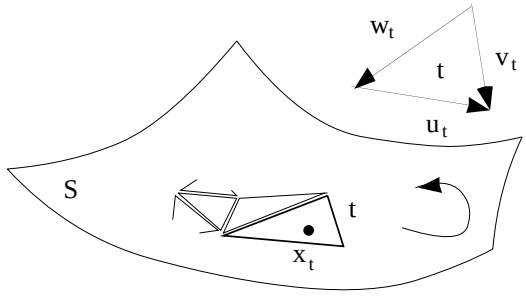

Figure 4. Subdividing an inner oriented surface S into triangular patches. A generic triangle t, and its assigned point x_t in S. Top right: three vectors associated, in an arbitrary way, with t's boundary.

Oriented is the keyword, as we shall see more clearly in dimension 2. Now, it's a 2-form b that we want to integrate over a surface S (Fig. 4), and the idea is to approximate S by a finite family $\mathcal{T}$ of tiny adjacent triangles, each with an associated surface point x_t and boundary edge vectors u_t, v_t, w_t. These have arbitrary orientation, but the inner orientation of S intervenes in the way we now build each term of the Riemann sum: the rule is to select any two vectors out of three, to *order them* so that they make a *positive frame* (this point is essential), and to let the covector $b(x_t)$ act on them. For instance, in the case of Fig. 4, a number which complies with this prescription is $\langle b(x_t); v_t, u_t\rangle/2$ (the 2 is a dimensional factor), and one easily sees that the same number would pop up if we had chosen vector-pairs $\{w_t, u_t\}$ or $\{w_t, v_t\}$, thanks to the properties of b as a bilinear alternating mapping. This number is thus a characteristic of the oriented triangle t, so we shall abbreviate it as $\langle b(x_t); t\rangle$. The rest is straightforward: form the Riemann sum

$$I(S; \mathcal{T}, b) = \sum_{t\in\mathcal{T}} \langle b(x_t), t\rangle,$$

and set $\int_S b = \lim_{\mathcal{T}\to\infty} I(S; \mathcal{T}, b)$.

Generalizing to a connected[5] p-manifold M is easy: To get the integral $\int_M \omega$, divide M into a family $\mathcal{S}$ of p-simplices, pick a point x_s in each of them, build vectors along the edges of s, select any p of them which form a *direct* frame, let $\omega(x_s)$ act on them, and divide by $p!$ to get $\langle \omega(x_s), s\rangle$. Then $\int_M \omega = \lim_{\mathcal{S}\to\infty} \sum_{s\in\mathcal{S}} \langle \omega(x_s), s\rangle$, as above.

The reader may want to work out the case $p = 3$, in 3D affine space. When $p = 0$, of course, the integral of a 0-form f over a 0-dimensional connected manifold, that is to say, a single point[6] x, is $f(x)$.

Remark 1. One should not confuse the integral of a 0-form, just defined, with the integral of the function f, in the common acception of the word, over a line, a surface, etc. This doesn't make sense in the present context, where there is no underlying measure. What may make sense, on the other hand, is the integration of some differential form *generated by* the function. We return to this point in a moment. ♢

3.3 "Old" vs "new" integration

We need to review the relationships between this new concept of integration and the standard one when dealing with differential forms in oriented Euclidean 3D space. From last installment (cf. Note 3), we know four kinds of such objects: the 0-form and 3-form ${}^0f$ and ${}^3f$

[4] Called the *Lebesgue measure*.

[5] No essential restriction there: The integrals can be evaluated over each connected component, then summed up. Note however that orientations could be chosen independently for each connected component of M.

[6] Which we assume, by convention, to be positively oriented. A point *can* be oriented: this amounts to give it a sign, + 1 or −1. But having no need for it, we don't consider this possibility. If however x was negatively oriented, the "integral" of ${}^0f$ would be $-f(x)$, by convention.

generated by a function f, and the 1-form and 2-form ${}^1u$ and ${}^2u$ generated by a vector field u.

We just saw what the integral of ${}^0f$ is. What comes next is the integral of ${}^1u$ on a smooth oriented curve c. Let's introduce a field $x \to \tau(x)$ of vectors tangent to c, all of length 1, all pointing in the direction of positive orientation. Terms of the Riemann sum (1) are of the form $\langle {}^1u(x_s), v_s\rangle = u(x_s)\cdot v_s \simeq u(x_s)\cdot\tau|v_s|$. Since the norm $|v_s|$ is also the Lebesgue measure of the segment s, this sum can be rewritten as $\sum_{s\in\mathcal{S}} \tau\cdot u(x_s)|v_s|$, which can be viewed as a Riemann sum for the classical integral of the *function* $x\to\tau\cdot u(x)$, on c, with respect to the metric-induced measure. Hence, at the limit,

$$\int_c {}^1u = \int_c \tau\cdot u, \tag{2}$$

where the second integral is a classical one (again with the Lebesgue measure understood), called the *circulation* of u along the oriented curve c.

It's an easy guess that, similarly

$$\int_S {}^2u = \int_S n\cdot u, \tag{3}$$

the *flux* of u through S. The field $x\to n(x)$ is made of vectors of length 1, orthogonal to S, and all oriented the same way. *Which* way? This is told by the orientation we had on E_3 in the first place: the crossing direction is the one which makes the intrinsic orientation of S and the ambient orientation "match", in the sense of the Ampère observer's rule. One may feel surprised that S needs an *inner* orientation at the left-hand side of (3), and (apparently) an *outer* orientation at the right-hand side. But this outer orientation is only meant to induce an inner one, in conjunction with the ambient orientation on E_3. And since the way u generates ${}^2u$ also depends on ambient orientation, the two effects of the latter cancel out. *No crossing direction is really involved in* (3), which makes the "flux" terminology a bit misleading.

Remark 2. So we may anticipate that when we really need to deal with a specific crossing direction, as will be the case for intensities, the integral of a 2-form will not be the adequate concept any longer. ♢

Remark 3. Would the flux of an *axial* vector[7] make sense? It does, over an *inner* oriented surface, even if the ambient Euclidean space is not oriented itself. Select one of the two representatives of the twisted vector, $\{u, Or\}$ say, use Or to determine which way n should point, then $\int_S n\cdot u$ is the flux. Again, we see the advantage of using twisted vectors: no need to specify an ambient orientation that will, anyway, be irrelevant to the final result (the value of the integral). But this is a tiny advantage: we still need a metric from which to build a Lebesgue measure, even though this measure is irrelevant to the final result. ♢

To complete the series, the integral of ${}^3f$ over an oriented three-dimensional domain D is $\pm\int_D f$, the integral in the standard sense of the function f, with the Lebesgue measure of volumes understood. The sign depends on whether the orientations of D and E_3 match or not.

There are other possibilities to build differential forms from functions and fields, but they always reduce to the above four ones in some way. For instance, what is often denoted $\mathrm{d}x$ is the one-form $v\to v^x$, where v^x is the x-component of vector v in an x–y–z system of local coordinates. The integral $\int_c \mathrm{d}x$ over a curve c is simply the circulation along c of the field of x-directed basis vectors.

Conversely, one could argue that any integral actually concerns some differential form: whatever one finds under a summation sign is, in some way, a form. For instance, the "double integral" $\iint_S f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ of classical calculus is the integral of a 2-form, here denoted $f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ (and sometimes, in a way which helps reinforce this interpretation, $f(x,y)\,\mathrm{d}x\wedge\mathrm{d}y$), built from f and from the coordinate-related 1-forms $\mathrm{d}x$ and $\mathrm{d}y$. (By definition, $\langle \mathrm{d}x\wedge\mathrm{d}y\,;v,w\rangle = v^x w^y - v^y w^x$. This is called the "wedge product" of $\mathrm{d}x$ and $\mathrm{d}y$.)

3.4 The Stokes theorem

Knowing how to integrate forms, one may wish to be able to differentiate them, too.

Start from a system $\{v_1, ..., v_{p+1}\}$ of p+1 vectors at a point x. They form a $(p+1)$-simplex s, say a triangle in the case $p=1$, to which the order in which the vectors have been presented confers an inner orientation (Fig. 5). The boundary ∂s of this simplex has an outer orientation, as we remarked earlier: "from inside to outside" defines a crossing direction. Using s

[7] Recall that I take "axial" and "twisted" as synonyms, but beware: other authors may use different conventions.

itself as "ambient space" in which ∂s is embedded, we derive from that a canonical inner orientation for all *faces* of s, i.e., the p-simplices that together constitute ∂s. So we may integrate the given p-form, ω say, over ∂s, hence a number $\int_{\partial s} \omega$.

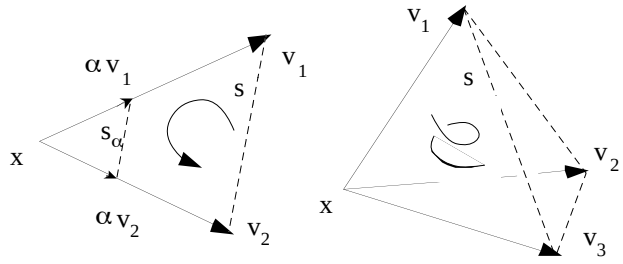

Figure 5. How $p + 1$ vectors, given in an ordered sequence, generate an inner oriented (p+1)-simplex. Left: construction of a homothetic triangle s_α.

Now let's define a new covector at x, denoted $\mathrm{d}\omega$, by the following limit process:

$$\mathrm{d}\omega(v_1, \ldots, v_{p+1}) = \lim_{\alpha \to 0} \frac{(p+1)!}{\alpha^{p+1}} \int_{\partial s_\alpha} \omega,$$

where s_α is homothetic to s as suggested by Fig. 5, and thus "shrinks to x" as α goes to 0. The new field $\mathrm{d}\omega$ of $(p+1)$-covectors thus obtained is called the *exterior differential* of the given ω.

This way, we have the near-equality $\int_{\partial s} \omega \simeq \int_s \mathrm{d}\omega$ for a small enough simplex s. Summing up over all simplices of a subdivision $\mathcal{S}$ of some manifold M, we get

$$\int_M \mathrm{d}\omega \simeq \sum_{s \in \mathcal{S}} \int_s \mathrm{d}\omega \simeq \sum_{s \in \mathcal{S}} \int_{\partial s} \omega \simeq \int_{\partial M} \omega,$$

where the first two near-equalities stem from things we already know to be true: The first one comes from approximating $\int_M \mathrm{d}\omega$ by a Riemann sum, the second one from the very definition of d. If the third one can be justified, we shall have proven the *Stokes theorem*,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega, \tag{4}$$

where ω is a p-form and M an oriented $(p+1)$-manifold. All it takes is passing to the limit as $\mathcal{S} \to \infty$.

Remark 4. Equation (4) shows that d and ∂ are *adjoint* to each other, in some way. Note also this: since $\partial\partial M$ is empty, one has dd = 0 as a corollary. $\diamondsuit$

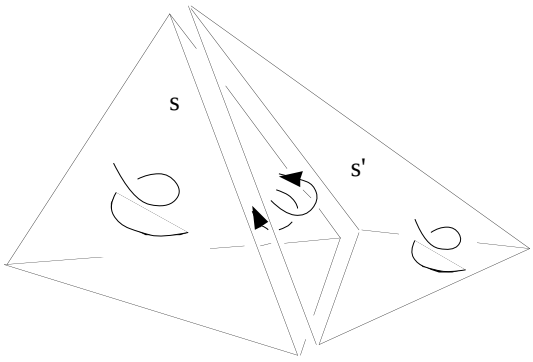

Figure 6. Because the orientation of M, whatever it is, induces compatible orientations on s and s', the two p-faces supported by the common triangle, which have opposite *outer* orientations, have opposite *inner* orientations, too.

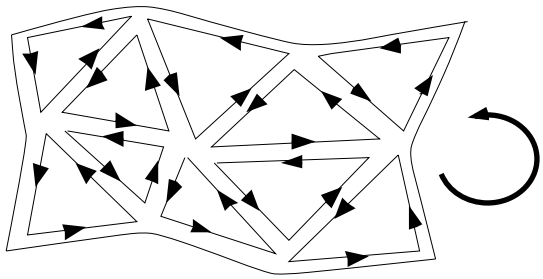

Figure 7. The same argument, in the case $p = 1$. Contributions of inner edges to the Riemann sum will cancel out.

The existence of an inner orientation of M, again, is the essential ingredient in this proof. First, because it gives sense to the first integral: if M is oriented, the outer orientation of its boundary (from inside to outside) inner-orients ∂M, as we already argued. Second, because of the cancellation two by two of the integrals $\int_\sigma \omega$ at the right-hand side of the equality

$$\sum_{s \in \mathcal{S}} \int_{\partial s} \omega = \sum_{s \in \mathcal{S}} \sum_{\sigma \in \partial s} \int_\sigma \omega, \tag{5}$$

where σ spans the set of faces of all simplices s of the subdivision $\mathcal{S}$. A face of s is an oriented p-simplex, whose orientation comes from that of s, the same way as above. But for two adjacent simplices of $\mathcal{S}$, say s and s' (Figures 6 and 7), it's two *opposite* orientations that are thus conferred on the two faces which coexist at the common boundary of s and s'. So all terms $\int_\sigma \omega$ in (5) disappear, except those corresponding to faces σ that belong to ∂M, and for these, the orientation is just right for

the corresponding terms to form a Riemann sum relative to $\int_{\partial M} \omega$.

Reviewing this proof, one will probably judge that the definition of the d operator was engineered *in order to* have Stokes' theorem hold. This is a quite defendable opinion. Indeed, the classical differential operators grad, rot, and div, are best defined in such a way, and we did that, earlier, bit by bit. So let's just review the whole thing systematically.

3.5 Grad, rot, and div, as avatars of d

Take E_3 as background, with metric and orientation Let be given a function f, and a curve c, represented by a parameterization[8] $t \to c(t)$, with $t \in [0,1]$. The gradient of f is the vector field $\operatorname{grad} f$ such that $\int_c \tau \cdot \operatorname{grad} f = f(c(0)) - f(c(1))$. This latter expression happens to be $\int_{\partial c} {}^0f$, so using Stokes, i.e. $\int_c \mathrm{d}\, {}^0f = \int_{\partial c} {}^0f$, we have that

$$(6) \qquad {}^1(\operatorname{grad} f) = \mathrm{d}\, {}^0f \equiv \mathrm{d}f$$

(we may drop the zero superscript, for there is only one way to turn a function into a 0-form). *The vector-field* $\operatorname{grad} f$ *is a proxy for the* 1*-form* $\mathrm{d}f$, as we saw last time.

Remark 5. In particular, if f is the function $\{x,y,z\} \to x$, in an x–y–z Cartesian system, then $\mathrm{d}f$ is what we denoted $\mathrm{d}x$ a moment ago. Having a unique notation d for what looked to be different notions of differentiation seems more logical at present. So there is some method in our madness. $\diamondsuit$

Now, you guessed it:

$$(7) \quad {}^2(\operatorname{rot} u) = \mathrm{d}\, {}^1u, \qquad (8) \quad {}^3(\operatorname{div} u) = \mathrm{d}\, {}^2u.$$

To prove (7), just compare

$$\int_S {}^2(\operatorname{rot} u) = \int_S n \cdot \operatorname{rot} u = \int_{\partial S} \tau \cdot u$$

$$= \int_{\partial S} {}^1u = \int_S \mathrm{d}\, {}^1u,$$

where all equalities hold either by definition or owing to the Stokes theorem. The proof of (8) is left as an exercise.

[8] This is how it's done in practice. Note that τ is then the field $t \to (\partial_t c)(c(t))/|(\partial_t c)(c(t))|$, and that an inner orientation is implied: from $c(0)$ to $c(1)$, that is, for increasing t, by convention.

What's nice about it is that it works both ways: you can define the d of a 1-form ${}^1u$ by (6) if you have the curl already, say by its definition in coordinates, or define $\operatorname{rot} u$ once you know what d is.

Still, there are subtleties in (7) and (8), worth mentioning. First, the dependence on metric. Change the dot product $\cdot$ for a new one $\bullet$, the gradient changes: the vector fields $\operatorname{grad} f$ and $\mathbf{grad} f$ such that $\operatorname{grad} f \cdot v = \mathbf{grad} f \bullet v = \langle \mathrm{d}f, v \rangle$, for all v, are of course distinct. (The notational distinction, bold versus plain, is ad hoc, and won't be used again.) A similar thing happens with rot and **rot**, respectively associated with $\cdot$ and $\bullet$. In terms of the transition matrix L such that $u \bullet v = Lu \cdot Lv$, one has $L^tL \operatorname{grad} = \mathbf{grad}$, obviously. If you solved the similar exercise regarding $\times$ (first paper of the series, §1.6), you won't find it too hard to work out the analogous relation between rot and **rot**, and to see why, as already mentioned, div = **div**.

The second remark is about the dependence on orientation. Observe that it intervenes twice in (8) (both superscripts 2 and 3 refer to orientation-dependent associations), once in (7), and not at all in (6). Therefore, rot is orientation-dependent, whereas neither grad nor div are. In more precise terms, if ${}^1u$ is the 1-form whose vector proxy is u, the vector proxy of its d, that is $\operatorname{rot} u$, depends on orientation.

Can that be cured? The illness comes from the fact that 2-forms, contrary to 1-forms, have two, opposite, vector proxies, one for each orientation. On the other hand, we know that a 2-form has a single *twisted* vector proxy. So let's represent the 2-form d^2u by a twisted vector field, denoted $\tilde{\operatorname{rot}} u$. This way, we have defined a differential operator, $\tilde{\operatorname{rot}}$, of a new type: it maps vector fields to *twisted* vector fields, instead of to plain vector fields. All this is so cumbersome that one may doubt the usefulness of such hair-splitting. So it's time to turn to a specific application: the various ways to mathematically describe (the ways to *encode*) Faraday's law.

3.6 Faraday's law

As established by Faraday's famous experiment, variations of the flux through a conducting loop create an electromotive force. A mathematical statement which is meant to express this law with maximum economy will there-

fore establish a link between the integral of b (the induction 2-form) over a fixed surface S and the integral of e (the electric field 1-form) over its boundary ∂S. Here it is:

$$\partial_t \int_S b + \int_{\partial S} e = 0, \tag{9}$$

where ∂_t denotes the time derivative. These numbers have dimension: webers for the first integral, volts for the second, which fits, since a volt is a weber per second.

Remark 6. Of course S and ∂S must be oriented (Fig. 8) for the integrals to make sense (and the orientation of ∂S must be the induced one, as earlier—otherwise change + to $-$ in (9)). But which orientation of S is selected is indifferent. (Physically, it's rather the orientation of ∂S which is selected: it corresponds to the two ways a galvanometer can be inserted in the circuit here abstracted as ∂S. Hence the orientation of S.) $\diamondsuit$

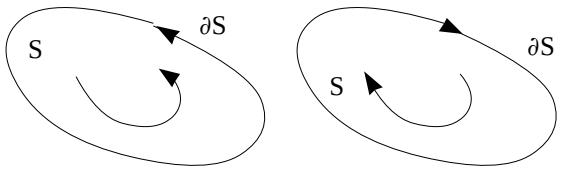

Figure 8. Both ways to orient S and ∂S in (9).

Putting metric and orientation in background (Fig. 9), and using vector proxies, we transform (9) into

$$\partial_t [\int_S n \cdot \mathbf{B}] + \int_{\partial S} \tau \cdot \mathbf{E} = 0, \tag{10}$$

the *integral* form of the law, saying that the rate of variation of the magnetic flux through a closed loop is compensated by the induced emf in this loop. By the Stokes theorem, (10) is equivalent to the *differential* version of the law:

$$\partial_t \mathbf{B} + \mathrm{rot}\mathbf{E} = 0, \tag{11}$$

and there we are in well charted territory.

But of course, one may as well apply Stokes to (9) directly, hence

$$\partial_t b + \mathrm{d}e = 0, \tag{12}$$

to be compared with (11). We have here a strictly *affine* expression of Faraday's law. Metric and orientation are not needed, and not invoked.

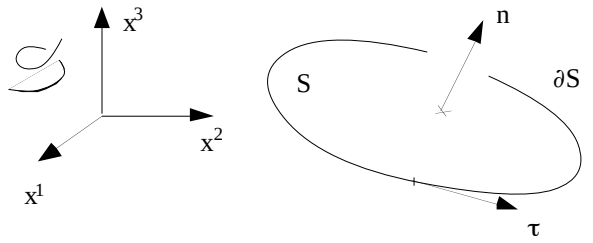

Figure 9. Notations used in formula (10). All vectors shown are of length 1.

By our earlier discussion of Lorentz force, we are prepared to see the third formalism, with no orientation but twisted vectors, stand in between. Indeed, in the "$\{\mathbf{E}$–$\tilde{\mathbf{B}}\}$-machinery", the differential expression of the Faraday law is

$$\partial_t \tilde{\mathbf{B}} + \tilde{\mathrm{rot}}\, \mathbf{E} = 0, \tag{13}$$

by using the above "twisted curl". Note the consistency: (13) is an equality between two objects of the same type—two twisted vector fields. There is of course an integral form (use Remark 3), which does not require, contrary to (10), an ambient orientation.

Is the simplification thus brought in by the use of an axial vector—the economy of the irrelevant ambient orientation—worth the trouble? Frankly, I don't think so. The metric, which is just as irrelevant, as (9) and (12) testify, is still present in (13). So why stop halfway? Either one decides to work in Euclidean space, with its metric and orientation, and the standard concept of vector field is enough, or one adopts the language of covectors and differential forms. We have not yet a compelling reason to take the latter stand (it's only when one deals with *deformable* conductors that one really appreciates the advantage of getting rid of the metric), but as for now, we find no outstanding advantage to the twisted vectors formalism.

Besides, these axial vectors pose notational problems when one wants to distinguish them from polar ones. This, which we managed to do here by using tildes, is often achieved by playing on the shape of the arrows above letters: thus $\breve{v}$ may denote an axial vector, whereas $\vec{v}$ is a polar one. Some electromagnetic treatises are graced at every page by such artwork. I am not very fond of the device: It's very difficult to use consistently, it generates cumbersome typesetting, and there are so many kinds of

geometrical objects to deal with (we have not exhausted the list!) that attempts to have a special graphism for each of them are hopeless. Better to explicitly declare the *type* of each entity when first introduced, and to use neutral typography.

We'll break here with our study of formalism, leaving the analogous discussion of Ampère's theorem for the next column. But before stopping, we need to discuss another issue: this restriction we made above by assuming a *fixed* surface S in our study of the integral form (9) of the law is a matter of concern.

3.7 Faraday's law for moving circuits

Since (9) encodes, in mathematical terms, a physical *experiment*, and since (9) and (12) are mathematically equivalent, the local expression (12) subsumes Faraday's law: it's the one on which one may *always* rely.

In many circumstances, however, we need to know the circuital emf generated by flux changes which are due, in part or in totality, to the *movement* of the circuit, a case *not* covered by laws (9) or (10). Though this situation goes beyond the limits fixed to this series (a single reference frame, no motion), the issue cannot be ignored (see [Co, CL, RS] for relevant discussions).

So, if we want a flux–emf relation for moving circuits, we must *derive* it from (12), or from one of its equivalents (11) or (13). This amounts to evaluate $\partial_t \int_S b$ when S moves, this motion being described by the velocity field V at instant t. There are two parts in this rate of variation: one is $\int_S \partial_t b$, the other is the value that $\partial_t \int_S b$ would assume if b was frozen at its instantaneous value $b(t)$, changes coming from the change of S with time. Approximating the derivative by a finite difference, we are led to a comparison between $\int_{S(t+\delta t)} b$ and $\int_{S(t)} b$ for a fixed field b. This can be done by considering the volume enclosed by a surface composed of $S(t+\delta t)$, $S(t)$, and the extrusion of ∂S in the direction of V, to which one applies Stokes' theorem, before letting δt go to 0. The computation (which unrolls just as the well-known one in vector formalism, to be found e.g. in [Co]) results in

$$\partial_t \int_S b + \int_{\partial S} (e - i_V b) = 0,$$

where one recognizes the Lorentz force felt by charges sitting on ∂S.

Discussing this would lead us too far astray. But let's stress the conclusion that the simplest form of Faraday's law we have found during this investigation of formalisms, $\partial_t b + \mathrm{d}e = 0$, is also the one which may be trusted in all circumstances.

Next, we may expect a parallel treatment of the Ampère relation, $-\partial_t d + \mathrm{d}h = j$, but an important difference will emerge: it's *twisted* differential forms that appear in this formula. And a new geometric object, the "Hodge operator", will be introduced as the right tool to model constitutive laws.

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(4): "Maxwell's house"

Last time, we were able to express Faraday's law and the Lorentz force in terms of new mathematical abstractions, namely the differential forms e and b, defined on affine 3D space without any metric or orientation. We now extend this approach to the full system of Maxwell equations. The metric of space will not be irrelevant. But we shall see exactly where and why it is needed, thus completing our layer-by-layer analysis of geometrical structures underlying Maxwell's equations. A synoptic presentation of the Maxwell equations ("Maxwell's house") can then be proposed.

4.1 The way ahead

In the previous article, we stopped at the point where Faraday's law could be expressed by $\partial_t b + \mathrm{d}e = 0$, instead of the familiar $\partial_t \mathbf{B} + \mathrm{rot}\,\mathbf{E} = 0$. In the newly arrived-at viewpoint, b and e are seen as more legitimate representatives of the (physical) electromagnetic field than their "vector proxies" **B** and **E**, because the latter need, in order to be defined, a metric and an orientation of ambient space: Both structures we recognized as irrelevant when discussing Faraday's law, whose physical content owes nothing to them. Lorentz force, also, proved representable by an affine object, the 2-covector $e - i_V b$. The correspondences $e = {}^1\mathbf{E}$, $b = {}^2\mathbf{B}$, and $i_V b = -{}^1(V \times \mathbf{B})$, where the superscripts 1 and 2 implicitly refer to a specific metric and orientation, made possible the passage from one formalism to the other.

We also acknowledged the existence of an intermediate formalism, in which ambient space need not be oriented, but still must be equipped with a metric. It accounts for Faraday's law in the rather awkward form $\partial_t \tilde{\mathbf{B}} + \tilde{\mathrm{rot}}\,\mathbf{E} = 0$, where $\tilde{\mathbf{B}}$ is a *twisted*, or *axial vector*, also acting as proxy for b, and $\tilde{\mathrm{rot}}$ is an orientation-independent variant of the curl operator, which maps ordinary (or *polar*) vectors to twisted ones.

Now it seems natural to go forward and to subject all other elements of the Maxwell system of equations to a similar transcription process. We'll find two main obstacles along the road: (1) It's not the same kind of integrals that appear in the integral forms of Faraday's and Ampère's law, (2) Metric reenters the stage when it comes to expressing constitutive laws such as $\mathbf{B} = \mu\mathbf{H}$, or $\mathbf{J} = \sigma\mathbf{E}$, and so on. Both obstacles will be overcome by the promotion of appropriate new entities ("twisted" forms, and the Hodge operator), and when we are through, a novel geometrical framework, called in jest "Maxwell's house", will be seen standing erect. We'll verify that this framework is "home", indeed, to all geometrical objects which contribute to the description of the electromagnetic field and to its dynamics.

4.2 Currents, twisted forms

Let's begin with the equation $\partial_t q + \mathrm{div}\,\mathbf{J} = \mathbf{0}$, which expresses the conservation of electric charge.

One is tempted to say, "Well, vector **J** looks very much like the proxy for a differential form j, namely $j = {}^2\mathbf{J}$; therefore, by mere imitation of what was done for Faraday's law, we guess that $\partial_t q + \mathrm{d}j = 0$ is the sought-for affine expression for charge conservation, provided we see q as a 3-form." But then what about the *integral* form of this law on some 3D domain D, that would be $\partial_t \int_D q + \int_{\partial D} j = 0$, thanks to the Gauss theorem? The integral $\int_{\partial D} j$ cannot immediately be recognized as one of the two quantities that would make physical sense, namely, the flux of charge *exiting from* D and the flux *entering* D, because integrals of this kind, as we saw last time, are only defined over *inner*-oriented surfaces, whereas such qualifiers as "exiting from" or "entering" imply a crossing direction through the boundary.

More generally, given a surface S (not necessarily closed), what we wish to capture is the notion of *intensity*, which relates to the question "how much electric charge crosses S, *in some definite direction*, per unit of time?" Intensity, therefore, refers to what we called last time an outer-oriented surface, that is, a surface endowed with a crossing direction. Note that ∂D, above, can be given one: either from

inside to outside, or the other way round, as we wish. It's a matter of convention on which we shall return.

So if we want an affine object that would properly represent intensities, it has to be something whose integral makes sense over outer-oriented surfaces. A two-form, which as we saw last time is intended to be integrated on inner-oriented surfaces, cannot do the job, unless—and this points to the solution—one has a way to infer an inner orientation of S from the given crossing direction. If ambient space is oriented, the crossing direction does imply an inner orientation of S, as we saw repeatedly (Fig. 1). But of course "exiting intensity" or (in the case of Fig. 1) "intensity from region $-$ to region +", as physical concepts, have objective meaning, which cannot depend on a mere convention about orientation. Therefore, the geometrical object entrusted to code the information about intensities across all possible outer-oriented surfaces must bring with it a tool to convert outer orientation into inner orientation.

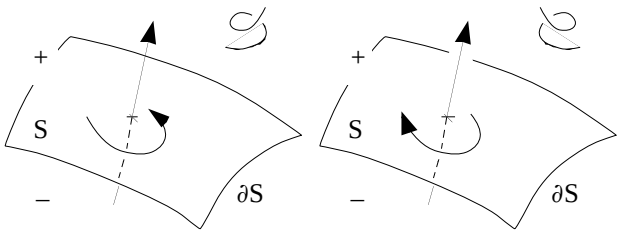

Figure 1. When a crossing direction is assigned to a surface (here from region "–" below to region "+" above), no inner orientation results, unless an ambient orientation of space is given. Then an inner orientation of S is induced. The figure displays the two possibilities in this respect.

And objects thus equipped we know about: *twisted* two-forms are it. A twisted 2-form $\tilde{\jmath}$ is an equivalence class of pairs $\{$*ordinary* 2*-form*, *orientation*$\}$: We take all pairs $\{j, Or\}$, where j is a 2-form and Or one of the two orientations possible in ambient space, decree that pair $\{j, Or\}$ is equivalent to pair $\{-j, -Or\}$, and to none other, and denote by $\tilde{\jmath}$ the equivalence class $\{\{j, Or\}, \{-j, -Or\}\}$ thus obtained. Of course, we may envision $\tilde{\jmath}$ also as a field of twisted 2-covectors: the value of $\tilde{\jmath}$ at point x is the twisted 2-covector $\tilde{\jmath}(x)$, which is itself a class $\{\{j(x), Or\}, \{-j(x), -Or\}\}$ of two equivalent pairs of type $\{$*ordinary* 2*-covector*, *orientation*$\}$.

Hence new geometrical entities which are duly integrable over outer-oriented surfaces: indeed, to compute $\int_S \tilde{\jmath}$, pick one of the pairs which compose $\tilde{\jmath}$, the pair $\{j, Or\}$ say, use Or to derive an inner orientation of S from its given outer one, as Fig. 1 explains, then integrate the ordinary form j over S, thus inner oriented. Hence a real value. Taking the other pair $\{-j, -Or\}$ would give the same value, because of two changes of sign which cancel out, so the number $\int_S \tilde{\jmath}$ is well defined by this recipe.

This is quite satisfying, for it corrects an imbalance which could be felt since we first characterized differential forms as "objects that can be integrated over inner-oriented manifolds". This left open the question "but then what about *outer*-oriented manifolds?", which we can answer now: It's twisted p-forms,

$$\tilde{\omega} = \{\{\omega, Or\}, \{-\omega, -Or\}\},$$

that can be integrated over outer-oriented p-manifolds. The integral of $\tilde{\omega}$ over the outer-oriented M is defined as $\int_M \omega$, with on M the inner orientation induced by its outer orientation in conjunction with Or.

In particular, when M is a domain of E_3, its outer orientation is a sign, + or $-$, and the inner orientation induced by Or is $+Or$ or $-Or$, depending on this sign. Hence $\int_M \tilde{\omega} = \pm \int_M \omega$, the outer orientation providing the sign.

Remark. An abstract manifold, though not necessarily orientable, can always be assigned an outer orientation (with respect to itself as its own "ambient space", so to speak), since this amounts to selecting a sign, + say. So, twisted forms directly defined over such a manifold can be integrated without ado. The weight of a Möbius band, for instance, is the integral of an appropriate 2-form, representing the paper's density. Another name for twisted forms of maximal degree is, appropriately, "densities". $\diamondsuit$

One may feel that the symmetry between ordinary forms and twisted ones is not as neat as the symmetry between inner- and outer-oriented manifolds, because of this cumbersome definition via classes.[1] But scales are balanced

[1] Earlier, we compared a p-covector with a machine with p numbered slots, in which one inserts vectors, v_1 in slot 1, v_2 in slot 2, etc., to get a real number in

again when one realizes that ordinary differential forms can be defined as equivalence classes, too: the form ω can be identified with the class $\{\{\tilde{\omega}, Or\}, \{-\tilde{\omega}, -Or\}\}$. One should therefore consider ordinary forms and twisted forms as objects of similar complexity.

Or should I say, as objects of similar *simplicity?* For we see now the possibility of defining differential forms, straight or twisted, in the most simple fashion, without having to introduce covectors first, as we did. Why not say that a (standard, twisted) p-form is just a *linear map* from (inner-oriented, outer-oriented) p-manifolds to real numbers? Only tradition, which nowadays reserves the label "differential form" to smooth fields of covectors, prevents us to do that.

Let me stress how easily this notion accounts for the notion of (physical) current density. For if we want a mathematical object that would describe electrical current, what do we need, *a minima*? Just to be able to tell which quantity of electric charge flows, per unit of time, across any given surface. This requires a machine of type *OUTER_ORIENTED_SURFACE* $\rightarrow$ *REAL*, a mapping that should obviously be linear. A twisted 2-form, in the sense of the last paragraph, fits this description. Nothing else, metric, orientation, or whatever, is needed, and the mathematical representation of the physical entity "current" could hardly be made simpler.

But let's end this digression. Next issue is the d of twisted forms and Stokes' theorem.

return. A twisted p-covector, in this spirit, is a *pair* of similar machines, set to yield opposite numbers when fed the same way, and which one is actually used would depend on ambient orientation. Thus described, straight and twisted covectors look like very dissimilar objects, indeed. But there is another metaphor for a p-covector: a machine with (1) a bin in which the vectors are thrown, unsorted, (2) a lever with which one selects an inner orientation of the p-dimensional subspace spanned by these vectors. Since inserting vectors in a definite order does point to an inner orientation (the one for which they form a positive frame), the two machines are easily seen to be equivalent. Now, the twisted p-covector machine has (1) a bin, of the same design, (2) a lever to select an *outer* orientation of the spanned subspace. The symmetry is thus restored. To give a concrete example, in dimension 3, a twisted 2-covector machine will accept two vectors (whose order doesn't count), plus a crossing direction of the parallelogram they form, and yield a number (think of an intensity, for instance). We'll return to this with Fig. 5, below.

4.3 The d, Stokes, and charge

Recall that an inner-oriented manifold M induces an orientation of its boundary ∂M (cf. Fig. 2). For shortness, we'll say that orientations of M and ∂M "match" when the orientation of ∂M is the one induced by M. Then, given a p-form ω, where p is the dimension of ∂M, one has $\int_M \mathrm{d}\omega = \int_{\partial M} \omega$. The d was defined, back in §3.4, in order to have this result, which holds whatever the orientation of M, provided the boundary's one does match.

