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Physics 8.962 Spring 2000

Introduction to Tensor Calculus for General Relativity

1 Introduction

There are three essential ideas underlying general relativity (GR). The first is that spacetime may be described as a curved, four-dimensional mathematical structure called a pseudo-Riemannian manifold. In brief, time and space together comprise a curved four-dimensional non-Euclidean geometry. Consequently, the practitioner of GR must be familiar with the fundamental geometrical properties of curved spacetime. In particular, the laws of physics must be expressed in a form that is valid independently of any coordinate system used to label points in spacetime.

The second essential idea underlying GR is that at every spacetime point there exist locally inertial reference frames, corresponding to locally flat coordinates carried by freely falling observers, in which the physics of GR is locally indistinguishable from that of special relativity. This is Einstein's famous strong equivalence principle and it makes general relativity an extension of special relativity to a curved spacetime. The third key idea is that mass (as well as mass and momentum flux) curves spacetime in a manner described by the tensor field equations of Einstein.

These three ideas are exemplified by contrasting GR with Newtonian gravity. In the Newtonian view, gravity is a force accelerating particles through Euclidean space, while time is absolute. From the viewpoint of GR as a theory of curved spacetime, there is no gravitational force. Rather, in the absence of electromagnetic and other forces, particles follow the straightest possible paths (geodesics) through a spacetime curved by mass. Freely falling particles define locally inertial reference frames. Time and space are not absolute but are combined into the four-dimensional manifold called spacetime.

In special relativity there exist global inertial frames. This is no longer true in the presence of gravity. However, there are *local* inertial frames in GR, such that within a suitably small spacetime volume around an event (just how small is discussed e.g. in MTW Chapter 1), one may choose coordinates corresponding to a nearly-flat spacetime. Thus, the *local* properties of special relativity carry over to GR. The mathematics of vectors and tensors applies in GR much as it does in SR, with the restriction that vectors and tensors are defined independently at each spacetime event (or within a sufficiently small neighborhood so that the spacetime is sensibly flat).

Working with GR, particularly with the Einstein field equations, requires some understanding of differential geometry. In these notes we will develop the essential mathematics needed to describe physics in curved spacetime. Many physicists receive their introduction to this mathematics in the excellent book of Weinberg (1972). Weinberg minimizes the geometrical content of the equations by representing tensors using component notation. We believe that it is equally easy to work with a more geometrical description, with the additional benefit that geometrical notation makes it easier to distinguish physical results that are true in any coordinate system (e.g., those expressible using vectors) from those that are dependent on the coordinates. Because the geometry of spacetime is so intimately related to physics, we believe that it is better to highlight the geometry from the outset. In fact, using a geometrical approach allows us to develop the essential differential geometry as an extension of vector calculus. Our treatment is closer to that Wald (1984) and closer still to Misner, Thorne and Wheeler (1973, MTW). These books are rather advanced. For the newcomer to general relativity we warmly recommend Schutz (1985). Our notation and presentation is patterned largely after Schutz. It expands on MTW Chapters 2, 3, and 8. The student wishing additional practice problems in GR should consult Lightman *et al.* (1975). A slightly more advanced mathematical treatment is provided in the excellent notes of Carroll (1997).

These notes assume familiarity with special relativity. We will adopt units in which the speed of light $c = 1$. Greek indices (μ, ν, etc., which take the range $\{0, 1, 2, 3\}$) will be used to represent components of tensors. The Einstein summation convention is assumed: repeated upper and lower indices are to be summed over their ranges, e.g., $A^\mu B_\mu \equiv A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$. Four-vectors will be represented with an arrow over the symbol, e.g., $\vec{A}$, while one-forms will be represented using a tilde, e.g., $\tilde{B}$. Spacetime points will be denoted in boldface type; e.g., $\mathbf{x}$ refers to a point with coordinates x^μ. Our metric has signature $+2$; the flat spacetime Minkowski metric components are $\eta_{\mu\nu} = \mathbf{diag}(-1, +1, +1, +1)$.

2 Vectors and one-forms

The essential mathematics of general relativity is differential geometry, the branch of mathematics dealing with smoothly curved surfaces (differentiable manifolds). The physicist does not need to master all of the subtleties of differential geometry in order to use general relativity. (For those readers who want a deeper exposure to differential geometry, see the introductory texts of Lovelock and Rund 1975, Bishop and Goldberg 1980, or Schutz 1980.) It is sufficient to develop the needed differential geometry as a straightforward extension of linear algebra and vector calculus. However, it is important to keep in mind the geometrical interpretation of physical quantities. For this reason, we will not shy from using abstract concepts like points, curves and vectors, and we will distinguish between a vector $\vec{A}$ and its components A^μ. Unlike some other authors (e.g., Weinberg 1972), we will introduce geometrical objects in a coordinate-free manner, only later introducing coordinates for the purpose of simplifying calculations. This approach requires that we distinguish vectors from the related objects called one-forms. Once the differences and similarities between vectors, one-forms and tensors are clear, we will adopt a unified notation that makes computations easy.

2.1 Vectors

We begin with vectors. A vector is a quantity with a magnitude and a direction. This primitive concept, familiar from undergraduate physics and mathematics, applies equally in general relativity. An example of a vector is $d\vec{x}$, the difference vector between two infinitesimally close points of spacetime. Vectors form a linear algebra (i.e., a vector space). If $\vec{A}$ is a vector and a is a real number (scalar) then $a\vec{A}$ is a vector with the same direction (or the opposite direction, if $a < 0$) whose length is multiplied by $|a|$. If $\vec{A}$ and $\vec{B}$ are vectors then so is $\vec{A} + \vec{B}$. These results are as valid for vectors in a curved four-dimensional spacetime as they are for vectors in three-dimensional Euclidean space.

Note that we have introduced vectors without mentioning coordinates or coordinate transformations. Scalars and vectors are invariant under coordinate transformations; vector components are not. The whole point of writing the laws of physics (e.g., $\vec{F} = m\vec{a}$) using scalars and vectors is that these laws do not depend on the coordinate system imposed by the physicist.

We denote a spacetime point using a boldface symbol: $\mathbf{x}$. (This notation is *not* meant to imply coordinates.) Note that $\mathbf{x}$ refers to a point, not a vector. In a curved spacetime the concept of a radius vector $\vec{x}$ pointing from some origin to each point $\mathbf{x}$ is not useful because vectors defined at two different points cannot be added straightforwardly as they can in Euclidean space. For example, consider a sphere embedded in ordinary three-dimensional Euclidean space (i.e., a two-sphere). A vector pointing east at one point on the equator is seen to point radially outward at another point on the equator whose longitude is greater by $90°$. The radially outward direction is undefined on the sphere.

Technically, we are discussing *tangent vectors* that lie in the *tangent space* of the manifold at each point. For example, a sphere may be embedded in a three-dimensional Euclidean space into which may be placed a plane tangent to the sphere at a point. A two-dimensional vector space exists at the point of tangency. However, such an embedding is not required to define the tangent space of a manifold (Wald 1984). As long as the space is smooth (as assumed in the formal definition of a manifold), the difference vector $d\vec{x}$ between two infinitesimally close points may be defined. The set of all $d\vec{x}$ defines the tangent space at $\mathbf{x}$. By assigning a tangent vector to every spacetime point, we can recover the usual concept of a vector field. However, without additional preparation one cannot compare vectors at different spacetime points, because they lie in different tangent spaces. In later notes we introduce will parallel transport as a means of making this comparison. Until then, we consider only tangent vectors at $\mathbf{x}$. To emphasize the status of a tangent vector, we will occasionally use a subscript notation: $\vec{A}_X$.

2.2 One-forms and dual vector space

Next we introduce one-forms. A one-form is defined as a linear scalar function of a vector. That is, a one-form takes a vector as input and outputs a scalar. For the one-form $\tilde{P}$, $\tilde{P}(\vec{V})$ is also called the scalar product and may be denoted using angle brackets:

$$\tilde{P}(\vec{V}) = \langle \tilde{P}, \vec{V} \rangle \ . \tag{1}$$

The one-form is a linear function, meaning that for all scalars a and b and vectors $\vec{V}$ and $\vec{W}$, the one-form $\tilde{P}$ satisfies the following relations:

$$\tilde{P}(a\vec{V} + b\vec{W}) = \langle \tilde{P}, a\vec{V} + b\vec{W} \rangle = a\langle \tilde{P}, \vec{V} \rangle + b\langle \tilde{P}, \vec{W} \rangle = a\tilde{P}(\vec{V}) + b\tilde{P}(\vec{W}) \ . \tag{2}$$

Just as we may consider any function $f(\ )$ as a mathematical entity independently of any particular argument, we may consider the one-form $\tilde{P}$ independently of any particular vector $\vec{V}$. We may also associate a one-form with each spacetime point, resulting in a one-form field $\tilde{P} = \tilde{P}_{\mathbf{X}}$. Now the distinction between a point a vector is crucial: $\tilde{P}_{\mathbf{X}}$ is a one-form at point $\mathbf{x}$ while $\tilde{P}(\vec{V})$ is a scalar, defined implicitly at point $\mathbf{x}$. The scalar product notation with subscripts makes this more clear: $\langle \tilde{P}_{\mathbf{X}}, \vec{V}_{\mathbf{X}} \rangle$.

One-forms obey their own linear algebra distinct from that of vectors. Given any two scalars a and b and one-forms $\tilde{P}$ and $\tilde{Q}$, we may define the one-form $a\tilde{P} + b\tilde{Q}$ by

$$(a\tilde{P} + b\tilde{Q})(\vec{V}) = \langle a\tilde{P} + b\tilde{Q}, \vec{V} \rangle = a\langle \tilde{P}, \vec{V} \rangle + b\langle \tilde{Q}, \vec{V} \rangle = a\tilde{P}(\vec{V}) + b\tilde{Q}(\vec{V}) \ . \tag{3}$$

Comparing equations (2) and (3), we see that vectors and one-forms are linear operators on each other, producing scalars. It is often helpful to consider a vector as being a linear scalar function of a one-form. Thus, we may write $\langle \tilde{P}, \vec{V} \rangle = \tilde{P}(\vec{V}) = \vec{V}(\tilde{P})$. The set of all one-forms is a vector space distinct from, but complementary to, the linear vector space of vectors. The vector space of one-forms is called the *dual* vector (or cotangent) space to distinguish it from the linear space of vectors (tangent space).

Although one-forms may appear to be highly abstract, the concept of dual vector spaces is familiar to any student of quantum mechanics who has seen the Dirac bra-ket notation. Recall that the fundamental object in quantum mechanics is the state vector, represented by a ket $|\psi\rangle$ in a linear vector space (Hilbert space). A distinct Hilbert space is given by the set of bra vectors $\langle\phi|$. Bra vectors and ket vectors are linear scalar functions of each other. The scalar product $\langle\phi|\psi\rangle$ maps a bra vector and a ket vector to a scalar called a probability amplitude. The distinction between bras and kets is necessary because probability amplitudes are complex numbers. As we will see, the distinction between vectors and one-forms is necessary because spacetime is curved.

3 Tensors

Having defined vectors and one-forms we can now define tensors. A tensor of rank (m, n), also called a (m, n) tensor, is defined to be a scalar function of m one-forms and n vectors that is linear in all of its arguments. It follows at once that scalars are tensors of rank $(0, 0)$, vectors are tensors of rank $(1, 0)$ and one-forms are tensors of rank $(0, 1)$. We may denote a tensor of rank $(2, 0)$ by $\mathsf{T}(\tilde{P}, \tilde{Q})$; one of rank $(2, 1)$ by $\mathsf{T}(\tilde{P}, \tilde{Q}, \vec{A})$, etc. Our notation will not distinguish a $(2, 0)$ tensor T from a $(2, 1)$ tensor T, although a notational distinction could be made by placing m arrows and n tildes over the symbol, or by appropriate use of dummy indices (Wald 1984).

The scalar product is a tensor of rank $(1, 1)$, which we will denote I and call the identity tensor:

$$\mathsf{I}(\tilde{P}, \vec{V}) \equiv \langle\tilde{P}, \vec{V}\rangle = \tilde{P}(\vec{V}) = \vec{V}(\tilde{P}) \ . \tag{4}$$

We call I the identity because, when applied to a fixed one-form $\tilde{P}$ and *any* vector $\vec{V}$, it returns $\tilde{P}(\vec{V})$. Although the identity tensor was defined as a mapping from a one-form and a vector to a scalar, we see that it may equally be interpreted as a mapping from a one-form to the same one-form: $\mathsf{I}(\tilde{P}, \cdot) = \tilde{P}$, where the dot indicates that an argument (a vector) is needed to give a scalar. A similar argument shows that I may be considered the identity operator on the space of vectors $\vec{V}$: $\mathsf{I}(\cdot, \vec{V}) = \vec{V}$.

A tensor of rank (m, n) is linear in all its arguments. For example, for $(m = 2, n = 0)$ we have a straightforward extension of equation (2):

$$\mathsf{T}(a\tilde{P} + b\tilde{Q}, c\tilde{R} + d\tilde{S}) = ac\,\mathsf{T}(\tilde{P}, \tilde{R}) + ad\,\mathsf{T}(\tilde{P}, \tilde{S}) + bc\,\mathsf{T}(\tilde{Q}, \tilde{R}) + bd\,\mathsf{T}(\tilde{q}, \tilde{S}) \ . \tag{5}$$

Tensors of a given rank form a linear algebra, meaning that a linear combination of tensors of rank (m, n) is also a tensor of rank (m, n), defined by straightforward extension of equation (3). Two tensors (of the same rank) are equal if and only if they return the same scalar when applied to all possible input vectors and one-forms. Tensors of different rank cannot be added or compared, so it is important to keep track of the rank of each tensor. Just as in the case of scalars, vectors and one-forms, tensor fields $\mathsf{T}_{\mathbf{X}}$ are defined by associating a tensor with each spacetime point.

There are three ways to change the rank of a tensor. The first, called the tensor (or outer) product, combines two tensors of ranks (m_1, n_1) and (m_2, n_2) to form a tensor of rank $(m_1 + m_2, n_1 + n_2)$ by simply combining the argument lists of the two tensors and thereby expanding the dimensionality of the tensor space. For example, the tensor product of two vectors $\vec{A}$ and $\vec{B}$ gives a rank $(2, 0)$ tensor

$$\mathsf{T} = \vec{A} \otimes \vec{B} \;, \quad \mathsf{T}(\tilde{P}, \tilde{Q}) \equiv \vec{A}(\tilde{P})\, \vec{B}(\tilde{Q}) \;. \tag{6}$$

We use the symbol $\otimes$ to denote the tensor product; later we will drop this symbol for notational convenience when it is clear from the context that a tensor product is implied. Note that the tensor product is non-commutative: $\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$ (unless $\vec{B} = c\vec{A}$ for some scalar c) because $\vec{A}(\tilde{P})\, \vec{B}(\tilde{Q}) \neq \vec{A}(\tilde{Q})\, \vec{B}(\tilde{P})$ for all $\tilde{P}$ and $\tilde{Q}$. We use the symbol $\otimes$ to denote the tensor product of any two tensors, e.g., $\tilde{P} \otimes \mathsf{T} = \tilde{P} \otimes \vec{A} \otimes \vec{B}$ is a tensor of rank $(2, 1)$. The second way to change the rank of a tensor is by contraction, which reduces the rank of a (m, n) tensor to $(m-1, n-1)$. The third way is the gradient. We will discuss contraction and gradients later.

3.1 Metric tensor

The scalar product (eq. 1) requires a vector and a one-form. Is it possible to obtain a scalar from two vectors or two one-forms? From the definition of tensors, the answer is clearly yes. Any tensor of rank $(0, 2)$ will give a scalar from two vectors and any tensor of rank $(2, 0)$ combines two one-forms to give a scalar. However, there is a particular $(0, 2)$ tensor field $\mathsf{g}_{\mathbf{X}}$ called the metric tensor and a related $(2, 0)$ tensor field $\mathsf{g}_{\mathbf{X}}^{-1}$ called the inverse metric tensor for which special distinction is reserved. The metric tensor is a symmetric bilinear scalar function of two vectors. That is, given vectors $\vec{V}$ and $\vec{W}$, g returns a scalar, called the dot product:

$$\mathsf{g}(\vec{V}, \vec{W}) = \vec{V} \cdot \vec{W} = \vec{W} \cdot \vec{V} = \mathsf{g}(\vec{W}, \vec{V}) \;. \tag{7}$$

Similarly, g^{-1} returns a scalar from two one-forms $\tilde{P}$ and $\tilde{Q}$, which we also call the dot product:

$$\mathsf{g}^{-1}(\tilde{P}, \tilde{Q}) = \tilde{P} \cdot \tilde{Q} = \tilde{P} \cdot \tilde{Q} = \mathsf{g}^{-1}(\tilde{P}, \tilde{Q}) \;. \tag{8}$$

Although a dot is used in both cases, it should be clear from the context whether g or g^{-1} is implied. We reserve the dot product notation for the metric and inverse metric tensors just as we reserve the angle brackets scalar product notation for the identity tensor (eq. 4). Later (in eq. 41) we will see what distinguishes g from other $(0, 2)$ tensors and g^{-1} from other symmetric $(2, 0)$ tensors.

One of the most important properties of the metric is that it allows us to convert vectors to one-forms. If we forget to include $\vec{W}$ in equation (7) we get a quantity, denoted $\tilde{V}$, that behaves like a one-form:

$$\tilde{V}(\,\cdot\,) \equiv \mathsf{g}(\vec{V},\,\cdot\,) = \mathsf{g}(\,\cdot\,,\vec{V})\ , \tag{9}$$

where we have inserted a dot to remind ourselves that a vector must be inserted to give a scalar. (Recall that a one-form is a scalar function of a vector!) We use the same letter to indicate the relation of $\vec{V}$ and $\tilde{V}$.