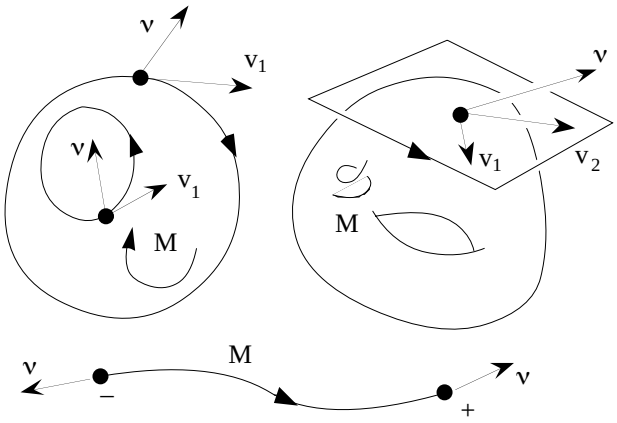

Figure 2. How the orientation of M induces one on ∂M. (M's dimension $p + 1$ is 2 on the left, 3 on the right, 1 at the bottom.) To know whether a frame of p vectors tangent at a point of ∂M is direct or skew, take a vector ν which points outwards with respect to M, list behind it the p vectors, and check whether the $(p + 1)$-frame $\{\nu, v_1, \ldots, v_p\}$ thus obtained is direct or skew with respect to the orientation of M. Note that ∂M may well be disconnected (left). Also notice the special icons (+ or $-$) for inner orientation of points.

Remark. As the caption of Fig. 2 explains, the matching rule relies on having chosen "inside to outside" as crossing direction through ∂M. This is the usual convention. Should one want to reverse it, there would be a change of sign in the Stokes theorem, then $\int_M \mathrm{d}\omega + \int_{\partial M} \omega = 0$. ♢

By definition, $\mathrm{d}\tilde{\omega}$ is the equivalence class $\{\{\mathrm{d}\omega, Or\}, \{-\mathrm{d}\omega, -Or\}\}$, which means, informally, "given a twisted form $\tilde{\omega}$, select a representative, whence a standard form to which one applies the d, and an orientation which one puts back, thus obtaining a representative of $\mathrm{d}\tilde{\omega}$".

This is engineered in such a way that Stokes' theorem be valid for twisted forms, too, when the outer orientations of M and ∂M match. We don't need a new convention to define this new kind of matching: Just select an orientation for

ambient space, then outer orientations of M and ∂M induce inner orientations, which do match or don't. If they do, it's evidence that the given outer orientations did. Figure 3 explains this.

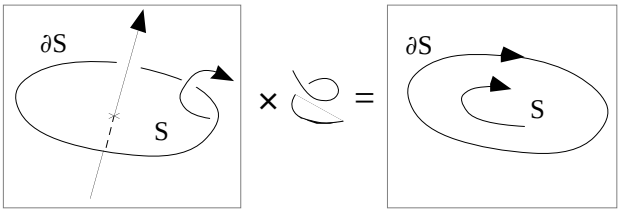

Figure 3. Matched outer orientations for a surface S and its boundary ∂S. To check that they do match, choose an ambient orientation, and derive from it inner orientations, which match, as one can see (cf. Fig. 2). (The symbol $\times$ may be understood as "compose the orientations on the left with" the icon that follows, which figures the orientation of ambient space.)

Outer-orienting a surface, therefore, induces a way to circle around its rim (not the same as turning around the surface along the rim!). Outer-orienting a 3D domain induces a crossing direction for (each of) its bounding surface(s). This outer orientation, being the inner orientation of a zero-dimensional vector space, is just a sign, plus or minus. If the sign is plus, the induced crossing direction is the conventional one (inside to outside). On the other hand, if we outer-orient the boundary from outside to inside, the outer orientation of D that matches this is the one with the minus sign. Please think all this over, because it's essential to the discussion we'll have in a moment about whether electric charge is, as physicists have it, "a pseudo-scalar or a true scalar" [Br]. Another useful exercise is to think about both kinds of orientation of the endpoints of a line (Fig. 4).

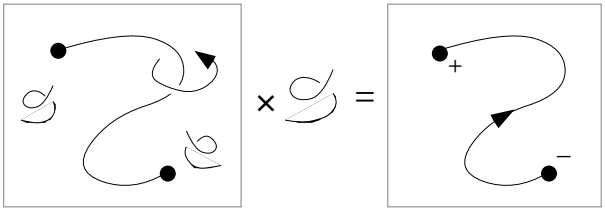

Figure 4. Matched outer orientations for a line and its endpoints (cf. Fig. 2). Outer orientation of a point is inner orientation of its 3D neighborhood, hence the icons attached to the endpoints.

Now, we may return to our equation and take a step forward. Since current density must be represented by the twisted form $\tilde{j}$, and knowing what we know about the Stokes theorem for twisted forms, charge conservation must be expressed by

$$\partial_t \tilde{q} + \mathrm{d}\tilde{j} = 0, \tag{1}$$

where the charge density $\tilde{q}$ must be a *twisted* 3-form, for consistency. Is it all right? Yes, if we define the *charge inside* D as $\int_D \tilde{q}$, with + for outer orientation of D. Then the matching outer orientation of ∂D is from inside to outside, so $\int_{\partial D} \tilde{j}$ is the outgoing current, and the expected conservation law, $\partial_t \int_D \tilde{q} + \int_{\partial D} \tilde{j}$, does result from (1) by Stokes.

What we need to fathom, now, is the relation between this $\tilde{q}$ and the ordinary charge density q, understood as a function. Recall that, if one assumes a metric and an orientation (call it Or), a function q generates a 3-form ${}^3q$, but in an orientation-dependent way, so there are, for a given metric structure, *two* opposite straight 3-forms coming from q, one for Or, one for $-Or$. In contrast, there is only one twisted 3-form associated with q, which is $\tilde{q} = \{\{{}^3q, Or\}, \{-{}^3q, -Or\}\}$, and the other way around. So, given a metric, there is a well defined function q associated with $\tilde{q}$, the twisted 3-form that stands for charge density. This function, which does not depend on the orientation of space, and is therefore a "true scalar", is what we understand by electric charge density usually. Its Lebesgue integral over a domain D, which is the same as $\int_D {}^3q$ when D's inner orientation is Or, coincides with the integral of $\tilde{q}$ over D when D's outer orientation is +.

So the "scalar proxy" of a twisted 3-form is a function. An ordinary 3-form can also be represented by a function, but the sign of the latter depends on orientation. We know how to give status to such a "pseudo scalar", as physicists say in their confusing lingo: Define a *twisted function* $\tilde{f}$ as the class $\{\{f, Or\}, \{-f, -Or\}\}$ of pairs of type $\{$*ordinary function*, *orientation*$\}$. In metricized, but non-oriented 3-space, the scalar proxy of a straight 3-form is a twisted function.

Remark. Electric charge is div **D**. By analogy, magnetic charge, if such a thing existed,[2] would be div **B** (and we would have to add some "magnetic current", $-$**K** say, on the right-hand

[2] which seems unlikely, as early observations of such charges were disconfirmed later [GT].

side of Faraday's law to account for nonzero values of div **B**). Since **B** stands for a 2-form b, magnetic charge is the straight 3-form db (let's call this m), and its scalar proxy would be a twisted (or pseudo, or axial, ...) function, so magnetic charge is not akin to electric charge in this respect. Its conservation would be expressed by

$$\partial_t m + \mathrm{d}k = 0, \tag{2}$$

of which db = 0 is a consequence when k = 0 and m = 0 at t = 0. There is a theory which sees the origin of electric charge in small-scale topological twists of space [So], based on an argument which applies to twisted forms but not to straight ones. Maybe, as it has been speculated, this is why magnetic monopoles don't show up, in spite of "grand unification" theories that seem to require them [GT]. ♢

4.4 The Maxwell equations

Next issue is Ampère's theorem. To simplify the discussion a little, let's first ignore displacement currents, and address the equation rot **H** = **J**, which is closer to the practice of low-frequency electrical engineering anyway.

No choice here: This translates as d$\tilde{h}$ = $\tilde{\jmath}$, in which $\tilde{h}$ must be a twisted form, like $\tilde{\jmath}$, and of degree 1. A magnetomotive force, therefore, is the result of integrating on an outer-oriented line (let's be careful not to say "along" the line, which was all right for electromotive forces, but connotes inner orientation). Figure 3, left, well illustrates how the crossing direction for currents and the "way of turning around" the line must match for the integral version of the theorem, $\int_S \tilde{\jmath} = \int_{\partial S} \tilde{h}$, to hold.

Finally, let's introduce displacement current. Being alike total current, it must be represented by a twisted 2-form, denoted $\tilde{d}$. The complete version of Ampère's theorem, in local differential form, is then $-\partial_t \tilde{d} + \mathrm{d}\tilde{h} = \tilde{\jmath}$. Applying d to both sides, and integrating in time, we get d$\tilde{d}$ = $\tilde{q}$, the transcription of div **D** = q in the new language, and the expected expression for the electric charge in terms of the electric induction.

So now we know them all! All the fields appearing in Maxwell equations, as formulated in oriented Euclidean space, have been replaced by differential forms in an orderly way: **E** and **B** by straight forms e and b of degrees 1 and 2, **H** and **D** by twisted forms $\tilde{h}$ and $\tilde{d}$ of degrees 1 and 2, **J** and q by twisted forms $\tilde{\jmath}$ and $\tilde{q}$ of degrees 2 and 3. Magnetic current and charge would be, if they existed, a 2-form $-k$ and a 3-form m, both straight. It's just as easy to guess about the potentials one may be led to use: Vector potential **A**, similar to **E**, becomes the 1-form a, electric potential is the 0-form ψ, and all this fits well with the representation $e = -\partial_t a - \mathrm{d}\psi$. Magnetic potential is a twisted 0-form $\tilde{\varphi}$, such that $\tilde{h}$ = d$\tilde{\varphi}$ holds (locally, at least ...) in current-free regions. Outside of such regions, one can introduce an electric vector potential—the one denoted by **T** in the context of the so-called "**T**– method" [Ca], here the twisted 1-form $\tilde{\tau}$, such that $\tilde{h} = \tilde{\tau} + \mathrm{d}\tilde{\varphi}$.

Of all these objects, only the twisted 0-form $\tilde{\varphi}$ is new to us. It is, as usual, an equivalence class $\{\{\varphi, Or\}, \{-\varphi, -Or\}\}$, where φ is a straight 0-form. As 0-forms are in direct correspondence with functions (no metric needed), we see that the magnetic potential is an instance of "pseudo scalar", one of these functions that, mysteriously, "change sign with orientation". We are now in a position to take a global view of this kind of phenomena. For this, let's round up all the geometrical objects that contribute to the field's description, in each of the three systems of representation, and display the relevant equations again.

In naked affine space, we have, in order of increasing degree, the straight forms ψ, e and a, b and k, and magnetic charge m, on one side, and the twisted forms $\tilde{\varphi}$, $\tilde{h}$ and $\tilde{\tau}$, $\tilde{d}$ and $\tilde{\jmath}$, and electric charge $\tilde{q}$, on the other side. The basic equations are

$$-\partial_t \tilde{d} + \mathrm{d}\tilde{h} = \tilde{\jmath}, \tag{3}$$

$$\partial_t b + \mathrm{d}e = -k, \tag{4}$$

where k, always null for all we know, is just a false window for symmetry. Conservation equations (1) and (2) derive from that, if one sets $\tilde{q}$ = d$\tilde{d}$ and m = db. Lorentz force on a unit electric charge with velocity V is the covector $e - i_V b$. (The reader is invited to work out a formula for the force that a hypothetical magnetic monopole would feel.)

When space is equipped with a metric and an orientation, we have vector and scalar proxies for all these entities, in terms of which these

equations rewrite as

$$(3') \qquad -\partial_t \mathbf{D} + \mathrm{rot}\,\mathbf{H} = \mathbf{J},$$

$$(4') \qquad \partial_t \mathbf{B} + \mathrm{rot}\,\mathbf{E} = -\mathbf{K},$$

and conservation equations as

$$(1') \qquad \partial_t q + \mathrm{div}\,\mathbf{J} = 0,$$

$$(2') \qquad \partial_t m + \mathrm{div}\,\mathbf{K} = 0.$$

The Lorentz force has $\mathbf{E} + V \times \mathbf{B}$ for vector proxy. If one chooses to work with the opposite orientation, one will describe the same physics by changing the signs of **B**, **H** (and hence, of **T** and φ, if they are used), **K**, and m, because rot and $\times$, both sensitive to orientation, "change sign" in the process, too.

If we keep the metric and shun the orientation, the equations become

$$(3'') \qquad -\partial_t \mathbf{D} + \mathrm{rot}\,\tilde{\mathbf{H}} = \mathbf{J},$$

$$(4'') \qquad \partial_t \tilde{\mathbf{B}} + \mathrm{r\tilde{o}t}\,\mathbf{E} = -\tilde{\mathbf{K}},$$

$$(1'') \qquad \partial_t q + \mathrm{div}\,\mathbf{J} = 0,$$

$$(2'') \qquad \partial_t \tilde{m} + \mathrm{div}\,\tilde{\mathbf{K}} = 0,$$

where **E**, **D**, **J** are vector fields, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{K}}$, twisted vector fields. Electric charge density q is here a function, magnetic charge density $\tilde{m}$ a twisted function (alike magnetic potential $\tilde{\varphi}$). The "twisted curl" rõt sends vectors to twisted vectors, while rot (which is thus subtly different from the rot of $(3')$) does the converse. The Lorentz force is the "polar" vector $\mathbf{E} + V\tilde{\times}\tilde{\mathbf{B}}$, where $\tilde{\times}$ denotes this orientation-independent cross product that was described in Fig. 2 of last installment.

As one sees, there is a clear correspondence between the two latter systems (equations numbered as (n') and (n'')): things denoted with a tilde in the orientation-free framework, that is, twisted (or axial, etc.) vectors and scalars, correspond to those fields in the standard framework (oriented Euclidean space) whose sign must be changed when one decides to change orientation.

Unfortunately, rules about the correspondence between equations in (n') and equations in (n), those of the metric-free system, are not so clearcut. In particular, and because of the vagaries of the curl operator, it is not true that proxies for twisted forms are, systematically, twisted vectors or scalars. The proxies of ψ and $\tilde{q}$ are scalar fields, while those of m and $\tilde{\varphi}$ are twisted scalar fields. It doesn't mean that there is no simple rule (there is one, as will be obvious on Fig. 6), but the tildes are no reliable mnemonics in this respect. I readily concede that calling "twisted" all geometrical objects that carry orientation with them may not have been such a good idea in the first place. Perhaps one should speak of twisted and straight *forms*, and of axial and polar *fields*? No accepted terminology has yet emerged, although one would think that the available vocabulary (odd vs even, axial vs polar, twisted vs plain or straight, pseudo versus true, and so forth) is rich enough for the needs of a rational taxonomy of our zoo of geometrical objects. Meanwhile, be wary of authors of papers or books who, misled by their own terminology, may misclassify some electromagnetic entities [BH]—and exert such caution against the present writer, too.

4.5 The Hodge operator

Let's now tackle constitutive laws. In the non-oriented Euclidean framework, we have

$$(5'') \qquad \tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}}, \qquad \mathbf{D} = \epsilon\mathbf{E}.$$

Ohm's law would be $\mathbf{J} = \sigma\mathbf{E}$. These are relations between objects of the same type—which is at it should be, since μ, ϵ, and σ are scalar entities.

Or are they? Let's not go too far and characterize such coefficients as scalar *invariants*, which they are not: they have dimension, they change value when the metric is changed, and besides, there is such a thing as anisotropy. So maybe the real nature of these parameters is still hidden by the formalism.

So let's see what their status can be in the minimal framework of differential forms—and it looks like a mess: We certainly *can't* write $b = \mu\tilde{h}$ and $\tilde{d} = \epsilon e$, or $\tilde{\jmath} = \sigma e$, because that would be trying to establish a proportionality relationship *between objects of different types*—different on two counts: forms of unequal degrees, which differ in orientation status (straight and twisted). So μ, ϵ and σ cannot be scalar multipliers, even if endowed with physical dimension. They have to be *operators*, linking

objects of different types. It happens that classical differential geometry has such an operator in store, that will prove perfectly apt to the task.

This so-called *Hodge operator*, or *star* operator in some countries, is a linear machine which maps p-covectors of one kind (twisted or straight) to $(n-p)$-covectors of the other kind, where n is the dimension of the underlying vector space. In affine space, a smooth field of similar machines, one at each point, will therefore map p-forms of one kind to $(n-p)$-forms of the other kind, which seems to be exactly what we need.

So let V_n be a real vector space of dimension n, endowed with a dot product " $\cdot$ " and an orientation Or. Let a p-covector ω be given. We denote by $\star\omega$ the $(n-p)$-covector such that, if the family of vectors $\{v_1, v_2, \ldots, v_n\}$ makes a *direct orthonormal frame*, then

$$\star\omega(v_{p+1}, \ldots, v_n) = \omega(v_1, \ldots, v_p). \tag{6}$$

This may look preposterous: does (6) really *define* a covector? Shouldn't we expect a formula that would give us the value $\star\omega(u_1, u_2, \ldots, u_{n-p})$ for *any* list of $n-p$ vectors? Such formulas can be given, but are not very instructive. Neither are they useful, for no actual computation is required. All we need is to make sure that $\star$ is well defined, and the above rule happens to be enough for that, thanks to the linearity of covectors with respect to their arguments and their alternation property (permute two factors, change the sign). Starting from a list of vectors $\{u_1, u_2, \ldots, u_{n-p}\}$, we may always apply the Gram–Schmidt orthogonalization method[3] to build a system of vectors all of length 1 and two-by-two orthogonal, and thus obtain a determinant, the value λ of which[4] we store. Call $\{v_{p+1}, v_{p+2}, \ldots, v_n\}$ the orthonormal system thus obtained. In the orthocomplement of the subspace it spans, pick p vectors $\{v_1, v_2, \ldots, v_p\}$ of length 1, orthogonal two-by-two, in such a way that the full list form a direct frame. Then $\star\omega(u_1, u_2, \ldots, u_{n-p}) = \lambda\omega(v_1, \ldots, v_p)$. The last objection, "but we could have selected a different system $\{v'_1, v'_2, \ldots, v'_p\}$," is countered by noting (same argument as in Note 5) that the determinant of $\{v'_1, v'_2, \ldots, v'_p\}$ with respect to the basis $\{v_1, v_2, \ldots, v_p\}$ would then be 1, by the rules. Note that $\star$ is a linear operator, in an obvious way.

Next, observe that not only metric played a role there, but orientation too, because of the stipulation that the v_i's should make a *direct* frame. Had we taken $-Or$ as orientation, the operator defined by (6) would have been the opposite. As we see, the star operator behaves very much like curl and the cross product, in this respect. Having already obtained, with (1′)–(5′), an expression of Maxwell's equations which is manifestly orientation-free, we can be sure that such an orientation-sensitive operator is not the right tool. But we also know how to fix it: define $\tilde{\star}$ by

$$\tilde{\star}\omega = \{\{\star\omega, Or\}, \{-\star\omega, -Or\}\}. \tag{7}$$

This maps p-covectors to twisted $(n-p)$-covectors, indeed. Finally, if a twisted p-covector $\tilde{\omega}$ is given, we select a representative $\{\omega, Or\}$ of $\tilde{\omega}$, apply to ω the $\star$ *as defined thanks to* Or (this is of course the key point), and $\tilde{\star}\tilde{\omega}$ is what results, a straight $(n-p)$-covector. Note that $\tilde{\star}$ is its own inverse, up to sign: one has $\tilde{\star}\tilde{\star}\omega = \pm\omega$, the sign depending on p and n.

This operator is the device by which we shall link b and $\tilde{h}$, $\tilde{d}$ and e, etc., like this:

$$b = \mu\tilde{\star}\tilde{h}, \qquad \tilde{d} = \epsilon\tilde{\star}e. \tag{5}$$

Let's show it by examining Ohm's law, which I claim is well expressed by $\tilde{\jmath} = \sigma\tilde{\star}e$.

To do this, select a point x, understood in what follows. Let e and $\tilde{\jmath}$ be the electric field and current density in the vicinity of x. Take three vectors $\{v_1, v_2, v_3\}$ at x, all of length one, mutually orthogonal, select as orientation the one which turns them into a direct frame (Fig. 5), and let j be the 2-covector at x that represents $\tilde{\jmath}$ for this orientation. Imagine a cube of metal of conductivity σ built on the three vectors. The intensity across the bottom of the cube is $I = j(v_2, v_3)$, by the very interpretation of the 2-covector j as current density. (Note that the chosen crossing direction does orient the cube's basis in such a way that $\{v_2, v_3\}$ is direct for the induced inner orientation.) The voltage drop from bottom to top is $V = e(v_1)$. And the resistance V/I of this unit cube is $1/\sigma$, so we have $j(v_2, v_3) = \sigma e(v_1)$. Compare this

[3] unless the u_i's are not independent—but then, the value of $\star\omega$ for such a list must be 0.

[4] Note that λ does not depend on which way the Gram–Schmidt procedure is performed.

with (6): it amounts to saying that $j = \sigma \star e$. Removing the orientation scaffolding, we get

$$\tilde{j} = \sigma \tilde{\star} e, \tag{8}$$

as promised.

So we are through, at last: Equations (1)–(5), plus Ohm's law (8) if needed, give a full description of electromagnetism (for linear materials and non-moving bodies). Orientation of space has been discarded entirely. As for metric, it's only at the level of constitutive laws that it has been invoked.

Is metric *necessary* at this level, or could we perhaps strip the framework even further? Apart from merging space and time, which is feasible and would lead us to the relativistic formalism of box (c) of Fig. 1 in the first paper of the series, there is little room left for such further improvement. For if we try to discard metric, the constitutive laws are still there and must somehow be described. We might directly introduce a Hodge operator as an affine entity, that would linearly map p-covectors to $(n-p)$-covectors. But then it can be proved [B3] that, as soon as we have such an operator in the case $p = 1$, it is the Hodge associated with some metric: *Hodge implies metric*.

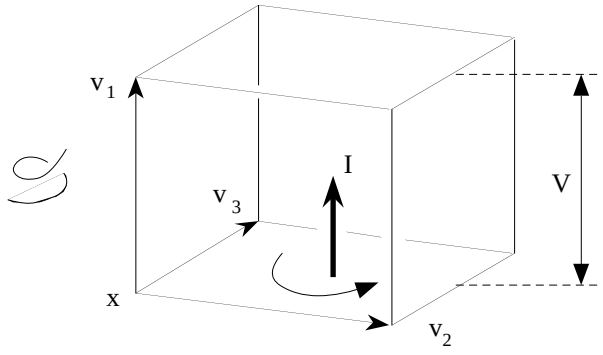

Figure 5. How $\tilde{j} = \sigma \tilde{\star} e$ expresses Ohm's law (see text).

Still, since at least two such operators would be needed, one for μ, one for ϵ, the question would arise whether there is some *common* metric in which they would take the forms $\mu\,\tilde{\star}$ and $\epsilon\,\tilde{\star}$. When this is so, we say that the material is *isotropic*, though not necessarily homogeneous since μ and ϵ may depend on position. There is no space left here to address such issues, but what precedes hints at the potential usefulness of what we have been doing: Questions such as "what do we mean by isotropy, exactly?" and other similar ones related to *material symmetries*, as distinct from the symmetries *of the equations*, and to what remains of such symmetries when the material is strained, do benefit from this dissection of structures. It's also useful is the investigation of *forces*, as suggested in [B2].

4.6 A synoptic conclusion

Let's gather all our findings in graphic form (Fig. 6). Since all relevant objects are differential forms of degrees 0 to 3, straight or twisted, and since time derivatives and, occasionally (cf. the example of $\tilde{\tau}$ on Fig. 6), primitives in time may have to be considered, we can group them in four similar categories, shown as vertical pillars on Fig. 6. Each pillar symbolizes the structure made by spaces of forms of all degrees, linked together by the d operator. Straight forms are on the left and twisted forms on the right. Differentiation or integration with respect to time links each pair of pillars (the front pillar and the back pillar) forming the sides of the building. Horizontal beams symbolize constitutive laws.

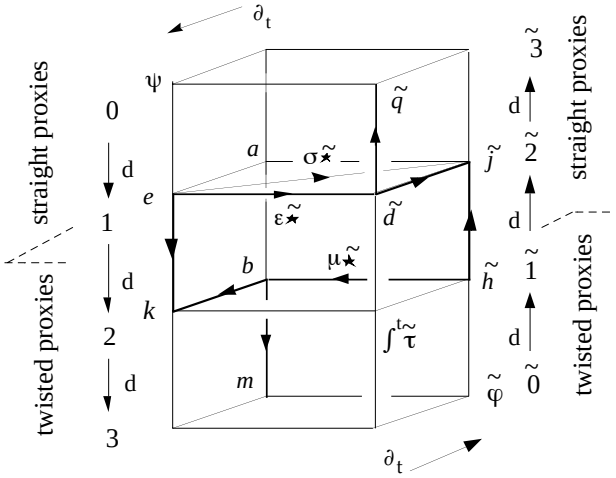

Figure 6. "Maxwell's house."

As one can see, each object has its own room in the building: b, a 2-form, at level 2 of the "straight" side, the 1-form a such that $b = \mathrm{d}a$ just above it, etc. Occasional asymmetries (e.g., the necessity to time-integrate $\tilde{\tau}$ before lodging it, the bizarre layout of Ohm's law ...) point to weaknesses which are less those of the diagram than those of the received nomenclature or (more ominously) to the incompatibility of Ohm's law with Einsteinian relativity. Most things mentioned up to now can be directly read

off from the diagram, up to sporadic sign inversions. An equation such as $\partial_t b + \mathrm{d}e = -k$, for instance, is obtained by gathering at the location of k the contributions of all adjacent niches, including k's, in the direction of the arrows. Note how the rule about which scalar- or vector-proxies must be twisted or straight is now apparent.

But the most important thing is probably the neat separation, in the diagram, between "vertical" relations, of purely affine character, and "horizontal" ones, which depend on metric. If this was not drawing too much on the metaphor, one could say that a change of metric (due for instance to a change in the local values of μ, σ, etc., because of a temperature modification or whatever) would shake the building horizontally but leave the vertical panels unscathed.

This points to a methodology for *discretizing* the Maxwell equations: The orderly structure of Fig. 6 should be preserved, if at all possible, in numerical simulations. Hence the search for finite elements *which fit differential forms*, and thus would allow to build a similar "discrete" structure. This search is not over, in spite of the existence of differential-geometric objects (Whitney forms, see e.g., [B1]) which are remarkably efficient as finite elements for forms, because the simultaneous discretization of straight *and* twisted forms, on the same mesh, and the concomitant construction of discrete Hodge operators, is still an open field of inquiry.

So maybe we'll have more to say about such things in future columns.

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Computational electromagnetism and geometry:
Building a finite-dimensional "Maxwell's house"

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(1): Network equations

INTRODUCTION

In this new series,* we shall highlight some benefits of a geometrical approach to the Maxwell equations as regards their *numerical treatment*.

The first series, devoted to a discussion of possible formalisms for the mathematical description of electromagnetism, focused on one of them, in which the Maxwell equations appear as

$$(1)\quad \partial_t b + \mathrm{d}e = 0, \qquad (2)\quad -\partial_t \tilde{d} + \mathrm{d}\tilde{h} = \tilde{\jmath},$$

$$(3)\qquad b = \mu \tilde{\star} \tilde{h}, \qquad (4)\qquad \tilde{d} = \epsilon \tilde{\star} e.$$

The meaning of this notation will soon be recalled in detail. For the time being, it's enough to remember that the above equations are just, thinly disguised, the familiar ones,

$$(1')\quad \partial_t \mathbf{B} + \mathrm{rot}\,\mathbf{E} = 0, \quad (2')\quad -\partial_t \mathbf{D} + \mathrm{rot}\,\mathbf{H} = \mathbf{J},$$

$$(3')\qquad \mathbf{B} = \mu \mathbf{H}, \qquad (4')\qquad \mathbf{D} = \epsilon \mathbf{E},$$

which hold for non-moving isotropic media and a given current density **J**. We want to "discretize" these equations, which implies that each field, at any instant, will be represented by a *finite* number of real parameters, time-dependent, and that ordinary differential equations, in equal number, will somehow be derived for these "degrees of freedom" (DoF).

A key remark in this respect: the main equations (1)(2) or $(1')(2')$, are "conservation laws" of sorts. More precisely, (1) and (2) are local, differential versions of the following integral forms of the Faraday and Ampère laws,

$$(1'')\qquad \partial_t \int_\Sigma b + \int_{\partial\Sigma} e = 0,$$

$$(2'')\qquad -\partial_t \int_{\tilde{\Sigma}} \tilde{d} + \int_{\partial\tilde{\Sigma}} \tilde{h} = \int_{\tilde{\Sigma}} \tilde{\jmath},$$

to be satisfied for all surfaces Σ or $\tilde{\Sigma}$, inner- and outer-oriented respectively. Equation $(2'')$, for instance, says that the flux of the total current $\partial_t \tilde{d} + \tilde{\jmath}$ across $\tilde{\Sigma}$ matches the magnetomotive force, relative to the rim $\partial\tilde{\Sigma}$, of the magnetic field $\tilde{h}$, for all smooth surfaces $\tilde{\Sigma}$. Such a conservation statement can be seen as *one* equation relative to $\tilde{d}$ and $\tilde{h}$—one equation for each surface—so we have an infinity of equations, as befits unknown entities which have infinite dimension.

This is enough to suggest a method: Instead of requesting $(1'')$ and $(2'')$ for *all* surfaces, we shall be content with enforcing these balance relations over a *finite* set of surfaces, those generated by the facets of some finite-element mesh, and we shall attribute one degree of freedom to each "cell" (facet or edge, as the case may be) of this mesh. "Network equations", discrete analogues to (1)(2), will thus be found—forced on us, in fact. (Their analogy with the Kirchhoff equations for ordinary networks will be obvious.)

Appropriate relations between DoF's will also be needed in order to transcribe the constitutive laws, (3)(4) or $(3')(4')$. In other words, some "discrete Hodge operator" will have to be defined, leading to what one may call "network constitutive laws". There, in contrast, we'll have a large freedom of choice: There are good and less good discrete Hodges, and hence, choosing one will be the only really difficult part of the whole process.

This being done, the number of independent equations will, as we shall see, automatically

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match the number of DoF's, hence an "equivalent network", described by a system of ordinary differential equations. The latter can be solved by standard methods. (We shall concentrate on one of them, a simple "leapfrog scheme", on the model of the well-known Yee scheme [Ye].) This will leave us with the following questions: (1) How do we recover the fields, or rather, approximations of the exact fields, from the computed degrees of freedom? (2) How far are these approximate fields from the true ones?

Such questions are the bread-and-butter of numerical analysis. They arise for all methods. What is special here is the way the discrete equations are derived: To a large extent, finite elements are *not* needed for this. Equations are set up in a way which is quite reminiscent of the old "finite differences" approach (or of its modern "finite volumes" avatar). Yet, instead of being confined to hexahedral bricks, we can use cells of complex shape, thus accommodating bodies with curved or contorted boundaries at will—best of both worlds. Moreover, the method can be explained to anyone: No previous familiarity with finite elements is required, to the point that a complete, explicit recipe—one that any competent programmer can implement—can be formulated without ever mentioning finite elements!

This does not mean they are useless. But their rôle is confined to, mainly, assessing the value of the discrete Hodge operator by way of error estimates, and, secondarily, helping to reconstruct fields from their degrees of freedom, in the post-processing phase.

This rapid description of the approach is enough to perceive the benefits of our previous geometrization. In standard theory of electrical networks, there is a neat distinction between what may be referred to as *topology* (the way the network is connected, encoded in the coefficients of the so-called "node equations" and "loop equations") and what belongs to *metric* in our sense, that is, the values of the impedances of the network branches. One has all reasons in the world to maintain such orderly distinctions, for instance when branch impedances may vary with time, whereas the structure of the network doesn't change. In the theory to be developed here (which can be understood as the construction of two interlocked networks), a similar separation exists: Network equations, stemming from the metric-free equations (1) and (2), only depend on the combinatorial properties of the underlying mesh(es), and network constitutive laws, like the Hodge operator from which they derive, encompass metric information, as well as material properties.

So here is, in a few words, what we may expect: a unification of field theory with network theory, via discretization methods. An enticing goal, worth a long journey. Let's take the first steps of it.

1. NETWORK EQUATIONS

It's a good thing to keep in mind a representative of the family of problems one wishes to model. Here, we shall have wave-propagation problems in view, and the following example is typical.

1.1 A model problem

In a closed cavity with metallic walls (Fig. 1), which has been free from any electromagnetic field till time t = 0, suppose a flow of electric charge is created in an enclosed antenna after this instant, by some unspecified agency. An electromagnetic field then develops, propagating at the speed of light towards the walls which, as soon as they are reached by the wavefront, begin to act as secondary antennas. Dielectric parts inside the cavity, too, may scatter waves. Hence a complex evolution, which one may imagine simulating by numerical means.

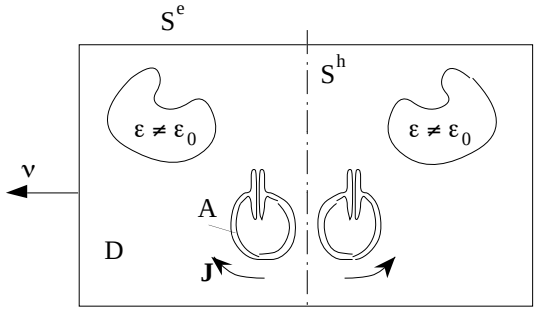

Figure 1. Situation and notation (dimension 3). Region D is the left half of the cavity. Its boundary S has a part S^e in the conductive wall and a part S^h in the symmetry plane. Region A, the left "antenna", is the support of the given current density **J** (mirrored on the right), for which some generator, not represented and not included in the modelling, is responsible.

For the sake of generality, let's assume a symmetry plane, and a symmetrically distributed current. The computation will thus be restricted

to a spatial domain D coinciding with one half of the cavity, on the left of the symmetry plane, say. Calling S its surface (in two parts, S^h and S^e, as Fig. 1 shows) and ν the outward directed field of normal unit vectors on S, the relevant equations, first expressed in standard notation, are $(1')$–$(4')$ above. Coefficients ϵ and μ are real, constant in time, but not necessarily equal to their vacuum values ϵ_0 and μ_0, and may therefore depend on variable x, which denotes the spatial position. (They could be tensors, too, without any serious extra difficulty.) The current density **J** is given, and assumed to satisfy $\mathbf{J}(t,x) = 0$, at all points x in D for $t \leq 0$. Other fields, unknown, are also supposed to be null before $t = 0$, hence initial conditions, $\mathbf{E}(0,x) = 0$ and $\mathbf{H}(0,x) = 0$ for all x. At the boundary,

$$(5')\ \ \nu \times \mathbf{E} = 0 \text{ on } S^e, \quad (6')\ \ \nu \times \mathbf{H} = 0 \text{ on } S^h.$$

Condition $(5')$ amounts to considering the cavity walls as perfect conductors, and $(6')$ comes from the mirror symmetry of **J**, which entails the *skew* symmetry of **H** ([B3], p. 28).

The mathematical theory of $(1')$–$(6')$ is not our concern here. Suffice it to say that, under reasonable assumptions about **J**, there is a unique solution $\{\mathbf{E}, \mathbf{H}\}$, that satisfies some standard requirements, apparently of mathematical nature, but actually dictated by physics. For instance, one wants the fields to have finite energy. This is translated, in mathematical terms,[1] by "**E**, at any time, should belong to the space, called $L^2(D)$, of square-summable vector fields over D", i.e., those for which the integral $\int_D |\mathbf{E}|^2$ is finite, and the same about **H**. For similar reasons, one requires $\mathrm{rot}\,\mathbf{E}$ and $\mathrm{rot}\,\mathbf{H}$ to be in $L^2(D)$, hence the convenience of the notation $L^2_{\mathrm{rot}}(D)$ for square-summable vector fields whose curl, too, is in $L^2(D)$. So both **E** and **H** are sought for in $L^2_{\mathrm{rot}}(D)$, at all times. Fields of this space happen to have a well-defined tangential part (one says a tangential "trace") on smooth surfaces. This gives sense to $(5')$–$(6')$, and entails the tangential continuity of **E** and **H** at material interfaces: Such "transmission conditions", often explicitly added to the set of equations, are here automatically enforced by the sole virtue of restricting the search to $L^2_{\mathrm{rot}}(D)$.