Thus, the metric g is a mapping from the space of vectors to the space of one-forms: $\mathsf{g} : \vec{V} \to \tilde{V}$. By definition, the inverse metric g^{-1} is the inverse mapping: $\mathsf{g}^{-1} : \tilde{V} \to \vec{V}$. (The inverse always exists for nonsingular spacetimes.) Thus, if $\tilde{V}$ is defined for any $\vec{V}$ by equation (9), the inverse metric tensor is defined by

$$\vec{V}(\,\cdot\,) \equiv \mathsf{g}^{-1}(\tilde{V},\,\cdot\,) = \mathsf{g}^{-1}(\,\cdot\,,\tilde{V})\ . \tag{10}$$

Equations (4) and (7)–(10) give us several equivalent ways to obtain scalars from vectors $\vec{V}$ and $\vec{W}$ and their associated one-forms $\tilde{V}$ and $\tilde{W}$:

$$\langle\tilde{V},\vec{W}\rangle = \langle\tilde{W},\vec{V}\rangle = \vec{V}\cdot\vec{W} = \tilde{V}\cdot\tilde{W} = \mathsf{I}(\tilde{V},\vec{W}) = \mathsf{I}(\tilde{W},\vec{V}) = \mathsf{g}(\vec{V},\vec{W}) = \mathsf{g}^{-1}(\tilde{V},\tilde{W})\ . \tag{11}$$

3.2 Basis vectors and one-forms

It is possible to formulate the mathematics of general relativity entirely using the abstract formalism of vectors, forms and tensors. However, while the geometrical (coordinate-free) interpretation of quantities should always be kept in mind, the abstract approach often is not the most practical way to perform calculations. To simplify calculations it is helpful to introduce a set of linearly independent basis vector and one-form fields spanning our vector and dual vector spaces. In the same way, practical calculations in quantum mechanics often start by expanding the ket vector in a set of basis kets, e.g., energy eigenstates. By definition, the dimensionality of spacetime (four) equals the number of linearly independent basis vectors and one-forms.

We denote our set of basis vector fields by $\{\vec{e}_{\mu\mathbf{x}}\}$, where μ labels the basis vector (e.g., $\mu = 0, 1, 2, 3$) and $\mathbf{x}$ labels the spacetime point. Any four linearly independent basis vectors at each spacetime point will work; we do not not impose orthonormality or any other conditions in general, nor have we implied any relation to coordinates (although later we will). Given a basis, we may expand any vector field $\vec{A}$ as a linear combination of basis vectors:

$$\vec{A}_{\mathbf{x}} = A^{\mu}{}_{\mathbf{x}}\,\vec{e}_{\mu\mathbf{x}} = A^0{}_{\mathbf{x}}\vec{e}_{0\mathbf{x}} + A^1{}_{\mathbf{x}}\vec{e}_{1\mathbf{x}} + A^2{}_{\mathbf{x}}\vec{e}_{2\mathbf{x}} + A^3{}_{\mathbf{x}}\vec{e}_{3\mathbf{x}}\ . \tag{12}$$

Note our placement of subscripts and superscripts, chosen for consistency with the Einstein summation convention, which requires pairing one subscript with one superscript. The coefficients A^μ are called the components of the vector (often, the contravariant components). Note well that the coefficients A^μ depend on the basis vectors but $\vec{A}$ does not!

Similarly, we may choose a basis of one-form fields in which to expand one-forms like $\tilde{A}_{\mathbf{X}}$. Although any set of four linearly independent one-forms will suffice for each spacetime point, we prefer to choose a special one-form basis called the *dual basis* and denoted $\{\tilde{e}^\mu{}_{\mathbf{X}}\}$. Note that the placement of subscripts and superscripts is significant; we never use a subscript to label a basis one-form while we never use a superscript to label a basis vector. Therefore, $\tilde{e}^\mu$ is *not* related to $\vec{e}_\mu$ through the metric (eq. 9): $\tilde{e}^\mu(\,\cdot\,) \neq \mathbf{g}(\vec{e}_\mu,\,\cdot\,)$. Rather, the dual basis one-forms are defined by imposing the following 16 requirements at each spacetime point:

$$\langle \tilde{e}^\mu{}_{\mathbf{X}}, \vec{e}_{\nu\,\mathbf{X}} \rangle = \delta^\mu{}_\nu \ , \tag{13}$$

where $\delta^\mu{}_\nu$ is the Kronecker delta, $\delta^\mu{}_\nu = 1$ if $\mu = \nu$ and $\delta^\mu{}_\nu = 0$ otherwise, with the same values for each spacetime point. (We must always distinguish subscripts from superscripts; the Kronecker delta always has one of each.) Equation (13) is a system of four linear equations at each spacetime point for each of the four quantities $\tilde{e}^\mu$ and it has a unique solution. (The reader may show that any nontrivial transformation of the dual basis one-forms will violate eq. 13.) Now we may expand any one-form field $\tilde{P}_{\mathbf{X}}$ in the basis of one-forms:

$$\tilde{P}_{\mathbf{X}} = P_{\mu\,\mathbf{X}}\, \tilde{e}^\mu{}_{\mathbf{X}} \ . \tag{14}$$

The component P_μ of the one-form $\tilde{P}$ is often called the covariant component to distinguish it from the contravariant component P^μ of the vector $\vec{P}$. In fact, because we have consistently treated vectors and one-forms as distinct, we should not think of these as being distinct "components" of the same entity at all.

There is a simple way to get the components of vectors and one-forms, using the fact that vectors are scalar functions of one-forms and vice versa. One simply evaluates the vector using the appropriate basis one-form:

$$\vec{A}(\tilde{e}^\mu) = \langle \tilde{e}^\mu, \vec{A} \rangle = \langle \tilde{e}^\mu, A^\nu \vec{e}_\nu \rangle = \langle \tilde{e}^\mu, \vec{e}_\nu \rangle A^\nu = \delta^\mu{}_\nu A^\nu = A^\mu \ , \tag{15}$$

and conversely for a one-form:

$$\tilde{P}(\vec{e}_\mu) = \langle \tilde{P}, \vec{e}_\mu \rangle = \langle P_\nu \tilde{e}^\nu, \vec{e}_\mu \rangle = \langle \tilde{e}^\nu, \vec{e}_\mu \rangle P_\nu = \delta^\nu{}_\mu P_\nu = P_\mu \ . \tag{16}$$

We have suppressed the spacetime point $\mathbf{x}$ for clarity, but it is always implied.

3.3 Tensor algebra

We can use the same ideas to expand tensors as products of components and basis tensors. First we note that a basis for a tensor of rank (m, n) is provided by the tensor product of m vectors and n one-forms. For example, a $(0,2)$ tensor like the metric tensor can be decomposed into basis tensors $\tilde{e}^\mu \otimes \tilde{e}^\nu$. The *components* of a tensor of rank (m, n), labeled with m superscripts and n subscripts, are obtained by evaluating the tensor using m basis one-forms and n basis vectors. For example, the components of the $(0,2)$ metric tensor, the $(2,0)$ inverse metric tensor and the $(1,1)$ identity tensor are

$$g_{\mu\nu} \equiv \mathsf{g}(\vec{e}_\mu, \vec{e}_\nu) = \vec{e}_\mu \cdot \vec{e}_\nu \ , \quad g^{\mu\nu} \equiv \mathsf{g}^{-1}(\tilde{e}^\mu, \tilde{e}^\nu) = \tilde{e}^\mu \cdot \tilde{e}^\nu \ , \quad \delta^\mu{}_\nu = \mathsf{I}(\tilde{e}^\mu, \vec{e}_\nu) = \langle \tilde{e}^\mu, \vec{e}_\nu \rangle \ . \tag{17}$$

(The last equation follows from eqs. 4 and 13.) The tensors are given by summing over the tensor product of basis vectors and one-forms:

$$\mathsf{g} = g_{\mu\nu}\, \tilde{e}^\mu \otimes \tilde{e}^\nu \ , \quad \mathsf{g}^{-1} = g^{\mu\nu}\, \vec{e}_\mu \otimes \vec{e}_\nu \ , \quad \mathsf{I} = \delta^\mu{}_\nu\, \vec{e}_\mu \otimes \tilde{e}^\nu \ . \tag{18}$$

The reader should check that equation (18) follows from equations (17) and the duality condition equation (13).

Basis vectors and one-forms allow us to represent any tensor equations using components. For example, the dot product between two vectors or two one-forms and the scalar product between a one-form and a vector may be written using components as

$$\vec{A} \cdot \vec{B} = g_{\mu\nu} A^\mu A^\nu \ , \quad \langle \tilde{P}, \vec{A} \rangle = P_\mu A^\mu \ , \quad \tilde{P} \cdot \tilde{Q} = g^{\mu\nu} P_\mu P_\nu \ . \tag{19}$$

The reader should prove these important results.

If two tensors of the same rank are equal in one basis, i.e., if all of their components are equal, then they are equal in any basis. While this mathematical result is obvious from the basis-free meaning of a tensor, it will have important physical implications in GR arising from the Equivalence Principle.

As we discussed above, the metric and inverse metric tensors allow us to transform vectors into one-forms and vice versa. If we evaluate the components of $\vec{V}$ and the one-form $\tilde{V}$ defined by equations (9) and (10), we get

$$V_\mu = \mathsf{g}(\vec{e}_\mu, \vec{V}) = g_{\mu\nu} V^\nu \ , \quad V^\mu = \mathsf{g}^{-1}(\tilde{e}^\mu, \tilde{V}) = g^{\mu\nu} V_\nu \ . \tag{20}$$

Because these two equations must hold for any vector $\vec{V}$, we conclude that the matrix defined by $g^{\mu\nu}$ is the inverse of the matrix defined by $g_{\mu\nu}$:

$$g^{\mu\kappa} g_{\kappa\nu} = \delta^\mu{}_\nu \ . \tag{21}$$

We also see that the metric and its inverse are used to lower and raise indices on components. Thus, given two vectors $\vec{V}$ and $\vec{W}$, we may evaluate the dot product any of four equivalent ways (cf. eq. 11):

$$\vec{V} \cdot \vec{W} = g_{\mu\nu} V^\mu W^\nu = V^\mu W_\mu = V_\mu W^\mu = g^{\mu\nu} V_\mu W_\nu \ . \tag{22}$$

In fact, the metric and its inverse may be used to transform tensors of rank (m, n) into tensors of any rank $(m+k, n-k)$ where $k = -m, -m+1, \ldots, n$. Consider, for example, a $(1,2)$ tensor T with components

$$T^\mu{}_{\nu\lambda} \equiv \mathsf{T}(\tilde{e}^\mu, \vec{e}_\nu, \vec{e}_\lambda) \ . \tag{23}$$

If we fail to plug in the one-form $\tilde{e}^\mu$, the result is the vector $T^\kappa{}_{\nu\lambda}\vec{e}_\kappa$. (A one-form must be inserted to return the quantity $T^\kappa{}_{\nu\lambda}$.) This vector may then be inserted into the metric tensor to give the components of a $(0,3)$ tensor:

$$T_{\mu\nu\lambda} \equiv \mathsf{g}(\vec{e}_\mu, T^\kappa{}_{\nu\lambda}\vec{e}_\kappa) = g_{\mu\kappa}T^\kappa{}_{\nu\lambda} \ . \tag{24}$$

We could now use the inverse metric to raise the third index, say, giving us the component of a $(1,2)$ tensor distinct from equation (23):

$$T_{\mu\nu}{}^\lambda \equiv \mathsf{g}^{-1}(\tilde{e}^\lambda, T_{\mu\nu\kappa}\tilde{e}^\kappa) = g^{\lambda\kappa}T_{\mu\nu\kappa} = g^{\lambda\kappa}g_{\mu\rho}T^\rho{}_{\nu\kappa} \ . \tag{25}$$

In fact, there are 2^{m+n} different tensor spaces with ranks summing to $m+n$. The metric or inverse metric tensor allow all of these tensors to be transformed into each other.

Returning to equation (22), we see why we must distinguish vectors (with components V^μ) from one-forms (with components V_μ). The scalar product of two vectors requires the metric tensor while that of two one-forms requires the inverse metric tensor. In general, $g^{\mu\nu} \neq g_{\mu\nu}$. The *only* case in which the distinction is unnecessary is in flat (Lorentz) spacetime with orthonormal Cartesian basis vectors, in which case $g_{\mu\nu} = \eta_{\mu\nu}$ is everywhere the diagonal matrix with entries $(-1, +1, +1, +1)$. However, gravity curves spacetime. (Besides, we may wish to use curvilinear coordinates even in flat spacetime.) As a result, it is impossible to define a coordinate system for which $g^{\mu\nu} = g_{\mu\nu}$ everywhere. We must therefore distinguish vectors from one-forms and we must be careful about the placement of subscripts and superscripts on components.

At this stage it is useful to introduce a classification of vectors and one-forms drawn from special relativity with its Minkowski metric $\eta_{\mu\nu}$. Recall that a vector $\vec{A} = A^\mu\vec{e}_\mu$ is called spacelike, timelike or null according to whether $\vec{A}\cdot\vec{A} = \eta_{\mu\nu}A^\mu A^\nu$ is positive, negative or zero, respectively. In a Euclidean space, with positive definite metric, $\vec{A}\cdot\vec{A}$ is never negative. However, in the Lorentzian spacetime geometry of special relativity, time enters the metric with opposite sign so that it is possible to have $\vec{A}\cdot\vec{A} < 0$. In particular, the four-velocity $u^\mu = dx^\mu/d\tau$ of a massive particle (where $d\tau$ is proper time) is a timelike vector. This is seen most simply by performing a Lorentz boost to the rest frame of the particle in which case $u^t = 1$, $u^x = u^y = u^z = 0$ and $\eta_{\mu\nu}u^\mu u^\nu = -1$. Now, $\eta_{\mu\nu}u^\mu u^\nu$ is a Lorentz scalar so that $\vec{u}\cdot\vec{u} = -1$ in any Lorentz frame. Often this is written $\vec{p}\cdot\vec{p} = -m^2$ where $p^\mu = mu^\mu$ is the four-momentum for a particle of mass m. For a massless particle (e.g., a photon) the proper time vanishes but the four-momentum is still well-defined with $\vec{p} \cdot \vec{p} = 0$: the momentum vector is null. We adopt the same notation in general relativity, replacing the Minkowski metric (components $\eta_{\mu\nu}$) with the actual metric $\mathbf{g}$ and evaluating the dot product using $\vec{A} \cdot \vec{A} = \mathbf{g}(\vec{A}, \vec{A}) = g_{\mu\nu} A^\mu A^\nu$. The same classification scheme extends to one-forms using $\mathbf{g}^{-1}$: a one-form $\tilde{P}$ is spacelike, timelike or null according to whether $\tilde{P} \cdot \tilde{P} = \mathbf{g}^{-1}(\tilde{P}, \tilde{P}) = g^{\mu\nu} P_\mu P_\nu$ is positive, negative or zero, respectively. Finally, a vector is called a unit vector if $\vec{A} \cdot \vec{A} = \pm 1$ and similarly for a one-form. The four-velocity of a massive particle is a timelike unit vector.

Now that we have introduced basis vectors and one-forms, we can define the contraction of a tensor. Contraction pairs two arguments of a rank (m, n) tensor: one vector and one one-form. The arguments are replaced by basis vectors and one-forms and summed over. For example, consider the $(1, 3)$ tensor R, which may be contracted on its second vector argument to give a $(0, 2)$ tensor also denoted R but distinguished by its shorter argument list:

$$\mathsf{R}(\vec{A}, \vec{B}) = \delta^\lambda{}_\kappa \, \mathsf{R}(\tilde{e}^\kappa, \vec{A}, \vec{e}_\lambda, \vec{B}) = \sum_{\lambda=0}^{3} \mathsf{R}(\tilde{e}^\lambda, \vec{A}, \vec{e}_\lambda, \vec{B}) \ . \tag{26}$$

In later notes we will define the Riemann curvature tensor of rank $(1, 3)$; its contraction, defined by equation (26), is called the Ricci tensor. Although the contracted tensor would appear to depend on the choice of basis because its definition involves the basis vectors and one-forms, the reader may show that it is actually invariant under a change of basis (and is therefore a tensor) as long as we use dual one-form and vector bases satisfying equation (13). Equation (26) becomes somewhat clearer if we express it entirely using tensor components:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \ . \tag{27}$$

Summation over λ is implied. Contraction may be performed on any pair of covariant and contravariant indices; different tensors result.

3.4 Change of basis

We have made no restrictions upon our choice of basis vectors $\vec{e}_\mu$. Our basis vectors are simply a linearly independent set of four vector fields allowing us to express any vector as a linear combination of basis vectors. The choice of basis is not unique; we may transform from one basis to another simply by defining four linearly independent combinations of our original basis vectors. Given one basis $\{\vec{e}_{\mu\mathbf{X}}\}$, we may define another basis $\{\vec{e}_{\mu'\mathbf{X}}\}$, distinguished by placing a prime on the labels, as follows:

$$\vec{e}_{\mu'\mathbf{X}} = \Lambda^\nu{}_{\mu'\mathbf{X}}\vec{e}_{\nu\mathbf{X}} \ . \tag{28}$$

The prime is placed on the index rather than on the vector only for notational convenience; we do not imply that the new basis vectors are a permutation of the old ones.