Equations $(1')$ and $(2')$ then imply that both **D** and **B** are in $L^2(D)$ with a divergence also in $L^2(D)$, and again the notation $L^2_{\mathrm{div}}(D)$ for such fields comes handy.[2] As a consequence, normal continuity of **D** and **B** at interfaces is enforced. Notice—it will be important later—that *exact* enforcement of the equations is necessary for this.

Now, assuming both **J** and $\mathrm{div}\,\mathbf{J}$ in $L^2(D)$, at all times, plus some smoothness of **J** with respect to time (details on these side issues can be found in [B1]), one can prove existence and uniqueness of the $\{\mathbf{E}, \mathbf{H}\}$ pair. Notice that $\mathrm{div}\,\mathbf{J} = 0$ is *not* assumed: some electric charge may accumulate at places in the antenna, in accordance with the charge-conservation equation $\partial_t q + \mathrm{div}\,\mathbf{J} = 0$, which results from (1) and from the relation $q = \mathrm{div}\,\mathbf{D}$, where q denotes charge density. This charge is not an independent data, but derives from **J** by integration in time, $q(t,x) = -\int_0^t (\mathrm{div}\,\mathbf{J})(s,x)ds$.

The relevance of $(1')$–$(6')$ as a realistic model-problem may be discussed, on several counts. For one, a term $\sigma\mathbf{E}$ might be introduced at the right-hand side of $(2')$ in order to account for the presence of conductive bodies inside the cavity. We refrain from this easy generalization for the sake of simplicity. Perhaps the assumption of a given current density in the antenna (which is routinely done) is a more serious issue, because the antenna is not insensitive to the reaction of its own radiated field, so we can't, in full rigor, know the antenna current in advance.

This point is made in [SS] in a comment on [UM]. Should one then feel compelled to model the dynamics of whatever drives charges in the antenna, including possibly the electrical network in background, and why not the whole universe? Authors of [UM] sensibly argue against that in their rejoinder [Um], and justify their informed guess of **J**. But even when **J** cannot be guessed about in advance with enough accuracy, $(1')$–$(6')$ can be considered

[1] There is no denying that mathematicians express things this way, in part, for their own comfort, because "Hilbert spaces", of which $L^2(D)$ is a well-known example, have nice properties. Notwithstanding, the close adequation of such abstractions to physics, often marvelled about [Wi], is especially obvious in the case of Maxwell's equations.

[2] Note that the same symbol, $L^2(D)$, serves here for a space of *scalar* fields and a space of *vector* fields, which is tolerable abuse.

as part of a coupled problem, for which it's legitimate to adopt **J** as an "interface parameter", the solution of $(1')$–$(6')$ thus becoming a subroutine in some higher-level iterative loop. All things considered, our model problem appears realistic enough, while being as simple as possible, as wave propagation problems go.

1.2 The model problem, in terms of differential forms

Next, let us translate this problem in the geometric language we acquired in the previous columns. Instead of the "proxy" vector fields **E** and **H**, we consider the differential forms e and $\tilde{h}$ they stand for.

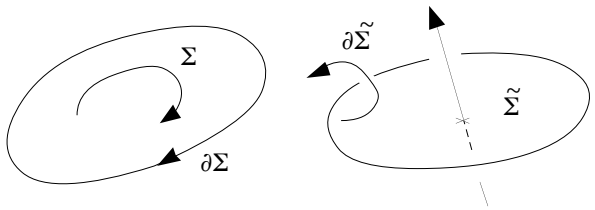

Figure 2. Inner- and outer-oriented surfaces Σ and $\tilde{\Sigma}$, with boundaries oriented in accordance.

Differential forms (DF's), one will remember, come in two varieties: the straight ones, like e and b, meant to be integrated over lines and surfaces with inner orientation, and the twisted ones, whose integrals make sense over lines or surfaces equipped with an outer orientation (a direction across the surface, in the case of $\tilde{d}$ and $\tilde{\jmath}$, a "way to turn around" lines, in the case of $\tilde{h}$). Therefore, eqs. $(1'')$ and $(2'')$ should be satisfied for all *inner* oriented surfaces Σ and *outer* oriented surfaces $\tilde{\Sigma}$ respectively (Fig. 2). As we know, these equations are equivalent, thanks to the Stokes theorem, to (1) and (2), themselves a translation of $(1')$–$(2')$. The nice thing about them is the absence of any reference to metric concepts. The latter are isolated in (3) and (4), where the Hodge operator $\tilde{\star}$, whose knowledge is equivalent to knowing the scalar product, maps ordinary p-forms to twisted $(n-p)$-forms and the other way round, where n is the dimension of ambient space.

What precedes is enough to let us carry on, and the rest of the present Section can be skipped. A few points about the relationships between DF's and their proxies, however, may be at their right place here.

In differential-geometric language, conditions $(5')$ and $(6')$ can be expressed very compactly, and without the recourse to metric (and orientation) that the " $\times$ " symbol may seem to imply. Starting from a 1-form e, consider the covectors defined, at points x of S, by $v \to \langle e(x), v\rangle$ for only those vectors v at x which are tangent to S. This defines a DF of degree 1, living on S, called the *trace* of e, and denoted $t_S e$, or of course just te in non-ambiguous cases. Its vector proxy is the tangential part of **E**, that we shall denote by $\mathbf{E}_S$, equal to $-\nu \times (\nu \times \mathbf{E})$. Same considerations about **H**. Conditions $(5')$–$(6')$ thus amount to

$$(5)\quad te = 0 \quad \text{on } S^e, \quad (6)\quad t\tilde{h} = 0 \quad \text{on } S^h.$$

This is more natural than $(5')$–$(6')$, in fact. Physically, it's indeed $\mathbf{E}_S$ that must vanish at a perfect conductor's boundary (and $\mathbf{H}_S$ at "magnetic walls", another case where $(6')$ would hold). People use "$\nu \times \mathbf{E} = 0$" as a way to say "$\mathbf{E}_S = 0$" without having to break pace to introduce this notation, or to use the more accurate but clumsy "$-\nu \times (\nu \times \mathbf{E}) = 0$".

The trace tb of b is, similarly, the field of 2-covectors $\{v, w\} \to \langle b(x); v, w\rangle$, defined for tangent vectors v and w. As a 2-form living on a 2-dimensional manifold, tb must have a *scalar* proxy, which one easily identifies as $\nu \cdot \mathbf{B}$, since $\langle b(x); v, w\rangle = \langle {}^2\mathbf{B}; v, w\rangle = \mathbf{B} \cdot v \times w = (\nu \cdot \mathbf{B})\ \nu \cdot (v \times w)$. Note the *unification* we have achieved: tangential part of this, normal part of that, are proxies for one and the same thing actually, the trace of a form. Imposing the trace of a form on a surface, as in (5) or (6), is a generalization to DF's of the "Dirichlet boundary condition" for functions, and thus deserves to be called that.

As an exercise, let us investigate the relations between t and d, in the case of 1-forms. (It goes the same way for higher degrees, and for twisted forms.) Let e be the 1-form, and Σ some surface. By the very definition of the integral (cf. [B3], p. 235), $\int_\Sigma t\,\mathrm{d}e = \int_\Sigma \mathrm{d}e$. By Stokes, this is $\int_{\partial\Sigma} e$, again equal, the way the integral was defined, to $\int_{\partial\Sigma} te$, which is $\int_\Sigma \mathrm{d}\,te$, by Stokes again. Since this holds for all smooth 1-forms and surfaces, we conclude that

$$(7)\qquad t\mathrm{d} = \mathrm{d}t.$$

Applying this to (5), and t to (1), we find (integrate in time and assume $b = 0$ at time 0) that $tb = 0$ on S^e: A perfectly conductive surface ($te = 0$) is also a barrier to the magnetic

flux ($tb = 0$). A nice example of notational tidying up!

Exercise. The reader is invited to work out the concept of "Neumann boundary condition", i.e., to try and define the *normal component* of a form on one side of a surface. (Hint: Play with the expression $\tilde{\star}t\tilde{\star}$.)

Although this is not crucial either to what follows, let's take the time to say how a priori restrictions such as "$\mathbf{E} \in L^2_{\mathrm{rot}}(D)$", and so forth, are expressed in terms of DF's. Taking the wedge product of the 1-form e with the twisted 2-form $\tilde{\star}e$, we obtain a twisted 3-form $e \wedge \tilde{\star}e$, which can be integrated over the 3-dimensional region D, hence a number $\int_D e \wedge \tilde{\star}e$, the square root of which is defined as the *norm* of the 1-form e. This works for all degrees (and of course relies on the metric). Notation $L^2(D)$ then takes on a new meaning: it's the Hilbert space of square-integrable p-forms. One may then define $L^2_{\mathrm{d}}(D)$ as the space of p-forms ω in $L^2(D)$ such that $\mathrm{d}\omega \in L^2(D)$. This is isomorphic, by passing to the vector proxies, to $L^2_{\mathrm{rot}}(D)$ if $p = 1$, and to $L^2_{\mathrm{div}}(D)$ if $p = 2$. Having no existence proofs in view, we shall not develop this, but it's comforting to know that all theorems, based on the Lax–Milgram lemma and similar things, that one can prove about vector-field solutions of Maxwell's equations, have precise counterparts, often much more compactly stated and proved, in terms of differential forms.

1.3 Primal mesh

Let's define what we shall call a "cellular paving". This is hardly different from a finite-element mesh, just a bit more general, but we need to be more fussy than usual about some details. Let's recall that V_n and A_n denote the real vector space of dimension n and its affine associate, and that when V_n is endowed with a dot product, whence a norm $|v| = \sqrt{v \cdot v}$, the distance this induces in A_n turns it into E_n, Euclidean space. Of course $n = 3$ in the sequel. By $\overline{A}$, we mean the *closure* of a set A in E_n, i.e., the set formed of all points whose distance to A is 0. *Closed* sets are those such that $\overline{A} = A$. Open sets are their complements, and the largest open set contained in A is its *interior*.

Start from the *open unit ball* B_p in V_p, that is, all vectors v such that $|v| < 1$. A *cell* of dimension p, or p-cell, for $1 \leq p \leq n$, is the image of B_p in A_n under some mapping c, piecewise smooth in both directions. (Notation c may refer to the image of the ball, or to the mapping itself, as convenient.) "Piecewise" leaves room for some irregularity: a 1-cell can well be a broken line, a 2-cell may have the shape of a triangle, a 3-cell the shape of a brick, etc. (Fig. 3).[3] Note that cells are *not* closed, for they don't contain their own boundaries. The case $p = 0$ is special: a 0-cell, by definition, is just a point,[4] that we shall call a *node*. A 1-cell will be an *edge*, a 2-cell a *facet* (we reserve "face" for another usage), and a 3-cell a *volume*.

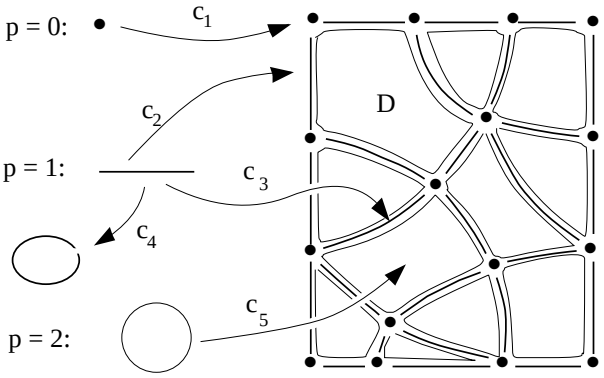

Figure 3. A few p-cells (caution, c_4 is *not* one of them), contributing to a closed cellular paving of D. (This should be imagined in dimension 3.)

Now, a *cellular paving* of some region R of space is a finite set of p-cells such that (1) Two distinct cells never intersect, (2) The union of all cells is R, (3) If the closures of two cells c and c' meet, their intersection is the closure of some (unique) cell c''. It may well happen that c'' is c, or c'. In such a case, e.g., if $\overline{c} \cap \overline{c'} = \overline{c}$, we say that c is *a face of* c'. For instance, on Fig. 3, c_3 is a face of c_5. If c is a face of c' which itself is a face of c'', then c is a face of c''.

We'll say we have a *closed* paving if R is closed. (Fig. 3 gives a two-dimensional example, where $R = \overline{D}$.) But it need not be so. Closed pavings are not necessarily what is needed in practice, as one may rather wish to discard some cells in order to deal with boundary conditions. Hence the usefulness of

[3] This notion of cell is slightly more restrictive than topologists would have it [HW]. For instance, to map B_1 (the segment $]-1,+1[$) to a closed loop minus one point (cf. c_4 in Fig. 3) would *not* make a cell in our sense.

[4] No inconsistency there: V_0 reduces to a single element, the null vector, and $B_0 \equiv V_0$.

the following notion of "relative closedness": C being a closed part of R, we shall say that a paving of R is *closed modulo* C if it can be obtained by removing, from some closed paving, all the cells which map into C. Fig. 4 displays the case we shall actually need, of a paving of $R = \overline{D} - S^e$ which is closed modulo S^e. Informally said, "pave $\overline{D}$ first, then remove all cells from the electric boundary".

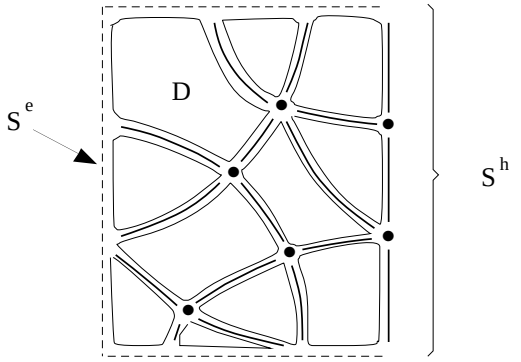

Figure 4. A culled paving, now "closed relative to" S^e. This is done in anticipation of the modelling we have in mind, in which cells of S^e would carry null DoF's, so they won't be missed.

Each cell is provided with an inner orientation of its own: Each edge has a "forward direction", each face has a notion of "turning clockwise" in it, each volume its own "corkscrew rule". These orientations are arbitrary and independent. For reasons soon to be disclosed, nodes must be oriented, too (a possibility we mentioned earlier, without using it). This consists in attributing a sign, $+$ or $-$, to each of them. (For simplicity, we may assume they all bear a $+$ sign.) We shall denote by $\mathcal{N}, \mathcal{E}, \mathcal{F}, \mathcal{V}$, the sets of oriented p-cells of the paving, and by N, E, F, V the number of cells in each of them.

Two cells σ and c, of respective dimensions p and $p+1$, are assigned an *incidence number*, equal to ± 1 if σ is a face of c, and to 0 otherwise. The sign, $+$ or $-$, depends on whether orientations "match" or not, a concept we have met before (cf. [B3], Section 3.3).[5] The boundary of c has a natural *outer* orientation, corresponding to the "inside to outside" crossing direction. Since σ belongs to this boundary, the inner orientation of c and this induced outer orientation of σ cooperate[5] in defining an *inner* orientation of σ. If this coincides with σ's *own* inner orientation, we say that the orientations of σ and c *match* (see Fig. 5). This understood, the sign rule is: $+$ if orientations match, $-$ if they don't.

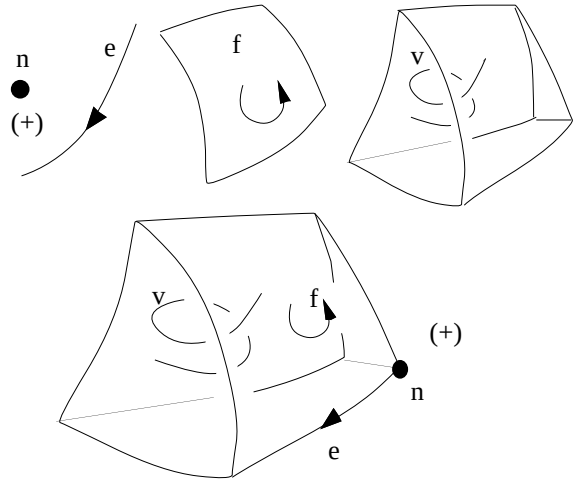

Figure 5. Top: individual oriented cells. Bottom: the same, as part of a paving, showing respective orientations. Here, orientations of v and f match, those of f and e, or of e and n, don't. So $\mathbf{G}_{ne} = -1, \mathbf{R}_{fe} = -1, \mathbf{D}_{vf} = 1$.

Collecting these numbers in arrays, we obtain rectangular matrices **G, R, D**, called *incidence matrices* of the tesselation. For instance (Fig. 5), the incidence number for edge e and facet f is denoted $\mathbf{R}_{fe}$, and makes one entry in matrix **R**, whose rows and columns are indexed over facets and edges, respectively. The entry $\mathbf{G}_{en}$ of **G** is -1, as explained in Note 5. Symbols **G, R, D** are of course intentionally reminiscent of grad, rot, div, but we still have a long way to go to fully understand the connection. Yet, one thing should be conspicuous already: contrary to grad, rot, div, the incidence matrices are *metric-independent* entities, so the analogy cannot be complete. Matrices **G, R, D** are more akin to the (metric-independent) operator d from this viewpoint, and the generic symbol **d**, indexed by the dimension p if needed, would make cleaner notation: $\mathbf{d}_0$ = **G**, $\mathbf{d}_1$ = **R**, $\mathbf{d}_2$ =

[5] Let's recall the essentials. An inner orientation of σ is a way to decide whether p independent vectors, tangent to σ at one of its points, form a direct frame of a skew frame. Take, at this point, a vector which goes *outward* with respect to c's boundary (an unambiguous notion), and list the p given vectors behind it. If the $(p+1)$-frame thus obtained is direct, with respect to the inner orientation of c, we decide the original p-frame was direct, and the other way round. If $p = 0$, σ is a node, c is an edge, and the rule specializes as follows: attribute the sign $+$ to σ if the outgoing vector tangent to c, at the end-point σ, goes in the same direction as c itself.

D. The mnemonic value of **G, R, D**, however, justifies the abuse.

Let's only point out that, just as rot ∘ grad = 0 and div ∘ rot = 0, one has **GR** = 0 and **DR** = 0. Indeed, for an edge e and a volume v, the $\{v, e\}$-entry of **DR** is $\sum_{f \in \mathcal{F}} \mathbf{D}_{vf}\mathbf{R}_{fe}$. Nonzero terms occur, in this sum over facets, only for facets which at once contain e and are a face of v, which happens only if e belongs to $\overline{v}$. In that case, there are exactly two facets f and g of v hinging on e (Fig. 6), and hence two nonzero terms. As Fig. 6 shows, they have opposite signs, whatever the orientations of the individual cells, hence the result, **DR** = 0. By a similar proof, **RG** = 0, and more generally, $\mathbf{d}_{p+1}\mathbf{d}_p = 0$.[6]

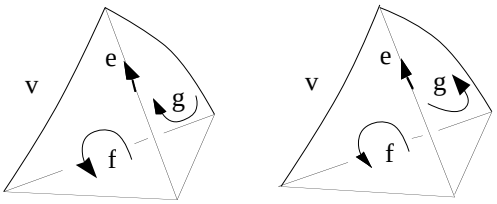

Figure 6. Opposition of incidence numbers, leading to **DR** = 0, whatever the orientations.

Remark. The answer to the natural question, "then, is the kernel **R** equal to the range of **G** ?", is "yes" here, because $\overline{D} - S^e$ has simple topology. Otherwise, this would lead us far into *homology*, a branch of topology which studies the global topological properties of manifolds by first chopping them into cells, then looking at the algebraic properties of the incidence matrices. (See, e.g., [Ar].) ♢

The whole algebraic structure composed of the sets $\mathcal{N}, \mathcal{E}, \mathcal{F}, \mathcal{V}$ and of the incidence matrices is called a "cell complex". This, plus the (later quite necessary) details about each cell's map, forms what we shall denote $\mathcal{M}$, and call, informally, a *mesh of* D, the *primal* mesh in the theory.

1.4 Dual mesh

The *dual* mesh of D is also a cellular paving, though not of the same region exactly, and with *outer* orientation of cells. Let's explain.

To each p-cell c of the primal mesh, we assign a unique $(n-p)$-cell, meeting c at a single point, called the *dual* of c, and denoted $\tilde{c}$. Hence a 1–1 correspondence between cells of complementary dimensions. Thus, for instance, facet f is pierced by the dual edge $\tilde{f}$ (a line), node n is inside the dual volume $\tilde{n}$, and so forth. Since, at the common point, the tangent spaces to the primal cell c and the dual cell $\tilde{c}$ are complementary ([B3], p. 26), the inner orientation of c provides an outer orientation for $\tilde{c}$ (Fig. 7). Incidence matrices $\tilde{\mathbf{G}}, \tilde{\mathbf{R}}, \tilde{\mathbf{D}}$ can then be defined, as above, the sign of each nonzero entry depending on whether outer orientations match or not.

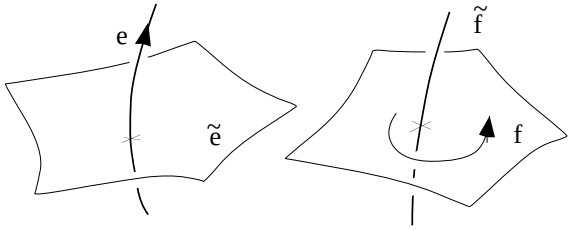

Figure 7. Inner orientations of edge e and facet f, respectively, give crossing direction through $\tilde{e}$ and gyratory sense around $\tilde{f}$.

Moreover, it is required that, when c is a face of c', the dual $\tilde{c}'$ be a face of $\tilde{c}$, and the other way round. This has two consequences. First, we don't really need new names for the dual incidence matrices. Indeed, consider for instance edge e and facet f, and suppose $\mathbf{R}_{fe} = 1$, i.e., e is a face of f and their orientations match: Then the dual edge $\tilde{f}$ is a face of the dual facet $\tilde{e}$, whose outer orientations match, too. So what we would otherwise denote $\tilde{\mathbf{R}}_{\tilde{e}\tilde{f}}$ is equal to $\mathbf{R}_{fe}$. Same reasoning with the opposite signs, and for other kinds of cells, from which we conclude that the would-be dual incidence matrices $\tilde{\mathbf{G}}, \tilde{\mathbf{R}}, \tilde{\mathbf{D}}$ are just the transposes $\mathbf{D}^t, \mathbf{R}^t, \mathbf{G}^t$ of the primal ones.

Second consequence, there is no gap between dual cells, which thus form a cellular paving of a connected region $\tilde{R}$, the interior $\tilde{D}$ of which is nearly D, but not quite (Fig. 8). A part of its boundary is paved by dual cells: We name it $\tilde{S}^e$, owing to its kinship with S^e (not so obvious on our coarse drawing! but the finer the mesh,[7]

[6] It's no accident if this evokes the proof of Stokes theorem we saw in [B3], Section 3.4. The same basic observation, "the boundary of a boundary is zero" [TW, KW], underlies both proofs.

[7] A *refinement* of a paving is another paving of the same

the closer $\tilde{S}^e$ and S^e will become). The other part is denoted $\tilde{S}^h$. So the cellular paving we now have is closed modulo $\tilde{S}^h$, whereas the primal one was closed modulo S^e. The whole structure, again, is called the *dual mesh*, denoted by $\widetilde{\mathcal{M}}$.

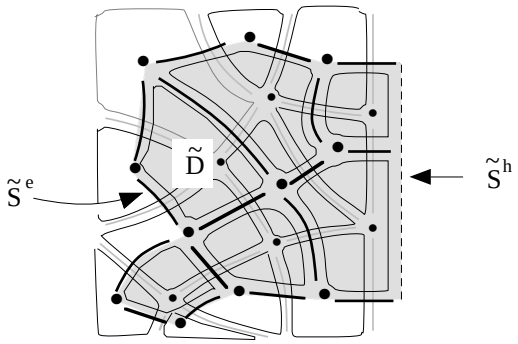

Figure 8. A dual paving, overlaid on the primal one.

Given $\mathcal{M}$, all its duals have the same *combinatorial* structure (the same incidence matrices), but can differ as regards *metric*, which leaves much leeway to construct dual meshes. Two approaches are noteworthy, which lead to the "barycentric dual" and the "Voronoi–Delaunay dual". We shall present them as special cases of slightly more general procedures, the "star construction" and the "orthogonal construction" of meshes in duality. We shall consider only *polyhedral* meshes (those with polyhedral 3-cells), which is not overly restrictive in practice.

The orthogonal construction is an old idea, developed by the young Maxwell[8] [Ma]. It works for "straight" primal cells, i.e., with straight edges and plane polygonal facets. Let dual cells be straight, too, each orthogonal to its primal partner. Figure 9 gives a 2D example. A particular case is the *Voronoi–Delaunay* tesselation: Start from a set $\mathcal{N}$ of would-be primal nodes; for each node n, located at point x_n, build the *Voronoi cell*

$$\tilde{n} = \{x : |x - x_n| < |x - x_m| \ \ \forall\, m \neq n\},$$

comprising all points closer to n than to other nodes;[9] for two nodes m and n, take

$$\{x : |x-x_n| = |x-x_m| < |x-x_l| \ \forall\, l \neq n, l \neq m\}$$

which, if non-empty, will be the dual 2-cell for edge $\{m, n\}$, and so on. If things go well, two meshes in duality are thus obtained. (See [B2], p. 107, for more details and references.) Generically,[10] the primal mesh is then a *simplicial* one, as its volumes are tetrahedra.

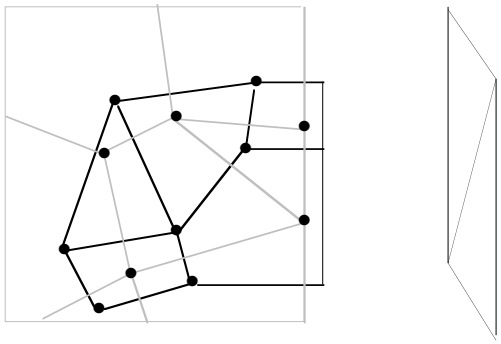

Figure 9. Left: Orthogonal dual mesh. (Same graphic conventions as in Fig. 8, slightly simplified.) Right: Likely the simplest example of a 2D mesh with no orthogonal dual.

Alas, such pavings, whose virtues will be obvious when we study the discrete Hodge operator, are notoriously difficult to build. Even the less stringent condition of orthogonality can be impossible to satisfy, if the primal mesh is imposed (Fig. 9). If one starts from a simplicial primal with only *acute* dihedral angles,[11] all goes well. But this property, which we shall see is desirable, is not so easily obtained, and certainly not warranted by common mesh generators.

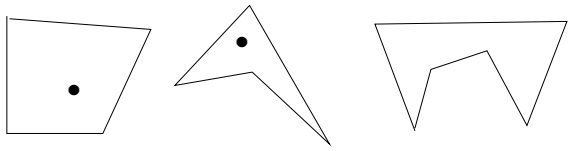

Figure 10. Two star-shaped polygons (left), and one which is not (right). Dots mark eligible centers.

Hence the usefulness of the star construction, more general, because it applies to any primal mesh with star-shaped cells. Recall that a part A of E_3 is *star-shaped* (Fig. 10) if it contains

region, which restricts to a proper cellular paving of each original cell.

[8] Working on elasticity, Maxwell had done the same kind of separation between non-metric and metric notions that we rediscover nowadays in analysing Maxwell equations. Read the masters ...

[9] The idea dates back to Dirichlet [Di].

[10] Meaning that, if not so, a slight displacement of nodes will make it so.

[11] The angles between facets.

a point a, that we shall call a *center*, such that the whole segment $[a, x]$ belongs to A when x belongs to A. Now, take a center for each primal cell (the center of a primal node is itself), and join it to centers of all faces of its cell. This way, simplicial *subcells* are obtained (tetrahedra and their faces, in 3D). One gets the dual mesh by rearranging them, as follows: for each primal cell c, build its dual by putting together all k-subcells, $k \leq n - p$, which have one of their vertices at c's center, and other vertices at centers of cells incident on c. Figures 11 and 12 give the idea.

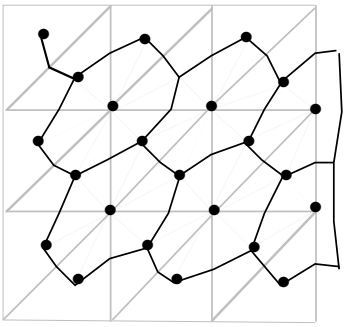

Figure 11. Star construction of a dual mesh (quite close, here, to a barycentric mesh). Notice the isolated dual edge, and the arbitrariness in shaping dual cells beyond S^h.

Remark. The recipe is imprecise about cells dual to those of S^h, whose shape outside D can be as one fancies (provided the requirements about duality are satisfied). Nothing there to worry about: Such choices are just as arbitrary as the selection of the cell centers. It's all part of the unavoidable approximation error, which can be reduced at will by refinement (insofar as computing resources are there). $\diamondsuit$

In the case of a simplicial mesh, cell barycenters make convenient centers, hence the *barycentric dual*. It's a well-known structure, but visualizing it in three dimensions may be not so easy. Cf. Fig. 12.

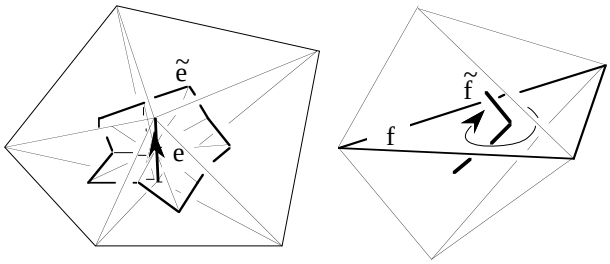

Figure 12. A dual facet and a dual edge, in the case of a simplicial mesh and of its barycentric dual. Observe the orientations.

Remark. If the primal mesh has been obtained by restriction of a closed one, as suggested above ("pave $\overline{D}$ first ..."), subcells built from the latter form a refinement of *both* the primal mesh and the dual mesh. The existence of this "underlying simplicial complex" will be important later. $\diamondsuit$

1.5 The network equations

We now want to apply the principle described in the Introduction: Satisfy the balance equations $(1'')$–$(2'')$ for a selected *finite* family of surfaces.

Let's first adopt a finite, approximate representation of the fields. Consider b, for instance. As a 2-form, it is meant to be integrated over oriented surfaces. So one may consider the integrals $\int_f b$, denoted $\mathbf{b}_f$, for all facets f, as a kind of "sampling" of b, and take the "DoF-vector" $\{\mathbf{b} = \mathbf{b}_f : f \in \mathcal{F}\}$, indexed over primal facets, as a finite representation of b. This does not tell us about the *value* of the field at any given point, of course. But is that the objective? Indeed, all we know about a field is what we can measure, and we don't measure point values. These are abstractions. What we do measure is the *flux* of b—or rather, its variations, but never mind—by reading off the induced emf along the loop of a small enough magnetic probe. The above sampling thus consists in having each facet of the mesh play the role of such a probe, and the smaller the facets, the better we know the field. Conceivably, the mesh may be made so fine that the $\mathbf{b}_f$'s are *sufficient information* about the field, in practice. So one may be content with a method that would yield the four meaningful arrays of degrees of freedom, listing

- the edge emf's, $\mathbf{e} = \{\mathbf{e}_e : e \in \mathcal{E}\}$,
- the facet fluxes, $\mathbf{b} = \{\mathbf{b}_f : f \in \mathcal{F}\}$,
- the dual-edge mmf's, $\tilde{\mathbf{h}} = \{\tilde{\mathbf{h}}_f : f \in \mathcal{F}\}$,
- and the dual-face displacement currents, $\tilde{\mathbf{d}} = \{\tilde{\mathbf{d}}_e : e \in \mathcal{E}\}$,

all that from a similar sampling, across dual facets, of the given current $\tilde{j}$, encoded in the DoF array $\tilde{\mathbf{j}} = \{\tilde{\mathbf{j}}_e : e \in \mathcal{E}\}$.

Next, suppose the surface Σ in $(1'')$ is the simplest possible one in the present context, that is, a *single* primal facet, f. The (inner) orientation of f confers an orientation to its boundary ∂f. The integral of e along ∂f,

by linearity, would be the sum of its integrals along edges that make f, if the orientations of these were compatible with the orientation of ∂f. But when orientations don't match, multiplying by $\mathbf{R}_{fe}$ restores the right value, by the very definition of these incidence numbers. Therefore, eq. $(1'')$ applied to f yields

$$\partial_t \mathbf{b}_f + \sum_{e \in \mathcal{E}} \mathbf{R}_{fe} \mathbf{e}_e = 0. \tag{8}$$

There is one equation like this for each facet of the primal mesh, that is—thanks for having discarded facets in S^e, for which the flux is known to be 0—one for each genuinely unknown facet-flux of b. We may now express (8) in matrix form, like this:

$$\partial_t \mathbf{b} + \mathbf{Re} = 0, \tag{9}$$

the first group of our *network* differential *equations*.

Finally, the same reasoning about each dual facet $\tilde{e}$ (the simplest possible outer-oriented surface that $\tilde{\Sigma}$ in $(2'')$ can be) yields

$$-\partial_t \tilde{\mathbf{d}}_e + \sum_{f \in \mathcal{F}} \mathbf{R}_{fe} \tilde{\mathbf{h}}_f = \tilde{\mathbf{j}}_e, \tag{10}$$

for all e in $\mathcal{E}$, i.e., in matrix form,

$$-\partial_t \tilde{\mathbf{d}} + \mathbf{R}^t \tilde{\mathbf{h}} = \tilde{\mathbf{j}}, \tag{11}$$

the second group of network equations.

If a field $e, b, \tilde{h}, \tilde{d}$, described by its DoF-arrays $\mathbf{e}, \mathbf{b}, \tilde{\mathbf{h}}, \tilde{\mathbf{d}}$, satisfies (10) and (11), it automatically satisfies the balance equations $(1'')$–$(2'')$ for, respectively, all inner-oriented surfaces made of 2-cells and outer-oriented surfaces made of dual 2-cells. Our first objective is thus achieved.

WHAT NEXT?

There are F scalar equations in (10), and E in (11), for $2(E+F)$ unknowns. So we miss $E+F$ equations: one relation between $\mathbf{e}$ and $\tilde{\mathbf{d}}$ for each edge, one relation between $\mathbf{b}$ and $\tilde{\mathbf{h}}$ for each facet. How to get them most simply? This is the problem of the discrete Hodge operator, next column's subject.

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(2): Network constitutive laws

Where we stand

Our objective, in this series, is to solve the Maxwell equations in a closed cavity, in presence of a given current density, starting from no field at time zero. What we aim at is a *numerical scheme*, an algorithm, so that the numbers issued by the number-cruncher as the computation proceeds can be converted into usable information about the evolution of the field. This task is what one calls, in common parlance, the "discretization" of the equations.

In this respect, what we achieved so far is a "spatial" discretization of the Faraday and Ampère relations,

$$\partial_t b + \mathrm{d}e = 0, \qquad -\partial_t \tilde{d} + \mathrm{d}\tilde{h} = \tilde{j},$$

in the form of a system of differential equations (eqs. (9) and (11) in the previous issue):

$$(1) \qquad \partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0, \quad (2) \quad -\partial_t \tilde{\mathbf{d}} + \mathbf{R}^t \tilde{\mathbf{h}} = \tilde{}.$$

There, **e** and **b**, $\tilde{\mathbf{d}}$ and $\tilde{\mathbf{h}}$, are "DoF-arrays": arrays of real numbers, the so-called "degrees of freedom", associated each with a specific geometrical element (edge or facet) of two interlocked "cellular pavings". Figure 1 reminds this association: assuming, here, tetrahedra as primal cells, and a barycentric construction of the dual paving, we assign the degrees of freedom $\mathbf{e}_e$ and $\mathbf{b}_f$ to the edge labelled e and to the facet labelled f, while $\tilde{\mathbf{h}}_f$ and $\tilde{\mathbf{d}}_e$ are affixed to the "dual edge" $\tilde{f}$ and to the "dual facet" $\tilde{e}$. Single letters with bold face, like for instance **b**, denote arrays such as $\{\mathbf{b}_f : f \in \mathcal{F}\}$, indexed over sets of so-called "p-cells", like here the set $\mathcal{F}$ of primal facets ($p = 2$). Same interpretation for $\tilde{} = \{\tilde{}_{\tilde{e}} : e \in \mathcal{E}\}$, indexed over edges, except that $\tilde{}$ is not a DoF-array but an array of *data*, the intensities across the dual facets. These are easily computed from the current density, which we considered as a given function of time, in the model problem we have in view.