Any linearly independent linear combination of old basis vectors may be selected as the new basis vectors. That is, any nonsingular four-by-four matrix may be used for $\Lambda^\nu{}_{\mu'}$. The transformation is not restricted to being a Lorentz transformation; the local reference frames defined by the bases are not required to be inertial (or, in the presence of gravity, freely-falling). Because the transformation matrix is assumed to be nonsingular, the transformation may be inverted:

$$\vec{e}_{\nu\mathbf{X}} = \Lambda^{\mu'}{}_{\nu\mathbf{X}}\vec{e}_{\mu'\mathbf{X}} \ , \quad \Lambda^\mu{}_{\kappa'\mathbf{X}}\Lambda^{\kappa'}{}_{\nu\mathbf{X}} \equiv \ \delta^\mu{}_\nu \ . \tag{29}$$

Comparing equations (28) and (29), note that in our notation the inverse matrix places the prime on the other index. Primed indices are never summed together with unprimed indices.

If we change basis vectors, we must also transform the basis one-forms so as to preserve the duality condition equation (13). The reader may verify that, given the transformations of equations (28) and (29), the new dual basis one-forms are

$$\tilde{e}^{\mu'}{}_{\mathbf{X}} = \Lambda^{\mu'}{}_{\nu\mathbf{X}}\tilde{e}^\nu{}_{\mathbf{X}} \ . \tag{30}$$

We may also write the transformation matrix and its inverse by scalar products of the old and new basis vectors and one-forms (dropping the subscript $\mathbf{X}$ for clarity):

$$\Lambda^\nu{}_{\mu'} = \langle\tilde{e}^\nu, \vec{e}_{\mu'}\rangle \ , \quad \Lambda^{\mu'}{}_\nu = \langle\tilde{e}^{\mu'}, \vec{e}_\nu\rangle \ . \tag{31}$$

Apart from the basis vectors and one-forms, a vector $\vec{A}$ and a one-form $\tilde{P}$ are, by definition, invariant under a change of basis. Their components are not. For example, using equation (29) or (31) we find

$$\vec{A} = A^\nu \vec{e}_\nu = A^{\mu'} \vec{e}_{\mu'} \ , \quad A^{\mu'} = \langle \tilde{e}^{\mu'}, \vec{A} \rangle = \Lambda^{\mu'}{}_\nu A^\nu \ . \tag{32}$$

The vector components transform oppositely to the basis vectors (eq. 28). One-form components transform like basis vectors, as suggested by the fact that both are labeled with a subscript:

$$\tilde{A} = A_\nu \tilde{e}^\nu = A_{\mu'} \tilde{e}^{\mu'} \ , \quad A_{\mu'} = \langle \tilde{A}, \vec{e}_{\mu'} \rangle = \Lambda^\nu{}_{\mu'} A_\nu \ . \tag{33}$$

Note that if the components of two vectors or two one-forms are equal in one basis, they are equal in any basis.

Tensor components also transform under a change of basis. The new components may be found by recalling that a (m, n) tensor is a function of m vectors and n one-forms and that its components are gotten by evaluating the tensor using the basis vectors and one-forms (e.g., eq. 17). For example, the metric components are transformed under the change of basis of equation (28) to

$$g_{\mu'\nu'} \equiv \mathsf{g}(\vec{e}_{\mu'}, \vec{e}_{\nu'}) = g_{\alpha\beta}\, \tilde{e}^\alpha(\vec{e}_{\mu'}) \tilde{e}^\beta(\vec{e}_{\nu'}) = g_{\alpha\beta}\, \Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} \ . \tag{34}$$

(Recall that "evaluating" a one-form or vector means using the scalar product, eq. 1.) We see that the covariant components of the metric (i.e., the lower indices) transform exactly like one-form components. Not surprisingly, the components of a tensor of rank (m, n) transform like the product of m vector components and n one-form components. If the components of two tensors of the same rank are equal in one basis, they are equal in any basis.

3.5 Coordinate bases

We have made no restrictions upon our choice of basis vectors $\vec{e}_\mu$. Before concluding our formal introduction to tensors, we introduce one more idea: a coordinate system. A coordinate system is simply a set of four differentiable *scalar* fields $x^\mu{}_{\mathbf{x}}$ (*not* one *vector* field — note that μ labels the coordinates and not vector components) that attach a unique set of labels to each spacetime point $\mathbf{x}$. That is, no two points are allowed to have identical values of all four scalar fields and the coordinates must vary smoothly throughout spacetime (although we will tolerate occasional flaws like the coordinate singularities at $r = 0$ and $\theta = 0$ in spherical polar coordinates). Note that we impose no other restrictions on the coordinates. The freedom to choose different coordinate systems is available to us even in a Euclidean space; there is nothing sacred about Cartesian coordinates. This is even more true in a non-Euclidean space, where Cartesian coordinates covering the whole space do not exist.

Coordinate systems are useful for three reasons. First and most obvious, they allow us to label each spacetime point by a set of numbers (x^0, x^1, x^2, x^3). The second and more important use is in providing a special set of basis vectors called a coordinate basis. Suppose that two infinitesimally close spacetime points have coordinates x^μ and $x^\mu + dx^\mu$. The infinitesimal difference vector between the two points, denoted $d\vec{x}$, is a vector defined at $\mathbf{x}$. We define the coordinate basis as the set of four basis vectors $\vec{e}_\mu\mathbf{x}$ such that the components of $d\vec{x}$ are dx^μ:

$$d\vec{x} \equiv dx^\mu \vec{e}_\mu \quad \text{defines } \vec{e}_\mu \text{ in a coordinate basis} . \tag{35}$$

From the trivial appearance of this equation the reader may incorrectly think that we have imposed no constraints on the basis vectors. However, that is not so. According to equation (35), the basis vector $\vec{e}_0\mathbf{x}$, for example, must point in the direction of increasing x^0 at point $\mathbf{x}$. This corresponds to a unique direction in four-dimensional spacetime just as the direction of increasing latitude corresponds to a unique direction (north) at a given point on the earth. In more mathematical treatments (e.g. Walk 1984), $\vec{e}_\mu$ is associated with the directional derivative $\partial/\partial x^\mu$ at $\mathbf{x}$.

It is worth noting the transformation matrix between two coordinate bases:

$$\Lambda^{\bar{\alpha}}{}_{\mu} = \frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} . \tag{36}$$

Note that not all bases are coordinate bases. If we wanted to be perverse we could define a non-coordinate basis by, for example, permuting the labels on the basis vectors but not those on the coordinates (which, after all, are not the components of a vector). In this case $\langle \tilde{e}^\mu, d\vec{x} \rangle$, the component of $d\vec{x}$ for basis vector $\vec{e}_\mu$, would not equal the coordinate differential dx^μ. This would violate nothing we have written so far except equation (35). Later we will discover more natural ways that non-coordinate bases may arise.

The coordinate basis $\{\vec{e}_\mu\}$ defined by equation (35) has a dual basis of one-forms $\{\tilde{e}^\mu\}$ defined by equation (13). The dual basis of one-forms is related to the gradient. We obtain this relation as follows. Consider any scalar field $f_\mathbf{X}$. Treating f as a function of the coordinates, the difference in f between two infinitesimally close points is

$$df = \frac{\partial f}{\partial x^\mu}\, dx^\mu \equiv \partial_\mu f \, dx^\mu . \tag{37}$$

Equation (37) may be taken as the definition of the components of the gradient (with an alternative brief notation for the partial derivative). However, partial derivatives depend on the coordinates, while the gradient (covariant derivative) should not. What, then, is the gradient — is it a vector or a one-form?

From equation (37), because df is a scalar and dx^μ is a vector component, $\partial f/\partial x^\mu$ must be the component of a one-form, not a vector. The notation ∂_μ, with its covariant (subscript) index, reinforces our view that the partial derivative is the component of a one-form and not a vector. We denote the gradient one-form by $\tilde{\nabla}$. Like all one-forms, the gradient may be decomposed into a sum over basis one-forms $\tilde{e}^\mu$. Using equation (37) and equation (13) as the requirement for a dual basis, we conclude that the gradient is

$$\tilde{\nabla} \equiv \tilde{e}^\mu \, \partial_\mu \quad \text{in a coordinate basis .} \tag{38}$$

Note that we must write the basis one-form to the left of the partial derivative operator, for the basis one-form itself may depend on position! We will return to this point in Section 4 when we discuss the covariant derivative. In the present case, it is clear from equation (37) that we must let the derivative act only on the function f. We can now rewrite equation (37) in the coordinate-free manner

$$df = \langle \tilde{\nabla} f, d\vec{x} \rangle \ . \tag{39}$$

If we want the directional derivative of f along any particular direction, we simply replace $d\vec{x}$ by a vector pointing in the desired direction (e.g., the tangent vector to some curve). Also, if we let $f_{\mathbf{X}}$ equal one of the coordinates, using equation (38) the gradient gives us the corresponding basis one-form:

$$\tilde{\nabla} x^\mu = \tilde{e}^\mu \quad \text{in a coordinate basis .} \tag{40}$$

The third use of coordinates is that they can be used to describe the distance between two points of spacetime. However, coordinates alone are not enough. We also need the metric tensor. We write the squared distance between two spacetime points as

$$ds^2 = |d\vec{x}\,|^2 \equiv \mathbf{g}(d\vec{x}, d\vec{x}) = d\vec{x} \cdot d\vec{x} \ . \tag{41}$$

This equation, true in any basis because it is a scalar equation that makes no reference to components, is taken as the *definition* of the metric tensor. Up to now the metric could have been any symmetric $(0,2)$ tensor. But, if we insist on being able to measure distances, given an infinitesimal difference vector $d\vec{x}$, only one $(0,2)$ tensor can give the squared distance. We define the metric tensor to be that tensor. Indeed, the squared magnitude of *any* vector $\vec{A}$ is $|\vec{A}\,|^2 \equiv \mathsf{g}(\vec{A},\vec{A})$.

Now we specialize to a coordinate basis, using equation (35) for $d\vec{x}$. In a coordinate basis (and *only* in a coordinate basis), the squared distance is called the line element and takes the form

$$ds^2 = g_{\mu\nu\,\mathbf{X}}\,dx^\mu dx^\nu \quad \text{in a coordinate basis .} \tag{42}$$

We have used equation (17) to get the metric components.

If we transform coordinates, we will have to change our vector and one-form bases. Suppose that we transform from $x^\mu{}_{\mathbf{X}}$ to $x^{\mu'}{}_{\mathbf{X}}$, with a prime indicating the new coordinates. For example, in the Euclidean plane we could transform from Cartesian coordinate $(x^1 = x, x^2 = y)$ to polar coordinates $(x^{1'} = r, x^{2'} = \theta)$: $x = r\cos\theta$, $y = r\sin\theta$. A one-to-one mapping is given from the old to new coordinates, allowing us to define the Jacobian matrix $\Lambda^{\mu'}{}_\nu \equiv \partial x^{\mu'}/\partial x^\nu$ and its inverse $\Lambda^\nu{}_{\mu'} = \partial x^\nu/\partial x^{\mu'}$. Vector components transform like $dx^{\mu'} = (\partial x^{\mu'}/\partial x^\nu)\,dx^\nu$. Transforming the basis vectors, basis one-forms, and tensor components is straightforward using equations (28)–(34). The reader should verify that equations (35), (38), (40) and (42) remain valid after a coordinate transformation.

We have now introduced many of the basic ingredients of tensor algebra that we will need in general relativity. Before moving on to more advanced concepts, let us reflect on our treatment of vectors, one-forms and tensors. The mathematics and notation, while straightforward, are complicated. Can we simplify the notation without sacrificing rigor?

One way to modify our notation would be to abandon ths basis vectors and one-forms and to work only with components of tensors. We could have defined vectors, one-forms and tensors from the outset in terms of the transformation properties of their components. However, the reader should appreciate the clarity of the geometrical approach that we have adopted. Our notation has forced us to distinguish physical objects like vectors from basis-dependent ones like vector components. As long as the definition of a tensor is not forgotten, computations are straightforward and unambiguous. Moreover, adopting a basis did not force us to abandon geometrical concepts. On the contrary, computations are made easier and clearer by retaining the notation and meaning of basis vectors and one-forms.

3.6 Isomorphism of vectors and one-forms

Although vectors and one-forms are distinct objects, there is a strong relationship between them. In fact, the linear space of vectors is isomorphic to the dual vector space of one-forms (Wald 1984). Every equation or operation in one space has an equivalent equation or operation in the other space. This isomorphism can be used to hide the distinction between one-forms and vectors in a way that simplifies the notation. This approach is unusual (I haven't seen it published anywhere) and is not recommended in formal work but it may be pedagogically useful.

As we saw in equations (9) and (10), the link between the vector and dual vector spaces is provided by $\mathbf{g}$ and $\mathbf{g}^{-1}$. If $\vec{A} = \vec{B}$ (components $A^\mu = B^\mu$), then $\tilde{A} = \tilde{B}$ (components $A_\mu = B_\mu$) where $A_\mu = g_{\mu\nu}A^\nu$ and $B_\mu = g_{\mu\nu}B^\nu$. So, why do we bother with one-forms when vectors are sufficient? The answer is that tensors may be functions of both one-forms and vectors. However, there is also an isomorphism among tensors of different rank. We have just argued that the tensor spaces of rank $(1,0)$ (vectors) and $(0,1)$ are isomorphic. In fact, all 2^{m+n} tensor spaces of rank (m,n) with fixed $m+n$ are isomorphic. The metric and inverse metric tensors link together these spaces, as exemplified by equations (24) and (25).

The isomorphism of different tensor spaces allows us to introduce a notation that unifies them. We could effect such a unification by discarding basis vectors and one-forms and working only with components, using the components of the metric tensor and its inverse to relate components of different types of tensors as in equations (24) and (25). However, this would require sacrificing the coordinate-free geometrical interpretation of vectors. Instead, we introduce a notation that replaces one-forms with vectors and (m,n) tensors with $(m+n,0)$ tensors in general. We do this by replacing the basis one-forms $\tilde{e}^\mu$ with a set of *vectors* defined as in equation (10):

$$\vec{e}^{\,\mu}(\,\cdot\,) \equiv \mathbf{g}^{-1}(\tilde{e}^\mu, \,\cdot\,) = g^{\mu\nu}\vec{e}_\nu(\,\cdot\,) \ . \tag{43}$$

We will refer to $\vec{e}^{\,\mu}$ as a *dual basis vector* to contrast it from both the basis vector $\vec{e}_\mu$ and the basis one-form $\tilde{e}^\mu$. The dots are present in equation (43) to remind us that a one-form may be inserted to give a scalar. However, we no longer need to use one-forms. Using equation (43), given the components A_μ of any one-form $\tilde{A}$, we may form the vector $\vec{A}$ defined by equation (10) as follows:

$$\vec{A} = A_\mu \vec{e}^{\,\mu} = A_\mu g^{\mu\nu}\vec{e}_\nu = A^\mu \vec{e}_\mu \ . \tag{44}$$

The reader should verify that $\vec{A} = A_\mu \vec{e}^{\,\mu}$ is invariant under a change of basis because $\vec{e}^{\,\mu}$ transforms like a basis one-form.

The isomorphism of one-forms and vectors means that we can replace all one-forms with vectors in any tensor equation. Tildes may be replaced with arrows. The scalar product between a one-form and a vector is replaced by the dot product using the metric (eq. 10 or 43). The only rule is that we must treat a dual basis vector with an upper index like a basis one-form:

$$\vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu} \ , \quad \vec{e}^{\,\mu} \cdot \vec{e}_\nu = \langle \tilde{e}^{\,\mu}, \vec{e}_\nu \rangle = \delta^\mu{}_\nu \ , \quad \vec{e}^{\,\mu} \cdot \vec{e}^{\,\nu} = \tilde{e}^{\,\mu} \cdot \tilde{e}^{\,\nu} = g^{\mu\nu} \ . \tag{45}$$

The reader should verify equations (45) using equations (17) and (43). Now, if we need the contravariant component of a vector, we can get it from the dot product with the dual basis vector instead of from the scalar product with the basis one-form:

$$A^\mu = \vec{e}^{\,\mu} \cdot \vec{A} = \langle \tilde{e}^\mu, \vec{A} \rangle \ . \tag{46}$$

We may also apply this recipe to convert the gradient one-form $\tilde{\nabla}$ (eq. 38) to a vector, though we must not allow the dual basis vector to be differentiated:

$$\vec{\nabla} = \vec{e}^{\,\mu}\partial_\mu = g^{\mu\nu}\vec{e}_\mu\,\partial_\nu \quad \text{in a coordinate basis} \ . \tag{47}$$

It follows at once that the dual basis vector (in a coordinate basis) is the vector gradient of the coordinate: $\vec{e}^{\,\mu} = \vec{\nabla}x^\mu$. This equation is isomorphic to equation (40).

The basis vectors and dual basis vectors, through their tensor products, also give a basis for higher-rank tensors. Again, the rule is to replace the basis one-forms with the corresponding dual basis vectors. Thus, for example, we may write the rank $(2,0)$ metric tensor in any of four ways:

$$\mathbf{g} = g_{\mu\nu}\vec{e}^{\,\mu} \otimes \vec{e}^{\,\nu} = g^\mu{}_\nu\vec{e}_\mu \otimes \vec{e}^{\,\nu} = g_\mu{}^\nu\vec{e}^{\,\mu} \otimes \vec{e}_\nu = g^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu \ . \tag{48}$$

In fact, by comparing this with equation (18) the reader will see that what we have written is actually the *inverse* metric tensor $\mathbf{g}^{-1}$, which is isomorphic to $\mathbf{g}$ through the replacement of $\tilde{e}^\mu$ with $\vec{e}^{\,\mu}$. But, what are the mixed components of the metric, $g^\mu{}_\nu$ and $g_\mu{}^\nu$? From equations (13) and (43), we see that they both equal the Kronecker delta $\delta^\mu{}_\nu$. Consequently, the metric tensor is isomorphic to the identity tensor as well as to its inverse! However, this is no miracle; it was guaranteed by our definition of the dual basis vectors and by the fact we defined $\mathbf{g}^{-1}$ to invert the mapping from vectors to one-forms implied by $\mathbf{g}$. The reader may fear that we have defined away the metric by showing it to be isomorphic to the identity tensor. However, this is not the case. We need the metric tensor components to obtain $\vec{e}^{\,\mu}$ from $\vec{e}_\mu$ or A^μ from A_μ. We cannot take advantage of the isomorphism of different tensor spaces without the metric. Moreover, as we showed in equation (41), the metric plays a fundamental role in giving the squared magnitude of a vector. In fact, as we will see later, the metric contains all of the information about the geometrical properties of spacetime. Clearly, the metric must play a fundamental role in general relativity.

3.7 Example: Euclidean plane

We close this section by applying tensor concepts to a simple example: the Euclidean plane. This flat two-dimensional space can be covered by Cartesian coordinates (x, y) with line element and metric components

$$ds^2 = dx^2 + dy^2 \quad \Rightarrow \quad g_{xx} = g_{yy} = 1 \ , \ g_{xy} = g_{yx} = 0 \ . \tag{49}$$

We prefer to use the coordinate names themselves as component labels rather than using numbers (e.g. g_{xx} rather than g_{11}). The basis vectors are denoted $\vec{e}_x$ and $\vec{e}_y$, and their use in plane geometry and linear algebra is standard. Basis one-forms appear unnecessary because the metric tensor is just the identity tensor in this basis. Consequently the dual basis vectors (eq. 43) are $\vec{e}^{\,x} = \vec{e}_x$, $\vec{e}^{\,y} = \vec{e}_y$ and no distinction is needed between superscripts and subscripts.

However, there is nothing sacred about Cartesian coordinates. Consider polar coordinates (ρ, θ), defined by the transformation $x = \rho\cos\theta$, $y = \rho\sin\theta$. A simple exercise in partial derivatives yields the line element in polar coordinates:

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 \quad \Rightarrow \quad g_{\rho\rho} = 1 \ , \ g_{\theta\theta} = \rho^2 \ , \ g_{\rho\theta} = g_{\theta\rho} = 0 \ . \tag{50}$$

This appears eminently reasonable until, perhaps, one considers the basis vectors $\vec{e}_\rho$ and $\vec{e}_\theta$, recalling that $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$. Then, while $\vec{e}_\rho \cdot \vec{e}_\rho = 1$ and $\vec{e}_\rho \cdot \vec{e}_\theta = 0$, $\vec{e}_\theta \cdot \vec{e}_\theta = \rho^2$: $\vec{e}_\theta$ is not a unit vector! The new basis vectors are easily found in terms of the Cartesian basis vectors and components using equation (28):

$$\vec{e}_\rho = \frac{x}{\sqrt{x^2+y^2}}\,\vec{e}_x + \frac{y}{\sqrt{x^2+y^2}}\,\vec{e}_y \ , \quad \vec{e}_\theta = -y\,\vec{e}_x + x\,\vec{e}_y \ . \tag{51}$$

The polar *unit* vectors are $\hat{\rho} = \vec{e}_\rho$ and $\hat{\theta} = \rho^{-1}\vec{e}_\theta$.

Why does our formalism give us non-unit vectors? The answer is because we insisted that our basis vectors be a *coordinate* basis (eqs. 35, 38, 40 and 42). In terms of the orthonormal unit vectors, the difference vector between points (ρ, θ) and $(\rho + d\rho, \theta + d\theta)$ is $d\vec{x} = \hat{\rho}\,d\rho + \hat{\theta}\,\rho d\theta$. In the coordinate basis this takes the simpler form $d\vec{x} = \vec{e}_\rho\,d\rho + \vec{e}_\theta\,d\theta = dx^\mu \vec{e}_\mu$. In the coordinate basis we don't have to worry about normalizing our vectors; all information about lengths is carried instead by the metric. In the *non*-coordinate basis of orthonormal vectors $\{\hat{\rho}, \hat{\theta}\}$ we have to make a separate note that the distance elements are $d\rho$ and $\rho d\theta$.

In the non-coordinate basis we can no longer use equation (42) for the line element. We must instead use equation (41). The metric components in the non-coordinate basis $\{\hat{\rho}, \hat{\theta}\}$ are

$$g_{\hat{\rho}\hat{\rho}} = g_{\hat{\theta}\hat{\theta}} = 1 \ , \ g_{\hat{\rho}\hat{\theta}} = g_{\hat{\theta}\hat{\rho}} = 0 \ . \tag{52}$$

The reader may also verify this result by transforming the components of the metric from the basis $\{\vec{e}_\rho, \vec{e}_\theta\}$ to $\{\hat{\rho}, \hat{\theta}\}$ using equation (34) with $\Lambda^\rho{}_{\hat{\rho}} = 1$, $\Lambda^\theta{}_{\hat{\theta}} = \rho^{-1}$. Now, equation (41) still gives the distance squared, but we are responsible for remembering $d\vec{x} = \hat{\rho}\, d\rho + \hat{\theta}\, \rho d\theta$. In a non-coordinate basis, the metric will not tell us how to measure distances in terms of coordinate differentials.

With a non-coordinate basis, we must sacrifice equations (35) and (42). Nonetheless, for some applications it proves convenient to introduce an orthonormal non-coordinate basis called a tetrad basis. Tetrads are discussed by Wald (1984) and Misner et al (1973).

The use of non-coordinate bases also complicates the gradient (eqs. 38, 40 and 47). In our polar coordinate basis (eq. 50), the inverse metric components are

$$g^{\rho\rho} = 1 \ , \ g^{\theta\theta} = \rho^{-2} \ , \ g^{\rho\theta} = g^{\theta\rho} = 0 \ . \tag{53}$$

(The matrix $g_{\mu\nu}$ is diagonal, so its inverse is also diagonal with entries given by the reciprocals.) The basis one-forms obey the rules $\tilde{e}^\mu \cdot \tilde{e}^\nu = g^{\mu\nu}$. They are isomorphic to the dual basis vectors $\vec{e}^{\,\mu} = g^{\mu\nu}\vec{e}_\nu$ (eq. 43). Thus, $\vec{e}^{\,\rho} = \vec{e}_\rho = \hat{\rho}$, $\vec{e}^{\,\theta} = \rho^{-2}\vec{e}_\theta = \rho^{-1}\hat{\theta}$. Equation (38) gives the gradient one-form as $\tilde{\nabla} = \tilde{e}^\rho\,(\partial/\partial\rho) + \tilde{e}^\theta\,(\partial/\partial\theta)$. Expressing this as a vector (eq. 47) we get

$$\vec{\nabla} = \vec{e}^{\,\rho}\frac{\partial}{\partial\rho} + \vec{e}^{\,\theta}\frac{\partial}{\partial\theta} = \hat{\rho}\,\frac{\partial}{\partial\rho} + \hat{\theta}\,\frac{1}{\rho}\frac{\partial}{\partial\theta} \ . \tag{54}$$

The gradient is simpler in the coordinate basis. The coordinate basis has the added advantage that we can get the dual basis vectors (or the basis one-forms) by applying the gradient to the coordinates (eq. 47): $\vec{e}^{\,\rho} = \vec{\nabla}\rho$, $\vec{e}^{\,\theta} = \vec{\nabla}\theta$.

From now on, unless otherwise noted, we will assume that our basis vectors are a coordinate basis. We will use one-forms and vectors interchangeably through the mapping provided by the metric and inverse metric (eqs. 9, 10 and 43). Readers who dislike one-forms may convert the tildes to arrows and use equations (45) to obtain scalars from scalar products and dot products.

4 Differentiation and Integration

In this section we discuss differentiation and integration in curved spacetime. These might seem like a delicate subjects but, given the tensor algebra that we have developed, tensor calculus is straightforward.

4.1 Gradient of a scalar

Consider first the gradient of a scalar field $f_{\mathbf{X}}$. We have already shown in Section 2 that the gradient operator $\tilde{\nabla}$ is a one-form (an object that is invariant under coordinate transformations) and that, in a coordinate basis, its components are simply the partial derivatives with respect to the coordinates:

$$\tilde{\nabla} f = (\partial_\mu f)\,\tilde{e}^\mu = (\nabla_\mu f)\,\tilde{e}^\mu \ , \tag{55}$$

where $\partial_\mu \equiv (\partial/\partial x^\mu)$. We have introduced a second notation, ∇_μ, called the *covariant derivative* with respect to x^μ. By definition, the covariant derivative behaves like the component of a one-form. But, from equation (55), this is also true of the partial derivative operator ∂_μ. Why have we introduced a new symbol?

Before answering this question, let us first note that the gradient, because it behaves like a tensor of rank $(0,1)$ (a one-form), changes the rank of a tensor field from (m,n) to $(m,n+1)$. (This is obviously true for the gradient of a scalar field, with $m=n=0$.) That is, application of the gradient is like taking the tensor product with a one-form. The difference is that the components are not the product of the components, because ∇_μ is not a number. Nevertheless, the resulting object must be a tensor of rank $(m,n+1)$; e.g., its components must transform like components of a $(m,n+1)$ tensor. The gradient of a scalar field f is a $(0,1)$ tensor with components $(\partial_\mu f)$.

4.2 Gradient of a vector: covariant derivative

The reason that we have introduced a new symbol for the derivative will become clear when we take the gradient of a vector field $\vec{A}_{\mathbf{X}} = A^{\mu}{}_{\mathbf{X}}\vec{e}_{\mu\,\mathbf{X}}$. In general, the basis vectors are functions of position as are the vector components! So, the gradient must act on both. In a coordinate basis, we have

$$\tilde{\nabla}\vec{A} = \tilde{\nabla}(A^{\nu}\vec{e}_{\nu}) = \tilde{e}^{\mu}\partial_{\mu}(A^{\nu}\vec{e}_{\nu}) = (\partial_{\mu}A^{\nu})\,\tilde{e}^{\mu}\vec{e}_{\nu} + A^{\nu}\tilde{e}^{\mu}\,(\partial_{\mu}\vec{e}_{\nu}) \equiv (\nabla_{\mu}A^{\nu})\,\tilde{e}^{\mu}\vec{e}_{\nu}\ . \qquad (56)$$

We have dropped the tensor product symbol $\otimes$ for notational convenience although it is still implied. Note that we must be careful to preserve the ordering of the vectors and tensors and we must not confuse subscripts and superscripts. Otherwise, taking the gradient of a vector is straightforward. The result is a $(1,1)$ tensor with components $\nabla_{\mu}A^{\nu}$. But now $\nabla_{\mu} \neq \partial_{\mu}$! This is why we have introduced a new derivative symbol. We reserve the covariant derivative notation ∇_{μ} for the actual components of the gradient of a tensor. We note that the alternative notation $A^{\nu}{}_{;\mu} = \nabla_{\mu}A^{\nu}$ is often used, replacing the comma of a partial derivative $A^{\nu}{}_{,\mu} = \partial_{\mu}A^{\nu}$ with a semicolon for the covariant derivative. The difference seems mysterious only when we ignore basis vectors and stick entirely to components. As equation (56) shows, vector notation makes it clear why there is a difference.

Equation (56) by itself does not help us evaluate the gradient of a vector because we do not yet know what the gradients of the basis vectors are. However, they are straightforward to determine in a coordinate basis. First we note that, geometrically,

$\partial_\mu \vec{e}_\nu$ is a vector at $\mathbf{x}$: it is the difference of two vectors at infinitesimally close points, divided by a coordinate interval. (The easiest way to tell that $\partial_\mu \vec{e}_\nu$ is a vector is to note that it has one arrow!) So, like all vectors, it must be a linear combination of basis vectors at $\mathbf{x}$. We can write the most general possible linear combination as

$$\partial_\nu \vec{e}_\mu \mathbf{x} \equiv \Gamma^\lambda{}_{\mu\nu}\mathbf{x}\, \vec{e}_\lambda \mathbf{x} \ . \tag{57}$$

4.3 Christoffel symbols

We have introduced in equation (57) a set of coefficients, $\Gamma^\lambda{}_{\mu\nu}$, called the connection coefficients or *Christoffel symbols*. (Technically, the term Christoffel symbols is reserved for a coordinate basis.) It should be noted at the outset that, despite their appearance, the Christoffel symbols are not the components of a $(1,2)$ tensor. Rather, they may be considered as a set of four $(1,1)$ tensors, one for each basis vector $\vec{e}_\nu$, because $\tilde{\nabla}\vec{e}_\mu = \Gamma^\lambda{}_{\mu\nu}\tilde{e}^\nu \vec{e}_\lambda$. However, it is not useful to think of the Christoffel symbols as tensor components for fixed ν because, under a change of basis, the basis vectors $\vec{e}_\nu$ themselves change and therefore the four $(1,1)$ tensors must also change. So, forget about the Christoffel symbols defining a tensor. They are simply a set of coefficients telling us how to differentiate basis vectors. Whatever their values, the components of the gradient of $\vec{A}$, known also as the covariant derivative of A^ν, are, from equations (56) and (57),

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\lambda\mu} A^\lambda \ . \tag{58}$$

How does one determine the values of the Christoffel symbols? That is, how does one evaluate the gradients of the basis vectors? One way is to express the basis vectors in terms of another set whose gradients are known. For example, consider polar coordinates (ρ, θ) in the Cartesian plane as discussed in Section 2. The polar coordinate basis vectors were given in terms of the Cartesian basis vectors in equation (51). We know that the gradients of the Cartesian basis vectors vanish and we know how to transform from Cartesian to polar coordinates. It is a straightforward and instructive exercise from this to compute the gradients of the polar basis vectors:

$$\tilde{\nabla}\vec{e}_\rho = \frac{1}{\rho}\,\tilde{e}^\theta \otimes \vec{e}_\theta \ , \quad \tilde{\nabla}\vec{e}_\theta = \frac{1}{\rho}\,\tilde{e}^\rho \otimes \vec{e}_\theta - \rho\,\tilde{e}^\theta \otimes \vec{e}_\rho \ . \tag{59}$$

(We have restored the tensor product symbol as a reminder of the tensor nature of the objects in eq. 59.) From equations (57) and (59) we conclude that the nonvanishing Christoffel symbols are

$$\Gamma^\theta{}_{\theta\rho} = \Gamma^\theta{}_{\rho\theta} = \rho^{-1} \ , \quad \Gamma^\rho{}_{\theta\theta} = -\rho \ . \tag{60}$$

It is instructive to extend this example further. Suppose that we add the third dimension, with coordinate z, to get a three-dimensional Euclidean space with cylindrical coordinates (ρ, θ, z). The line element (cf. eq. 50) now becomes $ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$. Because $\vec{e}_\rho$ and $\vec{e}_\theta$ are independent of z and $\vec{e}_z$ is itself constant, no new non-vanishing Christoffel symbols appear. Now consider a related but different manifold: a cylinder. A cylinder is simply a surface of constant ρ in our three-dimensional Euclidean space. This two-dimensional space is mapped by coordinates (θ, z), with basis vectors $\vec{e}_\theta$ and $\vec{e}_z$. What are the gradients of these basis vectors? They vanish! But, how can that be? From equation (59), $\partial_\theta \vec{e}_\theta = -\rho \vec{e}_\rho$. Have we forgotten about the $\vec{e}_\rho$ direction?

This example illustrates an important lesson. We cannot project tensors into basis vectors that do not exist in our manifold, whether it is a two-dimensional cylinder or a four-dimensional spacetime. A cylinder exists as a two-dimensional mathematical surface whether or not we choose to embed it in a three-dimensional Euclidean space. If it happens that we can embed our manifold into a simpler higher-dimensional space, we do so only as a matter of calculational convenience. If the result of a calculation is a vector normal to our manifold, we must discard this result because this direction does not exist in our manifold. If this conclusion is troubling, consider a cylinder as seen by a two-dimensional ant crawling on its surface. If the ant goes around in circles about the z-axis it is moving in the $\vec{e}_\theta$ direction. The ant would say that its direction is *not* changing as it moves along the circle. We conclude that the Christoffel symbols indeed all vanish for a cylinder described by coordinates (θ, z).