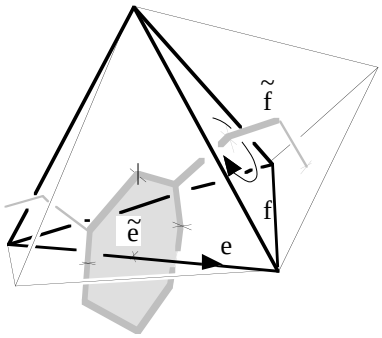

Figure 1. A few typical cells, in the case of what we called last time the "star construction" of a dual mesh, based on a simplicial primal mesh. Each primal edge or facet has its own inner orientation, which induces an outer orientation of its dual associate: for instance, the forward direction along edge e is taken as "crossing direction" of the dual facet $\tilde{e}$, etc.

These DoFs are what will eventually be known about the fields, once the computation is over: $\mathbf{e}_e$ will be the electromotive force[1] along edge e, $\tilde{\mathbf{d}}_e$ will be the flux of displacement current across the dual facet $\tilde{e}$, etc.

As we saw last time, the "network equations" (1) and (2) are not enough to determine **e**, **b**, $\tilde{\mathbf{h}}$, and $\tilde{\mathbf{d}}$. They need to be complemented by "network constitutive laws", that would relate **e** to $\tilde{\mathbf{d}}$ and **b** to $\tilde{\mathbf{h}}$. Finding such laws, i.e., discretizing the constitutive laws

$$b = \mu \tilde{\star} \tilde{h}, \qquad \tilde{d} = \epsilon \tilde{\star} e,$$

is the order of the day.

2.1 A generalized Yee scheme

Let's write these desired discrete relations as follows:

$$(3) \qquad \mathbf{b} = \tilde{\star}_\mu \tilde{\mathbf{h}}, \qquad (4) \qquad \tilde{\mathbf{d}} = \tilde{\star}_\epsilon \mathbf{e}.$$

[1] or rather, the best estimate of this voltage that we can achieve by using these meshes. Of course, some error occurs, and we shall not avoid the issue. We pointed out last time that everything one wants to know about the field can be obtained from the degrees of freedom, and that pointwise values of the field have secondary interest. Yet, one may have need for them, so we shall not dodge this issue either.

Here, $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ denote square matrices, of respective dimensions F and E (the numbers of active primal facets and edges[2]), which stand as finite-dimensional approximations of the above $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$. Building them is our objective, but we shall carry on for a while *as if* we knew them. This will point to a number of desirable properties of the discrete Hodges, and thus help in their construction.

The first, fairly obvious, requirement is that $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ should be *regular* matrices. If so, eqs. (1)–(4) can be rewritten as

$$\partial_t \mathbf{b} + \mathbf{R}\tilde{\star}_\epsilon^{-1}\tilde{\mathbf{d}} = 0, \qquad -\partial_t \tilde{\mathbf{d}} + \mathbf{R}^t \tilde{\star}_\mu^{-1}\mathbf{b} = \tilde{j},$$

a system of ODE's in terms of $\mathbf{b}$ and $\tilde{\mathbf{d}}$. This, plus the initial conditions $\mathbf{b}(0) = 0$ and $\tilde{\mathbf{d}}(0) = 0$ at time $t = 0$, constitutes a discretization "in space" of the original equations, to which we have thus substituted a finite-dimensional *dynamical system*. Obvious equivalent forms of this system will come to mind, since there are four ways to eliminate one variable out of two in each group, **e**–$\tilde{\mathbf{d}}$ and **b**–$\tilde{\mathbf{h}}$. For instance, the following:

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0, \qquad -\partial_t \tilde{\star}_\epsilon \mathbf{e} + \mathbf{R}^t \tilde{\star}_\mu^{-1}\mathbf{b} = \tilde{j},$$

which we shall adopt for definiteness.

The next step is a discretization "in time". Introducing a time-step $\delta t > 0$, let's approximate the time-dependent DoF array $\mathbf{b}$ by linear interpolation between successive values at times $t_k = k\delta t$, which we denote by $\mathbf{b}^k$, with $k = 0, 1, \ldots$. This way, the quotient $(\mathbf{b}^{k+1} - \mathbf{b}^k)/\delta t$ constitutes a natural approximation of the time-derivative $\partial_t \mathbf{b}$ at time $t_{k+1/2} = (k+1/2)\delta t$, which instructs us to approximate $\mathbf{e}$ by linear interpolation between values at such "half-integer times". These values we denote by $\mathbf{e}^{k+1/2}$. The obvious thing to do, now, is to let the $\mathbf{b}^k$s and $\mathbf{e}^{k+1/2}$s satisfy the following system of equalities,

$$\mathbf{b}^0 = 0, \quad \mathbf{e}^{-1/2} = 0,$$

$$(5) \quad \frac{\mathbf{e}^{k+1/2} - \mathbf{e}^{k-1/2}}{\delta t} = \tilde{\star}_\epsilon^{-1}(\mathbf{R}^t \tilde{\star}_\mu^{-1}\mathbf{b}^k - \tilde{j}^k),$$

$$(6) \quad \frac{\mathbf{b}^{k+1} - \mathbf{b}^k}{\delta t} + \mathbf{R}\mathbf{e}^{k+1/2} = 0,$$

for successive values $k = 0, 1, \ldots$, of the "discrete time" parameter. Clearly, this "leapfrog scheme" is an *algorithm* to solve eqs. (1)(4): it produces the $\mathbf{b}^k$s and $\mathbf{e}^{k+1/2}$s, step by step, when fed with the succession of known values of $\tilde{j}^k$. These, of course, are obtained by taking the flux of the known current density $\tilde{j}(t)$ at time t_k across dual facets.

Note that we start with $k = 0$, so the first required value of $\tilde{j}$ is $\tilde{j}^0$, which as a rule will be zero, but may not be: the algorithm can cope with sudden jumps of the current density (unphysical as these may be). More interestingly, let's remark that if $\tilde{j}(t)$ was a succession of steps, with $\tilde{j}(t)$ equal to some steady current density between times $(k-1/2)\delta t$ and $(k+1/2)\delta t$, the solution of (1)–(4) would be *exactly* what one obtains by linear interpolation in time between the successive $\mathbf{b}^k$s and $\mathbf{e}^{k+1/2}$s, as output by the numerical scheme, namely,

$$\mathbf{b}_{\delta t}(t) = [(t_k - t)\mathbf{b}^{k-1} + (t - t_{k-1})\mathbf{b}^k]/\delta t$$

for $t \in [t_{k-1}, t_k]$. This, though not making a proof, strongly suggests that the time-varying DoF arrays $\mathbf{b}_{\delta t}$ and $\mathbf{e}_{\delta t}$ thus built do converge towards the solution[3] $\{t \to \mathbf{b}(t), t \to \mathbf{e}(t)\}$ of (1)–(4) when δt tends to zero.

This property is a well-known feature (see., e.g., [MS]) of the classical Yee scheme [Ye], also known as the "finite difference in time-domain (FDTD) method", to which (5)(6) reduces when both pavings are made of brick-shaped cells, with facets parallel to the coordinate planes of an orthogonal Cartesian frame. Figure 2 is a reminder of the way vector components were assigned to grid-points in Yee's approach, 30 years ago, in the simplified 2D framework one had to assume in this age of limited computer resources.[4] (We consider

[2] This difference in dimensions explains why we have $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ instead of $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$. There is no *single* discrete Hodge operator $\tilde{\star}$, but one discrete Hodge for each constitutive law. In fact, we should also replace $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$ by $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ in the case of *anisotropic* materials, as discussed earlier (*JSAEM, 6*, 4, 1998, p. 325).

[3] I seize this pretext to recall that "$t \to f(t)$" means "the function that maps t to the value $f(t)$". (Cf. (*JSAEM, 6*, 2, 1998, p. 119.) This notation often helps to lift some ambiguities, as in the present case, where **b**, **e**, and their approximations $\mathbf{b}_{\delta t}$ and $\mathbf{e}_{\delta t}$ are not conceived as DoF-arrays, but as functions of time, whose instant values are DoF-arrays.

[4] A fully 3D modern avatar is the MAFIA code [W&], which can handle millions of DoFs on staggered cellular grids. (The "Finite Integration Technique" of [W&] is what we described last time when deriving eqs. (1) and (2).)

here the case of a horizontal electric field **E** = $\{\mathrm{E}^x, \mathrm{E}^y\}$ and of a vertical magnetic induction, whose single scalar component is denoted B.)

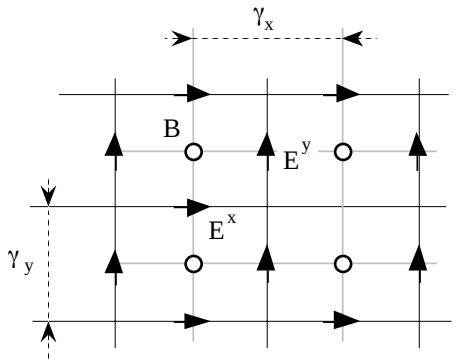

Figure 2. Assignment of field components to grid-points, in a 2D "transverse electric" Yee scheme. (We assume a uniform primal grid. Then all 2-cells have the same size, $\gamma_x \times \gamma_y$.)

It takes little imagination to see the relation between these components and our edge- and facet-based degrees of freedom. (It may be slightly easier in three dimensions, cf. Fig. 3, which corresponds to Fig. 1 in [Ye].) Yee's staggered grids are an instance of what we called last time the "orthogonal construction", with the additional feature that edges pierce their associated facets in their exact middles, and are divided by them into equal parts. All primal edges are oriented along the coordinate axes. All facets are oriented according to the usual conventions, so that for instance, facet f of Fig. 3, which lies in a horizontal plane, has the standard counterclockwise orientation.

There is an inessential difference, however: The degree of freedom $\mathbf{b}_f$ of facet f (cf. the caption of Fig. 3) is *not* the component B^z at f's center, i.e., at point $\{i+1/2, j+1/2, k+1\}$, but corresponds to the flux $\gamma_x\gamma_y\mathrm{B}^z$, and $\mathbf{e}_e$ is $\gamma_z\mathrm{E}^z$. Thus taking the vector-proxy components as DoFs makes the analogue of our network equations a bit cumbersome, because of the appearance of the edge-lengths γ_i in formulas, and of the heavy labelling (e.g., our $\mathbf{b}_f$ for facet $\{i + 1/2, j + 1/2, k + 1\}$ is $\gamma_x\gamma_y\mathrm{B}^z_{i+1/2,j+1/2,k+1}$). In compensation, network constitutive laws are most natural in Yee's scheme: Just set

$$\mathrm{B}^z_{i+1/2,j+1/2,k+1} = \mu\,\mathrm{H}^z_{i+1/2,j+1/2,k+1}, \tag{7}$$

and so forth.

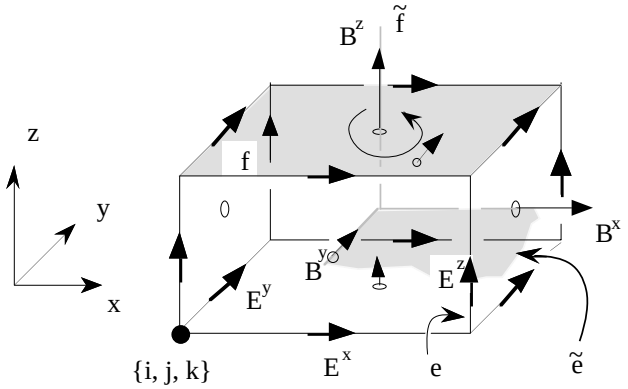

Figure 3. Connection with the present approach, in 3D. One face f and its dual $\tilde{f}$ are highlighted, as well as one edge e and a part of its dual facet $\tilde{e}$. It's convenient, with such grids, to label primal nodes with integer triples $\{i, j, k\}$, hence a natural labelling for all cells: For instance, the f of this figure is facet $\{i + 1/2, j + 1/2, k + 1\}$ (the label that one would stick to its center), edge e is $\{i + 1, j, k + 1/2\}$, etc.

Since $\gamma_x\gamma_y$ is the area of f and γ_z the length of its dual edge $\tilde{f}$, we may rewrite (7) in our notation as follows:

$$\mathbf{b}_f = \mu\,\frac{\mathrm{area}(f)}{\mathrm{length}(\tilde{f})}\,\tilde{\mathbf{h}}_f. \tag{8}$$

So, we have with (5)(6) a generalization of Yee's scheme, which only differs from it in details,[5] mostly notational, as far as the network equations are concerned. But—and here lies, from the point of view adopted here, Yee's achievement—both instances of *the discrete Hodge operator take on an ideally simple form*, in FDTD, thanks to the adoption of a system of staggered uniform grids. They are expressed by *diagonal* matrices $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$, whose entries are given by simple formulas[6] such as (8).

Looking back at (5), we may appreciate how essential to the efficiency of the scheme this property of diagonality can be. Because of well-known concerns about stability on which we soon return, δt has to be small. Having to solve one or two linear systems at each time

[5] Among these: In [Ye], **E** was evaluated at integer times, and **B** at half-integer times.

[6] If μ is non-uniform but smooth, just replace it in (8) by $\mu_{i+1/2,j+1/2,k+1}$, with obvious notation. We shall return on the case when point $\{i + 1/2, j + 1/2, k+1\}$ happens to be at a material interface. Note, on the other hand, that anisotropic tensors μ and ϵ *can* be accommodated if their principal directions go along coordinate axes.

step, as (5) may seem to require, would therefore make the computation too slow. Thanks to diagonality of the Hodges, Yee's scheme is *explicit*, not requiring any linear system solution. Diagonality of $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ is thus, we see, highly desirable.

Yet, we can be content with less, for the main point is to keep the number of arithmetic operations in steps (5) and (6) as low as possible. In (6), we have to multiply the E-dimensional vector $\mathbf{e}^{k+1/2}$ by a sparse matrix $\mathbf{R}$, anyway. A similar multiplication, by $\mathbf{R}^t$, intervenes in (5). So if both matrices $\tilde{\star}_\mu^{-1}$ and $\tilde{\star}_\epsilon^{-1}$ can be made sparse to a comparable degree, it will be an acceptable state of affairs. This should be achievable, since the operators these matrices are meant to approach, the inverses of $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$, have a *local* character.

2.2 Wedge product, energy

So, a regular matrix which purports to approximate a Hodge operator should, if not diagonal, at least be sparse, or have a sparse inverse. A connection with the notion of *energy* of the field will suggest that it should be *symmetric*, too.

This is the place to go deeper into the notion of wedge product of forms, that was only treated incidentally up to now. Let ω and η be a p-covector and a q-covector. We shall define a new covector, of degree $p+q$, denoted $\omega \wedge \eta$, and call it their *wedge product*, or external product. The notion of wedge product for p- and q-forms, i.e., for fields of covectors, will then follow naturally.

Recall that a p-covector ω was conceived as a machine with p slots, in which p vectors $\{v_1, \ldots, v_p\}$ can be inserted, in a definite order. The machine then outputs a real number, which linearly depends on all factors (ω is a "multilinear" map), and changes sign if two of them are interchanged ("alternating" map). Now, have two machines of this kind, with p and q slots respectively, and a sequence of $p+q$ vectors, $\{v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}\}$, to process. As these arrive, we may assign them to one or the other machine, so that p of them go to the first machine and the remaining q to the other one. Hence two numbers, of which we take the product. This satisfies the multilinearity condition. But since there is no criterion to allot vectors to one machine or the other, we must consider *all* ways to do that, and add the results, with appropriate sign changes in order to satisfy the alternation condition. For instance, if $p = q = 1$, this suggests the following, which we already know,

$$(9) \quad (\omega \wedge \eta)(v_1, v_2) = \omega(v_1)\eta(v_2) - \omega(v_2)\eta(v_1),$$

as definition of $\wedge$. If $p = 1$ and $q = 2$, the same idea leads to

$$(10) \quad (\omega \wedge \eta)(v_1, v_2, v_3) = \omega(v_1)\eta(v_2, v_3) + \omega(v_2)\eta(v_3, v_1) + \omega(v_3)\eta(v_1, v_2).$$

To comfortably generalize that, let's denote by σ a map from the set of integers $[1, \ldots, p]$ into the larger set $[1, \ldots, p+q]$, such that $\sigma(i) < \sigma(i+1)$ for all $i < p$. The set of positions not occupied in $[1, \ldots, p+q]$ determines in an obvious way (Fig. 4) a *complementary* map ς with similar properties. Now, we define

$$(11) \quad (\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \sum_\sigma \mathrm{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\, \eta(v_{\varsigma(1)}, \ldots, v_{\varsigma(q)}),$$

where $\mathrm{sgn}(\sigma) = \pm 1$, as explained on Fig. 4, the sum being taken with respect to all possible σs. (Note there are as many such σs as ways to choose p integers out of $p+q$.)

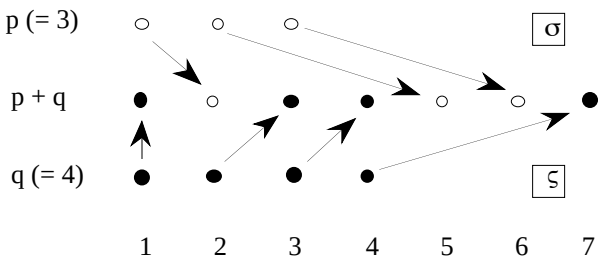

Figure 4. An increasing injection σ from $[1, \ldots, p]$ into $[1, \ldots, p+q]$ and its complement ς. We call *signature* of σ (or of ς), denoted sgn, the parity of the number of swaps between black and white spots that will bring all the white ones to the left (or all the black ones to the right). Here, this takes 7 swaps, so $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\varsigma) = -1$ in this example.

We won't really use this formula, given here for the sake of completeness. It's enough to know that covectors can thus be wedge-multiplied two by two,[7] and hence, differential forms, which are fields of covectors, have wedge-products, too. One or the factors, or

[7] The operation is associative. As for commutativity, one has $\eta \wedge \omega = (-1)^{pq}\, \omega \wedge \eta$, as detailed examination of (11) will show.

both, can be a *twisted* covector (*JSAEM, 6*, 2, 1998, p. 123). Rules to this effect are obvious, and tedious.[8]

As an immediate application of this remark, it makes sense to wedge-multiply the electric field e, a 1-form, by the current-density $\tilde{j}$, a twisted 2-form, thus obtaining a 3-form $e \wedge \tilde{j}$ which, being a *twisted* form, can be integrated over 3D space, irrespective of ambient orientation. As our previous discussion of the Lorentz force should make clear, the number thus obtained is, up to sign, the rate of work involved in moving electric charges in the antenna the prescribed way, which entails working against the electric field created by these very charges. With correct sign, $\int e \wedge \tilde{j}$ is the *power yielded by* the electromagnetic field to the rest of the world.

Rather than offer a proof of this, which would lead us astray, let's cash in on our knowledge that the density of such power is the dot product $\mathbf{E} \cdot \mathbf{J}$. So it's a matter of proving that, if $\mathbf{E}$ and $\mathbf{J}$ are the vector proxies of a covector e and a 2-covector j, one has $e \wedge j = \mathbf{E} \cdot \mathbf{J}\, vol$, where vol is the 3D volume form, $vol(v_1, v_2, v_3) = v_1 \cdot (v_2 \times v_3)$.

Indeed, let's apply (10) to e and j, knowing that $e(v) = \mathbf{E} \cdot v$ and $j(v, w) = \mathbf{J} \cdot v \wedge w = vol(\mathbf{J}, v, w)$, by the very definition of vector proxies. What comes out is $(e \wedge j)(v_1, v_2, v_3) = \mathbf{E} \cdot v_1 vol(\mathbf{J}, v_2, v_3) + \ldots$, with circular permutation, an expression in which one will recognize $\mathbf{E} \cdot \mathbf{J}$ times $vol(v_1, v_2, v_3)$. In compact form, using the notation introduced in *JSAEM, 6*, 3 (1998), p. 233,

$$^1\mathbf{E} \wedge {}^2\mathbf{J} = {}^3(\mathbf{E} \cdot \mathbf{J}).$$

The case $p = q = 1$ is even easier: Applying (9) to $e = {}^1\mathbf{E}$ and $h = {}^1\mathbf{H}$, we get $(e \wedge h)(v, w) = (\mathbf{E} \cdot v)(\mathbf{H} \cdot w) - (\mathbf{E} \cdot w)(\mathbf{H} \cdot v)$, equal to $(\mathbf{E} \times \mathbf{H}) \cdot (v \times w)$, by a well-known formula. So

$$^1\mathbf{E} \wedge {}^1\mathbf{H} = {}^2(\mathbf{E} \times \mathbf{H}).$$

Hail to thee, Poynting vector!! Indeed, taking account of the orientation issues neglected in what precedes, what plays the role of "Poynting field" in the geometric approach is the twisted 2-form $e \wedge \tilde{h}$, whose integral over a closed surface (outer-oriented from inside to outside) is the exiting power.

Now let us consider a product such as $\mathbf{B} \cdot \mathbf{H}$, which is, up to a factor 2, the density of magnetic energy. We see that ${}^3(\mathbf{B} \cdot \mathbf{H}) = b \wedge \tilde{h}$, and since $b = \mu\, \tilde{\star} \tilde{h}$, this density is the twisted 3-form $\mu/2\, \tilde{\star} \tilde{h} \wedge \tilde{h}$. One may verify, as an easy exercise,[9] the Poynting theorem:

$$(12) \quad d_t W_\Omega(e, \tilde{h}) + \int_{\partial\Omega} e \wedge \tilde{h} = -\int_\Omega \tilde{j} \wedge e,$$

where $W_\Omega(e, \tilde{h}) = \int_\Omega (\mu/2\, \tilde{\star} \tilde{h} \wedge \tilde{h} + \epsilon/2\, \tilde{\star} e \wedge e)$ is the part of the field energy ascribed to region Ω. This expresses energy conservation.

The equality $b \wedge \tilde{h} = {}^3(\mathbf{B} \cdot \mathbf{H})$ has something else to tell us. Replace $\tilde{h}$ by a different field $\tilde{h}'$, and set $b = \mu\, \tilde{\star} \tilde{h}$. Then $\mu\, \tilde{\star} \tilde{h} \wedge \tilde{h}' = {}^3(\mu\, \mathbf{H} \cdot \mathbf{H}')$, a symmetrical expression. Hence something we didn't pay attention to so far: the *symmetry* of the Hodge operator acting on forms, as expressed by $\int \mu\, \tilde{\star} \tilde{h} \wedge \tilde{h}' = \int \mu\, \tilde{\star} \tilde{h}' \wedge \tilde{h}$ for all $\tilde{h}, \tilde{h}'$. Moreover,[10] $\int \mu \tilde{\star} \tilde{h} \wedge \tilde{h} = \int \mu \mathbf{H} \cdot \mathbf{H} > 0$ for $\tilde{h}$ not identically 0, so $\tilde{\star}$ is positive definite. The matrices $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ by which we purport to approximate $\mu\, \tilde{\star}$ and $\epsilon\, \tilde{\star}$ should, consistently, be symmetric and strictly positive definite, too.

2.3 "Discrete" energy, and stability

To discuss the implications of this remark, let's introduce a notation for such sums as $\sum_{f \in \mathcal{F}} \mathbf{b}_f \tilde{\mathbf{h}}_f$, which one might construe as a dot product between two vectors, $\mathbf{b}$ and $\tilde{\mathbf{h}}$, of common dimension F. Rather than using a dot, however, we shall denote this[11] by

$$\langle \mathbf{b}, \tilde{\mathbf{h}} \rangle = \sum_{f \in \mathcal{F}} \mathbf{b}_f \tilde{\mathbf{h}}_f.$$

---

[8] For instance, if the pair $\{\omega, Or\}$ represents the twisted covector $\tilde{\omega}$, its wedge product by the straight η is the twisted covector represented by the pair $\{\omega \wedge \eta, Or\}$. The product with $\tilde{\eta}$ = (the class of) $\{\eta, Or'\}$ is (the class of) $\{\omega \wedge \eta, Or\, Or'\}$. Observe that the product of two twisted covectors is straight.

[9] Start from (1) and (2), wedge-multiply by $\tilde{h}$ and $-e$, add, and integrate over Ω, using Stokes and the formula $\mathrm{d}(e \wedge \tilde{h}) = \mathrm{d}e \wedge \tilde{h} - e \wedge \mathrm{d}\tilde{h}$.

[10] The integral concerns the whole region of space under consideration, i.e., the whole domain D in our model problem. Note (as a follow-up to a previous remark, *JSAEM, 7*, 2, 1999, p. 154) that $\int \mu\, \tilde{\star} \tilde{h} \wedge \tilde{h}'$ can now be understood as a *scalar product* on the functional space of (twisted) 1-forms, which can thereby be turned into a Hilbert space.

[11] Because this bra–ket notation traditionally connotes duality products between objects of *different* types, as was the case for instance in $\langle \omega, v \rangle$, which we used

Similarly, $\langle \tilde{\wr}, \mathbf{e} \rangle$ is defined as $\sum_{e \in \mathcal{E}} \tilde{\wr}_e \mathbf{e}_e$. One checks that

$$\langle \mathbf{Re}, \tilde{\mathbf{h}} \rangle = \sum_{e,f} \mathbf{R}_{ef} \mathbf{e}_e \tilde{\mathbf{h}}_f = \langle \mathbf{e}, \mathbf{R}^t \tilde{\mathbf{h}} \rangle,$$

a key formula in what follows.

Next, let's do what was suggested in Note 9, but applied to the dynamical system (1)–(4) instead of to the original equations: Take the $\langle\ ,\ \rangle$-product of (1) with $\tilde{\mathbf{h}}$, of (2) with $-\mathbf{e}$, add, apply the previous formula, replace $\mathbf{b}$ by $\tilde{\star}_\mu \tilde{\mathbf{h}}$ and $\tilde{\mathbf{d}}$ by $\tilde{\star}_\epsilon \mathbf{e}$. What results,

$$d_t(\,{}^1\!/_2 \langle \tilde{\star}_\epsilon \mathbf{e}, \mathbf{e} \rangle + {}^1\!/_2 \langle \tilde{\star}_\mu \tilde{\mathbf{h}}, \tilde{\mathbf{h}} \rangle) = -\langle \tilde{\wr}, \mathbf{e} \rangle$$

looks so much like (12)[12] that we can't avoid calling ${}^1\!/_2 \langle \tilde{\star}_\epsilon \mathbf{e}, \mathbf{e} \rangle$ and ${}^1\!/_2 \langle \tilde{\star}_\mu \tilde{\mathbf{h}}, \tilde{\mathbf{h}} \rangle$ the *electric* and *magnetic discrete energy*, respectively, and $\langle \tilde{\wr}, \mathbf{e} \rangle$ the *discrete power* leaving the system. But of course (don't forget we have *no* explicit definition of $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$ yet!), this has no justification till we establish some link between discrete and continuous energy or power (for instance by proving that discrete power and energy converge, in an appropriate sense, towards their continuous counterparts).

Now what about the *time-discretized* dynamical system (5)(6)? Would it conserve discrete energy too? No such luck. Start from (5)(6), where we suppose all $\tilde{\star}$s = 0, to simplify a little. (In compensation, suppose a nonzero initial situation $\{\mathbf{b}^0, \mathbf{e}^{-1/2}\}$.) Use $\tilde{\mathbf{h}}$ for $\tilde{\star}_\mu^{-1} \mathbf{b}$. Right-multiply (5) by $\mathbf{e}^{k+1/2}$, (6) by $\tilde{\mathbf{h}}^k$, subtract to check that

$$\langle \tilde{\star}_\epsilon (\mathbf{e}^{k+1/2} - \mathbf{e}^{k-1/2}, \mathbf{e}^{k+1/2} \rangle + \langle \tilde{\star}_\mu (\tilde{\mathbf{h}}^{k+1} - \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^k \rangle$$

vanishes, repeat this with k changed to $k-1$ in (6), add the results, to finally obtain

$$\langle \tilde{\star}_\epsilon \mathbf{e}^{k+1/2}, \mathbf{e}^{k+1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^{k+1}, \tilde{\mathbf{h}}^k \rangle =$$

$$\langle \tilde{\star}_\epsilon \mathbf{e}^{k-1/2}, \mathbf{e}^{k-1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^{k-1} \rangle.$$

So it's not discrete energy which is conserved. Only a quantity which looks like it (imagine δt tending to 0) happens not to depend on k.

The computation could have been done differently, keeping the same value of k in (6) and changing it in (5), this time showing the conservation of

$$\langle \tilde{\star}_\epsilon \mathbf{e}^{k+1/2}, \mathbf{e}^{k-1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^k \rangle$$

(which actually can be proved equal to the previous quantity). Setting $\mathbf{e}^k = (\mathbf{e}^{k+1} + \mathbf{e}^k)/2$ and $\tilde{\mathbf{h}}^{k+1/2} = (\tilde{\mathbf{h}}^k + \tilde{\mathbf{h}}^{k+1})/2$, we see that what is conserved is the more symmetrical-looking

$$\langle \tilde{\star}_\epsilon \mathbf{e}^k, \mathbf{e}^{k+1/2} \rangle + \langle \tilde{\star}_\mu \tilde{\mathbf{h}}^k, \tilde{\mathbf{h}}^{k+1/2} \rangle,$$

but ... so what? If $k + 1/2$, there, could be replaced by k, we would conclude that the algorithm is *stable*,[13] *under the condition of positive definiteness* of the matrices $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$. But this little difference voids the argument of any value.

So let's try harder. Take (6), with $\mathbf{b}$ written as $\tilde{\star}_\mu \tilde{\mathbf{h}}$, substitute $k-1$ for k, subtract the two equalities thus obtained, and use (5) (again, no $\tilde{\star}$ at the righ-hand side) to eliminate $\mathbf{e}$, hence

$$\tilde{\star}_\mu (\tilde{\mathbf{h}}^{k+1} - 2\tilde{\mathbf{h}}^k + \tilde{\mathbf{h}}^{k-1}) + \delta t^2\, \mathbf{R} \tilde{\star}_\epsilon^{-1} \mathbf{R}^t \tilde{\mathbf{h}}^k = 0.$$

The question is: can $\tilde{\mathbf{h}}^k$, defined by this recurrence, blow up for some initial conditions?

The *modal analysis* technique to answer it is well known. Let's use the generalized eigenmodes $\{\lambda_j, \tilde{\mathbf{w}}_j\}$, solutions of

$$(13) \qquad \mathbf{R} \tilde{\star}_\epsilon^{-1} \mathbf{R}^t\, \tilde{\mathbf{w}} = \lambda^2 \tilde{\star}_\mu\, \tilde{\mathbf{w}},$$

with $\lambda \geq 0$ real, as a basis for the E-dimensional space to which $\tilde{\mathbf{h}}$ belongs. *If* both $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ are symmetric and strictly positive definite, as we assume, there is such a basis (including all $\tilde{\mathbf{w}}_j$s for which $\lambda_j = 0$, which span the nullspace, i.e., the kernel of $\mathbf{R}^t$), made of "$\langle , \rangle_\mu$-orthogonal" vectors, in the sense that $\langle \tilde{\star}_\mu \tilde{\mathbf{w}}_i, \tilde{\mathbf{w}}_j \rangle = 0$ for $i \neq j$. One can then write $\tilde{\mathbf{h}}^k = \sum_j h_j^k \tilde{\mathbf{w}}_j$, bring that into the recurrence

earlier, whereas the two arguments in a dot product are of the *same* type. Here, indeed, $\mathbf{b}$ and $\tilde{\mathbf{h}}$ don't belong to the same type, due to their respective associations with inner- and outer-oriented cells, and we don't consider an expression such as $\langle \mathbf{b}, \mathbf{b} \rangle$ as legitimate. Dimensional analysis makes this plain: While $\langle \tilde{\wr}, \mathbf{e} \rangle$ is a sum of factors expressed in ampères × volts, and hence a power, $\langle \mathbf{b}, \mathbf{b} \rangle$ would be in "squared webers", a preposterous unit.

[12] There is no analogue of a surface term because, when defining the incidence matrices, we deleted from the boundary S of the domain all cells which, a priori, bear a null DoF, owing to boundary conditions. One might extend the theory in order to have such a "discrete Poynting flux".

[13] To check stability for linear systems of difference equations, one looks whether an initial state gets amplified in the absence of right-hand side, so it was all right to dismiss $\tilde{\wr}$.

relation, and obtain that $h_j^{k+1} = r_j h_j^k$, where r_j is a root of the characteristic equation

$$r^2 - (2 - \lambda_j^2 \delta t^2) r + 1 = 0. \qquad (14)$$

So it's all right (no blow up) if both solutions, whose product is 1, lie on the unit circle, which happens (cf. Fig. 5) when

$$\lambda_j \delta t < 2 \text{ for all } j.$$

This is the condition for stability of the generalized Yee scheme (5)(6).

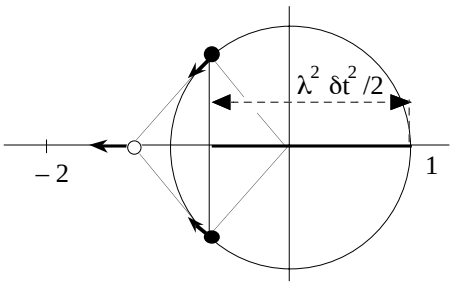

Figure 5. Why the $\lambda\delta t < 2$ condition. (The white spot lies at the sum of roots, i.e., $2 - \lambda^2 \delta t^2$. When it passes left to -2, as δt increases, the roots become real, one of them exiting the unit circle.)

In the case of the original Yee scheme, eigenvalues could explicitly be found, hence the well-know relation [Ye] between the maximum possible value of δt and the lengths of the cell sides. For general grids, we have no explicit formulas, but the thumbrule is the same: δt should be small enough for a signal travelling at the speed of light (in the medium under study) not to cross more than one cell during this lapse of time.

A contrario, having one of the matrices $\tilde{\star}_\mu$ or $\tilde{\star}_\epsilon$ *not* positive definite would destroy stability. Suppose this happens to $\tilde{\star}_\epsilon$ alone. Then, (13) will have imaginary solutions λ_j, for which $2 - \lambda^2 \delta t^2 > 2$ whatever δt, hence instability.

The conclusion is neat: a good discrete Hodge operator is a symmetric, nearly diagonal, positive-definite one. It's time we show this can be achieved.

2.4 A diagonal, positive-definite Hodge

The idea (which has been independently developed by many people) is bold and simple: Use the orthogonal construction, and apply formula (8), the same as in FDTD. So diagonal entries of $\tilde{\star}_\mu$ are

$$\tilde{\star}_\mu^{ff} = \mu_f \frac{\text{area}(f)}{\text{length}(\tilde{f})}, \qquad (15)$$

where f is a primal facet, $\tilde{f}$ its dual edge (Fig. 6), and μ_f the value of μ at the meeting point, if well defined. (More below on this. For the moment, let us assume a uniform μ.) All other entries $\tilde{\star}_\mu^{ff'}$ are set to zero. By the virtues of the orthogonal construction, this is a diagonal positive definite matrix, the ideal situation. The construction of $\tilde{\star}_\epsilon$ is similar.

But why should they be good *as approximations* of $\mu\tilde{\star}$ and $\epsilon\tilde{\star}$? After all, one could imagine multiplying the above number $\tilde{\star}_\mu^{ff}$ by any arbitrary positive factor, and still satisfy the requirements. But let's consider a uniform field $\mathbf{H}$, and abuse the notation by also calling f the vectorial area of f, and $\tilde{f}$ the vector along $\tilde{f}$, which allows us to write

$$f = \frac{\text{area}(f)}{\text{length}(\tilde{f})} \tilde{f},$$

thanks to the orthogonality property. Then $\mathbf{b}_f = \mathbf{B} \cdot f$, and $\tilde{\mathbf{h}}_f = \mathbf{H} \cdot \tilde{f}$. Since $\mathbf{B} = \mu\mathbf{H}$, the ratio $\mathbf{b}_f / \tilde{\mathbf{h}}_f$ is $\tilde{\star}_\mu^{ff}$ *whatever* $\mathbf{H}$, so (15) is the right coefficient for uniform fields. This is the main point in favor of the orthogonal construction.