4.4 Gradients of one-forms and tensors

Later we will return to the question of how to evaluate the Christoffel symbols in general. First we investigate the gradient of one-forms and of general tensor fields. Consider a one-form field $\tilde{A}_{\mathbf{X}} = A_{\mu\mathbf{X}} \tilde{e}^\mu{}_{\mathbf{X}}$. Its gradient in a coordinate basis is

$$\tilde{\nabla} \tilde{A} = \tilde{\nabla}(A_\nu \tilde{e}^\nu) = \tilde{e}^\mu \partial_\mu (A_\nu \tilde{e}^\nu) = (\partial_\mu A_\nu)\, \tilde{e}^\mu \tilde{e}^\nu + A_\nu \tilde{e}^\mu\, (\partial_\mu \tilde{e}^\nu) \equiv (\nabla_\mu A_\nu)\, \tilde{e}^\mu \tilde{e}^\nu \; . \tag{61}$$

Again we have defined the covariant derivative operator to give the components of the gradient, this time of the one-form. We cannot assume that ∇_μ has the same form here as in equation (58). However, we can proceed as we did before to determine its relation, if any, to the Christoffel symbols. We note that the partial derivative of a one-form in equation (61) must be a linear combination of one-forms:

$$\partial_\mu \tilde{e}^\nu{}_{\mathbf{X}} \equiv \Pi^\nu{}_{\lambda\mu\mathbf{X}}\, \tilde{e}^\lambda{}_{\mathbf{X}} \; , \tag{62}$$

for some set of coefficients $\Pi^\nu{}_{\lambda\mu}$ analogous to the Christoffel symbols. In fact, these coefficients are simply related to the Christoffel symbols, as we may see by differentiating the scalar product of dual basis one-forms and vectors:

$$0 = \partial_\mu \langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \Pi^\nu{}_{\kappa\mu} \langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle + \Gamma^\kappa{}_{\lambda\mu} \langle \tilde{e}^\nu, \vec{e}_\kappa \rangle = \Pi^\nu{}_{\lambda\mu} + \Gamma^\nu{}_{\lambda\mu} \; . \tag{63}$$

We have used equation (13) plus the linearity of the scalar product. The result is $\Pi^{\nu}{}_{\lambda\mu} = -\Gamma^{\nu}{}_{\lambda\mu}$, so that equation (62) becomes, simply,

$$\partial_\mu \tilde{e}^{\nu}{}_{\mathbf{X}} = -\Gamma^{\nu}{}_{\lambda\mu\,\mathbf{X}}\, \tilde{e}^{\lambda}{}_{\mathbf{X}} \ . \tag{64}$$

Consequently, the components of the gradient of a one-form $\tilde{A}$, also known as the covariant derivative of A_ν, are

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^{\lambda}{}_{\nu\mu} A_\lambda \ . \tag{65}$$

This expression is similar to equation (58) for the covariant derivative of a vector except for the sign change and the exchange of the indices ν and λ on the Christoffel symbol (obviously necessary for consistency with tensor index notation). Although we still don't know the values of the Christoffel symbols in general, at least we have introduced no more unknown quantities.

We leave it as an exercise for the reader to show that extending the covariant derivative to higher-rank tensors is straightforward. First, the partial derivative of the components is taken. Then, one term with a Christoffel symbol is added for every index on the tensor component, with a positive sign for contravariant indices and a minus sign for covariant indices. That is, for a (m, n) tensor, there are m positive terms and n negative terms. The placement of labels on the Christoffel symbols is a straightforward extension of equations (58) and (65). We illustrate this with the gradients of the $(0, 2)$ metric tensor, the $(1, 1)$ identity tensor and the $(2, 0)$ inverse metric tensor:

$$\tilde{\nabla}\mathsf{g} = (\nabla_\lambda g_{\mu\nu})\, \tilde{e}^{\lambda} \otimes \tilde{e}^{\mu} \otimes \tilde{e}^{\nu} \ , \quad \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^{\kappa}{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^{\kappa}{}_{\nu\lambda} g_{\mu\kappa} \ , \tag{66}$$

$$\tilde{\nabla}\mathsf{I} = (\nabla_\lambda \delta^{\mu}{}_{\nu})\, \tilde{e}^{\lambda} \otimes \vec{e}_{\mu} \otimes \tilde{e}^{\nu} \ , \quad \nabla_\lambda \delta^{\mu}{}_{\nu} = \partial_\lambda \delta^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\kappa\lambda} \delta^{\kappa}{}_{\nu} - \Gamma^{\kappa}{}_{\nu\lambda} \delta^{\mu}{}_{\kappa} \ , \tag{67}$$

and

$$\tilde{\nabla}\mathsf{g}^{-1} = (\nabla_\lambda g^{\mu\nu})\, \tilde{e}^{\lambda} \otimes \vec{e}_{\mu} \otimes \vec{e}_{\nu} \ , \quad \nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma^{\mu}{}_{\kappa\lambda} g^{\kappa\nu} + \Gamma^{\nu}{}_{\kappa\lambda} g^{\mu\kappa} \ . \tag{68}$$

Examination of equation (67) shows that the gradient of the identity tensor vanishes identically. While this result is not surprising, it does have important implications. Recall from Section 2 the isomorphism between g, I and g^{-1} (eq. 48). As a result of this isomorphism, we would expect that all three tensors have vanishing gradient. Is this really so?

For a smooth (differentiable) manifold the gradient of the metric tensor (and the inverse metric tensor) indeed vanishes. The proof is sketched as follows. At a given point $\mathbf{x}$ in a smooth manifold, we may construct a locally flat orthonormal (Cartesian) coordinate system. We define a *locally flat coordinate system* to be one whose coordinate basis vectors satisfy the following conditions in a finite neighborhood around $\mathbf{X}$: $\vec{e}_{\mu\,\mathbf{X}} \cdot \vec{e}_{\nu\,\mathbf{X}} = 0$ for $\mu \neq \nu$ and $\vec{e}_{\mu\,\mathbf{X}} \cdot \vec{e}_{\mu\,\mathbf{X}} = \pm 1$ (with no implied summation).

The existence of a locally flat coordinate system may be taken as the definition of a smooth manifold. For example, on a two-sphere we may erect a Cartesian coordinate system $x^{\bar{\mu}}$, with orthonormal basis vectors $\vec{e}_{\bar{\mu}}$, applying over a small region around $\mathbf{x}$. (We use a bar to indicate the locally flat coordinates.) While these coordinates cannot, in general, be extended over the whole manifold, they are satisfactory for measuring distances in the neighborhood of $\mathbf{x}$ using equation (42) with $g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = g^{\bar{\mu}\bar{\nu}}$, where $\eta_{\bar{\mu}\bar{\nu}}$ is the metric of a flat space or spacetime with orthonormal coordinates (the Kronecker delta or the Minkowski metric as the case may be). The key point is that this statement is true not only at $\mathbf{x}$ but also in a small neighborhood around it. (This argument relies on the absence of curvature singularities in the manifold and would fail, for example, if it were applied at the tip of a cone.) Consequently, the metric must have vanishing first derivative at $\mathbf{x}$ in the locally flat coordinates: $\partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$. The gradient of the metric (and the inverse metric) vanishes in the locally flat coordinate basis. But, the gradient of the metric is a tensor and tensor equations are true in any basis. Therefore, for any smooth manifold,

$$\tilde{\nabla}\mathbf{g} = \tilde{\nabla}\mathbf{g}^{-1} = 0 \ . \tag{69}$$

4.5 Evaluating the Christoffel symbols

We can extend the argument made above to prove the symmetry of the Christoffel symbols: $\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}$ for any coordinate basis. At point $\mathbf{x}$, the basis vectors corresponding to our locally flat coordinate system have vanishing derivatives: $\partial_{\bar{\mu}}\vec{e}_{\bar{\nu}} = 0$. From equation (57), this implies that *the Christoffel symbols vanish at a point in a locally flat coordinate basis.* Now let us transform to any other set of coordinates x^{μ}. The Jacobian of this transformation is $\Lambda^{\kappa}{}_{\bar{\mu}} = \partial x^{\kappa}/\partial x^{\bar{\mu}}$ (eq. 36). Our basis vectors transform (eq. 28) according to $\vec{e}_{\bar{\mu}} = \Lambda^{\kappa}{}_{\bar{\mu}}\vec{e}_{\kappa}$. We now evaluate $\partial_{\bar{\mu}}\vec{e}_{\bar{\nu}} = 0$ using the new basis vectors, being careful to use equation (57) for their partial derivatives (which do *not* vanish in non-flat coordinates):

$$0 = \partial_{\bar{\mu}}\vec{e}_{\bar{\nu}} = \frac{\partial^2 x^{\kappa}}{\partial x^{\bar{\mu}}\partial x^{\bar{\nu}}}\,\vec{e}_{\kappa} + \frac{\partial x^{\kappa}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\lambda}}{\partial x^{\bar{\nu}}}\,\Gamma^{\sigma}{}_{\kappa\lambda}\vec{e}_{\sigma} = 0 \ . \tag{70}$$

Exchanging $\bar{\mu}$ and $\bar{\nu}$ we see that

$$\Gamma^{\sigma}{}_{\kappa\lambda} = \Gamma^{\sigma}{}_{\lambda\kappa} \quad \text{in a coordinate basis} \ , \tag{71}$$

implying that our connection is torsion-free (Wald 1984).

We can now use equations (66), (69) and (71) to evaluate the Christoffel symbols in terms of partial derivatives of the metric coefficients in any coordinate basis. We write $\nabla_{\lambda} g_{\mu\nu} = 0$ and permute the indices twice, combining the results with one minus sign and using the inverse metric at the end. The result is

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(\partial_{\mu}g_{\nu\kappa} + \partial_{\nu}g_{\mu\kappa} - \partial_{\kappa}g_{\mu\nu}\right) \quad \text{in a coordinate basis} \ . \tag{72}$$

Although the Christoffel symbols vanish at a point in a locally flat coordinate basis, they do not vanish in general. This confirms that the Christoffel symbols are not tensor components: If the components of a tensor vanish in one basis they must vanish in all bases.

We can now summarize the conditions defining a locally flat coordinate system $x^{\bar{\mu}}{}_{\mathbf{X}}$ about point $\mathbf{x_0}$: $g_{\bar{\mu}\bar{\nu}\,\mathbf{X_0}} = \eta_{\bar{\mu}\bar{\nu}}$ and $\Gamma^{\bar{\mu}}{}_{\bar{\kappa}\bar{\lambda}\,\mathbf{X_0}} = 0$ or, equivalently, $\partial_{\bar{\mu}} g_{\bar{\kappa}\bar{\lambda}\,\mathbf{X_0}} = 0$.

4.6 Transformation to locally flat coordinates

We have derived an expression for the Christoffel symbols beginning from a locally flat coordinate system. The problem may be turned around to determine a locally flat coordinate system at point $\mathbf{x_0}$, given the metric and Christoffel symbols in any coordinate system. The coordinate transformation is found by expanding the components $g_{\mu\nu\,\mathbf{X}}$ of the metric in the non-flat coordinates x^{μ} in a Taylor series about $\mathbf{x_0}$ and relating them to the metric components $\eta_{\bar{\mu}\bar{\nu}}$ in the locally flat coordinates $x^{\bar{\mu}}$ using equation (34):

$$g_{\mu\nu\,\mathbf{X}} = g_{\mu\nu\,\mathbf{X_0}} + (x^{\lambda} - x_0^{\lambda})\,\partial_{\lambda} g_{\mu\nu\,\mathbf{X_0}} + O(x - x_0)^2 = \eta_{\bar{\mu}\bar{\nu}} \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}} \frac{\partial x^{\bar{\nu}}}{\partial x^{\nu}} + O(x - x_0)^2 \ . \quad (73)$$

Note that the partial derivatives of $\eta_{\bar{\mu}\bar{\nu}}$ vanish as do those of any correction terms to the metric in the locally flat coordinates at $x^{\bar{\mu}} = x_0^{\bar{\mu}}$. Equation (73) imposes the two conditions required for a locally flat coordinate system: $g_{\bar{\mu}\bar{\nu}\,\mathbf{X_0}} = \eta_{\bar{\mu}\bar{\nu}}$ and $\partial_{\bar{\mu}} g_{\bar{\kappa}\bar{\lambda}\,\mathbf{X_0}} = 0$. However, the *second* partial derivatives of the metric do not necessarily vanish, implying that we cannot necessarily make the derivatives of the Christoffel symbols vanish at $\mathbf{x_0}$. Quadratic corrections to the flat metric are a manifestation of curvature. In fact, we will see that all the information about the curvature and global geometry of our manifold is contained in the first and second derivatives of the metric. But first we must see whether general coordinates x^{μ} can be transformed so that the zeroth and first derivatives of the metric at $\mathbf{x_0}$ match the conditions implied by equation (73).

We expand the desired locally flat coordinates $x^{\bar{\mu}}$ in terms of the general coordinates x^{μ} in a Taylor series about the point $\mathbf{x_0}$:

$$x^{\bar{\mu}} = x_0^{\bar{\mu}} + A^{\bar{\mu}}{}_{\kappa}(x^{\kappa} - x_0^{\kappa}) + B^{\bar{\mu}}{}_{\kappa\lambda}(x^{\kappa} - x_0^{\kappa})(x^{\lambda} - x_0^{\lambda}) + O(x - x_0)^3 \ , \quad (74)$$

where $x_0^{\bar{\mu}}$, $A^{\bar{\mu}}{}_{\kappa}$ and $B^{\bar{\mu}}{}_{\kappa\lambda}$ are all constants. We leave it as an exercise for the reader to show, by substituting equations (74) into equations (73), that $A^{\bar{\mu}}{}_{\kappa}$ and $B^{\bar{\mu}}{}_{\kappa\lambda}$ must satisfy the following constraints:

$$g_{\kappa\lambda\,\mathbf{X_0}} = \eta_{\bar{\mu}\bar{\nu}} A^{\bar{\mu}}{}_{\kappa} A^{\bar{\nu}}{}_{\lambda} \ , \quad B^{\bar{\mu}}{}_{\kappa\lambda} = \frac{1}{2} A^{\bar{\mu}}{}_{\mu} \Gamma^{\mu}{}_{\kappa\lambda\,\mathbf{X_0}} \ . \quad (75)$$

If these constraints are satisfied then we have found a transformation to a locally flat coordinate system. It is possible to satisfy these constraints provided that the metric and the Christoffel symbols are finite at $\mathbf{x_0}$. This proves the consistency of the assumption underlying equation (69), at least away from singularities. (One should not expect to find a locally flat coordinate system centered on a black hole.)

From equation (75), we see that for a given matrix $A^{\bar{\mu}}{}_{\kappa}$, $B^{\bar{\mu}}{}_{\kappa\lambda}$ is completely fixed by the Christoffel symbols in our nonflat coordinates. So, the Christoffel symbols determine the quadratic corrections to the coordinates relative to a locally flat coordinate system. As for the $A^{\bar{\mu}}{}_{\kappa}$ matrix giving the linear transformation to flat coordinates, it has 16 independent coefficients in a four-dimensional spacetime. The metric tensor has only 10 independent coefficients (because it is symmetric). From equation (75), we see that we are left with 6 degrees of freedom for any transformation to locally flat spacetime coordinates. Could these 6 have any special significance? Yes! Given any locally flat coordinates in spacetime, we may rotate the spatial coordinates by any amount (labeled by one angle) about any direction (labeled by two angles), accounting for three degrees of freedom. The other three degrees of freedom correspond to a rotation of one of the space coordinates with the time coordinate, i.e., a Lorentz boost! This is exactly the freedom we would expect in defining an inertial frame in *special* relativity. Indeed, in a locally inertial frame general relativity reduces to special relativity by the Equivalence Principle.

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Physics 8.962 Spring 2002

Tensor Calculus, Part 2

1 Introduction

The first set of 8.962 notes, *Introduction to Tensor Calculus for General Relativity*, discussed tensors, gradients, and elementary integration. The current notes continue the discussion of tensor calculus with orthonormal bases and commutators (§2), parallel transport and geodesics (§3), and the Riemann curvature tensor (§4).

2 Orthonormal Bases, Tetrads, and Commutators

A vector basis is said to be orthonormal at point $\mathbf{X}$ if the dot product is given by the Minkowski metric at that point:

$$\{\vec{e}_{\hat{\mu}}\} \text{ is orthonormal if and only if } \vec{e}_{\hat{\mu}} \cdot \vec{e}_{\hat{\nu}} = \eta_{\mu\nu} \ . \tag{1}$$

(We have suppressed the implied subscript $\mathbf{X}$ for clarity.) Note that we will always place a hat over the index for any component of an orthonormal basis vector. The smoothness properties of a manifold imply that it is always possible to choose an orthonormal basis at any point in a manifold. One simply choose a basis that diagonalizes the metric g and furthermore reduces it to the normalized Minkowski form. Indeed, there are infinitely many orthonormal bases at $\mathbf{X}$ related to each other by Lorentz transformations. Orthonormal bases correspond to locally inertial frames.

For each basis of orthonormal vectors there is a corresponding basis of orthonormal one-forms related to the basis vectors by the usual duality condition:

$$\left\langle \tilde{e}^{\hat{\mu}}, \vec{e}_{\hat{\nu}} \right\rangle = \delta^{\mu}{}_{\nu} \ . \tag{2}$$

The existence of orthonormal bases at one point is very useful in providing a locally inertial frame in which to present the components of tensors measured by an observer at rest in that frame. Consider an observer with 4-velocity $\vec{V}$ at point $\mathbf{X}$. Since $\vec{V}\cdot\vec{V} = -1$, the observer's rest frame has timelike orthonormal basis vector $\vec{e}_{\hat{0}} = \vec{V}$. The observer has a set of orthonormal space axes given by a set of spatial unit vectors $\vec{e}_{\hat{i}}$. For a given $\vec{e}_{\hat{0}}$, there are of course many possible choices for the spatial axes that are related by spatial rotations. Each choice of spatial axes, when combined with the observer's 4-velocity, gives an orthonormal basis or tetrad. Thus, an observer carries along an orthonormal bases that we call the **observer's tetrad**. This basis is the natural one for splitting vectors, one-forms, and tensors into timelike and spacelike parts. We use the observer's tetrad to extract physical, measurable quantities from geometric, coordinate-free objects in general relativity.

For example, consider a particle with 4-momentum $\vec{P}$. The energy in the observer's instantaneous inertial local rest frame is $E = -\vec{V}\cdot\vec{P} = -\vec{e}_{\hat{0}}\cdot\vec{P} = \langle\tilde{e}^{\hat{0}}, \vec{P}\rangle$. The observer can define a $(2,0)$ projection tensor

$$\mathsf{h} \equiv \mathsf{g}^{-1} + \vec{V}\otimes\vec{V} \tag{3}$$

with components (in any basis) $h^{\alpha\beta} = g^{\alpha\beta} + V^\alpha V^\beta$. This projection tensor is essentially the inverse metric on spatial hypersurfaces orthogonal to $\vec{V}$; the corresponding $(0,2)$ tensor is $h_{\mu\nu} = g_{\alpha\mu}g_{\beta\nu}h^{\alpha\beta}$. The reader can easily verify that $h_{\mu\nu}V^\mu = h_{\mu\nu}V^\nu = 0$, hence in the observer's tetrad, $h^{\hat{\mu}\hat{\nu}} = h_{\hat{\mu}\hat{\nu}} = \mathbf{diag}(0,1,1,1)$. Then, the spatial momentum components follow from $P^{\hat{i}} = \langle\tilde{e}^{\hat{i}}, \vec{P}\rangle = P_{\hat{i}} = \vec{e}_{\hat{i}}\cdot\vec{P}$. (Normally it is meaningless to equate components of one-forms and vectors since they cannot be equal in all bases. Here we are restricting ourselves to a single basis — the observer's tetrad — where it happens that spatial components of one-forms and vectors are equal.) Note that $P^{\hat{i}}\vec{e}_{\hat{i}} = \mathsf{h}(\mathsf{g}(\vec{P}\,))$: the spatial part of the momentum is extracted using h. Thus, in any basis, $P^\mu = EV^\mu + h^\mu{}_\nu P^\nu$ splits $\vec{P}$ into parts parallel and perpendicular to $\vec{V}$. (Note $h^\mu{}_\nu \equiv g_{\kappa\nu}h^{\mu\kappa}$.)

2.1 Tetrads

If one can define an orthonormal basis for the tangent space at any point in a manifold, then one can define a set of orthonormal bases for **every** point in the manifold. In this way, equation (1) applies everywhere. At all spacetime points, the dot product has been reduced to the Minkowski form: $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$. One then has an orthonormal basis, or tetrad, for all points of spacetime.

If spacetime is not flat, how can we reduce the metric at every point to the Minkowski form? Doesn't that require a globally flat, Minkowski spacetime? How can one have the Minkowski metric without having Minkowski spacetime?

The resolution of this paradox lies in the fact that the metric we introduced in a coordinate basis has at least three different roles, and only one of them is played by $\eta_{\hat{\mu}\hat{\nu}}$. First, the metric gives the dot product: $\vec{A}\cdot\vec{B} = g_{\mu\nu}A^\mu B^\nu = \eta_{\hat{\mu}\hat{\nu}}A^{\hat{\mu}}B^{\hat{\nu}}$. Both $g_{\mu\nu}$

and $\eta_{\hat{\mu}\hat{\nu}}$ fulfill this role. Second, the metric components in a coordinate basis give the connection through the well-known Christoffel formula involving the partial derivatives of the metric components. Obviously since $\eta_{\hat{\mu}\hat{\nu}}$ has zero derivatives, it cannot give the connection. Third, the metric in a coordinate basis gives spacetime length and time through $d\vec{x} = dx^\mu \vec{e}_\mu$. Combining this with the dot product gives the line element, $ds^2 = d\vec{x} \cdot d\vec{x} = g_{\mu\nu} dx^\mu dx^\nu$. This formula is true only in a coordinate basis!

Usually when we speak of "metric" we mean the metric in a coordinate basis, which relates coordinate differentials to the line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. An orthonormal basis, unless it is also a coordinate basis, does not have enough information to provide the line element (or the connection). To determine these, we must find a linear transformation from the orthonormal basis to a coordinate basis:

$$\vec{e}_\mu = E^{\hat{\mu}}{}_\mu \vec{e}_{\hat{\mu}} \ . \tag{4}$$

The coefficients $E^{\hat{\mu}}{}_\mu$ are called the **tetrad components**. Note that $\hat{\mu}$ labels the (tetrad) basis vector while μ labels the component in some coordinate system (which may have no relation at all to the orthonormal basis). For a given orthonormal basis, $E^{\hat{\mu}}{}_\mu$ may be regarded as (the components of) a set of 4 one-form fields, one one-form $\tilde{E}^{\hat{\mu}} = E^{\hat{\mu}}{}_\mu \tilde{e}^\mu$ for each value of $\hat{\mu}$. Note that the tetrad components are *not* the components of a (1,1) tensor because of the mixture of two different bases.

The tetrad may be inverted in the obvious way:

$$\vec{e}_{\hat{\mu}} = E^\mu{}_{\hat{\mu}} \vec{e}_\mu \quad \text{where} \quad E^\mu{}_{\hat{\mu}} E^{\hat{\mu}}{}_\nu = \delta^\mu{}_\nu \ . \tag{5}$$

The dual basis one-forms are related by the tetrad and its inverse as for any change of basis: $\tilde{e}^\mu = E^\mu{}_{\hat{\mu}} \tilde{e}^{\hat{\mu}}$, $\tilde{e}^{\hat{\mu}} = E^{\hat{\mu}}{}_\mu \tilde{e}^\mu$,

The metric components in the coordinate basis follow from the tetrad components:

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu = \eta_{\hat{\mu}\hat{\nu}} E^{\hat{\mu}}{}_\mu E^{\hat{\nu}}{}_\nu \tag{6}$$

or $g = E^T \eta E$ in matrix notation. Sometimes the tetrad is called the "square root of the metric." Equation (6) is the key result allowing us to use orthonormal bases in curved spacetime.

To discuss the curvature of a manifold we first need a connection relating nearby points in the manifold. If there exists any basis (orthonormal or not) such that $\langle \tilde{e}^\lambda, \widetilde{\nabla} \vec{e}_\mu \rangle \equiv \Gamma^\lambda{}_{\mu\nu} \tilde{e}^\nu = 0$ everywhere, then the manifold is indeed flat. However, the converse is not true: if the basis vectors rotate from one point to another even in a flat space (e.g. the polar coordinate basis in the plane) the connection will not vanish. Thus we will need to compute the connection and later look for additional quantities that give an invariant (basis-free) meaning to curvature. First we examine a more primitive object related to the gradient of vector fields, the commutator.

2.2 Commutators

The difference between an orthonormal basis and a coordinate basis arises immediately when one considers the commutator of two vector fields, which is a vector that may symbolically be defined by

$$[\vec{A},\vec{B}\,] \equiv \nabla_A \nabla_B - \nabla_B \nabla_A \tag{7}$$

where ∇_A is the directional derivative ($\nabla_A = A^\mu \partial_\mu$ in a coordinate basis). Equation (7) introduces a new notation and new concept of a vector since the right-hand side consists solely of differential operators with no arrows! To interpret this, we rewrite the right-hand side in a coordinate basis using, e.g., $\nabla_A \nabla_B f = A^\mu \partial_\mu (B^\nu \partial_\nu f)$ (where f is any twice-differentiable scalar field):

$$[\vec{A},\vec{B}\,] = \left(A^\mu \frac{\partial B^\nu}{\partial x^\mu} - B^\mu \frac{\partial A^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu} \ . \tag{8}$$

This is equivalent to a vector because $\{\partial/\partial x^\nu\}$ provide a coordinate basis for vectors in the formulation of differential geometry introduced by Cartan. Given our heuristic approach to vectors as objects with magnitude and direction, it seems strange to treat a partial derivative as a vector. However, Cartan showed that directional derivatives form a vector space isomorphic to the tangent space of a manifold. Following him, differential geometry experts replace our coordinate basis vectors $\vec{e}_\mu$ by $\partial/\partial x^\mu$. (MTW introduce this approach in Chapter 8. On p. 203, they write $\vec{e}_\alpha = \partial \mathcal{P}/\partial x^\alpha$ where $\mathcal{P}$ refers to a point in the manifold, as a way to indicate the association of the tangent vector and directional derivative.) With this choice, vectors become differential operators (e.g. $\vec{A} = A^\mu \partial_\mu$) and thus the commutator of two vector fields involves derivatives. However, we need not follow the Cartan notation. It is enough for us to define the commutator of two vectors by its components in a coordinate basis,

$$[\vec{A},\vec{B}\,] = (A^\mu \partial_\mu B^\nu - B^\mu \partial_\mu A^\nu)\vec{e}_\nu \quad \text{in a coordinate basis,} \tag{9}$$

where the partial derivative operators act only on B^ν and A^ν but not on $\vec{e}_\nu$.

Equation (9) implies

$$[\vec{A},\vec{B}\,] = \nabla_A \vec{B} - \nabla_B \vec{A} + T^\mu{}_{\alpha\beta} A^\alpha B^\beta \vec{e}_\mu \ , \tag{10}$$

where $T^\mu{}_{\alpha\beta} \equiv \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}$ in a coordinate basis is a quantity called the torsion tensor. The reader may easily show that the torsion tensor also follows from the commutator of covariant derivatives applied to any twice-differentiable scalar field,

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) f = T^\mu{}_{\alpha\beta} \nabla_\mu f \tag{11}$$

This equation shows that the torsion is a tensor even though the connection is not. The torsion vanishes by assumption in general relativity. This is a statement of physics, not mathematics. Other gravity theories allow for torsion to incorporate possible new physical effects beyond Einstein gravity.

The basis vector fields $\vec{e}_\mu(x)$ are vector fields, so let us examine their commutators. From equation (9) or (10), in an coordinate basis, the commutators vanish identically (even if the torsion does not vanish):

$$[\vec{e}_\mu, \vec{e}_\nu] = 0 \quad \text{in a coordinate basis} . \tag{12}$$

The vanishing of the commutators occurs because the coordinate basis vectors are dual to an integrable basis of one-forms: $\tilde{e}^\mu = \widetilde{\nabla} x^\mu$ for a set of 4 scalar fields x^μ. It may be shown that this integrability condition (i.e. that the basis one-forms may be integrated to give functions) is equivalent to equation (12) (see Wald 1984, problem 5 of Chapter 2).

Now let us examine the commutator for an orthonormal basis. We use equation (9) by expressing the tetrad components in a coordinate basis using equation (5). The result is

$$[\vec{e}_{\hat{\mu}}, \vec{e}_{\hat{\nu}}] = \partial_{\hat{\mu}} \vec{e}_{\hat{\nu}} - \partial_{\hat{\nu}} \vec{e}_{\hat{\mu}} \equiv \omega^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} \vec{e}_{\hat{\alpha}} , \tag{13}$$

where $\partial_{\hat{\mu}} \equiv E^\mu{}_{\hat{\mu}} \partial_\mu$. Equation (13) defines the **commutator basis coefficients** $\omega^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}}$ (cf. MTW eq. 8.14). Using equations (5), (12), and (13), one may show

$$\omega^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} = E^{\hat{\alpha}}{}_\alpha \left(\nabla_{\hat{\mu}} E^\alpha{}_{\hat{\nu}} - \nabla_{\hat{\nu}} E^\alpha{}_{\hat{\mu}} \right) = E^\mu{}_{\hat{\mu}} E^\nu{}_{\hat{\nu}} \left(\partial_\mu E^{\hat{\alpha}}{}_\nu - \partial_\nu E^{\hat{\alpha}}{}_\mu \right) . \tag{14}$$

In general the commutator basis coefficients do not vanish. Despite the appearance of a second (coordinate) basis, the commutator basis coefficients are independent of any other basis besides the orthonormal one. The coordinate basis is introduced solely for the convenience of partial differentiation with respect to the coordinates.

The commutator basis coefficients carry information about how the tetrad rotates as one moves to nearby points in the manifold. It is useful practice to derive them for the orthonormal basis $\{\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}\}$ in the Euclidean plane.

2.3 Connection for an orthonormal basis

The connection for the basis $\{\vec{e}_{\hat{\mu}}\}$ is defined by

$$\partial_{\hat{\nu}} \vec{e}_{\hat{\mu}} \equiv \Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} \vec{e}_{\hat{\alpha}} . \tag{15}$$

(The placement of the lower subscripts on the connection agrees with MTW but is reversed compared with Wald and Carroll.) From the local flatness theorem (metric compatibility with covariant derivative) discussed in the first set of notes,

$$\nabla_{\hat{\alpha}} g_{\hat{\mu}\hat{\nu}} = E^\alpha{}_{\hat{\alpha}} \partial_\alpha g_{\hat{\mu}\hat{\nu}} - \Gamma^{\hat{\beta}}{}_{\hat{\mu}\hat{\alpha}} g_{\hat{\beta}\hat{\nu}} - \Gamma^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}} g_{\hat{\mu}\hat{\beta}} = 0 . \tag{16}$$

In an orthonormal basis, $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ is constant so its derivatives vanish. We conclude that, in an orthonormal basis, the connection is antisymmetric on its first two indices:

$$\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} = -\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} \ , \quad \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} \equiv g_{\hat{\mu}\hat{\beta}}\Gamma^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}} = \eta_{\hat{\mu}\hat{\beta}}\Gamma^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}} \ . \tag{17}$$

In an orthonormal basis, the connection is *not*, in general, symmetric on its last two indices. (That is true only in a coordinate basis.)

Another equation for the connection coefficients comes from combining equations (13) with equation (15):

$$\omega_{\hat{\alpha}\hat{\mu}\hat{\nu}} = -\Gamma_{\hat{\alpha}\hat{\mu}\hat{\nu}} + \Gamma_{\hat{\alpha}\hat{\nu}\hat{\mu}} \ , \quad \omega_{\hat{\alpha}\hat{\mu}\hat{\nu}} \equiv g_{\hat{\alpha}\hat{\beta}}\omega^{\hat{\beta}}{}_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\alpha}\hat{\beta}}\omega^{\hat{\beta}}{}_{\hat{\mu}\hat{\nu}} \ . \tag{18}$$

Combining these last two equations yields

$$\Gamma_{\hat{\alpha}\hat{\mu}\hat{\nu}} = \frac{1}{2}\left(\omega_{\hat{\mu}\hat{\alpha}\hat{\nu}} + \omega_{\hat{\nu}\hat{\alpha}\hat{\mu}} - \omega_{\hat{\alpha}\hat{\mu}\hat{\nu}}\right) \quad \text{in an orthonormal basis.} \tag{19}$$

The connection coefficients in an orthonormal basis are also called Ricci rotation coefficients (Wald) or the spin connection (Carroll).

It is straightforward to generalize the results of this section to general bases that are neither orthonormal nor coordinate. The commutator basis coefficients are defined as in equation (12). Dropping the carets on the indices, the general connection is (MTW eq. 8.24b)

$$\Gamma_{\alpha\mu\nu} \equiv g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} + \omega_{\mu\alpha\nu} + \omega_{\nu\alpha\mu} - \omega_{\alpha\mu\nu}\right) \quad \text{in any basis.} \tag{20}$$

The results for coordinate bases (where $\omega_{\alpha\mu\nu} = 0$) and for orthonormal bases (where $\partial_\alpha g_{\mu\nu} = 0$) follow as special cases.

3 Parallel transport and geodesics

3.1 Differentiation along a curve

As a prelude to parallel transport we consider another form of differentiation: differentiation along a curve. A curve is a parametrized path through spacetime: $\mathbf{x}(\lambda)$, where λ is a parameter that varies smoothly and monotonically along the path. The curve has a tangent vector $\vec{V} \equiv d\vec{x}/d\lambda = (dx^\mu/d\lambda)\,\vec{e}_\mu$. Here one must be careful about the interpretation: x^μ are not the components of a vector; they are simply 4 scalar fields. However, $\vec{V} = d\vec{x}/d\lambda$ is a vector (i.e. a tangent vector in the manifold).

If we wish, we could make $\vec{V}$ a unit vector (provided $\vec{V}$ is non-null) by setting $d\lambda = |d\vec{x} \cdot d\vec{x}\,|^{1/2}$ to measure path length along the curve. However, we will impose no such restriction in general.

Now, suppose that we have a scalar field $f_{\mathbf{X}}$ defined along the curve. We define the derivative along the curve by a simple extension of equations (36) and (38) of the first set of lecture notes:

$$\frac{df}{d\lambda} \equiv \nabla_V f \equiv \langle \tilde{\nabla} f, \vec{V} \rangle = V^\mu \partial_\mu f \ , \quad \vec{V} = \frac{d\vec{x}}{d\lambda}. \tag{21}$$

We have introduced the symbol ∇_V for the **directional derivative**, i.e. the covariant derivative along $\vec{V}$, the tangent vector to the curve $\mathbf{x}(\lambda)$. This is a natural generalization of ∇_μ, the covariant derivative along the basis vector $\vec{e}_\mu$.

For the derivative of a scalar field, ∇_V involves just the partial derivatives ∂_μ. Suppose, however, that we differentiate a vector field $\vec{A}_{\mathbf{X}}$ along the curve. Now the components of the gradient $\nabla_\mu A^\nu$ are not simply the partial derivatives but also involve the connection. The same is true when we project the gradient onto the tangent vector $\vec{V}$ along a curve:

$$\frac{d\vec{A}}{d\lambda} \equiv \frac{DA^\mu}{d\lambda}\,\vec{e}_\mu \equiv \nabla_V \vec{A} \equiv \langle \tilde{\nabla}\vec{A}, \vec{V} \rangle = V^\nu (\nabla_\nu A^\mu)\,\vec{e}_\mu = \left(\frac{dA^\mu}{d\lambda} + \Gamma^\mu{}_{\kappa\nu} A^\kappa V^\nu \right) \vec{e}_\mu \ . \tag{22}$$

We retain the symbol ∇_V to indicate the covariant derivative along V but we have introduced the new notation $D/d\lambda = V^\mu \nabla_\mu \neq d/d\lambda = V^\mu \partial_\mu$.