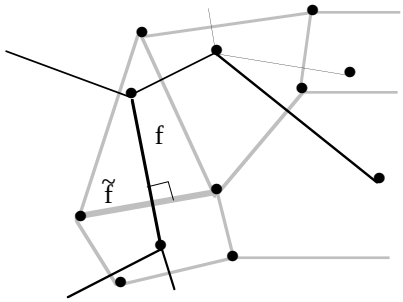

Figure 6. A piece of the pavings, in the case of the orthogonal construction (to be imagined in dimension 3). Essential features are that each dual edge $\tilde{f}$ is orthogonal to its associated primal facet f, does meet it, and that dual nodes are inside primal cells. Under these conditions, $\tilde{f}$ has positive length.

Since all smooth fields will appear uniform at the scale of a cell when the meshes are refined, one may imagine building on this an argument [To] that would lead to a comparison between discrete energy and energy. (Try it: the volume "controlled by" f and $\tilde{f}$ is $\text{area}(f) \times \text{length}(\tilde{f})/3$, the average projection of $\mathbf{H}$ onto a random unitary vector squares to $|\mathbf{H}|^2/3 \ldots$) But we are still far from that, which will require a serious convergence proof.

Finally, let's consider the case (Fig. 7) when two adjacent primal volumes T_1 and T_2, with

common facet f, have permeabilities μ_1 and μ_2, different. Call $\tilde{f}_1$ and $\tilde{f}_2$ the vectors along both parts of $\tilde{f}$. Then, instead of (15),

$$\tilde{\star}_\mu^{ff} = \frac{\mu_1\,\mu_2\,\text{area}(f)}{\mu_2\,\text{length}(\tilde{f}_1) + \mu_1\,\text{length}(\tilde{f}_2)}.$$

This is easily justified: let **u** and **v** be arbitrary vectors, normal and tangent to f respectively, and let $\mathbf{H}_1 = \mathbf{u} + \mathbf{v}$ in T_1. Transmission conditions across f determine a unique uniform field $\mathbf{B}_2 = \mu_1\mathbf{u} + \mu_2\mathbf{v}$ in T_2. Then $\mathbf{b}_f = \mu_1\, f \cdot \mathbf{u}$, and $\mu_2\tilde{\mathbf{h}}_f = \mu_2\,\tilde{f}_1 \cdot \mathbf{u} + \mu_1\,\tilde{f}_2 \cdot \mathbf{u}$. As f, $\tilde{f}_1$, and $\tilde{f}_2$ are collinear, **u** disappears from the quotient, as before.

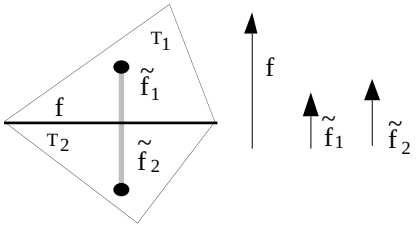

Figure 7. The case of a discontinuous permeability.

WHAT NEXT?

With this realization of the discrete Hodge operator, all elements of the theory have now a discrete counterpart, hence a "discrete Maxwell house". But is it safe to inhabit? This is the question of convergence. Before that, however, the main practical concern is, "can orthogonal meshes easily be produced?", and the answer, unfortunately, is "no, not always". So there is a need for alternatives. We'll see that the Galerkin method, which implies the use of finite elements as ways to reconstruct fields from DoF arrays, offers one. Finite elements will also be instrumental as regards convergence.

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(3): Convergence

Where we stand

Trying to discretize the Maxwell equations by using a pair of interlocked meshes, we have obtained a system of ordinary differential equations

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0, \quad -\partial_t \tilde{\star}_\epsilon \mathbf{e} + \mathbf{R}^t \tilde{\star}_\mu^{-1} \mathbf{b} = \tilde{\mathbf{j}}. \tag{1}$$

There, **e** and **b** are arrays of degrees of freedom, meant to approximate the emf's and magnetic fluxes associated with edges and facets of the primal mesh, and **R** is the edge-to-facets incidence matrix. Intensities through dual facets, which compose the array $\tilde{\mathbf{j}}$, are supposed to be known, as functions of time. Symbols $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ denote square symmetric positive-definite matrices which encode the constitutive laws in "discrete" form.

We chose such strange-looking symbols in order to emphasize the connection with the Hodge operator $\tilde{\star}$ of differential geometry, which appears in the "continuous" form of the constitutive laws,

$$b = \mu \tilde{\star} \tilde{h}, \qquad \tilde{d} = \epsilon \tilde{\star} e.$$

We shall rewrite this as $b = \tilde{\star}_\mu \tilde{h}$ and $\tilde{d} = \tilde{\star}_\epsilon e$ from now on, to stress the parallel between the compound operators $\tilde{\star}_\mu = \mu\tilde{\star}$ and $\tilde{\star}_\epsilon = \epsilon\tilde{\star}$ and the "discrete Hodges" $\tilde{\star}_\mu$ and $\tilde{\star}_\epsilon$.

The form (1) of the discrete equations was practically forced on us—as soon as we decided to use meshes in duality, that is. But what $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ should be, in detail, was left to our choice. We found several criteria about what constitutes a "good discrete Hodge". In the case of mutually orthogonal meshes (the "orthogonal construction" of Part 1, cf. *JSAEM, 7,* 2, 1999, pp. 150-9), a likely candidate was identified: a diagonal discrete Hodge, which in the case of $\tilde{\star}_\mu$ had the following diagonal entries (indexed over facets f of the primal mesh):

$$\tilde{\star}_\mu^{ff} = \mu \frac{\text{area}(f)}{\text{length}(\tilde{f})}, \tag{2}$$

where $\tilde{f}$ is the edge dual to f in the dual mesh (Fig. 1). Similarly, $\tilde{\star}_\epsilon^{ee} = \epsilon\,\text{area}(\tilde{e})/\text{length}(e)$. Let's immediately emphasize that there are other ways to build a discrete Hodge, and we'll consider a few of them later. But this one, being particularly simple, should be a good test of the validity of the whole approach: Can we prove that, when the meshes are refined in some well-specified way, the solution of (1) converges, in some reasonable sense, towards the solution of the Maxwell equations? This is the subject of the present installment.

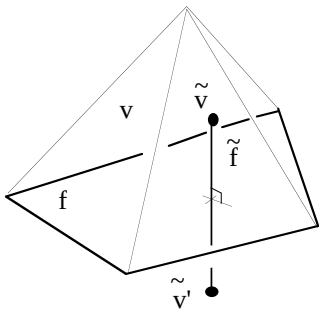

Figure 1. A facet f and its dual edge $\tilde{f}$ in the orthogonal construction ($\tilde{v}$ and $\tilde{v}'$ are the dual nodes which lie inside the volumes v and v' just above and just below f). From $\tilde{v}$, all boundary facets of v can directly be seen at right angle, but we don't require more: $\tilde{v}$ is neither v's barycenter nor the center of its circumscribed sphere, if there is such a sphere.

3.1 The static case

Obviously, the discrete Hodge of (2) would have no virtue if it didn't work satisfactorily in *static* situations. So let's begin with that, which will take the major part of this paper. Then we shall briefly return to the full Maxwell system.

Magnetostatics, in the context of our original model problem, is this: Given a time-independent current density **J** in the bounded

domain of Fig. 2, find **B** and **H** such that

$$(3') \qquad \begin{array}{c} \mathrm{div}\,\mathbf{B} = 0,\ \mathbf{B} = \mu\,\mathbf{H},\ \mathrm{rot}\,\mathbf{H} = \mathbf{J}, \\ \nu \cdot \mathbf{B} = 0 \text{ on } S^e,\ \nu \times \mathbf{H} = 0 \text{ on } S^h, \end{array}$$

which our geometric language of differential forms expresses as

$$(3) \qquad \begin{array}{c} \mathrm{d}b = 0,\ b = \tilde{\star}_\mu \tilde{h},\ \mathrm{d}\tilde{h} = \tilde{\jmath}, \\ tb = 0 \text{ on } S^e,\ t\tilde{h} = 0 \text{ on } S^h. \end{array}$$

The corresponding network equations are

$$(4) \qquad \mathbf{Db} = 0,\ \mathbf{b} = \tilde{\star}_\mu \tilde{\mathbf{h}},\ \mathbf{R}^t \tilde{\mathbf{h}} = \tilde{\mathbf{j}},$$

with $\mathbf{b} = \{\mathbf{b}_f : f \in \mathcal{F}\}$ and $\tilde{\mathbf{h}} = \{\tilde{\mathbf{h}}_f : f \in \mathcal{F}\}$ indexed over the set $\mathcal{F}$ of "active" facets, i.e., all of them except those lying in S^e. We might establish (4) from first priciples, as we did for the network equations (1). But since we have these already, let's rather derive (4) from (1), as the steady-state equations for infinite t, under the hypothesis that $\tilde{\mathbf{j}}$ does not depend on time. For this, we let the facets-to-volumes incidence matrix **D** act on the first eq. (1): since **DR** = 0, $\partial_t(\mathbf{Db}) = 0$, hence **Db** = 0 at all times, since **b** was null at time zero.

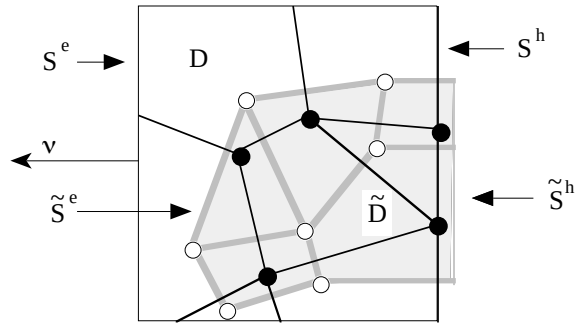

Figure 2 (to be imagined in dimension 3). A reminder of our notations: Domain D, bounded by S, stands for the left half of the cavity. D is paved by the primal cells. The dual cells pave a slightly different domain $\tilde{D}$. Dual $(n-p)$-cells and primal p-cells are orthogonal, two by two. Primal cells in the electric boundary S^e, as well as dual cells in the magnetic boundary $\tilde{S}^h$, are discarded, because degrees of freedom they would bear are a priori zero. (The primal and dual pavings are "closed modulo" S^e and $\tilde{S}^h$, cf. *JSAEM, 7*, 2, 1999, p. 155.) The ν of (3′) is the field of outgoing normals.

Our first concern is whether (4) determines a unique pair $\{\mathbf{b}, \tilde{\mathbf{h}}\}$. Let's denote by N, E, F, V the numbers of active primal nodes, edges, facets, and volumes. (Again, active nodes and edges are those not in S^e.) In (4), we count $2F$ unknowns, the components of **b** and $\tilde{\mathbf{h}}$, and $V + E + F$ equations, for **D** has V rows (one per volume), **R** has E columns, indexed over the set $\mathcal{E}$ of active edges, and $\mathbf{b} = \tilde{\star}_\mu \tilde{\mathbf{h}}$ provides F equations. By a basic result of topology, the *Euler–Poincaré formula*, we know that, whichever way the primal mesh was constructed,

$$(5) \qquad N - E + F - V = \chi,$$

where χ is a constant (equal to 0 in the case we consider) which only depends on the topology of D, S^e, and S^h. This leaves us with $V + E + F \equiv N + 2F$ equations for $2F$ unknowns. So it's not so obvious that (4) has a unique solution.

However, there are precisely N constraints on the data $\tilde{\mathbf{j}}$, owing to current conservation: For each dual volume, that is, for each active primal node n, currents entering this volume should cancel out, hence N relations on the $\tilde{\mathbf{j}}_e$s, of the form $\sum_{e \in \mathcal{E}} \mathbf{G}_{en}\, \tilde{\mathbf{j}}_e = 0$, where **G** denotes the (primal, and active) nodes-to-edges incidence matrix. They can simultaneously be expressed as $\mathbf{G}^t \tilde{\mathbf{j}} = 0$. (We could as well have derived this necessary condition from the last eq. (4), since $\mathbf{G}^t \mathbf{R}^t = 0$, by transposing the combinatorial relation **RG** = 0.)

So by the removal of N redundant data, and of the corresponding equations, we could fall back on a square system, which we should still prove regular. This can be done, but the following indirect approach will be more instructive: We shall construct a linear system *equivalent* to (4), the regularity of which will be obvious.

Let's recall that, with the simple topology we assume here, not only **RG** = 0 and **DR** = 0, but the kernels ker(**R**) and ker(**D**) coincide with the ranges of **G** and **R**. By transposition, ker($\mathbf{G}^t$) is the range of $\mathbf{R}^t$, and ker($\mathbf{R}^t$) is the range of $\mathbf{D}^t$. So if $\mathbf{G}^t \tilde{\mathbf{j}} = 0$, there exists an $\mathcal{F}$-indexed array $\tilde{\mathbf{h}}^{\mathbf{j}}$ such that $\mathbf{R}^t \tilde{\mathbf{h}}^{\mathbf{j}} = \tilde{\mathbf{j}}$. (It's not unique, and need not be explicitly constructed, though that would be a trivial task. That there be one is enough for our purpose.) Now, $\mathbf{R}^t(\tilde{\mathbf{h}} - \tilde{\mathbf{h}}^{\mathbf{j}}) = 0$, so there is a DoF-array $\tilde{\varphi}$, indexed over volumes, such that $\tilde{\mathbf{h}} = \tilde{\mathbf{h}}^{\mathbf{j}} + \mathbf{D}^t \tilde{\varphi}$, and (4) reduces to

$$(6) \qquad \mathbf{D} \tilde{\star}_\mu \mathbf{D}^t \tilde{\varphi} = -\mathbf{D} \tilde{\star}_\mu \tilde{\mathbf{h}}^{\mathbf{j}}.$$

Now this is a square symmetric linear system, with respect to $\tilde{\varphi}$, with a regular matrix,

because $\tilde{\star}_\mu$ is regular on the one hand, and (this is the non-obvious part) $\ker(\mathbf{D}^t) = \{0\}$ on the other hand. Indeed, $\mathbf{D}^t\tilde{\psi} = 0$ means that $\sum_v \mathbf{D}_{vf}\tilde{\psi}_v = 0$ for all primal facets f. But for each such f, there are at most two incident volumes v and v', one on each side of f, and their incidence numbers $\mathbf{D}_{vf}$ and $\mathbf{D}_{v'f}$ have opposite signs. Therefore, $\tilde{\psi}_v = \tilde{\psi}_{v'}$, and $\mathbf{D}^t\tilde{\psi} = 0$ implies that all components of $\tilde{\psi}$ are equal, as soon as the paved domain is connected. Moreover (and now, this is a characteristic of the present situation, where $\tilde{S}^h$ is not empty, not an always valid property), there are facets f with only one adjacent volume (Fig. 2), hence this common value must be zero for all $\tilde{\psi}_v$s. So $\tilde{\varphi} = 0$ in (6) if $\tilde{\mathbf{h}}^{\mathbf{j}} = 0$.

Equation (6) thus appears as a way to solve (4), with guaranteed existence and uniqueness: having $\tilde{\varphi}$, we set $\tilde{\mathbf{h}} = \tilde{\mathbf{h}}^{\mathbf{j}} + \mathbf{D}^t\tilde{\varphi}$, and $\mathbf{b} = \tilde{\star}_\mu\tilde{\mathbf{h}}$. This is known as the *finite volume* approach to magnetostatics, with one degree of freedom per volume of the (primal) mesh, which one may of course interpret as the value of a magnetic potential at the dual node. Many researchers have analyzed the convergence of (6), for various mesh designs and various choices of $\tilde{\star}_\mu$. (See, e.g., [2, 8, 10, 11, 14, 19].) Why then not rely on their results? Because system (4), with its symmetrical and balanced treatment of $\mathbf{b}$ and $\tilde{\mathbf{h}}$, will lend itself to a much simpler error analysis than (6), and one which does the job for several apparently distinct formulations, in one stroke.

For there are other systems equivalent to (4), that we shall indicate before carrying on, by following up on this symmetry idea. Since $\ker(\mathbf{D})$ is the range of $\mathbf{R}$, one may look for $\mathbf{b}$, which has to be in it, in the form $\mathbf{b} = \mathbf{R}\mathbf{a}$, where the DoF-array $\mathbf{a}$ is indexed over $\mathcal{E}$. Then (4) is equivalent to the following linear system, in terms of $\mathbf{a}$,

$$\mathbf{R}^t\tilde{\star}_\mu^{-1}\mathbf{R}\mathbf{a} = \tilde{\mathbf{j}}. \tag{7}$$

No uniqueness, this time,[1] because $\ker(\mathbf{R})$ does not reduce to 0, but there are solutions, thanks to the condition $\mathbf{G}^t\tilde{\mathbf{j}} = 0$, which guarantees that $\tilde{\mathbf{j}} = 0$ lies in the range of $\mathbf{R}^t$, and $\mathbf{b} = \mathbf{R}\mathbf{a}$ is the same for all these solutions. So solving (7), thus getting a unique $\mathbf{b}$, and setting $\tilde{\mathbf{h}} = \tilde{\star}_\mu^{-1}\mathbf{b}$, is equivalent to solving (4).

This is not all. If we refrain to eliminate $\tilde{\mathbf{h}}$ in the reduction of (4) to (7), but still use $\mathbf{b} = \mathbf{R}\mathbf{a}$, we get an intermediate two-equation system,

$$\begin{pmatrix} -\tilde{\star}_\mu & \mathbf{R} \\ \mathbf{R}^t & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{h}} \\ \mathbf{a} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\mathbf{j}} \end{pmatrix}, \tag{8}$$

often called a *mixed* algebraic system. The same manipulation in the other direction (eliminating $\tilde{\mathbf{h}}$ by $\tilde{\mathbf{h}} = \tilde{\mathbf{h}}^{\mathbf{j}} + \mathbf{D}^t\tilde{\varphi}$, but keeping $\mathbf{b}$) gives

$$\begin{pmatrix} -\tilde{\star}_\mu^{-1} & \mathbf{D}^t \\ \mathbf{D} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \tilde{\varphi} \end{pmatrix} = \begin{pmatrix} -\tilde{\mathbf{h}}^{\mathbf{j}} \\ 0 \end{pmatrix}. \tag{9}$$

Systems (6), (7), (8), and (9) differ in size and in sparsity, but give the same solution pair $\{\mathbf{b}, \tilde{\mathbf{h}}\}$, so which one effectively to solve is a matter of algorithmics that need not concern us here.[2] The important point is, the error analysis we shall perform applies to all of them.

3.2 Consistency

A notational point, before going further. Last time, we used $\mathcal{M}$ to denote the primal mesh. We shall subscript by $\mathcal{M}$, when necessary, all mesh-related entities. For instance, the largest diameter of all cells, primal and dual, will be denoted $\gamma_\mathcal{M}$ (with a mild abuse, since it also depends on the metric of the dual mesh, $\widetilde{\mathcal{M}}$), and called the "grain" of the pair of meshes. The computed solution $\{\mathbf{b}, \tilde{\mathbf{h}}\}$ will be $\{\mathbf{b}_\mathcal{M}, \tilde{\mathbf{h}}_\mathcal{M}\}$ when we wish to mark its dependence on the mesh-pair. And so on.

Our purpose can informally be stated as "study $\{\mathbf{b}_\mathcal{M}, \tilde{\mathbf{h}}_\mathcal{M}\}$ when $\gamma_\mathcal{M}$ tends to 0". Alas, this lacks definiteness, because how the *shape*

[1] Whether to "gauge" $\mathbf{a}$ in this method, that is, to impose a condition such as $\mathbf{G}^t\mathbf{a} = 0$ that would select a unique solution, remains to these days a contentious issue. It depends on which method is used to solve (7), and on how well the necessary condition $\mathbf{G}^t\tilde{\mathbf{j}} = 0$ is implemented. With iterative methods such as the conjugate gradient and its variants, and if one takes care to set up an array $\tilde{\mathbf{h}}^{\mathbf{j}}$ such that $\mathbf{R}^t\tilde{\mathbf{h}}^{\mathbf{j}} = \tilde{\mathbf{j}}$, and to use $\mathbf{R}^t\tilde{\mathbf{h}}^{\mathbf{j}}$ instead of $\tilde{\mathbf{j}}$ in (7), then it's better *not* to gauge. See [18].

[2] Assigning b to *dual* facets and $\tilde{h}$ to *primal* edges would generate a similar family of equivalent systems, but not equivalent to (6)–(9), thus yielding *complementary* information. See [5], Chap. 6.

of the cells changes in the process matters a lot. In the case of triangular 2D meshes, for instance, there are well-known counter-examples [1] showing that, if one tolerates too much "flattening" of the triangles as the grain tends to 0, convergence may not occur. Hence the following definition: A family $\mathcal{M}$ of (pairs of interlocked) meshes is *uniform* if there is a *finite* catalogue of "model cells" such that any cell in any $\mathcal{M}$ or $\widetilde{\mathcal{M}}$ of the family is similar to one of them. The notation "$\mathcal{M} \to 0$" will then refer to a sequence of meshes, all belonging to some definite uniform family, and such that their $\gamma_{\mathcal{M}}$s tend to zero. Now we redefine our objective: Show that the error, whatever one means by that, incurred by taking $\{\mathbf{b}_{\mathcal{M}}, \tilde{\mathbf{h}}_{\mathcal{M}}\}$ as a substitute for the real field $\{b, \tilde{h}\}$, tends to zero when $\mathcal{M} \to 0$.

Practical implications of achieving this are as follows. If, for a given $\mathcal{M}$, the computed solution $\{\mathbf{b}_{\mathcal{M}}, \tilde{\mathbf{h}}_{\mathcal{M}}\}$ is not deemed satisfactory, one must *refine* the mesh and redo the computation, again and again. If the refinement rule guarantees that all meshes such a process can generate will belong to some definite uniform family, then the convergence result means "you may get as good an approximation as you wish by refining this way", a state of affairs we are more or less happy to live with.[3]

Fortunately, such refinement rules do exist (this is an active area of research [3, 4, 7, 15]). Given a pair of coarse meshes to start with, there are ways to subdivide the cells so as to keep bounded the number of different cell-shapes that appear in the process, hence a potential infinity of refined meshes, which do constitute a uniform family. (A refinement process for tetrahedra is illustrated by Fig. 3. As one can see, at most five different shapes can occur, for each tetrahedral shape present in the original coarse mesh.)

[3] Effective error *bounds*, for a given $\mathcal{M}$, would of course be more satisfying. Such bounds can be obtained with the complementarity methods alluded to in Note 2.

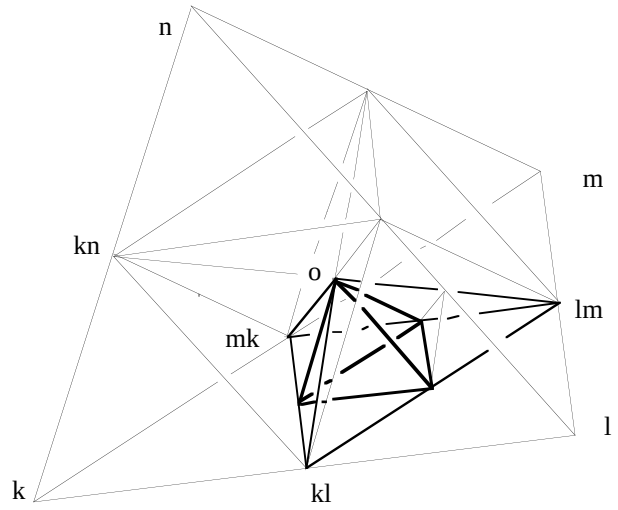

Figure 3. Subdivision rule for a tetrahedron T = $\{k, l, m, n\}$. (Mid-edges are denoted kl, lm, etc., and o is the barycenter.) A first halving of edges generates four small tetrahedra and a core octahedron, which itself can be divided into eight "octants" such as O = $\{o, kl, lm, mk\}$, of at most four different shapes. Now, octants like O should be subdivided as follows: divide the facet in front of o into four triangles, and join to o, hence a tetrahedron similar to T, and three peripheral tetrahedra. These, in turn, are halved, as shown for the one hanging from edge $\{o, lm\}$. Its two parts are similar to O and to the neighbor octant $\{o, kn, kl, mk\}$ respectively.

Back to the comparison between $\{\mathbf{b}_{\mathcal{M}}, \tilde{\mathbf{h}}_{\mathcal{M}}\}$ and $\{b, \tilde{h}\}$, a natural idea is to compare the computed DoF arrays, $\mathbf{b}_{\mathcal{M}}$ and $\tilde{\mathbf{h}}_{\mathcal{M}}$, with arrays of the same kind, $r_{\mathcal{M}} b = \{\int_f b : f \in \mathcal{F}\}$ and $r_{\mathcal{M}} \tilde{h} = \{\int_{\tilde{f}} \tilde{h} : f \in \mathcal{F}\}$, composed of the fluxes and mmf's of the (unknown) solution of (3). This implicitly defines two operators with the same name, $r_{\mathcal{M}}$: one that acts on 2-forms, giving an array of facet-fluxes, one that acts on twisted 1-forms, giving an array of dual-edge mmf's. (No risk of confusion, since the name of the operand, b or $\tilde{h}$, reveals its nature.)

Since $\mathrm{d}b = 0$, the flux of b across the boundary of any primal 3-cell v must vanish, hence the sum of facet fluxes $\sum_f \mathbf{D}_{vf} \int_f b$ must vanish for all v. Similarly, $\mathrm{d}\tilde{h} = \tilde{\jmath}$ yields the relation $\sum_f \mathbf{R}_{fe} \int_{\tilde{f}} \tilde{h} = \int_{\tilde{e}} \tilde{\jmath}$, by integration over a dual 2-cell. In matrix form, all this becomes

$$\mathbf{D} r_{\mathcal{M}} b = 0, \quad \mathbf{R}^t r_{\mathcal{M}} \tilde{h} = \tilde{\mathbf{j}}, \tag{10}$$

since the entries of $\tilde{\mathbf{j}}$ are precisely the intensities across the dual facets. Comparing with (4), we obtain

$$\mathbf{D}(\mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b) = 0, \quad \mathbf{R}^t(\tilde{\mathbf{h}}_{\mathcal{M}} - r_{\mathcal{M}} \tilde{h}) = 0, \tag{11}$$

and

$$(\mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b) - \tilde{\star}_\mu(\tilde{\mathbf{h}}_{\mathcal{M}} - r_{\mathcal{M}} \tilde{h}) = (\tilde{\star}_\mu r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\mu)\tilde{h} \equiv \tilde{\star}_\mu(r_{\mathcal{M}} \tilde{\star}_\nu - \tilde{\star}_\nu r_{\mathcal{M}})b. \tag{12}$$

(Here, of course, ν and $\tilde{\star}_\nu$ stand for the inverses of μ and $\tilde{\star}_\mu$, and $\tilde{\star}_\nu$ for $\tilde{\star}_\mu^{-1}$. We have no more use for ν as a normal vector, so no confusion should ensue.)

Now some algebra, which requires further notation. Last time, we chose to denote by $\langle \mathbf{b}, \tilde{\mathbf{h}} \rangle$ a sum such as $\sum_{f \in \mathcal{F}} \mathbf{b}_f \tilde{\mathbf{h}}_f$. We shall use the shorthands $|\tilde{\mathbf{h}}|_\mu$ and $|\mathbf{b}|_\nu$ for the square roots of the quantities $\langle \tilde{\star}_\mu \tilde{\mathbf{h}}, \tilde{\mathbf{h}} \rangle$ and $\langle \mathbf{b}, \tilde{\star}_\nu \mathbf{b} \rangle$, and call them the "$\mu$-norm" and "$\nu$-norm" of these DoF arrays. (Notice their connection with what we called last time "discrete energy".) We want to compute the μ-norm of both sides of (12).

Doing this, "square" and "rectangle" terms appear, as usual. The rectangle term for the left-hand side is $2\langle \mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b, \tilde{\mathbf{h}}_{\mathcal{M}} - r_{\mathcal{M}} \tilde{h} \rangle$, but since $\mathbf{D}(\mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b) = 0$ implies the existence of some $\mathbf{a}$ such that $\mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b = \mathbf{R}\mathbf{a}$, we have

$$\langle \mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b, \tilde{\mathbf{h}}_{\mathcal{M}} - r_{\mathcal{M}} \tilde{h} \rangle = \langle \mathbf{R}\mathbf{a}, \tilde{\mathbf{h}}_{\mathcal{M}} - r_{\mathcal{M}} \tilde{h} \rangle = \langle \mathbf{a}, \mathbf{R}^t(\tilde{\mathbf{h}}_{\mathcal{M}} - r_{\mathcal{M}} \tilde{h}) \rangle = 0,$$

after (11), by the same transposition trick as last time. Only square terms remain, and we get

$$|\mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b|_\nu^2 + \|\tilde{\mathbf{h}}_{\mathcal{M}} - r_{\mathcal{M}} h|_\mu^2 = \|(\tilde{\star}_\nu r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\nu)b\|_\mu^2 \equiv |(\tilde{\star}_\mu r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\mu)\tilde{h}\|_\nu^2, \tag{13}$$

which will be the cornerstone of the convergence proof.

So at last we have found a plausible measure for what we called earlier "the error incurred by taking $\mathbf{b}_{\mathcal{M}}$ as a substitute for the real field b": the μ-norm of $\mathbf{b}_{\mathcal{M}} - r_{\mathcal{M}} b$. Components of this array are what can be called the "residual fluxes" $\mathbf{b}_f - \int_f b$, i.e., the difference between the computed flux across face f and the genuine (but unknown) flux $\int_f b$. It makes sense to try and *bound* this norm. (Parallel considerations apply to $\tilde{h}$, with mmf's along $\tilde{f}$ instead of fluxes.) So let's focus on the right-hand side of (13), for instance on its second expression, in terms of $\tilde{h}$.

By definition of $r_{\mathcal{M}}$, the f-component of $r_{\mathcal{M}} \tilde{\star}_\mu \tilde{h}$ is the flux of $b = \tilde{\star}_\mu \tilde{h}$ across f. On the other hand, the flux-array $\tilde{\star}_\mu r_{\mathcal{M}} \tilde{h}$ is the result of applying the discrete Hodge operator to the mmf-array $r_{\mathcal{M}} \tilde{h}$, so the compound operators $r_{\mathcal{M}} \tilde{\star}_\mu$ and $\tilde{\star}_\mu r_{\mathcal{M}}$ will not be equal: they give different fluxes when applied to a generic $\tilde{h}$. This contrasts with the equalities $(\mathbf{D} r_{\mathcal{M}} - r_{\mathcal{M}} \mathrm{d})b = 0$ and $(\mathbf{R}^t r_{\mathcal{M}} - r_{\mathcal{M}} \mathrm{d})\tilde{h} = 0$, which stem from the Stokes theorem. The mathematical word to express such equalities is "conjugacy": $\mathbf{D}$ and d are conjugate via $r_{\mathcal{M}}$, and so are $\mathbf{R}^t$ and d, too.

Thus, $\tilde{\star}_\mu$ and $\tilde{\star}_\mu$ are *not* conjugate via $r_{\mathcal{M}}$—and this is, of course, the reason why discretizing entails some error. Yet, in the case we are examining (the diagonal Hodge defined by (2)), $r_{\mathcal{M}} \tilde{\star}_\mu$ and $\tilde{\star}_\mu r_{\mathcal{M}}$ *do* coincide for *some* $\tilde{h}$s, those that have piecewise constant vector proxies, since this is how formula (2) was motivated. Since all smooth fields look constant at a small enough scale, we may expect "asymptotic conjugacy", in the sense that the right-hand side of (13) will tend to 0 with $\mathcal{M}$, for a smooth b or $\tilde{h}$. This property, which we rewrite informally but suggestively as

$$\begin{aligned} &\tilde{\star}_\nu r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\nu \to 0 \text{ when } \mathcal{M} \to 0, \\ &\tilde{\star}_\mu r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\mu \to 0 \text{ when } \mathcal{M} \to 0 \end{aligned} \tag{14}$$

(two equivalent statements), is called *consistency* of an approximation scheme in Numerical Analysis (approximation of $\tilde{\star}_\mu$ and $\tilde{\star}_\nu$ by $\tilde{\star}_\mu$ and $\tilde{\star}_\nu$, here). To prove it, we need to estimate the right-hand side of (13).

This can be done by estimating the contribution of a single facet f, that is

$$\tilde{\star}_\nu^{ff}(\tilde{\star}_\mu^{ff} \int_{\tilde{f}} \tilde{h} - \int_f \tilde{\star}_\mu \tilde{h})^2, \tag{15}$$

and we may even pretend that $\tilde{f}$ is entirely in the volume v to do so (cf. Fig. 1), since there are two parts in the contribution of f, one for each adjacent volume. Then we may assume a constant μ inside v, and work in terms of the vector proxies $\mathbf{H}$ and $\mathbf{B} = \mu\mathbf{H}$. Now $P \equiv \int_{\tilde{f}} \tilde{h}$ is the circulation of $\mathbf{H}$ along $\tilde{f}$ and $Q \equiv \int_f \tilde{\star}_\mu \tilde{h}$ is μ times the flux of $\mathbf{H}$ across f. Since (15) vanishes for a constant $\mathbf{H}$ we may, suppose that $\mathbf{H} = 0$ vanishes at the intersection $f \cap \tilde{f}$ (Fig. 1), and select this point as origin. Then $\mathbf{H}(x)$ is bounded by $C\gamma_{\mathcal{M}}$ over v, where C is a constant which depends on $\mathbf{H}$, but not

on the mesh. (All such constants, whatever their value, will uniformly be denoted by C from now on.) A very crude[4] bound for Q is then $C\mu\gamma_{\mathcal{M}}$area(f). Similarly, P is bounded by $C\gamma_{\mathcal{M}}$length($\tilde{f}$), and hence $|\tilde{\star}_\mu^{ff}P - Q| \leq C\mu\gamma_{\mathcal{M}}$ area(f), after (2). Now, to obtain the desired estimate, square this, divide by $\tilde{\star}_\mu^{ff}$, which gives C length($\tilde{f}$) area(f)$\gamma_{\mathcal{M}}^2$, and finally, sum over f, hence $C\gamma_{\mathcal{M}}^2$ volume(D) as bound for the right-hand side of (13). This proves (14).

Going back to (13), we conclude that both the ν-norm of the residual flux-array and the μ-norm of the residual mmf-array tend to 0 as fast as $\gamma_{\mathcal{M}}$.

3.3 Stability

Although this is considered by many as sufficient in practice, we can't be satisfied with such "discrete energy" estimates. To really prove convergence, one should build from the DoF-arrays $\mathbf{b}_{\mathcal{M}}$ and $\tilde{\mathbf{h}}_{\mathcal{M}}$ an approximation $\{b_{\mathcal{M}}, \tilde{h}_{\mathcal{M}}\}$ of the pair of differential forms $\{b, \tilde{h}\}$, and prove that both the magnetic energy of the discrepancy $b_{\mathcal{M}} - b$ and the magnetic coenergy of $\tilde{h}_{\mathcal{M}} - \tilde{h}$ tend to 0 with $\mathcal{M}$.

To deal with such things, let us denote by $|\tilde{h}|_\mu$ and $|b|_\nu$, on the model of the previous $|\tilde{\mathbf{h}}|_\mu$ and $|\mathbf{b}|_\nu$, the square roots of the quantities $\int_D \tilde{\star}_\mu \tilde{h} \wedge \tilde{h}$ and $\int_D b \wedge \tilde{\star}_\nu b$.

So we are after some map, that we shall denote by $p_{\mathcal{M}}$, that would transform a flux-array $\mathbf{b}$ into a 2-form $p_{\mathcal{M}}\mathbf{b}$ and an mmf-array $\tilde{\mathbf{h}}$ into a twisted 1-form $p_{\mathcal{M}}\tilde{\mathbf{h}}$, and a satisfactory result would be that both $|b - p_{\mathcal{M}}\mathbf{b}_{\mathcal{M}}|_\nu$ and $|\tilde{h} - p_{\mathcal{M}}\tilde{\mathbf{h}}_{\mathcal{M}}|_\mu$ tend to 0 with $\mathcal{M}$ (convergence "in energy"). As next paragraph will show, sufficient conditions on $p_{\mathcal{M}}$ to this effect are the obvious consistency conditions:

$$(16)\qquad \begin{array}{l} p_{\mathcal{M}} r_{\mathcal{M}} b \to b, \text{ in energy, when } \mathcal{M} \to 0, \\ p_{\mathcal{M}} r_{\mathcal{M}} \tilde{h} \to \tilde{h}, \text{ in energy, when } \mathcal{M} \to 0, \end{array}$$

and the following inequalities:

$$(17)\qquad \alpha |p_{\mathcal{M}}\mathbf{b}|_\nu \leq |\mathbf{b}|_\nu, \quad \alpha |p_{\mathcal{M}}\tilde{\mathbf{h}}|_\mu \leq |\tilde{\mathbf{h}}|_\mu$$

for all $\mathbf{b}$ and $\tilde{\mathbf{h}}$, where the constant $\alpha > 0$ does not depend on $\mathcal{M}$. Since $|\mathbf{b}|_\nu$ and $|\tilde{\mathbf{h}}|_\mu$ depend on the discrete Hodge, this is a property of the approximation scheme, called *stability*.

Indeed, with both (14)(16) and (17), convergence is straightforward, thanks to (13): First, $p_{\mathcal{M}}(\mathbf{b} - r_{\mathcal{M}} b) \to 0$, by (17), then $p_{\mathcal{M}}\mathbf{b} \to b$, thanks to (16), all that "in energy". Same argument about $\tilde{h}$. This is Lax's celebrated folk theorem: *consistency + stability = convergence*.

So what about $p_{\mathcal{M}}$? Later, we shall find a systematic way to construct it, at least in the case of a tetrahedral primal mesh, the so-called *Whitney map*. If we don't insist right now on generality, there is an easy way to find this map in the case of DoF arrays $\mathbf{b}$ that satisfy $\mathbf{Db} = 0$, and luckily, only these do matter. The idea is to find a vector proxy $\mathbf{B}$ which be uniform inside each tetrahedron and such that its flux across each facet f be equal to $\mathbf{b}_f$. (Then, div$\mathbf{B} = 0$ all over D.) This, which would not be possible with cells of arbitrary shapes, can be done with tetrahedra, for there are, for each tetrahedral volume v, three unknowns (the components of $\mathbf{B}$) to four fluxes linked by one linear relation, $\sum_f \mathbf{D}_{vf}\mathbf{b}_f = 0$, so the problem has a solution. Hence $p_{\mathcal{M}}\mathbf{b}$.

We do have $p_{\mathcal{M}} r_{\mathcal{M}} b \to b$, then. This was proved long ago [9], by an argument which relies on mesh uniformity [17], and is very close to the one we now invoke to establish the stability condition (17). One has $|p_{\mathcal{M}}\mathbf{b}|_\nu^2 = \int_D \mu^{-1}|\mathbf{B}|^2$, which is obviously some quadratic form with respect to the facet fluxes, which we may therefore denote by $\langle \mathbf{b}, \mathbf{Nb} \rangle$, with $\mathbf{N}$ some square regular matrix. Now, suppose first a *single* tetrahedron in the mesh $\mathcal{M}$, and consider the Rayleigh-like quotient $\langle \mathbf{b}, \tilde{\star}_\nu \mathbf{b} \rangle / \langle \mathbf{b}, \mathbf{Nb} \rangle$. Its lower bound, strictly positive, depends only on the *shape* of the tetrahedron, not on its size. Uniformity of the family of meshes, then, allows us to take for α in (17) the smallest of these lower bounds, which is strictly positive and independent of $\mathcal{M}$. We may thereby conclude that $p_{\mathcal{M}}\mathbf{b}_{\mathcal{M}}$ converges towards b in energy.

No similar construction on the side of $\tilde{h}$ is

[4] Any known regularity of the mesh can be exploited, at this level, to obtain sharper bounds. In particular, for the kind of paving that Yee used, or generalizations of it [13], not only the constant part but the *linear* part of $\mathbf{H}$ gives a vanishing contribution. This accounts for a higher order of convergence in FDTD ($\gamma_{\mathcal{M}}^2$ rather than $\gamma_{\mathcal{M}}$, typically [16]) than what we find here.

available, but this is not such a handicap: if $p_{\mathcal{M}}\mathbf{b}_{\mathcal{M}} \to b$, then $\tilde{\star}_\nu p_{\mathcal{M}}\mathbf{b}_{\mathcal{M}} \to \tilde{h}$. This amounts to setting $p_{\mathcal{M}}$ on the dual side equal to $\tilde{\star}_\nu p_{\mathcal{M}}\tilde{\star}_\mu$. The problem with that is, $p_{\mathcal{M}}\tilde{\mathbf{h}}_{\mathcal{M}}$ fails to have the continuity properties we expect from a magnetic field: its vector proxy **H** is not tangentially continuous across facets, so one cannot take its curl. (One says of such a $p_{\mathcal{M}}$ that it constitutes a "non-conformal" approximation.) But never mind: In the case of a tetrahedral primal mesh, we have succeeded in proving the convergence in energy of $b_{\mathcal{M}}$ and $\tilde{h}_{\mathcal{M}}$ to b and $\tilde{h}$, which was our objective. And no Sobolev space has been invoked!

3.4 The dynamic case

Let us finish with a sketch of the convergence proof for the generalized Yee scheme of last issue.

First, linear interpolation in time between the values of the DoF arrays, as output by the Yee scheme, provides DoF-array-valued functions of time which converge, when δt tends to zero, towards the solution of the "spatially discretized" equations (1). This is not difficult.

Next, linearity of the equations permits to pass from the time domain to the frequency domain, via a Laplace transformation. Instead of studying (1), therefore, we may examine the behavior of the solution of

$$-p\,\tilde{\mathbf{D}} + \mathbf{R}^t\tilde{\mathbf{H}} = \tilde{\mathbf{J}}, \quad p\,\mathbf{B} + \mathbf{R}\mathbf{E} = 0, \tag{18}$$

$$\tilde{\mathbf{D}} = \tilde{\star}_\epsilon \mathbf{E}, \quad \mathbf{B} = \tilde{\star}_\mu \tilde{\mathbf{H}}, \tag{19}$$

when $\mathcal{M} \to 0$. Here, $p = \xi + \mathrm{i}\omega$, with $\xi > 0$, and small capitals denote Laplace transforms, which are arrays of *complex*-valued DoFs. If one can prove uniform convergence with respect to ω (which the requirement $\xi > 0$ makes possible), convergence of the solution of (1) will ensue, by inverse Laplace transformation. The main problem, therefore, is to compare E, B, $\tilde{\mathrm{H}}$, $\tilde{\mathrm{D}}$, as given by (18)(19), with $r_{\mathcal{M}}\mathrm{E}, r_{\mathcal{M}}\mathrm{B}, r_{\mathcal{M}}\tilde{\mathrm{H}}, r_{\mathcal{M}}\tilde{\mathrm{D}}$, where small capitals, again, denote Laplace transforms, but of differential forms this time.

The approach is similar to what we did in statics. First establish that

$$\begin{aligned} p\tilde{\star}_\mu(\tilde{\mathbf{H}} - r_{\mathcal{M}}\tilde{\mathrm{H}}) + \mathbf{R}(\mathbf{E} - r_{\mathcal{M}}\mathrm{E}) \\ = p(r_{\mathcal{M}}\tilde{\star}_\mu - \tilde{\star}_\mu r_{\mathcal{M}})\tilde{\mathrm{H}}, \end{aligned} \tag{20}$$

$$\begin{aligned} -p\tilde{\star}_\epsilon(\mathbf{E} - r_{\mathcal{M}}\mathrm{E}) + \mathbf{R}^t(\tilde{\mathbf{H}} - r_{\mathcal{M}}\tilde{\mathrm{H}}) \\ = -p(r_{\mathcal{M}}\tilde{\star}_\epsilon - \tilde{\star}_\epsilon r_{\mathcal{M}})\mathrm{E}. \end{aligned} \tag{21}$$

Then, right-multiply (20) by $(\tilde{\mathbf{H}} - r_{\mathcal{M}}\tilde{\mathrm{H}})^*$ and the conjugate of (21) by $-(\mathbf{E} - r_{\mathcal{M}}\mathrm{E})$, add. The middle terms (in $\mathbf{R}$ and $\mathbf{R}^t$) cancel out, and energy estimates follow. The similarity between the right-hand sides of (12), on the one hand, and (20)(21), on the other hand, shows that no further consistency requirements emerge. Stability, thanks to $\xi > 0$, holds there if it held in statics. What is a good Hodge discrete operator in statics, therefore, is a good one in transient situations. We may tentatively promote this remark as a heuristic principle:

As regards discrete constitutive laws, *what makes a convergent scheme for* static *problems will, as a rule, make one for the Maxwell evolution equations as well.*

From the perspective of the present series ("to build a finite-dimensional Maxwell house"), this is noteworthy. The idea was to replace all infinite-dimensional entities by finite-dimensional ones in consistent fashion: differential forms by DoF arrays, operator d by **G**, **R**, **D** and their transposes, depending on the degree, and the μ- and ϵ-related Hodge stars by appropriate square symmetric, positive definite matrices. All that in the hope that simple substitution of such "discrete" objects to "continuous" ones in the equations would generate valid approximation schemes. This working programme has succeeded, to some extent: We have a consistent diagonal discrete Hodge, at least for orthogonal meshes, and a convergence proof for the Yee-like scheme based on it, at least for tetrahedral primal meshes.

But the weak spot in all that is now apparent: We need a systematic way to pass from DoF arrays to differential forms—the $p_{\mathcal{M}}$ operator. Not only to interpolate inside volumes (this we could do without), but as a way to assess stability, in the above sense. Whitney forms, which will now enter the scene, provide this mechanism.

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(4): From degrees of freedom to fields

Where we stand

"Building a finite-dimensional 'Maxwell's house'", so was the subtitle of this series. The goal is now in sight, as we have in hand a systematic discretizing procedure for all equations derived from Maxwell's. Solving the algebraic system that results, we can find time-dependent arrays of degrees of freedom (DoF arrays), which give an approximate picture of the fields by telling us about fluxes through faces, about emf's along edges, etc. What we don't know yet is how to "interpolate" from DoF's, i.e., how to reconstruct fields from computed DoF arrays.

4.1 Why interpolants?

One may wonder, *must* this be done, really? After all, as we argued earlier, all one can hope to measure about the electromagnetic field is, precisely, fluxes and circulations, so what's wrong with a method that would provide only those? First answer: We may want them at a much smaller scale than what the computational mesh provides, hence the necessity of interpolation. Post-processing, in the same spirit, may require the determination of the fields inside mesh cells. But these reasons are not the most compelling ones. As we began to see last time, field reconstruction is needed, basically, to *assess the validity* of the numerical method.

To stress this point anew, let's review the discretization process. Having built a mesh, with its incidence matrices **G**, **R**, and **D**, we assign unknowns to primal or dual cells, according to their geometrical nature (Fig. 1), and replace fields, in the equations, by arrays **e**, **b**, $\tilde{\mathbf{d}}$, $\tilde{\mathbf{h}}$, of such DoF's. We replace the operators rot and div by **R** and **D** (or by $\mathbf{R}^t$ and $\mathbf{G}^t$, according to what Fig. 1 suggests), the time-derivatives by finite differences (at integer or half-integer time-steps), and the material coefficients ϵ and μ by our so-called "discrete Hodge operators", the square matrices denoted $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ in past issues. Hence algebraic equations, that a computer can solve for us.

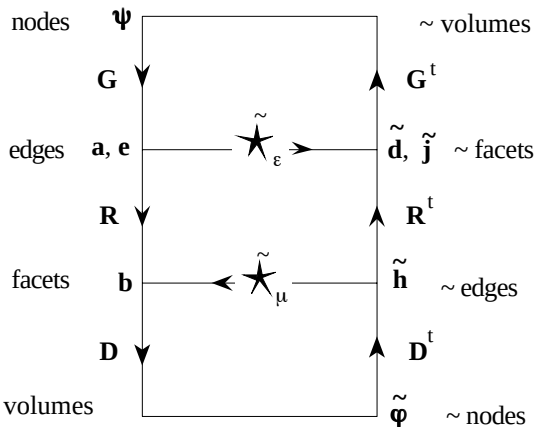

Figure 1. A discretizing machine for Maxwell's equations, showing where degrees of freedom sit ("∼" is short for "dual") and which matrices link the various DoF arrays. (Time discretization, not shown, is done via the replacements $\partial_t \mathbf{b} \simeq (\mathbf{b}^{k+1} - \mathbf{b}^k)/\delta t$ and $\partial_t \tilde{\mathbf{d}} \simeq (\tilde{\mathbf{d}}^{k+1/2} - \tilde{\mathbf{d}}^{k-1/2}/\delta t$.) Compare with Fig. 6, p. 326, *JSAEM, 6,* 4 (1998).

We realized that building these matrices $\tilde{\star}_\epsilon$ and $\tilde{\star}_\mu$ is the central problem in this approach: Assessing their quality mandates a convergence proof. We argued that it was enough to have one in the static case, and we identified two steps in this proof: checking *consistency*, i.e.,

(1) $\qquad \tilde{\star}_\mu r_{\mathtt{M}} - r_{\mathtt{M}} \tilde{\star}_\mu \to 0$ when $\mathtt{M} \to 0$

(and a similar thing about $\tilde{\star}_\epsilon$),[1] and showing *stability*, a property of the approximation

[1] See *JSAEM, 7,* 4 (1999), for the notation. Let's recall that $r_{\mathtt{M}}$ denotes the operator which maps a field to a DoF array of the appropriate kind. For instance, in the case of $\tilde{h}$, it maps to the array $r_{\mathtt{M}}\,\tilde{h}$ of mmf's $\tilde{\mathbf{h}}_f = \int_{\tilde{f}} \tilde{h}$ along dual edges, indexed over the set $\mathcal{F}$ of primal faces. As $b = \tilde{\star}_\mu \tilde{h}$, the array $r_{\mathtt{M}}\,\tilde{\star}_\mu \tilde{h}$ contains the facet-fluxes of b. This is not quite the same as $\tilde{\star}_\mu r_{\mathtt{M}}\,\tilde{h}$, hence $\tilde{\star}_\mu r_{\mathtt{M}} - r_{\mathtt{M}}\,\tilde{\star}_\mu \neq 0$. But we made sure, in defining $\tilde{\star}_\mu$,

scheme which we wrote, in the case of **b**, as

$$\alpha\,|p_{\mathtt{M}}\,\mathbf{b}|_\nu \le |\mathbf{b}|_\nu . \tag{2}$$

On the right, there,[2] the "discrete energy" of the DoF array **b** (up to squaring and halving). On the left, the magnetic energy of the approximation $p_{\mathtt{M}}\,\mathbf{b}$ to b that one can produce, given **b**, thanks to the interpolation operator $p_{\mathtt{M}}$. We had, last time, such an operator ready-made, thanks to the fact that **Db** = 0.[3] But that would not work for volumes with more than four facets. Besides, to go beyond magnetostatics, we need (2) for other fields than b, for which such a lucky accident as this relation **Db** = 0 will not occur.

So, to sum up: For each kind of DoF array, we need an operator, generically denoted by $p_{\mathtt{M}}$, which maps it to a differential form (DF) of the appropriate kind: $p_{\mathtt{M}}\,\mathbf{e}$, starting from an edge-based DoF array **e**, should be a 1-form; $p_{\mathtt{M}}\,\mathbf{b}$, obtained from a facet-based **b**, should be a 2-form, and so forth. We want the emf of $p_{\mathtt{M}}\,\mathbf{e}$ along edge e, that is to say $(r_{\mathtt{M}}\,p_{\mathtt{M}}\,\mathbf{e})_e$, to equal $\mathbf{e}_e$, for each edge, therefore $r_{\mathtt{M}}\,p_{\mathtt{M}} = 1$ should hold. On the other hand, if the array, **u** let's say, comes from a DF u by $\mathbf{u} = r_{\mathtt{M}}\,u$ (i.e., by taking fluxes or circulations of u, as the case may be), we don't expect $p_{\mathtt{M}}\,\mathbf{u}$ to equal u, but the difference should be small in the proper energy norm, and the finer the mesh, the smaller it should become. So $p_{\mathtt{M}}$ should behave as follows in relation with $r_{\mathtt{M}}$:

$$r_{\mathtt{M}}\,p_{\mathtt{M}} = 1, \quad p_{\mathtt{M}}\,r_{\mathtt{M}} \to 1 \quad \text{when } {\mathtt{M}} \to 0. \tag{3}$$

As for the stability property (2), we saw last time that it would automatically be satisfied in the case of a *uniform* family of meshes, defined as one in which only a finite number of cell-shapes can exist, for both the primal and the dual[4] mesh. Our objective, therefore, is to satisfy (3) when building $p_{\mathtt{M}}$ for all possible kinds of DoF arrays.

4.2 Interpolating from nodal values

We know a solution to this problem in the case of (primal-) *node-based* DoF arrays, at least for a simplicial primal mesh.[5] Such arrays correspond to straight DF's of degree 0, i.e., to functions, and interpolating a function from its nodal values is what *finite elements* are about, so we are on familiar ground there. To capitalize on this knowledge, we shall need to look at this kind of interpolation from an unusual angle, however.

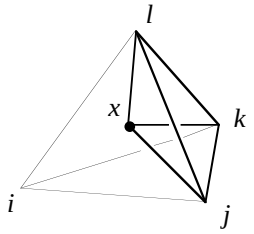

Figure 2. The weight $\lambda^i(x)$ is the relative volume of the tetrahedron $\{x, j, k, l\}$ (cf. (5) below).

Let's first consider a single tetrahedron in 3D space (Fig. 2), with nodes labelled i, j, k, l, located at points x_i, x_j, x_k, x_l (not all of them in the same plane). As one knows, there are four functions, $\lambda^i, \lambda^j, \lambda^k, \lambda^l$, such that

$$\sum_{n\in\{i,j,k,l\}} \lambda^n(x)\,(x - x_n) = 0 \tag{4}$$

for all points x. Known as the *barycentric coordinates* of x, they satisfy the equality $\sum_n \lambda^n = 1$, and have the familiar geometric

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that they *were* the same when b is uniform, a condition to which one tends, locally, when the mesh is refined; hence the ease with which we proved (1) for the specific discrete Hodge operator of last issue, the one with entries μ area(f)/length($\tilde{f}$) on the diagonal. ($\tilde{\star}_\nu$ is the inverse of $\tilde{\star}_\mu$.)

[2] Again, let's recall that $|\mathbf{b}|_\nu$ (with boldface vertical bars), called the "ν-norm" of **b**, is the square root of the energy-related quantity $\langle \mathbf{b}, \tilde{\star}_\nu \mathbf{b}\rangle$, where $\langle \mathbf{b}, \tilde{\mathbf{h}}\rangle$ stands for the sum $\sum_{f\in\mathcal{F}} \mathbf{b}_f \tilde{\mathbf{h}}_f$. The similar quantity $|b|_\nu$ was defined as the square root of $\int_D \mu^{-1}|\mathbf{B}|^2$, where **B** is the vector proxy of b.

[3] We simply took the proxy **B** uniform inside each tetrahedron, by adjusting three of its facet-fluxes to the given $\mathbf{b}_f$s. The fourth DoF is automatically right, because of the linear constraint **Db** = 0 (which also entails div**B** = 0).

[4] This implies that the centers of primal cells, as used in the orthogonal construction, occupy the same relative position within all cells of similar shape. This property *was* used last time (*JSAEM, 7,* 4 (1999), p. 407) to show that the α of (2) is > 0, but I should have put more emphasis on this important point.

[5] The case of hexahedral 3-cells, also well understood, won't be discussed here for the sake of brevity.

interpretation that Fig. 2 recalls:

$$\lambda^i(x) = \frac{\mathrm{vol}(\{x,j,k,l\})}{\mathrm{vol}(\{i,j,k,l\})}. \tag{5}$$

Clearly, at least one $\lambda^n(x)$ is negative for x outside the tetrahedron.

Remark. There is a unique set of "barycentric weights" $\lambda^n(x)$ for which (4) holds, so (4) does define the λ^ns, which are therefore *affine* objects. Formula (5), which needs the extraneous notion of volume to make sense, is not their definition but a consequence. There is no measure of volumes in an affine space, yet *ratios* of volumes are a meaningful affine concept. (Check that the ratio (5) is invariant with respect to a change of metric, i.e., the same for two different dot products.) ♢

Remark. Actually, volume in an affine space can be defined without a dot product in background. It suffices to introduce a reference tetrahedron (a reference d-simplex, in dimension d), which by convention has volume 1. Then, any tetrahedron T can be mapped onto the reference one by some affine transformation, the determinant of which is, by definition, vol(T). (It's a signed number; take its absolute value to get a positive volume; chop bodies of more general shape into tetrahedra to extend the notion of volume to them.) The mapping T $\to$ vol(T) is called "a volume". Possible volumes, in this sense, differ by a multiplicative constant. Note that giving a volume says nothing about areas or lengths: In this respect, the structure "affine space + volume" lies between "(naked) affine space" and "Euclidean space" (affine space + dot product). ♢

Since $\sum_n \lambda^n(x) = 1$ for all x, another way to write (4) can be, with a very mild notational abuse,

$$x = \sum_{n\in\{i,j,k,l\}} \lambda^n(x)\, x_n, \tag{4'}$$

which represents x as the barycenter of the vertices x_n, with weights $\lambda^n(x)$.

Next, if instead of a single tetrahedron we have a tetrahedral paving of some domain D (this will be our primal mesh, M), then a barycentric function λ^n_{T} can be defined for each node n and each tetrahedron T. Set[6]

$$w^n(x) = \max\{\lambda^n_{\mathrm{T}}(x) : \mathrm{T} \in \mathcal{T}\}. \tag{6}$$

This is a piecewise affine function (i.e., affine over each individual T), continuous in x, non-negative all over D (contrary to the λ^n_{T}s), and positive inside the set-union of tetrahedra that have n as one of their nodes. One will have recognized the standard[7] "hat function" of finite element theory.

On the model of (4′), we may write

$$x = \sum_{n\in\mathcal{N}} w^n(x)\, x_n, \tag{7}$$

again a fairly legitimate abuse of notation: (7) makes x appear as a weighted sum of *all* nodes of the mesh, but with all weights null, except for the nodes of the simplex that contains[8] x. This is a key observation, which we shall soon exploit.

Hat functions do serve as interpolants, as one knows: Starting from a node-based DoF array $\{\psi_n : n \in \mathcal{N}\}$, one may set

$$(p_{\mathtt{M}}\,\psi)(x) = \sum_{n\in\mathcal{N}} w^n(x)\boldsymbol{\psi}_n \tag{8}$$

to obtain a piecewise affine function whose value at x_n be $\boldsymbol{\psi}_n$. Hence a $p_{\mathtt{M}}$ for which $r_{\mathtt{M}}\, p_{\mathtt{M}} = 1$ holds. On the other hand, if we start from some continuous function ψ, and set $\boldsymbol{\psi} = r_{\mathtt{M}}\,\psi$, that is, $\boldsymbol{\psi}_n = \psi(x_n)$, we get an approximation $p_{\mathtt{M}}\, r_{\mathtt{M}}\,\psi$ to ψ, with the property that $\int_D |p_{\mathtt{M}}\, r_{\mathtt{M}}\,\psi - \psi|^2 \to 0$ when $\mathtt{M} \to 0$. (No need to prove this well-known fact here. See [2] for a proof in line with the present approach.)

[6] Recall that we denote by $\mathcal{N}, \mathcal{E}, \mathcal{F}, \mathcal{T}$ the sets of nodes, edges, etc., of the primal mesh—*all* of them: the restriction to "active" cells, introduced in previous columns, is irrelevant this time.

[7] Except for the unusual symbol w, that one may understand as a mnemonic for "weight". Later, we'll find another interpretation.

[8] It may be a p-simplex with $p < 3$. Remember that our primal cells are not supposed to contain their boundaries, unless $p = 0$. So if we say that "x belongs to edge $e = \{m,n\}$", for instance, we understand that $w_m(x) > 0$, $w^n(x) > 0$, and $w_i(x) = 0$ for all other nodes.

Well, what have we done there, that could show the way to generalization? Compare (8) to (7). The value we attribute to $p_{\mathtt{M}}\,\psi$ at point x is the weighted sum of the nodal values ψ_n, *the weights being the same as those by which x is expressed as a sum of nodes*. That's our clue: if we were somehow able to express a line as a weighted sum of mesh-edges, a surface as a weighted sum of mesh-facets, etc., we could generate a 1-form from an edge-based DoF array, a 2-form from a facet-based array, and so on.

4.3 Chains

To see how, suppose for a moment that we have (to be specific) a facet-based DoF array **b**, from which we want to build a (straight) 2-form $p_{\mathtt{M}}\,\mathbf{b}$. Straight 2-forms map inner-oriented surfaces to real numbers. So we will know $p_{\mathtt{M}}\,\mathbf{b}$ if we know its integrals $\int_S p_{\mathtt{M}}\,\mathbf{b}$ for all inner-oriented surfaces S. These we do know when S is one of the primal facets (remember they have inner orientation), since then, $\int_f p_{\mathtt{M}}\,\mathbf{b} = \mathbf{b}_f$. (This is what $r_{\mathtt{M}}\,p_{\mathtt{M}} = 1$ means in that case.) Now, suppose we have a sensible way to represent S as a sum of facets:

$$S \simeq \sum_{f\in\mathcal{F}} w^f(S)\,f, \tag{9}$$

with appropriate weights $w^f(S)$ (compare with (7)). Then, owing to the additivity of the integral,

$$\int_S p_{\mathtt{M}}\,\mathbf{b} = \sum_{f\in\mathcal{F}} w^f(S) \int_f p_{\mathtt{M}}\,\mathbf{b} \equiv \sum_{f\in\mathcal{F}} w^f(S)\,\mathbf{b}_f,$$

so it's just natural to set

$$\int_S p_{\mathtt{M}}\,\mathbf{b} = \sum_{f\in\mathcal{F}} w^f(S)\,\mathbf{b}_f. \tag{10}$$

Thus our problem is solved if we can make some sense out of formula (9).

Not an easy task, it seems, for the term on the right in (9) has no obvious meaning, on the face of it (what would be the result of "multiplying a facet by a number", and then, what would it mean to "add" such objects?) This is why we can't use an equal sign, hence the "$\simeq$", meaning "similar to". Yet we can give status to this expression $\sum_{f\in\mathcal{F}} w^f(S)\,f$, by considering it as just another way to denote the array of real values $\{w^f(S) : f \in \mathcal{F}\}$. Such an array (quite alike a DoF array, but conceptually different) is called a (simplicial) 2-*chain*. Needless to say, there are p-chains for all dimensions p of the underlying simplices.[9]

If $\mathbf{c} = \{c^f : f \in \mathcal{F}\}$ is such a 2-chain, writing it as a formal sum, $\mathbf{c} = \sum_{f\in\mathcal{F}} \mathbf{c}^f\,f$, is a convenient device, which makes chain addition, for instance, defined as $\mathbf{c}_1 + \mathbf{c}_2 = \sum_{f\in\mathcal{F}}(\mathbf{c}_1^f + \mathbf{c}_2^f)\,f$, look natural: just follow the rules of algebra. Further evidence of its usefulness will come.

So that's what we have at the right-hand side of (9): a 2-chain. Could S itself considered as a 2-chain in some way?

This is not so unlikely. A single facet f can be viewed as the chain with all weights 0, except $\mathbf{c}^f$, equal to 1. Therefore, an inner-oriented surface which is made of an assembly of facets can be viewed as a chain, too, that we shall call the *associate* chain: its coefficients are ± 1 for each facet of the assembly, with sign + or $-$ depending on whether orientations match or not, and 0 for all other facets (Fig. 3). For brevity, we shall call "M -surfaces" those composed of such assemblies. (There is a similar notion of "M -line", and we'll say "M -manifold" to cover all cases, p = 0 to 3.) If S in (9) is an M -surface, the pseudo equality begins to make sense: on the right, we have the 2-chain associated with S, with weights ± 1 or 0.

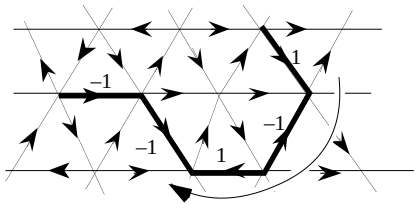

Figure 3. An M -line, oriented, embedded in a 2D-mesh, and the coefficients of its 1-chain associate. (Arrows indicate the orientation of each edge.) Imagine that for p = 2 in dimension 3.

Yet the two concepts don't coincide. Not all surfaces are M -surfaces, and coefficients of an arbitrary chain can take other values than 0 and

[9] Chains make sense for all kinds of cell-pavings, not only the simplicial ones to which we restrict here.

± 1. So it's better to use distinct notation for the $\mathtt{M}$ -surface S and for its associate: We shall denote the latter by $p^t_{\mathtt{M}}\, S$. (Why this symbol will soon be clear.) Now we may rewrite (9) as $p^t_{\mathtt{M}}\, S = \sum_f w^f(S)\, f$, with a legitimate equality, and the task at hand becomes better defined: To any inner-oriented surface S, made of mesh-facets or not, associate a 2-chain $p^t_{\mathtt{M}}\, S$. Then use the coefficients of this chain as weights in (10).

Before facing this task, a last concept relative to chains: boundaries. Taking the boundary of a chain is an algebraic operation, defined in order to correspond, as closely as possible, to taking the boundary of a manifold. The boundary of a p-chain $\mathbf{c}$ is a $(p-1)$-chain $\boldsymbol{\partial}\mathbf{c}$, with the following properties: First, linearity, $\boldsymbol{\partial}(\mathbf{c}_1 + \mathbf{c}_2) = \boldsymbol{\partial}\mathbf{c}_1 + \boldsymbol{\partial}\mathbf{c}_2$. Second, if M is an $\mathtt{M}$ -manifold with boundary ∂M, the boundary of the chain associated with M is the chain associated with the boundary of M ... Oh well, rather use symbols:

$$p^t_{\mathtt{M}}\,(\partial M) = \boldsymbol{\partial}(p^t_{\mathtt{M}}\, M), \tag{11}$$

that is, $p^t_{\mathtt{M}}\,\partial = \boldsymbol{\partial} p^t_{\mathtt{M}}$, another instance of $conjugacy$. Thanks to (11), $\boldsymbol{\partial}$ is known for all chains, by linearity, if it is known for those associated with simplices.

But this information is precisely what incidence matrices convey: For instance, $\boldsymbol{\partial} p^t_{\mathtt{M}}\,(f) = \sum_e \mathbf{R}_{fe}\, e$, where $\mathbf{R}$ is the edge-to-facets incidence matrix. From this, the boundary of a 2-chain $\mathbf{c}$ is seen to be $\mathbf{R}^t\mathbf{c}$. Matrix representations of $\boldsymbol{\partial}_p$, therefore, are $\mathbf{G}^t$, $\mathbf{R}^t$, $\mathbf{D}^t$, for p = 1, 2, 3. Remember that, as regards DoF arrays, we had a generic operator $\mathbf{d}_p$, realized as $\mathbf{G}$, $\mathbf{R}$, $\mathbf{D}$, for p = 0, 1, 2. ($JSAEM$, 7, p. 156.) So $\boldsymbol{\partial}_{p+1} = \mathbf{d}^t_p$. More and more, chains appear as $dual$ objects with respect to what we have called up to now DoF arrays.[10]

4.4 Interpolating from edge values

We know how to associate a 0-chain with a point: that's what formula (7) does, although we should rewrite it, in full rigor, $p^t_{\mathtt{M}}\, x = \sum_{n\in\mathcal{N}} w^n(x)\, x_n$ if the right-hand side is seen as a 0-chain, not as a barycenter. We won't bother with that, however, and drop the $p^t_{\mathtt{M}}$ in what follows.

Next step is p = 1, curves. How on earth can a curve c be expressed as a 1-chain? Easy: Chop it into small parts, replace them by straight segments, assign chains to these, sum all these chains, and go to the limit. So it boils down to being able to associate a 1-chain to any oriented segment, going from point x to point y (we shall write it xy). We may as well (Fig. 4) suppose that x and y belong to the same tetrahedron, $\{i, j, k, l\}$ say. (Otherwise, break xy into subsegments.)

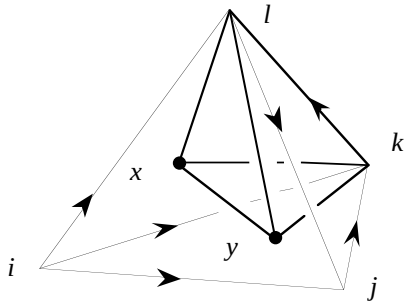

Figure 4. The weight of xy relative to $x_i x_j$ (see (13) below).

We now indulge for a while in a heuristic derivation (for which sloppy notation can be tolerated). Since $x = \sum_n \lambda^n(x)\, x_n$ and $y = \sum_n \lambda^n(y)\, x_n$ (where n spans $\{i, j, k, l\}$), one has, by deliberately confusing oriented segments such as xy with vectors such as $y - x$,

$$y - x = y - \sum_n \lambda^n(x)\, x_n = \sum_n \lambda^n(x)\,(y - x_n)$$

(since $\sum_n \lambda^n(x) = 1$), hence

$$\begin{aligned} xy &= \sum_n \lambda^n(x)\,(x_n - \sum_m \lambda^m(y)\, x_m) \\ &= \sum_n \lambda^n(x) \sum_m \lambda^m(y)\, x_m x_n, \end{aligned} \tag{12}$$

and that's a step forward: we have the segment xy as a weighted sum of segments which coincide with edges, but either with one or the other orientation. So each of the relevant edges (those marked by orientation arrows on Fig. 4) appears twice, with opposite signs, in the above

[10] Indeed, the received name for DoF arrays is "cochains". The duality pairing between a 2-chain $\mathbf{c}$ and a 2-cochain $\mathbf{b}$ yields the number $\sum_{f\in\mathcal{F}} \mathbf{b}_f \mathbf{c}^f$. Eq. (10) says this will coincide with $\int_S p_{\mathtt{M}}\, \mathbf{b}$ when $\mathbf{c} = p^t_{\mathtt{M}}\, S$. Here we see the rationale for the symbol $p^t_{\mathtt{M}}$: in ad hoc notation, $\langle p_{\mathtt{M}}\, \mathbf{b}, S\rangle = \langle \mathbf{b}, p^t_{\mathtt{M}}\, S\rangle$ —a transposition.

sum. Grouping these pairs of terms, we find

$$xy = (\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y))\, x_i x_j + \ldots, \tag{13}$$

where the dots stand for five similar expressions for the other edges—hence our weights, and the 1-chain which represents xy.

After (5), it's a good bet that

$$\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y) = \frac{\text{vol}(\{x, y, k, l\})}{\text{vol}(\{i, j, k, l\})},$$

as suggested by Fig. 4. To check this, it will be convenient to place the origin at x_k, which allows us to use the symbol x for the vector $x - x_k$, and thus to write $x = \lambda^i(x)\, x_i + \lambda^j(x)\, x_i + \lambda^l(x)\, x_l$. Let's also introduce a metric, via a dot product "$\cdot$", and an orientation, which makes a cross product "$\times$" available. The volume of $\{x, y, k, l\}$ is then $1/6(x \times y) \cdot x_l$, i.e. (terms in $\lambda^l x_l$, which would contribute nothing, are left out), one sixth of

$$((\lambda^i(x)\, x_i + \lambda^j(x)\, x_j) \times (\lambda^i(y)\, x_i + \lambda^j(y)\, x_j)) \cdot x_l$$

$$\equiv (\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y))(x_i \times x_j) \cdot x_l,$$

which equals $6\,\text{vol}(\{i, j, k, l\})$ when $x = x_i$ and $y = x_j$, hence the above expression of the weight as a ratio of volumes. Of course, this result does not depend on the metric used.