3.2 Parallel transport

The derivative of a vector along a curve leads us to an important concept called parallel transport. Suppose that we have a curve $\mathbf{x}(\lambda)$ with tangent $\vec{V}$ and a vector $\vec{A}(0)$ defined at one point on the curve (call it $\lambda = 0$). We define a procedure called parallel transport by defining a vector $\vec{A}(\lambda)$ along each point of the curve in such a way that $DA^\mu/d\lambda = 0$:

$$\nabla_V \vec{A} = 0 \quad \Leftrightarrow \quad \text{parallel transport of } \vec{A} \text{ along } \vec{V} . \tag{23}$$

Over a small distance interval this procedure is equivalent to transporting the vector $\vec{A}$ along the curve in such a way that the vector remains parallel to itself with constant length: $A(\lambda + \Delta\lambda) = A(\lambda) + O(\Delta\lambda)^2$. In a locally flat coordinate system, with the connection vanishing at $\mathbf{x}(\lambda)$, the components of the vector do not change as the vector is transported along the curve. If the space were globally flat and we used rectilinear coordinates (with vanishing connection everywhere), the components would not change at all no matter how the vector is transported. This is not the case in a curved space or in a flat space with curvilinear coordinates because in these cases the connection does not vanish everywhere.

3.3 Geodesics

Parallel transport can be used to define a special class of curves called *geodesics*. A geodesic curve is one that parallel-transports its own tangent vector $\vec{V} = d\vec{x}/d\lambda$, i.e., a curve that satisfies $\nabla_V \vec{V} = 0$. In other words, not only is $\vec{V}$ kept parallel to itself (with constant magnitude) along the curve, but locally the curve continues to point in the same direction all along the path. A geodesic is the natural extension of the definition of a "straight line" to a curved manifold. Using equations (22) and (23), we get a second-order differential equation for the coordinates of a geodesic curve:

$$\frac{DV^\mu}{d\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} V^\alpha V^\beta = 0 \quad \text{for a geodesic} \ , \quad V^\mu \equiv \frac{dx^\mu}{d\lambda} \ . \tag{24}$$

Indeed, in locally flat coordinates (such that the connection vanishes at a point), this is the equation of a straight line. However, in a curved space the connection cannot be made to vanish everywhere. A well-known example of a geodesic in a curved space is a great circle on a sphere.

There are several technical points worth noting about geodesic curves. The first is that $\vec{V} \cdot \vec{V} = \mathsf{g}(\vec{V}, \vec{V})$ is constant along a geodesic because $d\vec{V}/d\lambda = 0$ (eq. 24) and $\nabla_V \mathsf{g} = 0$ (metric compatibility with gradient). Therefore, a geodesic may be classified by its tangent vector as being either timelike ($\vec{V} \cdot \vec{V} < 0$), spacelike ($\vec{V} \cdot \vec{V} > 0$) or null ($\vec{V} \cdot \vec{V} = 0$). The second point is that a nonlinear transformation of the parameter λ will invalidate equation (24). In other words, if $x^\mu(\lambda)$ solves equation (24), $y^\mu(\lambda) \equiv x^\mu(\xi(\lambda))$ will not solve it unless $\xi = a\lambda + b$ for some constants a and b. Only a special class of parameters, called *affine parameters*, can parametrize geodesic curves.

The affine parameter has a special interpretation for a non-null geodesic. We deduce this relation from the constancy along the geodesic of $\vec{V} \cdot \vec{V} = (d\vec{x} \cdot d\vec{x})/(d\lambda^2) \equiv a$, implying $ds = a d\lambda$ and therefore $s = a\lambda + b$ where s is the path length ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$). For a non-null geodesic ($\vec{V} \cdot \vec{V} \neq 0$), all affine parameters are linear functions of path length (or proper time, if the geodesic is timelike). The linear scaling of path length amounts simply to the freedom to change units of length and to choose any point as $\lambda = 0$. Note that originally we imposed no constraints on the parameterization. However, the solutions of the geodesic equation automatically have λ being an affine parameter. There is no fundamental reason to use an affine parameter; one could always take a solution of the geodesic equation and reparameterize it or eliminate the parameter altogether by replacing it with one of the coordinates along the geodesic. For example, for a timelike trajectory, $x^i(t)$ is a perfectly valid description and is equivalent to $x^\mu(\lambda)$. But the spatial components as functions of $t = x^0$ clearly do not satisfy the geodesic equation for $x^\mu(\lambda)$.

Another interesting point is that the total path length is stationary for a geodesic:

$$\delta \int_A^B ds = \delta \int_A^B \left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right|^{1/2} d\lambda = 0 \tag{25}$$

if λ is an affine parameter. The δ refers to a variation of the integral arising from a variation of the curve, $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$, with fixed endpoints. The metric components are considered here to be functions of the coordinates. The variational principle is discussed in section 2 of the 8.962 notes "Hamiltonian Dynamics of Particle Motion," where it is shown that stationary path length implies the geodesic equation (24) if the parameterization is affine. Equation (25) is invariant under reparameterization, so its stationary solutions are a broader class of functions than the solutions of equation (24). In general, the tangent vector of the stationary solutions are not normalized: $|\vec{V} \cdot \vec{V}|^{1/2} = Q(\lambda) \neq$ constant, implying that λ is not affine. It is easy to show that any stationary solution may be reparameterized, $\lambda \to \tau$ through $d\tau/d\lambda = Q(\lambda)$, and that the resulting curve $x^\mu(\lambda(\tau))$ obeys the geodesic equation with affine parameter τ. This transformation replaces the unnormalized tangent vector $\vec{V}$ by $\vec{V}/Q(\lambda)$. For an affine parameterization, the tangent vector must always have constant length.

Equation (25) is a curved space generalization of the statement that a straight line is the shortest path between two points in flat space.

3.4 Integrals of motion and Killing vectors

Equation (24) is a set of four second-order nonlinear ordinary differential equations for the coordinates of a geodesic curve. One may ask whether the order of this system can be reduced by finding integrals of the motion. An integral, also called a conserved quantity, is a function of x^μ and $V^\mu = dx^\mu/d\lambda$ that is constant along any geodesic. At least one integral always exists: $\vec{V} \cdot \vec{V} = g_{\mu\nu} V^\mu V^\nu$. (For an affine parameterization, $\vec{V} \cdot \vec{V}$ is constant along the curve.) Are there others? Sometimes. One may show that equation (24) may be rewritten as an equation of motion for $V_\mu \equiv g_{\mu\nu} V^\nu$, yielding

$$\frac{dV_\mu}{d\lambda} = \frac{1}{2} (\partial_\mu g_{\alpha\beta}) V^\alpha V^\beta \ . \tag{26}$$

Consequently, if all of the metric components are independent of some particular coordinate x^μ, the corresponding component of the tangent one-form is constant along the geodesic. This result is very useful in reducing the amount of integration needed to construct geodesics for metrics with high symmetry. However, the condition $\partial_\mu g_{\alpha\beta} = 0$ is coordinate-dependent. There is an equivalent coordinate-free test for integrals, based on the existence of special vector fields $\vec{K}$ call *Killing vectors*. Killing vectors are, by definition, solutions of the differential equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \ . \tag{27}$$

(The Killing *vector* components are, of course, $K^\mu = g^{\mu\nu} K_\nu$.) The Killing equation (27) usually has no solutions, but for highly symmetric spacetime manifolds there may be one or more solutions. It is a nice exercise to show that each Killing vector leads to the integral of motion

$$\langle \tilde{V}, \vec{K} \rangle = K^\mu V_\mu = \text{constant along a geodesic} \ . \tag{28}$$

Note that if one of the basis vectors (for some basis) satisfies the Killing equation, then the corresponding component of the tangent one-form is an integral of motion. The test for integrals implied by equation (26) is a special case of the Killing vector test when the Killing vector is simply a coordinate basis vector.

The discussion here has focused on geodesics as curves. The notes "Hamiltonian Dynamics of Particle Motion" interprets them as worldlines for particles because, as we will see, a fundamental postulate of general relativity is that, in the absence of non-gravitational forces, particles move along geodesics. Given this fact, we are free to choose units of the affine parameter λ so that $dx^\mu/d\lambda$ is the 4-momentum P^μ, normalized by $\vec{P} \cdot \vec{P} = -m^2$ for a particle of mass m (instead of $dx^\mu/d\lambda = V^\mu$, $\vec{V} \cdot \vec{V} = -1$). Thus, the tangent vector, denoted $\vec{V}$ above, is equivalent to the particle 4-momentum vector. The affine parameter λ then measures proper time divided by particle mass. Although one might fear this makes no sense for a massless particle, in fact it is the only way to affinely parameterize null geodesics because the proper time change $d\tau$ vanishes along a null geodesic so $dx^\mu/d\tau$ is undefined. For a massless particle, one takes the limit $m \to 0$ starting from the solution for a massive particle, with the result that $d\lambda = d\tau/m$ is finite as $m \to 0$.

4 Curvature

We introduce curvature by considering parallel transport around a general (non-geodesic) closed curve. In flat space, in a globally flat coordinate system (for which the connection vanishes everywhere), parallel transport leaves the components of a vector unchanged. Thus, in flat space, transporting a vector around a closed curve returns the vector to its starting point unchanged. Not so in a nonflat space. This change under a closed cycle is called an "anholonomy."

Consider, for example, a sphere. Suppose that we have a vector pointing east on the equator at longitude 0°. We parallel transport the vector eastward on the equator by 180°. At each point on the equator the vector points east. Now the vector is parallel transported along a line of constant longitude over the pole and back to the starting point on the equator. At each point on this second part of the curve, the vector points at right angles to the curve, and its direction never changes. Yet, at the end of the curve, at the same point where the curve started, the vector points west!

The reader may imagine that the example of the sphere is special because of the sharp changes in direction made in the path. However, parallel transport around any

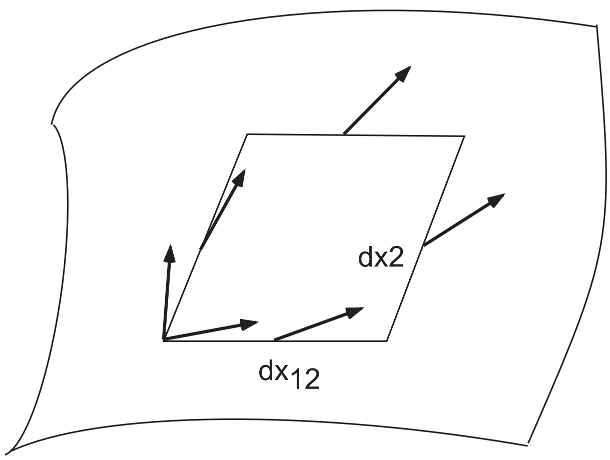

Figure 1: Parallel transport around a closed curve. The vector in the lower-left corner is parallel transported in a counter-clockwise direction along around 4 segments $d\vec{x}_1$, $d\vec{x}_2$, $-d\vec{x}_1$, and $-d\vec{x}_2$. At the end of the journey, the vector has been rotated. This mismatch ("anholonomy") does not occur for parallel transport in a flat space; its existence is the defining property of curvature.

smooth closed curve results in an anholonomy on a sphere. For example, consider a latitude circle away from the equator. Imagine you are an airline pilot flying East from Boston. If you were flying on a great circle route, you would soon be flying in a south-east direction. If you parallel transport a vector along a geodesic, its direction relative to the tangent vector (direction of motion) does not change, i.e. $\nabla_V(\vec{A}\cdot\vec{V}\,)=0$ for parallel transport of $\vec{A}$ along tangent $\vec{V}$. Parallel transport implies $\nabla_V\vec{A}=0$; moreover, $\nabla_V\vec{V}=0$ for a geodesic. However, a constant-latitude circle is not a geodesic, hence $\nabla_V\vec{V}\neq 0$. In order to maintain a constant latitude, you will have to constantly steer the airplane north compared with a great circle route. Consequently, the angle between $\vec{A}$ (which is parallel-transported) and the tangent changes: $\nabla_V(\vec{A}\cdot\vec{V}\,)=A\cdot(\nabla_V\vec{V}\,)$. A nonzero rotation accumulates during the trip, leading to a net rotation of $\vec{A}$ around a closed curve.

We can refine this into a definition of curvature as follows. Suppose that our closed curve consists of four infinitesimal segments: $d\vec{x}_1$, $d\vec{x}_2$, $-d\vec{x}_1$ and $-d\vec{x}_2$. In a flat space this would be called a parallelogram and the difference $d\vec{A}$ between the final and initial vectors would vanish. In a curved space we can create a parallelogram by taking two pairs of coordinate lines and choose $d\vec{x}_1$ and $d\vec{x}_2$ to point along the coordinate lines

(e.g. in directions $\vec{e}_1$ and $\vec{e}_2$). Parallel transport around a closed curve gives a change in the vector $d\vec{A}$ that must be proportional to $\vec{A}$, to $d\vec{x}_1$, and to $d\vec{x}_2$. Remarkably, it is proportional to nothing else. Therefore, $d\vec{A}$ is given by a rank $(1,3)$ tensor called the Riemann curvature tensor:

$$d\vec{A}(\,\cdot\,)\equiv-\mathsf{R}(\,\cdot\,,\vec{A},d\vec{x}_1,d\vec{x}_2)=-\vec{e}_\mu R^\mu{}_{\nu\alpha\beta}\,A^\nu\,dx_1^\alpha\,dx_2^\beta\ . \tag{29}$$

The dots indicate that a one-form is to be inserted; recall that a vector is a function of a one-form. The minus sign is purely conventional and is chosen for agreement with MTW. Note that the Riemann tensor must be antisymmetric on the last two slots because reversing them amounts to changing the direction around the parallelogram, i.e. swapping the final and initial vectors $\vec{A}$, hence changing the sign of $d\vec{A}$.

All standard GR textbooks show that equation (29) is equivalent to the following important result known as the **Ricci identity**

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\mu = R^\mu{}_{\nu\alpha\beta} A^\nu \quad \text{in a coordinate basis} \ . \tag{30}$$

In a non-coordinate basis, there is an additional term on the left-hand side, $-\nabla_C A$ where $\vec{C} \equiv [\vec{e}_\alpha, \vec{e}_\beta]$. This commutator vanishes for a coordinate basis (eq. 12).

μ

Equation (30) is a remarkable result. In general, there is no reason whatsoever that the derivatives of a vector field should be related to the vector field itself. Yet the difference of second derivatives is not only related to, but is linearly proportional to the vector field! This remarkable result is a mathematical property of metric spaces with connections. It is equivalent to the statement that parallel transport around a small closed parallelogram is proportional to the vector and the oriented area element (eq. 29).

Equation (30) is similar to equation (11). The torsion tensor and Riemann tensor are geometric objects from which one may build a theory of gravity in curved spacetime. In general relativity, the torsion is zero and the Riemann tensor holds all of the local information about gravity.

It is straightforward to determine the components of the Riemann tensor using equation (30) with $\vec{A} = \vec{e}_\nu$. The result is

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu{}_{\nu\beta} - \partial_\beta \Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\kappa\alpha} \Gamma^\kappa{}_{\nu\beta} - \Gamma^\mu{}_{\kappa\beta} \Gamma^\kappa{}_{\nu\alpha} \quad \text{in a coordinate basis} \ . \tag{31}$$

Note that some authors (e.g., Weinberg 1972) define the components of Riemann with opposite sign. Our sign convention follows Misner et al (1973), Wald (1984) and Schutz (1985).

Note that the Riemann tensor involves the first and second partial derivatives of the metric (through the Christoffel connection in a coordinate basis). Weinberg (1972) shows that the Riemann tensor is the only tensor that can be constructed from the metric tensor and its first and second partial derivatives and is linear in the second derivatives. Recall that one can always define locally flat coordinates such that $\Gamma^\mu{}_{\nu\lambda} = 0$ at a point. However, one cannot choose coordinates such that $\Gamma^\mu{}_{\nu\lambda} = 0$ everywhere unless the space is globally flat. The Riemann tensor vanishes everywhere if and only if the manifold is globally flat. This is a very important result.

If we lower an index on the Riemann tensor components we get the components of a $(0, 4)$ tensor:

$$R_{\mu\nu\kappa\lambda} = g_{\mu\alpha}R^\alpha{}_{\nu\kappa\lambda} = \frac{1}{2}\left(g_{\mu\lambda,\nu\kappa} - g_{\mu\kappa,\nu\lambda} + g_{\nu\kappa,\mu\lambda} - g_{\nu\lambda,\mu\kappa}\right) + g_{\alpha\beta}\left(\Gamma^\alpha{}_{\mu\lambda}\Gamma^\beta{}_{\nu\kappa} - \Gamma^\alpha{}_{\mu\kappa}\Gamma^\beta{}_{\nu\lambda}\right) \ , \tag{32}$$

where we have used commas to denote partial derivatives for brevity of notation: $g_{\mu\lambda,\nu\kappa} \equiv \partial_\kappa\partial_\nu g_{\mu\lambda}$. In this form it is easy to determine the following symmetry properties of the Riemann tensor:

$$R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu} = -R_{\nu\mu\kappa\lambda} = -R_{\mu\nu\lambda\kappa} \ , \quad R_{\mu\nu\kappa\lambda} + R_{\mu\kappa\lambda\nu} + R_{\mu\lambda\nu\kappa} = 0 \ . \tag{33}$$

It can be shown that these symmetries reduce the number of independent components of the Riemann tensor in four dimensions from 4^4 to 20.