Yet, while we have a metric, let's use it to express (13) a little differently. Since the gradients $\nabla\lambda^n$ are uniform fields, one has $\lambda^n(y) - \lambda^n(x) = \nabla\lambda^n \cdot xy$. Therefore,

$$\begin{aligned} &\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y) = \\ &\lambda^i(x)(\lambda^j(y) - \lambda^j(x)) - \lambda^j(x)(\lambda^i(y) - \lambda^i(x)) \\ &= (\lambda^i(x)\nabla\lambda^j - \lambda^j(x)\nabla\lambda^i) \cdot xy, \end{aligned}$$

also equal to $(\lambda^i(y)\nabla\lambda^j - \lambda^j(y)\nabla\lambda^i) \cdot xy$, by symmetry between x and y, and to

$$(\lambda^i(z)\nabla\lambda^j - \lambda^j(z)\nabla\lambda^i) \cdot xy$$

for all z on the segment xy, thanks to the affine character of the λ^ns. Averaging in z, one sees that $\lambda^i(x)\lambda^j(y) - \lambda^j(x)\lambda^i(y)$ equals the circulation along xy of the vector field $\lambda^i\nabla\lambda^j - \lambda^j\nabla\lambda^i$, in which one recognizes the familiar form of the "edge element".

Remark. Another useful form of it (easy to check: just dot-product by xy) is

$$(\lambda^i\nabla\lambda^j - \lambda^j\nabla\lambda^i)(x) = \frac{xx_k \times xx_l}{(x_i x_k \times x_i x_l) \cdot x_i x_j}.$$

(The denominator is 6 vol($\{i, j, k, l\}$), a constant, but be careful with its *sign*, orientation-dependent. In programming, the formula as given is safer.) Note that $xx_k \times xx_l = (x_k - x) \times (x_l - x) = (x_l - x_k) \times (x - x_k)$, hence the Nedelec $A + B \times x$ representation [4] of a vector field generated by edge elements,[11] the one used in early implementations [3]. ♢

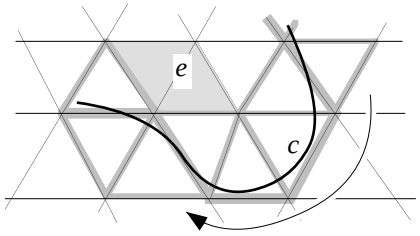

Figure 5. Edges with nonzero weight (highlighted) in the chain associated with c. The weight is null, for instance, for edge e, since the support of its Whitney form (shaded) is not traversed by c.

Back to where we started, a curve c inside D, its weight, with respect to an edge $e = \{m, n\}$ which goes from node m to node n, is the limit of the Riemann-like sum of contributions of small segments analogous to xy in which c is partitioned. So it's the circulation $w^e(c) = \int_c \tau \cdot \mathbf{W}^e$ along c (oriented by the choice of unit tangent vectors τ) of the vector field

$$\mathbf{W}^e = w^m\nabla w^n - w^n\nabla w^m. \tag{14}$$

Only edges of the tetrahedra traversed by c contribute, actually (Fig. 5), and $w^e(c)$ only depends on the part of c lying in the support of $\mathbf{W}^e$, i.e., in the cluster of tetrahedra that have edge e in common.

[11] Here, A and B denote 3D vectors, one pair for each tetrahedron, hence 6 parameters to match the six edge circulations. That was neat, at the time. But this expression looked so much like the field of velocities in the rotation of a solid (cf. a previous column, *JSAEM, 6*, 2 (1998), p. 115) that years were lost on this false track before the *affine* nature of the edge element (cf. (15) below) was recognized. Metric notions, such as "rigid-motion fields" or "orthogonal subspaces", though often met in discussions of edge elements, are irrelevant.

But we know better than resting with (14): the weight $w^e(c)$ must not depend on the metric. What we have in (14) is just a vector proxy for the real thing, the differential form

$$w^e = w^m \mathrm{d}w^n - w^n \mathrm{d}w^m, \tag{15}$$

whose integral $\int_c w^e$ is the weight $w^e(c)$. (Note that $\int_e w_e = 1$.) This is called a "Whitney form", and we have one for each edge. (In retrospect, w^n, a zero-form, was the Whitney form of node n, so now we may expect each simplex, whatever its dimension, to have a Whitney form of its own.)

Thus we have solved the problem of associating a chain with c: the associate is

$$p^t_{\mathtt{M}}\, c = \sum_e w^e(c)\; e \tag{16}$$

(compare with (7)). Correlatively, we have solved the problem of interpolating from edge values: what interpolates from the edge-DoF array **a** is the (straight) 1-form

$$p_{\mathtt{M}}\, \mathbf{a} = \sum_{e\in\mathcal{E}} \mathbf{a}_e w^e. \tag{17}$$

Note that $\int_{e'} w_e = 0$ for $e' \neq e$, because both w^n and w^m in (14) or (15) vanish on other edges than $e = \{m, n\}$. This shows (integrate both sides of (17) along e') that $r_{\mathtt{M}}\, p_{\mathtt{M}} = 1$ holds, again in this case. The convergence property, $p_{\mathtt{M}}\, r_{\mathtt{M}} \to 1$ when $\mathtt{M} \to 0$, also holds,[12] under the condition of mesh uniformity.

4.5 The complex of Whitney forms

How to associate a chain with a tiny triangle xyz, and hence, how to interpolate from facet values, is now an easy guess.

First, we notice a pattern in (4′) and (12), which suggests

$$xyz = \sum_{m,n,q} \lambda^m(x)\lambda^n(y)\lambda^q(z)\, x_m x_n x_q$$

as the next item in the sequence. Here, indices m, n, and q span the set $\{i, j, k, l\}$ independently, so it makes 64 terms, but many of them, like e.g., the term in $x_i x_i x_j$, don't correspond to facets, and thus "count for nothing". Others, such as $x_i x_j x_k$, $x_j x_k x_i$, $x_k x_j x_i$, etc., correspond to the same facet, but with both possible orientations. Grouping them, we find that the weight of xyz with respect to facet ijk is

$$\lambda^i(x)\lambda^j(y)\lambda^k(z) + \ldots = \frac{\mathrm{vol}(\{x, y, z, l\})}{\mathrm{vol}(\{i, j, k, l\})},$$

where the dots stand for 5 other terms, obtained by index permutation and (for the three odd permutations) sign-change. If this sounds like a determinant, no surprise: indeed, $\mathrm{vol}(\{x, y, z, l\})$ is one sixth of the determinant of vectors $x_l x$, $x_l y$, $x_l z$. The consistency, so far (Figs 2, 4, and 6), of these interpretations of weights as "relative volumes" shows that we have the general rule, as summarized by Fig. 6.

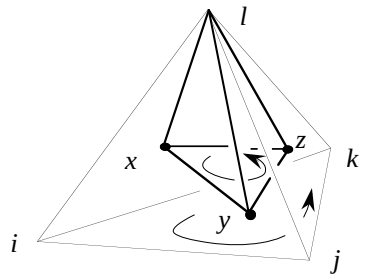

Figure 6. Representing a "small" p-simplex (xyz, here, $p = 2$) as a weighted sum of p-faces of the d-simplex $\{i, j, k, l, \ldots\}$ (spatial dimension $d = 3$ here). Note how the very ordering of points, x, y, z, inner-orients xyz. Its weight with respect to $\{i, j, k\}$ is the ratio of the volumes of $\{x, y, z, l, \ldots\}$ and of $\{i, j, k, l, \ldots\}$. (Recall that volumes can be negative, and that a *ratio* of volumes is a purely affine notion.)

Next, a manipulation similar[13] to what led us to the above form $\lambda^i \nabla\lambda^j - \lambda^j \nabla\lambda^i$ of the vector proxy leads this time to

$$\mathbf{W}^{ijk}(x) = 2\,(\lambda^i(x)\nabla\lambda^j \times \nabla\lambda^k + \ldots + \ldots)$$

[12] Refs [9] and [17] of last issue contain a general proof, valid for all dimensions p. Let's just sketch an elementary one for $p = 1$, using vector proxies. The task reduces to evaluate $\int_{\mathrm{T}} |\mathbf{A} - \mathbf{A}_{\mathtt{M}}|^2$, where $\mathbf{A}$ and $\mathbf{A}_{\mathtt{M}}$ are proxies for a and $p_{\mathtt{M}}\, r_{\mathtt{M}}\, a$, for each tetrahedron T. There, one has $\mathbf{A}_{\mathtt{M}}(x) = A_{\mathrm{T}} + B_{\mathrm{T}} \times (x - x_{\mathrm{T}})$, where x_{T} is some point inside T. The Taylor expansion of $\mathbf{A} - \mathbf{A}_{\mathtt{M}}$ about x_{T} contains terms of degree 2 and higher, which contribute to the integral a term in $\gamma_{\mathtt{M}}^4\,\mathrm{vol(T)}$ (where $\gamma_{\mathtt{M}}$ is the grain of the mesh), and the gradient of a quadratic function which one may assume *vanishing at nodes* (by adjusting A_{T}). Thanks to this, and to uniformity (necessary, as the counter-example in [1] shows), its contribution is in $\gamma_{\mathtt{M}}^2\,\mathrm{vol(T)}$, hence $\int_D |\mathbf{A} - \mathbf{A}_{\mathtt{M}}|^2 \leq C(\mathbf{A})\gamma_{\mathtt{M}}$.

[13] but a bit too lengthy to be included; the trick is to consider xyz as a vector (the vectorial area) and to express it as half the cross product $xy \times xz$.

(two other terms, by circular permutation on i, j, k), of which another form is

$$\frac{2\,x x_l}{(x_i x_k \times x_i x_l)\cdot x_i x_j} \equiv \frac{x x_l}{3\mathrm{vol}(\{i,j,k,l\})}.$$

As above, the weight of a surface S with respect to a facet $f = \{l, m, n\}$ is then the flux $w^f(S) = \int_S \nu \cdot \mathbf{W}^f$ of the vector field

$$\mathbf{W}^f = 2(w^l \nabla w^m \times \nabla w^n + \ldots + \ldots) \quad (18)$$

(again, two other terms, by circular permutation on l, m, n). Of course, $w^f(f') = 1$ if $f = f'$, 0 otherwise.

Remark. Be warned that the unit normal vector ν here is meant to $inner$-orient S, which it does in cooperation with the assumed orientation of 3D space. It does not correspond to an intrinsic, physically meaningful crossing direction. $\diamondsuit$

Finally, we infer the affine representation of the facet element from its vector proxy (18):

$$w^f = 2(w^l \mathrm{d}w^m \wedge \mathrm{d}w^n + \ldots + \ldots), \quad (19)$$

and from this, we infer the structure of Whitney forms in general: if $s = \{n_0, n_1, \ldots, n_p\}$ is a p-simplex (whose inner orientation is implied by the very order in which we list its nodes), its Whitney form [5] is

$$w^s = (-1)^i\, p! \sum_{i=0,\ldots,p} w^{n_i} \mathrm{d}w^{n_0} \wedge \ldots \langle i \rangle \ldots \wedge \mathrm{d}w^{n_p},$$

where the $\langle i \rangle$ means "omit the term $\mathrm{d}w^{n_i}$". But of course there is little use for such generality in the applications we have in view. Already when $p = 3$, in 3D, the "volume element" W^{T} for tetrahedron T is simply the function equal to 0 except on T, where it's equal to 1/vol(T). (This is the function proxy of a piecewise constant 3-form w^{T}, totally determined by the simple condition $\int_{\mathrm{T}'} w^{\mathrm{T}} = 1$ if $\mathrm{T} = \mathrm{T}'$, else 0.)

Anyway, we have enough to see the whole picture: To each primal p-simplex s (with p = 0 to d in d-dimensional affine space), there corresponds a Whitney form w^s of degree p, such that $\int_s w^s = 1$, and $\int_{s'} w^s = 0$ for other p-simplices of the mesh. The prolongator of a p-cochain $\mathbf{u}$ (the promised $Whitney\ map$) is

$$p_{\mathrm{M}}\,\mathbf{u} = \textstyle\sum_s \mathbf{u}_s w^s, \quad (20)$$

the p-chain associate of a p-manifold M is

$$p^t_{\mathrm{M}}\,M = \sum_s (\int_M w^s)\, s, \quad (21)$$

and (3) holds for a uniform family of meshes. Last, transposing (11) yields

$$\mathrm{d}p_{\mathrm{M}} = p_{\mathrm{M}}\,\mathbf{d}. \quad (22)$$

We shall adopt the notation $W^p(D)$ for the finite dimensional space generated by Whitney p-forms.

This quite satisfying uniformity of properties does not exhaust the subject. Remember that $\mathbf{d}_p \circ \mathbf{d}_{p-1} = 0$ and, in the case of a contractible[14] domain D, $\ker(\mathbf{d}_p) = \mathrm{cod}(\mathbf{d}_{p-1})$. Combining that with (22) we find, effortlessly, the following $structural\ property$ of the Whitney complex of forms: for p = 1 to d,

$$\mathrm{d}\,W^{p-1} \subset W^p, \quad \ker(\mathrm{d}\,; W^p) = \mathrm{d}\,W^{p-1}. \quad (23)$$

We'll see later how important this is (and hence, how important (11) is). But the next immediate task, now, is to use the above Whitney map for $Galerkin$-style discretization. We'll find, with not too much surprise, that this is just another way to derive discrete Hodge operators.

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[14] One in which any closed p-manifold can be shrunk to a point by continuous deformation. In dimension 3, this means a simply connected domain with connected boundary, or, more colloquially, "neither loops nor holes". (Holes, I understand, are aptly discussed by A. Kameari in *JSAEM*, *6*, 1 (1999), pp. 1-2.)

(5): The "Galerkin hodge"

Where we stand

With the "Whitney map" of last issue, we have a way to pass from degrees of freedom (DoF) to fields, in the case of a simplicial mesh. In particular, we may construct a vector field

$$\mathbf{E} = \sum_{e \in \mathcal{E}} \mathbf{e}_e \mathbf{W}^e \tag{1}$$

from a DoF array **e**, edge-based, thanks to the "edge element"

$$\mathbf{W}^e = w^m \nabla w^n - w^n \nabla w^m. \tag{2}$$

In this formula, m and n are the endpoints of edge e (cf. Fig. 1), and w^n is the "hat function" of standard finite element theory (equal to 1 at node n, 0 at other nodes, and linearly interpolating in between). $\mathbf{W}^e$ is a rightful finite element for the electric field, because it has tangential continuity across element interfaces, thus automatically conferring to **E** this essential physical property. Its circulation along edge e' is 1 if $e' = e$, else 0, which ensures, thanks to (1), that each DoF $\mathbf{e}_e$ is indeed the circulation of **E** along edge e.

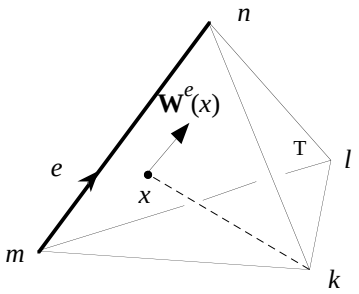

Figure 1. The edge element. One has $\mathbf{W}^e(x) = kl \times kx/(6\,\mathrm{vol}(\mathrm{T}))$, which makes it easy to visualize the field. Recall however that in spite of this apparently metric-dependent expression, the edge element is an affine object: $\mathbf{W}^e$ is just the vector proxy of the Whitney form $w^e = w^m \mathrm{d}w^n - w^n \mathrm{d}w^m$.

All this designates $\mathbf{W}^e$ as a suitable finite element for vector-valued entities associated with lines, such as the fields **E** and **H**. So why not use it as such for problems involving this kind of fields, by following the Galerkin variational approach? That was indeed the viewpoint 20 years ago, when the edge element made possible the solution in dimension 3 of eddy current problems, with **H** as unknown field. But nowadays we tend to see things in a different light: The Galerkin method using edge elements can be interpreted as a way—one among several possible ways—to build a discrete Hodge operator, which is what will occupy us in this installment of the series.

5.1 Model problem

The model problem this time, for a change, will be *electrostatics*, in the same cavity as usual (Fig. 2). Given a time-independent charge density q, find **D** and **E** such that

$$\begin{aligned} &\operatorname{div} \mathbf{D} = q,\ \mathbf{D} = \epsilon \mathbf{E},\ \operatorname{rot} \mathbf{E} = 0, \\ &\nu \cdot \mathbf{D} = 0 \text{ on } S^h,\ \nu \times \mathbf{E} = 0 \text{ on } S^e. \end{aligned} \tag{3'}$$

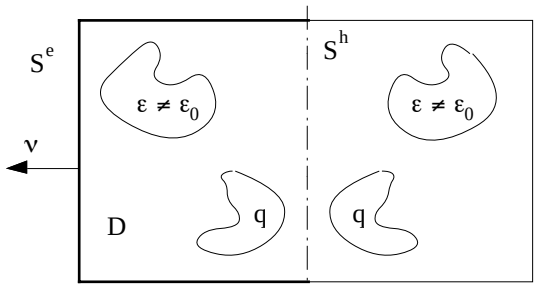

Figure 2. Position of the model problem: Same metallic cavity as before (*JSAEM, 7,* 1999, p. 151), but a steady cloud of electric charge where we formerly had an antenna. Symmetry of the cavity, and of the charge distribution, allow us to compute in domain D (the left half), with the boundary condition $\nu \cdot \mathbf{D} = 0$ on the symmetry plane S^h, where ν denotes the outward-directed unit normal vector.

In differential geometric form, this is

$$\begin{aligned} &\mathrm{d}\tilde{d} = \tilde{q},\ \tilde{d} = \tilde{\star}_\epsilon e,\ \mathrm{d}e = 0, \\ &t\tilde{d} = 0 \text{ on } S^h,\ te = 0 \text{ on } S^e, \end{aligned} \tag{3}$$

where $\tilde{q}$ stands for the twisted 3-form the scalar proxy of which is q.

We learned how to produce a discretization, at least in the case when an orthogonal dual mesh can be built. This gave us a diagonal matrix $\tilde{\star}_\epsilon$, whose entries, indexed over the set $\mathcal{E}$ of "active" edges (those not in S^e), were $\tilde{\star}_\epsilon^{ee} = \epsilon\,\text{area}(\tilde{e})/\text{length}(e)$, where $\tilde{e}$ is the 2-cell dual to e. Hence the numerical scheme:

$$-\mathbf{G}^t\tilde{\mathbf{d}} = \tilde{\mathbf{q}},\ \tilde{\mathbf{d}} = \tilde{\star}_\epsilon\mathbf{e},\ \mathbf{Re} = 0, \tag{4}$$

with $\mathbf{e} = \{\mathbf{e}_e : e \in \mathcal{E}\}$ and $\tilde{\mathbf{d}} = \{\tilde{\mathbf{d}}_e : e \in \mathcal{E}\}$, the electric fluxes across the $\tilde{e}$s. The data $\tilde{\mathbf{q}} = \{\tilde{\mathbf{q}}_n : n \in \mathcal{N}\}$ is obtained by integration of q over each dual 3-cell $\tilde{n}$. All we need to do, to get (4), is to replace each element of $(3')$ by its discrete counterpart: **E** by **e**, rot by **R**, div by $-\mathbf{G}^t$, etc., and ϵ by the discrete Hodge operator $\tilde{\star}_\epsilon$. Of course, one must have done the practical work leading to $\tilde{\star}_\epsilon$.

The Ritz–Galerkin approach proceeds differently, and one certainly does not recognize it in (4) at first glance. Yet, in depth, there are strong homologies. To argue this point, I shall first sketch a possible presentation of the Galerkin method, using standard finite elements, to a classroom of advanced students in computational electromagnetism. Then a crucial modification (cf. Prop. 1 below) will be introduced, leading back to (4).

5.2 The Galerkin method, with node-based scalar finite elements

The method is based on the so-called *weak formulation* of the equation div$\mathbf{D} = q$ and (both things in one stroke) the boundary condition $\nu \cdot \mathbf{D} = 0$ on S^h:

$$-\int_D \mathbf{D}\cdot\nabla\psi' = \int_D q\,\psi' \text{ for all } \psi' \text{ in } \Psi,$$

where Ψ is a space of *test functions*, characterized, in addition to a few technical requirements,[1] by the condition $\psi' = 0$ on S^e. Using the fact that the equations about **E**, i.e., rot$\mathbf{E} = 0$ and $\nu \times \mathbf{E} = 0$ on S^e, are equivalent to "$\mathbf{E} = -\nabla\psi$ for some ψ in Ψ", one is led to *find ψ in Ψ such that*

$$\int_D \epsilon\,\nabla\psi\cdot\nabla\psi' = \int_D q\,\psi'\ \forall\,\psi' \in \Psi. \tag{5}$$

This comprehensive weak formulation of the original problem is equivalent, as one easily shows, to minimizing[2] over Ψ the energy-related quantity

$$W(\psi') = \int_D \epsilon\,|\nabla\psi'|^2 - 2\int_D q\,\psi'.$$

At this stage, one points out that a function such as

$$\psi = \sum_{n\in\mathcal{N}} \boldsymbol{\psi}_n w^n \tag{6}$$

does belong to Ψ, since $\psi = 0$ at nodes of S^e, which have been excluded from the set $\mathcal{N}$. Such functions span a subspace of Ψ of finite dimension (equal to the number N of active nodes), which we denote by $\Psi_{\mathcal{M}}$, since it depends on the mesh. When "$\mathcal{M} \to 0$" (in the precise sense introduced last time, including uniformity of the family of meshes), $\Psi_{\mathcal{M}}$ "tends to fill-out" Ψ, which motivates the replacement of (5) by the approximation *find $\psi_{\mathcal{M}}$ in $\Psi_{\mathcal{M}}$ such that*

$$\int_D \epsilon\,\nabla\psi_{\mathcal{M}}\cdot\nabla\psi' = \int_D q\,\psi'\ \forall\,\psi' \in \Psi_{\mathcal{M}}. \tag{7}$$

This—one then proceeds to show—is actually a linear system with respect to the N components of the DoF array $\boldsymbol{\psi} = \{\boldsymbol{\psi}_n : n \in \mathcal{N}\}$. Indeed, introduce the matrix elements

$$\mathbf{A}^{nm} = \int_D \epsilon\,\nabla w^n\cdot\nabla w^m \tag{8}$$

and the N-vector **b**, say (nothing to do with the induction field), whose components are $\mathbf{b}^m =$

[1] They must belong to the Sobolev space $L^2_{\text{grad}}(D)$ of square-summable functions whose grad, too, is in $L^2(D)$. Then, $\nabla\psi$ belongs to the space $L^2_{\text{rot}}(D)$ that we briefly encountered in the first column (*JSAEM, 7*, 1999, p. 152). The necessary recourse, if one wants to be thorough, to such difficult notions of functional analysis has acted as a serious deterrent against the popularization of the method.

[2] This constitutes a *variational principle*, as so often met in physics. If there is such a variational characterization of the solution, one can derive a weak formulation from it. But many important problems don't correspond to the minimization of anything, and still can be cast in an appropriate weak form that makes them eligible to application of the Galerkin method. This is why the variational aspects of the method, although historically decisive, are downplayed in the presentation suggested here.

$\int_D q\, w^m$. Then, using (6) and replacing $\psi^{\prime}$ by the test function w^m (eligible, when $m \in \mathcal{N}$, since it then belongs to $\Psi_{\mathcal{M}}$), one derives from (7) the N algebraic equations

$$(9) \qquad \sum_{n\in\mathcal{N}} \mathbf{A}^{mn}\psi_n = \mathbf{b}^m \text{ for all } m \text{ in } \mathcal{N},$$

or $\mathbf{A}\psi = \mathbf{b}$, in compact form. Matrix $\mathbf{A}$, obviously symmetric and positive definite, is called the "stiffness matrix" of the problem, due to a mechanical analogy of little concern for us.

Assuming one has already covered a matrix algebra curriculum, including algorithms to solve $\mathbf{A}\psi = \mathbf{b}$, the last practical topic to treat is the effective computation of the stiffness matrix. This process, called "assembly" of $\mathbf{A}$, consists in first evaluating the "element matrices" $\mathbf{A}_{\mathrm{T}}$ defined, for each tetrahedron T, by $\mathbf{A}_{\mathrm{T}}^{nm} = \int_{\mathrm{T}} \epsilon \nabla w^n \cdot \nabla w^m$, then to form $\mathbf{A} = \sum_{\mathrm{T}} \mathbf{A}_{\mathrm{T}}$ by looping over T. Each $\mathbf{A}_{\mathrm{T}}$ is easily computed. There are *few* nonzero $\mathbf{A}_{\mathrm{T}}^{nm}$s, for one thing, and if ϵ is taken uniform inside T, then, after (8),

$$(10) \qquad \mathbf{A}_{\mathrm{T}}^{nm} = -\epsilon \cot(\theta_{kl}^{\mathrm{T}})\ \mathrm{length}(kl)/6$$

(Fig. 3), as some elementary geometry shows.

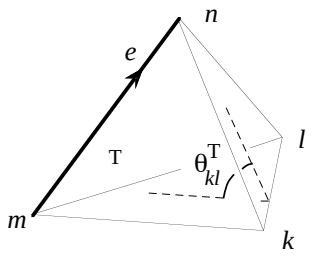

Figure 3. Notation for formula (10): θ_{kl}^{T} is the dihedral angle in front of edge $e = \{m, n\}$.

So the assembly problem is solved by formulas (8) and (10). These, however, hide some structural features of $\mathbf{A}$ which, though one can ignore them in a first course, are of some interest.

5.3 A reinterpretation

Let's introduce, for two edges e and $e^{\prime}$ of the mesh, the number

$$(11) \qquad \mathbf{M}^{ee^{\prime}} = \int_D \epsilon\, \mathbf{W}^e \cdot \mathbf{W}^{e^{\prime}},$$

and call "mass matrix" (again, due to an analogy) the square $(E \times E)$-matrix $\mathbf{M}_1(\epsilon)$, abbreviated as $\mathbf{M}$, as a rule, that these numbers form.[3] Now, $\mathbf{G}$ being the nodes-to-edges incidence matrix as usual,

Proposition 1. $\mathbf{A} = \mathbf{G}^t\mathbf{M}\mathbf{G}$.

Proof. This is a consequence of a major structural property of the Whitney complex, which appeared last time in the general form $\mathrm{d}p_{\mathcal{M}} = p_{\mathcal{M}}\mathbf{d}$ (*JSAEM*, *8*, 2000, p. 109, formula (22)). Its avatar of interest here is $\nabla w^n = \sum_{e\in\mathcal{E}} \mathbf{G}_{en}\mathbf{W}^e$ (the gradient of a hat function is a linear combination of edge elements, those of the edges that abut on the node, with weights ± 1 according to orientation). Bringing that into (8), we do get $\mathbf{A}^{nm} = \sum_{e,e^{\prime}} \mathbf{G}_{en}\mathbf{M}^{ee^{\prime}}\mathbf{G}_{e^{\prime}m}$, thanks to (11). $\diamondsuit$

Let's not misunderstand this result: The point is *not* to compute $\mathbf{A}$ by first computing $\mathbf{M}$, then using Prop. 1. Standard assembly, using (10), is cheaper. Here is the point: If we set $\mathbf{e} = -\mathbf{G}\psi$ and $\tilde{\mathbf{d}} = \mathbf{M}\mathbf{e}$, the equation $\mathbf{G}^t\mathbf{M}\mathbf{G}\psi = \mathbf{b}$ is equivalent to the system

$$(12) \qquad -\mathbf{G}^t\tilde{\mathbf{d}} = \mathbf{b},\ \tilde{\mathbf{d}} = \mathbf{M}\mathbf{e},\ \mathbf{R}\mathbf{e} = 0,$$

which looks very much like (4), and suggests to interpret $\mathbf{M}$ as a discrete Hodge operator similar to $\tilde{\star}_\epsilon$. Indeed, $\mathbf{M}$ has the right size (the number of active edges), and $\mathbf{e}$ is the array of edge-circulations of $\mathbf{E}$, as we remarked in the Introduction. But it's not so obvious that component $\tilde{\mathbf{d}}_e$ of $\tilde{\mathbf{d}}$ corresponds to the flux of $\mathbf{D}$ through some dual cell associated with e. As we shall see, this is indeed so, and $\tilde{\mathbf{d}}_e = \int_{\tilde{e}} \mathbf{D}$, where $\tilde{e}$ is the "barycentric dual" of edge e. There is also a sense in which the $\mathbf{b}$ of (12) deserves to be denoted by $\tilde{\mathbf{q}}$.

This will take some preparation (next Sections, 5.4 and 5.5). Meanwhile, let's remark that the mixed systems one can derive from the symmetric formulation (12), that is,

$$(13) \qquad \begin{pmatrix} -\mathbf{M} & \mathbf{R}^t \\ \mathbf{R} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \tilde{\mathbf{h}} \end{pmatrix} = \begin{pmatrix} -\tilde{\mathbf{d}}^q \\ 0 \end{pmatrix}$$

(where $\tilde{\mathbf{d}}^q$ is a DoF array such that $-\mathbf{G}^t\tilde{\mathbf{d}} = \mathbf{b}$, and $\tilde{\mathbf{h}}$ a kind of electric vector potential), and

$$(14) \qquad \begin{pmatrix} \mathbf{M}^{-1} & \mathbf{G} \\ \mathbf{G}^t & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{d}} \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{b} \end{pmatrix},$$

[3] The subscript refers to the simplices' dimension, 1 for edges. Similarly, there is a mass matrix $\mathbf{M}_2(\mu^{-1})$, with entries $\int_D \mu^{-1}\, \mathbf{W}^f \cdot \mathbf{W}^{f^{\prime}}$.

are less similar than their analogues in magnetostatics were (see (8) and (9) in *JSAEM, 7,* 1999, p. 403), because $\mathbf{M}^{-1}$ is here a full matrix. For the same reason, the facet-based "electric vector potential" formulation, $\mathbf{R}\mathbf{M}^{-1}\mathbf{R}^t\tilde{\mathbf{h}} = -\mathbf{R}\mathbf{M}^{-1}\tilde{\mathbf{d}}^q$ (compare with eq. (6), same source) has little appeal here.

5.4 A remarkable formula

We now introduce a new notion: "dyadic products", or simply "dyads". Remember the metaphor of a covector ω as a machine with one slot, in which one slips a vector v, to get in return a real number, denoted $\langle \omega, v \rangle$? We generalized that to p-covectors (machines with p slots, each of them meant to receive one vector). Now, our purpose is to define other similar machines, whose slots can accept geometrical objects of various types.

The simplest case is that of a machine with two slots, one on the left that can receive a covector, one on the right that can receive a vector, and a central dial that displays a real number, with the habitual linearity properties with respect to both arguments. Denoting by M the machine, we shall write the dial's reading in the form $\langle \omega, M, v \rangle$.

What can be the inner structure of such a machine? Note that if ω stays permanently in the left-hand slot, the machine behaves like a covector "when operated, single-handedly, from the right", and vice-versa. Hence the idea to build a machine of this kind by using as inner components a fixed vector and a fixed covector, w and η let's say, and to make the dial indicate $\langle \omega, w \rangle \langle \eta, v \rangle$ when ω and v are slipped into the slots, since this rule has the required linearity properties. Let's denote the machine thus obtained by the symbol $w\rangle\langle\eta$, and call that the *dyadic product* of w and η. So if we substitute $w\rangle\langle\eta$ for M in the above expression $\langle \omega, M, v \rangle$, we get this:

$$(15) \qquad \langle \omega, w \rangle \langle \eta, v \rangle = \langle \omega, w \rangle \langle \eta, v \rangle \; \forall\, \omega, v,$$

which in spite of looking like a notational joke is a bona-fide *definition* of $w\rangle\langle\eta$! It would not be difficult to show that the structure of the general machine—the right name for which, by the way, is *tensor*—is $M = \sum_i w_i \rangle\langle \eta_i$, but we'll dispense with the proof. (**Hint:** Take a basis, represent M as a matrix, and think of its decomposition as a sum of matrices of rank 1.) We shall not insist either on the generalization to dyadic products of p-vectors by q-covectors.

Among machines like M, one is special, the *unity*, denoted by 1, and defined by $\langle \omega, 1, v \rangle = \langle \omega, v \rangle$ for all ω and v.

With this, we are ready for

Proposition 2. $\sum_{e \in \mathcal{E}} \, e\rangle\langle w^e(x) = 1,$

where e is not only a label for edge e, but stands for the vector along edge e, while w^e is the associated Whitney form.

No proof is needed, for Prop. 2 is actually what we obtained last time (*JSAEM, 8,* 2000, p. 107): The Whitney forms w^e were constructed in such a way that a vector v anchored at x (then denoted xy), be equal to the sum $\sum_e \langle w^e(x), v \rangle \, e$ over edges of the mesh. This equality between vectors is equivalent to

$$\langle \omega, \sum_e \langle w^e(x), v \rangle \; e \rangle = \langle \omega, v \; \rangle$$

for any covector ω, and hence, by linearity, to

$$\sum_e \langle \omega, e \rangle \langle w^e(x), v \rangle = \langle \omega, v \; \rangle \;\; \forall \, \omega, v$$

which after (15) is exactly what Prop. 2 says. One may argue that the very definition of Whitney 1-forms (edge elements) was engineered in order to obtain Prop. 2.

Since similar considerations dictated the construction of Whitney forms for simplices of all dimensions, edges here have no privilege. There is such a formula for simplices of all dimensions: For facets, one has $\sum_{f \in \mathcal{F}} \, f\rangle\langle w^f(x) = 1$, where f is interpreted as a 2-vector. For nodes, $\sum_{n \in \mathcal{N}} \, n\rangle\langle w^n(x) = 1$. This one, like all formulas of the family, says two things: that $x = \sum_n w^n(x)\, x_n$ (formula (7) of last paper), and that $\sum_n w^n(x) = 1$, whatever x, the "partition of unity" property of hat functions. That's what is so remarkable about the formula of Prop. 2, and similar ones: They express the fact that *Whitney forms make a partition of unity*, for all degrees.

Remark. Some time ago (*JSAEM, 6,* 1998, pp. 121–ff), we discussed basis vectors ∂_i and basis covectors d^i, and noticed that $\langle \omega, v \rangle = \sum_i \omega_i v^i$,

in terms of the components. With the dyadic notation, we can rewrite this as $\sum_{i=1,\dots,n} \partial_i \rangle\langle \mathrm{d}^i = 1$, where n is the space dimension. A partition of unity, again, which makes Prop. 2 less surprising, and points at a deep analogy between "Cartesian frames" on the one hand, and the (local) "barycentric frames" provided by a simplicial mesh on the other hand. Pursuing this would lead us too far astray. $\diamondsuit$

Dyadic products are affine objects, but the idea of vector proxies also applies to them. For instance, if $\mathbf{H}$ is the proxy for η, the proxy for the dyad $w\rangle\langle\eta$ is the so-called "vector dyadic product" $w \otimes \mathbf{H}$, which we define by the formula

$$(15') \quad \Omega \cdot (w \otimes \mathbf{H}) \cdot v = (\Omega \cdot w)(\mathbf{H} \cdot v) \quad \forall\, \Omega, v,$$

where both Ω and v are vectors, this time. The slot for Ω may be left empty, which results in $(w \otimes \mathbf{H}) \cdot v = (\mathbf{H} \cdot v)\, w$ for all v. Alternatively, v may be left out. Note that if Ω is considered as the proxy for covector ω, $(15')$ is just an avatar of (15). Prop. 2 translates as $\sum_{e \in \mathcal{E}} e \otimes \mathbf{W}^e(x) = 1$, which amounts to

$$(16) \qquad \sum_{e \in \mathcal{E}} (\mathbf{W}^e(x) \cdot v)\, e = v \quad \forall\, v.$$

It will be a bit easier for us to use that, rather than the literal version of Prop. 2, for the computations that follow.

5.5 Matrix M and the dual mesh

So in this section, we shall use a definite metric (the one implicitly assumed here since the beginning, for which the edge element has the form (2)), and a definite orientation of ambient space (the "right-hand rule" one).

The dual mesh in consideration will be the one obtained by the barycentric construction (Fig. 4). In line with our convention to use e for the vector along edge e, we'll make $\tilde{e}$ serve not only as a label for the dual edge, but for its vectorial area, that is, the vector orthogonal to $\tilde{e}$, pointing in the same general direction as e, of length equal to the area of $\tilde{e}$. Generalizing that, we shall feel free to consider $\{k, l, m\}$, for instance, not only as a label for the facet spanned by these nodes, but as its vectorial area. (The vector points in the direction specified by the ordering of the nodes, in conjunction with Ampère's right-hand rule.)

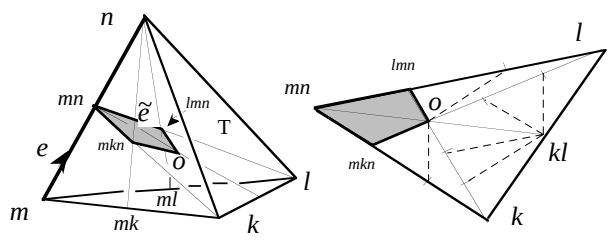

Figure 4. Left: The facet $\tilde{e}$ dual to edge e, in the barycentric construction. Labels such as mk, mkn, etc., point to centers of primal edges and facets. The tetrahedron's center is o. Right: The area of $\tilde{e}$ is one sixth that of $\{mn, k, l\}$. (All small triangles shown have the same area.)

Now, let us set $v = \epsilon \mathbf{W}^e$ in (16). Replacing the summation index e by e' to avoid confusion, we get

$$\sum_{e' \in \mathcal{E}} (\epsilon \mathbf{W}^e(x) \cdot \mathbf{W}^{e'}(x))\, e' = \epsilon \mathbf{W}^e(x),$$

which can be integrated over D, yielding, if one takes (11) into account,

$$\sum_{e' \in \mathcal{E}} \mathbf{M}^{ee'}\, e' = \int_D \epsilon \mathbf{W}^e.$$

Be careful here that we integrate a *vector*-valued function (the vector field $\epsilon \mathbf{W}^e$), so the result is a vector.

For the next step, let's suppose ϵ is the same all over D, and thus can be factored out, a simplifying hypothesis that we shall reconsider later. First,

Lemma 1. *The vector-valued integral of* $\mathbf{W}^e$ *over tetrahedron* T *is equal to the vectorial area of the part of* $\tilde{e}$ *contained in* T.

Proof. Let h^n be the length of the altitude falling from node n onto the opposite facet $\{k, l, m\}$. Since ∇w^n is a constant vector over T, of magnitude $1/h^n$, we have $\int_{\mathrm{T}} \nabla w^n = \{k, l, m\}/3$. The average of w^m being 1/4, we get $\int_{\mathrm{T}} \mathbf{W}^e = (\{k, l, m\} + \{k, l, n\})/12$. But this is $\{mn, k, l\}/6$, which, as Fig. 4 shows, is equal to the part of $\tilde{e}$ local to T. $\diamondsuit$

The result, obviously, does not depend on the particular arrangement of nodes on the figure: What counts is the fact that e and $\tilde{e}$ point in the same direction. Therefore, by adding contributions from all tetrahedra around edge e, we get this, where $\tilde{e}$ is now the whole dual facet:

Proposition 3. $\int_D \mathbf{W}^e = \tilde{e}$.

So when ϵ is constant, we arrive at the following relation:

$$\sum_{e' \in \mathcal{E}} \mathbf{M}^{ee'} e' = \epsilon\, \tilde{e}. \tag{17}$$

Back to the general case, now. If ϵ is uniform in each individual tetrahedron, which we generally can assume, then $\int_{\mathrm{T}} \epsilon \mathbf{W}^e = \epsilon_{\mathrm{T}}\, \tilde{e}_{\mathrm{T}}$, where $\tilde{e}_{\mathrm{T}}$ is the part of $\tilde{e}$ local to T. By adding contributions of all tetrahedra, we still find (17), provided $\epsilon\, \tilde{e}$ is understood as the sum $\sum_{\mathrm{T}} \epsilon_{\mathrm{T}}\, \tilde{e}_{\mathrm{T}}$, which we may call the "ϵ-related vectorial area" of the dual facet $\tilde{e}$.

Remark. This is not ad-hoc fancy. As repeatedly mentioned here, the coefficient ϵ of $(3')$ (which could as well be a tensor ϵ_{ij}) is just a proxy for the Hodge operator $\tilde{\star}_\epsilon$. The constitutive law $\tilde{d} = \tilde{\star}_\epsilon e$, as expressed in terms of differential forms, has an intrinsic character which its equivalent in terms of the proxies, $\mathbf{D}^i = \sum_j \epsilon_{ij} \mathbf{E}^j$, lacks. Change the metric, $\tilde{\star}_\epsilon$ stays the same, but ϵ_{ij} has to change. Could therefore the metric be selected so as to make ϵ_{ij} as simple as possible, that is, unity?

The answer to that is *yes*. This is the "Hodge implies metric" result alluded to in *JSAEM, 6,* 1998, p. 325: Given a linear map $\tilde{\star}_\epsilon$ from 1-forms to twisted 2-forms, with adequate properties of symmetry and positivity, one may construct a metric the Hodge operator of which is precisely $\tilde{\star}_\epsilon$. Let's call that the "ϵ-related" metric. In this metric, vector proxies for $\tilde{d}$ and e are $\mathbf{D}_\epsilon$ and $\mathbf{E}_\epsilon$, different from $\mathbf{D}$ and $\mathbf{E}$, and they are linked by $\mathbf{D}_\epsilon = \mathbf{E}_\epsilon$. When one works out the exercise on "what is the ϵ-related vectorial area of $\tilde{e}$?", one does find the above expression $\sum_{\mathrm{T}} \epsilon_{\mathrm{T}}\, \tilde{e}_{\mathrm{T}}$, where ϵ_{T} may be understood as a tensor acting on the vector $\tilde{e}_{\mathrm{T}}$. $\diamondsuit$

Thus (17) has validity beyond the particular metric we use. Comforted by that, let's carry on with this particular metric to see the implications of this formula.

5.6 The Galerkin-induced discrete hodge

Suppose now a piecewise uniform vector field $\mathbf{E}$ (i.e., constant over each primal tetrahedron), not necessarily the physical solution, but a field which does have tangential continuity across facets. It's the proxy of some 1-form e. Let $\mathbf{D} = \epsilon\mathbf{E}$, also piecewise uniform by our assumptions about ϵ, and the proxy of $\tilde{d} = \tilde{\star} e$. Take the dot product of both sides with $\mathbf{E}$. Then the array of circulations $\mathbf{e} = \{\mathbf{E} \cdot e \,:\, e \in \mathcal{E}\}$ is what we denoted $r_{\mathcal{M}} e$ in recent issues. The left-hand side thus yields the sum $\sum_{e'} \mathbf{M}^{ee'} \mathbf{e}_{e'}$, i.e., $(\mathbf{M} r_{\mathcal{M}} e)_e$. On the right-hand side, we get the flux of $\mathbf{D}$ through $\tilde{e}$, that is, the component at edge e of what we called $r_{\mathcal{M}} \tilde{\star} e$. We may therefore assert that $\tilde{\star}_\epsilon r_{\mathcal{M}} e = r_{\mathcal{M}} \tilde{\star} e$ for any piecewise constant 1-form e when $\tilde{\star}_\epsilon$ is taken equal to $\mathbf{M}$.

This, we know, is the essential property for a would-be discrete hodge, which allows us to conclude, after a process of local Taylor expansion similar to what was done earlier,

$$\tilde{\star}_\epsilon r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\epsilon \to 0 \text{ when } \mathcal{M} \to 0, \tag{18}$$

the consistency property for $\tilde{\star}_\epsilon \equiv \mathbf{M}(\epsilon)$.

So now, we are justified to consider the edge-element mass matrix $\mathbf{M}(\epsilon)$ of (11) as a realization of the discrete Hodge operator $\tilde{\star}_\epsilon$. It links edge circulations with fluxes through dual facets, indeed, provided the latter are taken as the dual 2-cells in the barycentric construction (Fig. 4).

This "Galerkin hodge" gives a convergent scheme in statics, which is no news, but the interesting point is that it can be proven along the same lines as with the "orthogonal" hodge. Let's not go into this again. Note that stability is here a built-in property, because the (up to a factor 2) discrete energy $|\mathbf{e}|_\epsilon^2 = \langle \mathbf{M}\mathbf{e}, \mathbf{e} \rangle$ coincides here with the continuous energy $|p_{\mathcal{M}} \mathbf{e}|_\epsilon^2 = \int_D \epsilon |\mathbf{E}|^2$, by construction, hence $|p_{\mathcal{M}} \mathbf{e}|_\epsilon = \|\mathbf{e}|_\epsilon$ (cf. (17) p. 406 in *JSAEM, 7,* 1999). As we invoked mesh uniformity earlier in order to assess stability, one may wonder whether uniformity is necessary in the Galerkin approach. It is: the convergence property $p_{\mathcal{M}} r_{\mathcal{M}} e \to e$ relies on it, and will be lost for pathological refinement methods [1].

For facets, the analogue of Prop. 3 is $\int_D \mathbf{W}^f = \tilde{f}$. This has an unexpected application to the interpetation of the standard "method of moments" [5] as a Galerkin method, for which I must refer to [3]. The analogue for nodes is worth looking at, too: it's $\int_D w^n = \mathrm{vol}(\tilde{n})$, with the consequence that if q is piecewise constant,

then $\int_D q\, w^n$ is the total electric charge contained in the dual 3-cell $\tilde{n}$. JUst what we needed to identify the **b** of (12) and the $\tilde{\mathbf{q}}$ of (4): this is right when q is piecewise constant.

Remark. Be careful, it's *not* true that $\int_{\tilde{e}} \tilde{\star}_\epsilon w^{e'} = \int_{\tilde{e}'} \tilde{\star}_\epsilon w^e$. Therefore, simply taking the flux of $\epsilon \mathbf{W}^e$ through the dual face $\tilde{e}'$ does not provide a discrete Hodge operator. $\diamondsuit$

The analogue of (17) for facets, derived from $\sum_{f\in\mathcal{F}} (\mathbf{W}^f(x)\cdot v)\ f = v\ \forall\, v$, is

$$\sum_{f'\in\mathcal{F}} (\mathbf{M}_2(\mu^{-1}))^{ff'} f' = \mu^{-1}\, \tilde{f}, \tag{19}$$

hence a Galerkin hodge that discretizes $\tilde{\star}_{\mu^{-1}}$.

5.7 Dynamics, again

Thus in possession of the two required hodges, we may use them to solve the antenna problem, with given current density. Substitute $\mathbf{M}_1(\epsilon)$ for $\tilde{\star}_\epsilon$ and $\mathbf{M}_2(\mu^{-1})$ for $\tilde{\star}_\mu^{-1}$ in eqs. (5) and (6), p. 295 of *JSAEM, 7,* 1999, hence the following scheme [6, 7]: Starting from $\mathbf{b}^0 = 0$ and $\mathbf{e}^{-1/2} = 0$, find successive DoF arrays $\mathbf{b}^k$ and $\mathbf{e}^{k+1/2}$ such that, for $k = 0, 1, \ldots$, etc.,

$$\mathbf{M}_1(\epsilon)\frac{\mathbf{e}^{k+1/2} - \mathbf{e}^{k-1/2}}{\delta t} = \mathbf{R}^t \mathbf{M}_2(\mu^{-1})\mathbf{b}^k - \mathbf{j}^k,$$

$$\frac{\mathbf{b}^{k+1} - \mathbf{b}^k}{\delta t} + \mathbf{R}\mathbf{e}^{k+1/2} = 0. \tag{20}$$

This, which is the time-domain extension of an earlier proposal to use edge elements for the time-harmonic problem [2], has the advantage of avoiding the sometimes problematic construction of the orthogonal dual mesh. But the incurred penalty is heavy: Since $\mathbf{M}_1(\epsilon)$ is not diagonal, the scheme is not explicit. Fortunately [4],

Proposition 4. *The diagonal matrix* **H***, indexed over edges, whose entries are, for each edge* $e = \{m, n\}$*,*

$$\mathbf{H}^{ee} = -(\mathbf{G}^t \mathbf{M}_1(\epsilon)\mathbf{G})_{mn} \equiv -\mathbf{A}_{mn}$$

(the **A** of Prop. 1)*, verifies* $\mathbf{G}^t\mathbf{H}\mathbf{G} = \mathbf{G}^t\mathbf{M}_1\mathbf{G}$.

Proof. $(\mathbf{G}^t\mathbf{H}\mathbf{G})_{mn} = \sum_{e\in\mathcal{E}} \mathbf{G}_{em}\mathbf{H}^{ee}\mathbf{G}_{en}$. The only nonzero term in this sum obtains for edge e joining m to n, if there is such an edge, and then equals $-\mathbf{H}^{ee}$, since $\mathbf{G}_{em}\mathbf{G}_{en} = -1$. On the other hand, $(\mathbf{G}^t\mathbf{M}_1\mathbf{G})_{mn} = 0$ if m and n are not joined by an edge. (Note this is special to tetrahedral meshes. The proof would fail otherwise.) $\diamondsuit$

Thus **H** can replace **M** in statics, giving the same scheme. By a previously enunciated heuristic principle, we may substitute this "diagonally lumped" hodge **H** for $\mathbf{M}_1(\epsilon)$ in the dynamic scheme (20), and expect convergence. This works, as proven in [4].

Alas, there is a hitch: Nothing ensures that $\mathbf{H}_{ee} > 0$, which as we know is required for stability in (20). As formula (8) shows, this requires *acute* dihedral angles, a not so easily met condition. Nothing comes free!

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(6): Some questions and answers

Summing up—provisionally

As this Series, if not its subject matter, approaches its closure, it seems appropriate to summarize our observations so far, and to address some questions that were asked at various occasions, over e-mail or in recent meetings. To most of these questions, I have only partial answers, at best. But unanswered questions are not so bad a thing in science: they may foster further progress.

What we have described so far can be characterized as *a discretization toolkit.* Faced with the task of solving electromagnetic equations in some definite physical configuration, the modeller will proceed as follows:

• Partition the region of interest into small volumes, or "3-cells", hence an algebraic structure called "the primal mesh".

This is described by sets $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{V}$, of nodes, edges, etc., and by incidence matrices, **G**, **R**, **D**, which say how oriented p-dimensional faces, with $p \leq 3$, of said volumes, relate. (Some boundary conditions, one will recall, are taken into account by deletion of some p-cells.) We have seen repeatedly how a "dual mesh" can, conceptually at least, be associated with the primal one. (*Practical* aspects of this association are among questions to be answered.)

• Take as unknowns, or "degrees of freedom" (DoF), for the problem at hand, cell-based quantities.

As we described several times, such DoFs are the emf's along primal edges, the induction fluxes relative to primal facets, mmf's along dual edges, etc. Hence DoF arrays, **e**, **b**, $\tilde{\mathbf{h}}$, **d**, $\tilde{\mathbf{j}}$, which relate in definite ways to either the primal or the dual mesh. Some of these quantities may be imposed by the physical situation, which turns them into data. In particular, as we always assumed up to now, intensities $\tilde{\mathbf{j}}_f$ through dual facets can all be data, which one gets by (approximate) integration of the given current density $\tilde{\jmath}$ over the dual facet $\tilde{f}$. But arranging for some of them to be unknowns, linked with emf's via Ohm's law, is not difficult, as we shall see presently. Other data may come from the rest of the boundary conditions.

• Form the so-called "discrete Hodge operators".

These are square symmetric matrices, $\tilde{\star}_\mu$ and $\tilde{\star}_\varepsilon$, and their inverses. (This, for wave propagation problems. It's easy to imagine how a $\tilde{\star}_\sigma$ would intervene in an eddy-current problem, although we didn't yet address this issue: It's precisely by an equation of the form $\tilde{\mathbf{j}} = \tilde{\star}_\sigma \mathbf{e} + \tilde{\mathbf{j}}^{\mathrm{s}}$, where $\tilde{\mathbf{j}}^{\mathrm{s}}$ stands for the known "source"-current intensities, that one might account for Ohm's law.) In contrast to incidence matrices, discrete hodges require a knowledge of the metric of the mesh (lengths of edges, areas of facets, etc.), which one usually derives from a database containing nodal positions, and material properties of each elementary volume. (Such properties *are* part of the metric structure, in our view.) We return in a moment to the two main methods by which hodges can be constructed.

• Substitute **e**, **b**, $\tilde{\mathbf{h}}$, etc., for the fields e, b, $\tilde{h}$, etc., in the equations. Substitute **R** or $\mathbf{R}^t$ for rot, depending on whether the field to which this applies is on the primal side or on the dual side.

If things evolve in time, replace time-derivatives by appropriate finite differences. It may happen, in some modellings, that operators grad or div also appear in the equations. If so, substitute **G** and **D**, or $-\mathbf{D}^t$ and $-\mathbf{G}^t$, to grad and div, according to which side the DoF-array belongs in.

• Solve the algebraic equations thus obtained.

• Display the results, and discuss accuracy.

Both operations, we saw, require *interpolants*, by which one can climb back from DoF arrays to fields, and perform some error analysis that will justify the choice of Hodge operators, and hence, tell to which extent the results can be trusted. *Whitney forms*, including the well known "edge element", provide these, at least for simplicial primal meshes.

For definiteness, we'll refer to this line of attack as *Generalized Finite Differences* (GFD). Indeed, it reduces to finite differences in their basic and best known form, Yee's FDTD method, when the 3-cells are regular bricks (which are

of course, even in GFD, the recommended volume shape wherever possible; the point is that you don't easily model real objects, like a cellular phone near a human head, say, by laying only bricks).

6.1 Where is the beef?

Now, the questions.

Q. – *Though known as a supporter of edge elements, you advocate here an approach in which finite elements seem to play quite a modest role. Did you change your mind?*

A. Hardly. Whitney forms *are* the right finite elements for differential forms. Edge elements for e, facet-elements for b, this is the basic tenet. I sure continue to stand for that. But yes, the role of finite elements, or perhaps more accurately said, of variational formulations in numerical methods may have been exaggerated. For long, the dominant paradigm was: Give a variational form to your problem (minimize, or perhaps stationarize, this or that functional over this or that Hilbert space of a priori eligible fields). Restrict consideration to some finite-dimensional subspace, generated by a set of basis fields, called finite elements. (There is more in the notion, but let's not err.) Hence a system of equations, or an ODE system. Solve. Plot. Check. Assess accuracy.

The obvious difference there, with respect to GFD, is that one needs finite elements much earlier. They are essential. Without them, no discrete equations. Besides, the *technique* to prove that the method works, based on the variational method, forces one to adopt the variational method in the first place in order to just *describe* the numerical method. Conceptually, these two phases of the action should be distinct—as they are in GFD.

I would be the last to deny the virtues of the variational method: Among other things, it helps us to be *thorough* in the description of a problem: No way to forget some subtle constraint of topological origin, for instance, or to impose too many boundary conditions, because it would result in a flawed formulation (with either an infinity of solutions or none), something which can then (hopefully) be detected by theoretical reasoning—not discovered with embarrassment a few minutes before the client comes for the contract-clinching demo.

Yet, not being able to simply explain (to a programmer, for instance) what the method is, which recipe one should follow, without first explaining the mathematical foundations in painful detail, is a terrible burden. It faces teachers with the impossible challenge of dumping high-level mathematics into reluctant minds *as a precondition to* giving them the useful (and much easier to understand) stuff. It reduces the workforce in research groups, since only the mathematical élite seems qualified enough. And it forces users to rely on proprietary "codes", rather than using open-source software pieces that would correspond to elements of the above "toolkit".

In GFD, while the methodological advantages of the variational method are retained,[1] such obstacles are removed, because the need for finite elements is relaxed and postponed. The first thing they serve for, building a discrete Hodge operator, can be achieved without them in a different, much more transparent way, if one uses mutually orthogonal meshes. Then comes the minute of truth—did we get a convergent method of approximation? And there, finite elements are necessary. But at this late stage, the question is one for specialists. Error analysis, speed of convergence, are no less important issues in GFD. They still call for the

[1] The tools in the kit incoporate them, in some way. This point could not be explained in general without some deep forays in topology, but an example will perhaps do. Consider eddy currents induced in a conductive torus by temporal variations of the current in some exciting coil. Setting up eddy-current equations inside the torus, plus magnetostatic equations outside (of course parameterized by time), plus the obvious conditions of tangential and normal continuity at the boundary, will *not* make a well posed problem. There is something else to say, about the relation between total current and rate of change of the flux traversing the torus loop. This kind of "non-local" boundary condition is very easily overlooked, and is one of these things the variational approach helps not to forget about. In GFD, the existence of a problem is revealed by the fact that the range of **G** does not completely fill the kernel of **R**, something a software element can test for you, automatically, as soon as the mesh has been completed. Work is in progress, in some research groups [5, 12] to design such "watchdog" software tools.

attention of professionals (and much is left there to do), but don't stand in the way of teaching and training. Which is as it should be: Driving instructors don't teach you thermodynamics, do they?

Q. – *How effective is all this, in practice? Have you done numerical experiments?*

A. This question has been asked a couple of times in a way that seemed to imply that GFD was a novel proposal I would be making. Not so. GFD is an approach common to several groups of researchers, loosely connected if at all, who discovered its main elements independently. (See e.g., in addition to alrady cited work, [10] on Tonti's method, Shashkov et al. [8] on "mimetic discretization", Chew et al. [13] on "lattice" approaches.) One should turn to their publications, especially those of the Darmstadt group, which has the largest accumulated experience [2], for evidence about the effectiveness of the method.

Q. – *Well, then, so nothing new under the sun. But if so, what's the purpose of all this differential geometric apparatus? After all, this is not elementary mathematics either.*

A. No, the geometric approach by itself does not generate new *algorithms*. Even the fact that it brings in a new *understanding* of existing algorithms is not so compelling a reason to decide on seriously studying it. After all, as a friend recently observed, when people in good command of an efficient method understand what they are doing in their own way, they tend to care little for explanations of the, allegedly, "real nature" of what they do. Which benefit they may expect in getting acquainted with a new perspective, when the effort involved is not negligible, is then a legitimate prior question.

However, the Computational Electromagnetism (CEM) community as a whole may find such benefits in a synthesis of the different viewpoints. The very fact that GFD, or whatever name one eventually adopts for this approach,[1] emerged in almost identical form in independent research groups, is a problem. Take for instance, [11], a work I only recently became aware of, to realize that, apart from minor variations in the definition of degrees of freedom (densities instead of integrated quantities), they propose and analyze the same network equations we dealt with here. Conceivably, the same thing might well happen again next month, with another paper from researchers outside the CEM community, and no doubt it would be the same network equations again. When parallel and independent efforts result in proving the same *theorem*, no one is surprised: there was some kind of logical necessity at work. But when it results in the same *numerical technique*, what can it mean, if not that some necessity of the same kind lies underneath?

If so, we must dig and find out. That's the real point of conceiving Maxwell's equations as relations between differential forms, i.e., between objects meant to be integrated: As soon as we have decided for one DoF per cell, the network equations follow, with inner necessity. All those making this basic choice will find the same equations.

Now, about the discrete constitutive laws, the situation is different. Not only is there this great divide between Galerkin-inspired methods and those which use a diagonal hodge, but tiny variants can be observed in the latter category. The relevant question now is, why this variety, and how much of it is allowed? The geometric approach offers an answer: in order to meet the necessary consistency property

$$(1) \qquad \tilde{\star}_\epsilon r_{\mathcal{M}} - r_{\mathcal{M}} \tilde{\star}_\epsilon \to 0 \text{ when } \mathcal{M} \to 0,$$

the discrete Hodge operator must satisfy

$$(2) \qquad \sum_{e' \in \mathcal{E}} \tilde{\star}_\epsilon^{ee'} e' = \epsilon\, \tilde{e},$$

[1] "Generalized finite volumes" has been suggested [7]. Tonti prefers "finite formulation of electromagnetism" [14], which may reveal a more ambitious agenda than simply discretizing an underlying "continuous" formulation, as we have done here. Weiland et al. [2] use "Finite Integration Technique" (FIT), which in my opinion describes well the derivation of the *network* equations, but they consider the construction of their hodges (diagonal, as a rule) as a component of FIT, as well. In contrast, I would include in GFD the Galerkin method (which they would probably consider foreign to FIT), for the reasons exposed here so far: it does generate the same network equations, and differs in the way the hodges (then, non-diagonal) are built.

where e' is the vector along edge e' and $\tilde{e}$ the vectorial area[2] of the dual facet pierced by edge e.

This is a *criterion*, the form of which explains why the hodge and the dual mesh are so closely related: A discrete Hodge operator is acceptable, as an element of the toolkit, if there is a way to devise a dual mesh that will satisfy (2). We found two examples: (1) When dual cells are orthogonal to the primal ones, and $\tilde{\star}_\epsilon^{ee}$ equals the ratio between vectors $\tilde{e}$ and e, all off-diagonal terms of $\tilde{\star}_\epsilon$ null, (2) When $\tilde{\star}_\epsilon$ equals the mass matrix of edge elements, dual cells then being those of the barycentric dual. Clearly, there are other possibilities, which remain to be explored, but we now know the rules of the game: Satisfy (2).

6.2 Technical issues

Q. – *How to construct the dual mesh?*

A. The problem, it should be stressed, is to build the *primal* mesh in such a way that a suitable dual one will exist, and "suitable" depends on which hodge one wants to use. This remark, unfortunately, is of little help. If primal 3-cells are tetrahedra with all dihedral angles acute, there is a simple solution: join the circumcenters of the primal cells, as in the Voronoi-Delaunay construction, and lo the dual orthogonal mesh. In this case, the problem is with the primal mesh, since getting the dual is a simple *local* process. But it's a tough problem, because this condition on angles is very strong.

Observe, however, that (2) is a *local* condition. It means in particular that one may well use the diagonal construction of the hodge in large homogeneous regions, where a uniform mesh is all right, and the Galerkin one at places, such as material boundaries, where the shape of the primal mesh is severely constrained. This results in a non-diagonal hodge, but with so few off-diagonal terms that it may not be a problem.

[2] Recall that the vectorial area of a triangle is its area times the normal vector. The vectorial area of a polyhedral surface is defined as the sum of vector areas of its triangular facets. When ϵ is not uniform, $\epsilon\,\tilde{e}$ is the weighted sum of vector areas of triangles that constitute the dual facet $\tilde{e}$, with the values of ϵ as weights.

Also remark that (2) is not "necessary" in that strong a sense, since its purpose is to satisfy (1), an asymptotic condition only. If (2) is violated for a small proportion of edges, and if the (virtual) refinement process makes this proportion tend to zero, (1) can still hold. So it's tolerable to cheat on some edges or facets, as Fig. 1 suggests. (Similar procedures have been tested for the MAFIA codes [1].)

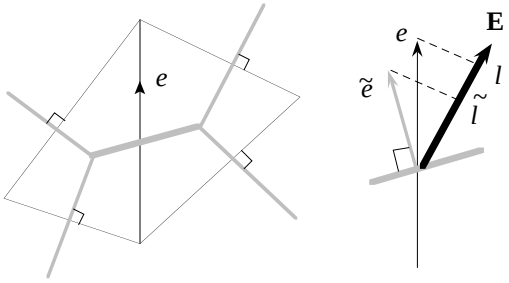

Figure 1. "Cheating", when an occasional dual facet cannot be made orthogonal to its primal edge (e here). Instead of ϵ area$(\tilde{e})$/length(e), set $\tilde{\star}_\epsilon^{ee}$ equal to $\epsilon\,(\mathbf{E}\cdot\tilde{e})/(\mathbf{E}\cdot e)$, i.e., to $\epsilon\,l'/l$, with the notations of the figure, where **E** is a guesstimate of the direction of the electric field in the vicinity. (Recall that e and $\tilde{e}$ stand for the vector along e and the vectorial area of $\tilde{e}$.) This amounts to replace length(e) by the length of its projection onto the support of **E**, and area$(\tilde{e})$ by the apparent area when looking parallel to **E**.

Q. – *Which elements for cells of general shape?*

A. We have had edge elements for hexahedra (not necessarily with plane facets) for long [15], and more recently, for shapes such as pyramids [4, 3, 6], compatible enough to ensure tangential continuity of the interpolated fields when used together. The problem is crucial if one wants the analytical form of the interpolants, in order for instance to compute the Galerkin hodge. If one uses the orthogonal construction and the diagonal hodge, it's no more a practical issue but a theoretical one: The only problem is to find *some* convergent $p_{\mathcal{M}}$, in last installment's notation. To this effect, one may rely on the convergence properties of simplicial Whitney forms, and build interpolants as weighted combinations of these.

To be specific, suppose each p-cell of the mesh $\mathcal{M}$, for all p, has been provided with a "center", in the precise sense of *JSAEM 7*, 2 (1999), p. 158, i.e., a point with respect to which the cell is star-shaped. Then, join the centers in order to obtain a simplicial re-

finement, $\overline{\mathcal{M}}$ say, where the new set $\overline{\mathcal{S}}_p$ of p-simplices contains the old one $\mathcal{S}_p$. In similar style, let $\mathbf{u}$ and $\overline{\mathbf{u}}$ stand for DoF arrays indexed over $\mathcal{S}_p$ and $\overline{\mathcal{S}}_p$ respectively, with $\overline{\mathbf{u}}_s = \mathbf{u}_s$ for all s in $\mathcal{S}_p$. Our problem is to define $p_{\mathcal{M}}\mathbf{u}$, knowing what $p_{\overline{\mathcal{M}}}\overline{\mathbf{u}}$ is. Isn't it obvious? Just take for $p_{\mathcal{M}}\mathbf{u}$ the *smallest*, in the energy norm, of the $p_{\overline{\mathcal{M}}}\overline{\mathbf{u}}$'s, with respect to all $\overline{\mathbf{u}}$'s compatible with $\mathbf{u}$.

The family of interpolants thus obtained is to our cellular mesh, for all purposes, what Whitney forms were to a simplicial mesh. Purists, however, will object against calling them "Whitney forms", because they are metric-dependent, unlike the standard Whitney forms. The same construction on the dual side will provide similar pseudo-whitneys on the dual mesh.

Q. – *What about higher-degree Whitney forms?*

A. When using the Galerkin method, finite elements of higher polynomial degree give schemes of higher accuracy, which more than compensates for the increased number of DoFs. Hence the interest for Whitney forms of higher polynomial degree. But a caveat about that: Our approach to network equations, as exposed so far, is in jeopardy if there is more than one DoF per cell, so it's not so clear what to do of higher-degree Whitney forms out of the rather restrictive context of the Galerkin method. Since the red thread in GFD has been, up to now, the duality between chains (formal sums of cells) and DoF arrays, these forms make full sense only if we can associate their DoFs with well identified geometric objects.

Thus, though this is a very promising area for future progress, it should be explored with the right equipment. A good compass, in my opinion, is this "partition of unity" property,

$$\sum\nolimits_{e\in\mathcal{E}} e\rangle\langle w^e(x) = 1,$$

in the esoteric notation of last time, or more clearly, in terms of vector proxies,

$$\sum\nolimits_{e\in\mathcal{E}} (\mathbf{W}^e(x)\cdot v)\, e = v,$$

at all points x and for all vectors v, where e, again, is the vector along edge e. From this, which generalizes the $\sum_{n\in\mathcal{N}} \lambda^n(x) = 1$ valid for nodal 0-forms,[3] and has counterparts for simplices of all degrees, we were able to prove that the mass matrix of edge elements does satisfy the criterion (2). This was, at the root, the reason why edge elements give a convergent scheme in the Galerkin approach. Therefore, this property should be taken seriously: whatever Whitney elements of higher degree are, *they must constitute a partition of unity*.

Now (a heuristic move, not a formal assertion, even less a proof), the product of two partitions of unity makes a partition of unity. Let us, therefore, take as second-degree edge elements, on a simplicial mesh, the products $\lambda^n\, w^e$, indexed over the set $\mathcal{N}\times\mathcal{E}$. Actually, let's do that for all simplicial degrees, in all dimensions: second-degree p-forms are the products $\lambda^n\, w^s$, where n spans the set of nodes, and s the set $\mathcal{S}_p$ of p-simplices.

Forms obtained in this manner have all the required properties. In particular, they constitute an exact sequence, i.e., if for instance $b = \sum_{n,f} \mathbf{b}_{nf}\lambda^n\, w^f$ has a divergence-free proxy ($\mathrm{d}b = 0$), then there are DoFs $\mathbf{a}_{ne}$ such that $b = \mathrm{d}(\sum_{n,e} \mathbf{a}_{ne}\lambda^n\, w^e)$.

The main problem with such forms is the interpretation of degrees of freedom such as $\mathbf{a}_{ne}$. With standard Whitney forms, the DoF $\mathbf{a}_e$ was the integral of the 1-form $a = \sum_{e'} \mathbf{a}_{e'} w^{e'}$ over edge e. Here, we cannot expect to find a family of simple 1-chains such that $\mathbf{a}_{ne}$ would be the integral of $a = \sum_{ne} \mathbf{a}_{ne}\lambda^n\, w^e$ over one of them, *and have a null integral over all other chains* of the family. Although such a family will exist, the emphasized condition makes it anything but simple. We must be content with less: 1-cells such that integrals of $\sum_{n,e} \mathbf{a}_{ne}\lambda^n\, w^e$ over them determine the $\mathbf{a}_{ne}$s, and in clear one-to-one correspondence with the basis forms $\lambda^n\, w^e$ (Fig. 2). Let's call such cells (introduced in [9]) "small" edges, an ad hoc terminology.

[3] I denote here by λ^n the same nodal hat functions that were called w^n last time. This is for readability only, $\lambda^n w^s$ being better than $w^n w^s$ in this respect.

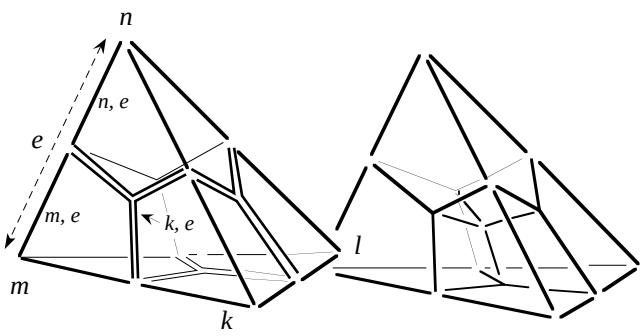

Figure 2. Left: Small edges (some of them are broken lines) associated with forms $\lambda^n\, w^e$. Right: One of the possible systems of chain elements in 1-1 correspondence with *independent* degrees of freedom.

A problem then emerges: There are 24 small edges, but the dimension of the space generated by the $\lambda^n\, w^e$, if the mesh reduces to this single tetrahedron, is only 20! This is due to the relation

$$\sum\nolimits_e \mathbf{R}_{fe} \lambda^{f-e} w^e = 0,$$

here written for edges and facets, but actually a general property of Whitney forms. (λ^{f-e} denotes, if $\mathbf{R}_{fe} \neq 0$, the hat function of the node opposite edge e in facet f.) Since each facet contributes one such constraint, the span of the forms $\lambda^n\, w^e$ has dimension $2(E+F)$.

We might just omit one small edge out of three on each facet, but this is an ugly solution. Fig. 2, right, suggests a better one. It shows 20 "chain elements" (12 half-edges, four inner segments, and the four three-pronged stars, which themselves are 1-chains formed of 3 small segments each). This time, 1-chains formed from these elements have the desired property (and no smaller set of chain elements can do): A second-order one-form whose integrals over all of them vanish must itself vanish.

The reader will easily guess about "small facets" and "small volumes", and may want to tackle the challenge of finding a nicely symmetric set of 2-chain elements (15 instead of 16; total dimension of the span $3(F+T)$).

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