4.1 Bianchi identities, Ricci tensor and Einstein tensor

We note here several more mathematical properties of the Riemann tensor that are needed in general relativity. First, by differentiating the components of the Riemann tensor one can prove the *Bianchi identities*:

$$\nabla_\sigma R^\mu{}_{\nu\kappa\lambda} + \nabla_\kappa R^\mu{}_{\nu\lambda\sigma} + \nabla_\lambda R^\mu{}_{\nu\sigma\kappa} = 0 \ . \tag{34}$$

Note that the gradient symbols denote the covariant derivatives and not the partial derivatives (otherwise we would not have a tensor equation). The Bianchi identities imply the vanishing of the divergence of a certain $(2, 0)$ tensor called the Einstein tensor. To derive it, we first define a symmetric contraction of the Riemann tensor, known as the Ricci tensor:

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} = R_{\nu\mu} = \partial_\kappa\Gamma^\kappa{}_{\mu\nu} - \partial_\mu\Gamma^\kappa{}_{\kappa\nu} + \Gamma^\kappa{}_{\kappa\lambda}\Gamma^\lambda{}_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda}\Gamma^\lambda{}_{\kappa\nu} \ . \tag{35}$$

One can show from equations (33) that any other contraction of the Riemann tensor either vanishes or is proportional to the Ricci tensor. The contraction of the Ricci tensor is called the Ricci scalar:

$$R \equiv g^{\mu\nu}R_{\mu\nu} \ . \tag{36}$$

Contracting the Bianchi identities twice and using the antisymmetry of the Riemann tensor one obtains the following relation:

$$\nabla_\nu G^{\mu\nu} = 0 \ , \quad G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = G^{\nu\mu} \ . \tag{37}$$

The symmetric tensor $G^{\mu\nu}$ that we have introduced is called the *Einstein tensor*. Equation (37) is a mathematical identity, not a law of physics. Through the Einstein equations it provides a deep illustration of the connection between mathematical symmetries and physical conservation laws.

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Physics 8.962 Spring 2002

Number-Flux Vector and Stress-Energy Tensor

1 Introduction

These notes supplement Section 3 of the 8.962 notes "Introduction to Tensor Calculus for General Relativity." Having worked through the formal treatment of vectors, one-forms and tensors, we are ready to evaluate two particularly useful and important examples, the number-flux four-vector and the stress-energy (or energy-momentum) tensor for a gas of particles. A good elementary discussion of these objects is given in chapter 4 of Schutz, *A First Course in General Relativity*; more advanced treatments are in chapters 5 and 22 of MTW. Some of the mathematical material presented here is formalized in Section 4 of the 8.962 notes; to avoid repetition we will present the computations here in a locally flat frame (orthonormal basis with locally vanishing connection) frame rather than in a general basis. However, the final results are tensor equations valid in any basis.

2 Number-Flux Four-Vector for a Gas of Particles

We wish to describe the fluid properties of a gas of noninteracting particles of rest mass m starting from a microscopic description. In classical mechanics, we would describe the system by giving the spatial trajectories $\underline{x}_a(t)$ where a labels the particle and t is absolute time. (An underscore is used for 3-vectors; arrows are reserved for 4-vectors. While the position $\underline{x}_a$ isn't a true tangent vector, we retain the common notation here.) The number density and number flux density are

$$n = \sum_a \delta^3(\underline{x} - \underline{x}_a(t)) \ , \quad \underline{J} = \sum_a \delta^3(\underline{x} - \underline{x}_a(t)) \frac{d\underline{x}_a}{dt} \tag{1}$$

where the Dirac delta function has its usual meaning as a distribution:

$$\int d^3x\, f(\underline{x})\, \delta^3(\underline{x} - \underline{y}) = f(\underline{y}) \ . \tag{2}$$

In order to get well-defined quantities when relativistic motions are allowed, we attempt to combine the number and flux densities into a four-vector $\vec{N}$. The obvious generalization of equation (1) is

$$\vec{N} = \sum_a \delta^3(\underline{x} - \underline{x}_a(t)) \frac{d\vec{x}_a}{dt} \ . \tag{3}$$

However, this is not suitable because time and space are explicitly distinguished: $(t, \underline{x})$. A first step is to insert one more delta function, with an integral (over time) added to cancel it:

$$\vec{N} = \sum_a \int dt'\, \delta^4(x - x_a(t')) \frac{d\vec{x}_a}{dt'} \ . \tag{4}$$

The four-dimensional Dirac delta function is to be understood as the product of the three-dimensional delta function with $\delta(t - t_a(t')) = \delta(x^0 - t')$:

$$\delta^4(x - y) \equiv \delta(x^0 - y^0)\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3) \ . \tag{5}$$

Equation (4) looks promising except for the fact that our time coordinate t' is frame-dependent. The solution is to use a Lorentz-invariant time for each particle — the proper time along the particle's worldline. We already know that particle trajectories in spacetime can be written $x^a(\tau)$. We can change the parametrization in equation (4) so as to obtain a Lorentz-invariant object, a four-vector:

$$\vec{N} = \sum_a \int d\tau\, \delta^4(x - x_a(\tau)) \frac{d\vec{x}_a}{d\tau} \ . \tag{6}$$

2.1 Lorentz Invariance of the Dirac Delta Function

Before accepting equation (6) as a four-vector, we should be careful to check that the delta function is really Lorentz-invariant. We can do this without requiring the existence of a globally inertial frame (something that doesn't exist in the presence of gravity!) because the delta function picks out a single spacetime point and so we may regard spacetime integrals as being confined to a small neighborhood over which locally flat coordinates may be chosen with metric $\eta_{\mu\nu}$ (the Minkowski metric).

To prove that $\delta^4(x - y)$ is Lorentz invariant, we note first that it is nonzero only if $x^\mu = y^\mu$. Now suppose we that perform a local Lorentz transformation, which maps dx^μ to $dx^{\bar\mu} = \Lambda^{\bar\mu}{}_\nu dx^\nu$ and d^4x to $d^4\bar{x} = |\det\Lambda|\, d^4x$. Clearly, $\delta^4(\bar{x} - \bar{y})$ is nonzero only if

$x^{\bar{\mu}} = y^{\bar{\mu}}$ and hence only if $x^\mu = y^\mu$. From this it follows that $\delta^4(\bar{x} - \bar{y}) = S\delta^4(x - y)$ for some constant S. We will show that $S = 1$.

To do this, we write the Lorentz transformation in matrix notation as $\bar{x} = \Lambda x$ and we make use the definition of the Dirac delta function:

$$f(\bar{y}) = \int d^4\bar{x}\, \delta^4(\bar{x} - \bar{y}) f(\bar{x}) = \int d^4x\, |\det \Lambda|\, S\delta^4(x - y) f(\Lambda x) = S\, |\det \Lambda|\, f(\bar{y}) \ . \tag{7}$$

Lorentz transformations are the group of coordinate transformations which leave the Minkowski metric invariant, $\eta = \Lambda^T \eta \Lambda$. Now, $\det \eta = -1$, from which it follows that $|\det \Lambda| = 1$. From equation (7), $S = 1$ and the four-dimensional Dirac delta function is Lorentz-invariant (a Lorentz scalar).

As an aside, $\delta^4(x)$ is *not* invariant under arbitrary coordinate transformations, because d^4x isn't invariant in general. (It is invariant only for those transformations with $|\det \Lambda| = 1$). In part 2 of the notes on tensor calculus we show that $|\det g|^{1/2} d^4x$ is fully invariant, so we should multiply the Dirac delta function by $|\det g|^{-1/2}$ to make it invariant under general coordinate transformations. In the special case of an orthonormal basis, $g = \eta$ so that $|\det g| = 1$.

3 Stress-Energy Tensor for a Gas of Particles

The energy and momentum of one particle is characterized by a four-vector. For a gas of particles, or for fields (e.g. electromagnetism), we need a rank $(2, 0)$ tensor which combines the energy density, momentum density (or energy flux — they're the same) and momentum flux or stress. The stress-energy tensor is symmetric and is defined so that

$$\mathsf{T}(\tilde{e}^\mu, \tilde{e}^\nu) = T^{\mu\nu} \text{ is the flux of momentum } p^\mu \text{ across a surface of constant } x^\nu \ . \tag{8}$$

It follows (Schutz chapter 4) that in an orthonormal basis T^{00} is the energy density, T^{0i} is the energy flux (energy crossing a unit area per unit time), and T^{ij} is the stress (i-component momentum flux per unit area per unit time crossing the surface $x^j =$ constant. The stress-energy tensor is especially important in general relativity because it is the source of gravity. It is important to become familiar with it.

The components of the number-flux four-vector $N^\nu = \vec{N}(\tilde{e}^\nu)$ give the flux of particle number crossing a surface of constant x^ν (with normal one-form $\tilde{e}^\nu$). From this, we can obtain the stress-energy tensor following equation (6). Going from number (a scalar) to momentum (a four-vector) flux is simple: multiply by $\vec{p} = m\vec{V} = m d\vec{x}/d\tau$. Thus,

$$\mathsf{T} = \sum_a \int d\tau\, \delta^4(x - x_a(\tau)) m \vec{V}_a \otimes \vec{V}_a \ . \tag{9}$$

4 Uniform Gas of Non-Interacting Particles

The results of equations (6) and (9) include a discrete sum over particles. To go to the continuum, or fluid, limit, we suppose that the particles are so numerous that the sum of delta functions may be replaced by its average over a small spatial volume. To get the number density measured in a locally flat (orthonormal) frame we must undo some of the steps leading to equation (6). Using the fact that $dt/d\tau = \gamma$, comparing equations (3) and (6) shows that we need to evaluate

$$\sum_a \int d\tau\, \delta^4(x - x_a(\tau)) = \sum_a {\gamma_a}^{-1}\, \delta^3(\underline{x} - \underline{x}_a(t)) \ . \tag{10}$$

Now, aside from the factor γ_a^{-1}, integrating equation (10) over a small volume ΔV and dividing by ΔV would yield the local number density. However, we must also keep track of the velocity distribution of the particles. Let us suppose that the velocities are randomly sampled from a (possibly spatially or temporally varying) three-dimensional velocity distribution $f(\underline{x}, \underline{v}, t)$ normalized so that, in an orthonormal frame,

$$\int d^3v\, f(\underline{x}, \underline{v}, t) = 1 \ . \tag{11}$$

To make the velocity distribution Lorentz-invariant takes a little more work which we will not present here; the interested reader may see problem 5.34 of the *Problem Book in Relativity and Gravitation* by Lightman, Press, Price, and Teukolsky.

In an orthonormal frame with flat spacetime coordinates, the result becomes

$$\sum_a \int d\tau\, \delta^4(x - x_a(\tau)) = n(x) \int d^3v\, \gamma^{-1}\, f(x, \underline{v}) \ . \tag{12}$$

Using $\vec{V} = \gamma(1, \underline{v})$ and substituting into equation (3), we obtain the number-flux four-vector

$$\vec{N} = (n, \underline{J}) \ , \quad \underline{J} = n(x) \int d^3v\, f(x, \underline{v})\underline{v} \ . \tag{13}$$

Although this result has been evaluated in a particular Lorentz frame, once we have it we could transform to any other frame or indeed to any basis, including non-orthonormal bases.

The stress-energy tensor follows in a similar way from equations (9) and (12). In a local Lorentz frame,

$$T^{\mu\nu} = mn(x) \int d^3v\, f(x, \underline{v})\, \frac{V^\mu V^\nu}{V^0} \ . \tag{14}$$

If there exists a frame in which the velocity distribution is isotropic (independent of the direction of the three-velocity), the components of the stress-energy tensor are particularly simple in that frame:

$$T^{00} \equiv \rho = \int d^3v\, f(x,\underline{v})\, \gamma m n(x) \ , \quad T^{0i} = T^{i0} = 0 \ ,$$
$$T^{ij} = p\delta^{ij} \text{ where } p \equiv \frac{1}{3}\int d^3v\, f(x,\underline{v}) \gamma m n(x) v^2 \ . \tag{15}$$

Here ρ is the energy density (γm is the energy of a particle) and p is the pressure.

Equation (15) isn't Lorentz-invariant. However, we can get it into the form of a spacetime tensor (an invariant) by using the tensor basis plus the spatial part of the metric:

$$\mathsf{T} = \rho \vec{e}_0 \otimes \vec{e}_0 + p\eta^{ij} \vec{e}_i \otimes \vec{e}_j \ . \tag{16}$$

We can make further progress by noting that the pressure term may be rewritten after defining the projection tensor

$$\mathsf{h} = \mathsf{g}^{-1} + \vec{e}_0 \otimes \vec{e}_0 \tag{17}$$

since $g^{\mu\nu} = \eta^{\mu\nu}$ in an orthonormal basis and therefore $h^{00} = \eta^{00} + 1 = 0$, $h^{0i} = h^{i0} = 0$ and $h^{ij} = \delta^{ij}$. The tensor h projects any one-form into a vector orthogonal to $\vec{e}_0$. Combining results, we get

$$\mathsf{T} = (\rho + p)\vec{e}_0 \otimes \vec{e}_0 + p\, \mathsf{g}^{-1} \ . \tag{18}$$

Equation (18) is in the form of a tensor, but it picks out a preferred coordinate system through the basis vector $\vec{e}_0$. To eliminate this remnant of our nonrelativistic starting point, we note that, for any four-velocity $\vec{U}$, there exists an orthonormal frame (the instantaneous local inertial rest frame) in which $\vec{U} = \vec{e}_0$. Thus, if we identify $\vec{U}$ as the fluid velocity, we obtain our final result, the stress-energy tensor of a perfect gas:

$$\mathsf{T} = (\rho + p)\vec{U} \otimes \vec{U} + p\, \mathsf{g}^{-1} \quad \text{or} \quad T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p\, g^{\mu\nu} \tag{19}$$

If the sleight-of-hand of converting $\vec{e}_0$ to $\vec{U}$ seems unconvincing (and it is worth checking!), the reader may apply an explicit Lorentz boost to the tensor of equation (18) with three-velocity U^i/U^0 to obtain equation (19). We must be careful to remember that ρ and p are scalars (the proper energy density and pressure in the fluid rest frame) and $\vec{U}$ is the fluid velocity four-vector.

From this result, one may be tempted to rewrite the number-flux four-vector as $\vec{N} = n\vec{U}$ where $\vec{U}$ is the same fluid 4-velocity that appears in the stress-energy tensor. This is valid for a perfect gas, whose velocity distribution is isotropic in a particular frame, where n would be the proper number density. However, in general T^{0i} is nonzero in the frame in which $N^i = 0$, because the energy of particles is proportional to γ but the number is not. Noting that the kinetic energy of a particle is $(\gamma - 1)m$, we could have a net flux of kinetic energy (heat) even if there is no net flux of momentum. In other words, energy may be conducted by heat as well as by advection of rest mass. This leads to a fluid velocity in the stress-energy tensor which differs from the velocity in the number-flux 4-vector.

Besides heat conduction, a general fluid has a spatial stress tensor differing from $p\delta^{ij}$ due to shear stress provided by, for example, shear viscosity.

An example where these concepts and techniques find use is in the analysis of fluctuations in the cosmic microwave background radiation. When the radiation (photon) field begins to decouple from the baryonic matter (hydrogen-helium plasma) about 300,000 years after the big bang, anisotropies in the photon momentum distribution develop which lead to heat conduction and shear stress. The stress-energy tensor of the radiation field must be computed by integrating over the full non-spherical momentum distribution of the photons. Relativistic kinetic theory is one of the ingredients needed in a theoretical calculation of cosmic microwave background anisotropies (Bertschinger & Ma 1995, *Astrophys. J.* **455**, 7).

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Physics 8.962 Spring 1999

Hamiltonian Dynamics of Particle Motion

1 Introduction

These notes present a treatment of geodesic motion in general relativity based on Hamilton's principle, illustrating a beautiful mathematical point of tangency between the worlds of general relativity and classical mechanics.

2 Geodesic Motion

Our starting point is the standard variational principle for geodesics as extremal paths. Adopting the terminology of classical mechanics, we make the action stationary under small variations of the parameterized spacetime path $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$ subject to fixed values at the endpoints. The action we use is the path length:

$$S_1[x(\tau)] = \int \left[g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2} d\tau \equiv \int L_1(x, dx/d\tau)\, d\tau \ . \tag{1}$$

Variation of the trajectory leads to the usual Euler-Lagrange equations

$$\frac{d}{d\tau} \left[\frac{\partial L}{\partial (dx^\mu/d\tau)} \right] - \frac{\partial L}{\partial x^\mu} = 0 \ , \tag{2}$$

from which one obtains the equation of motion

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} - \frac{1}{L_1} \frac{dL_1}{d\tau} \frac{dx^\mu}{d\tau} = 0 \ . \tag{3}$$

The last term arises because the action of equation (1) is invariant under arbitrary reparameterization. If the path length is taken to be proportional to path length, $d\tau \propto$

$ds = (g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$, then $L_1 = ds/d\tau =$ constant and the last term vanishes, giving the standard geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \ . \tag{4}$$

It may be shown that any solution of equation (3) can be reparameterized to give a solution of equation (4). Moreover, at the level of equation (4), we needn't worry about whether τ is an affine parameter; we will see below that for any solution of equation (4), τ is automatically proportional to path length. The full derivation of the geodesic equation and discussion of parameterization of geodesics can be found in most general relativity texts (e.g. Misner et al 1973, §13.4).

The Lagrangian of equation (1) is not unique. Any Lagrangian that yields the same equations of motion is equally valid. For example, equation (4) also follows from

$$S_2[x(\tau)] = \int \frac{1}{2}g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau \equiv \int L_2(x, dx/d\tau)\,d\tau \ . \tag{5}$$

Unlike equation (1), which is extremal for geodesic curves regardless of their parameterization, equation (5) is extremal for geodesics only when τ is an affine parameter, $d\tau/ds =$ constant. In other words, τ measures path length up to a linear rescaling.

The freedom to linearly rescale the affine parameter allows us to define τ so that $p^\mu = dx^\mu/d\tau$ gives the 4-momentum (vector) of the particle, even for massless particles for which the proper path length vanishes. One may easily check that $d\tau = ds/m$ where m is the mass.

With the form of the action given by equation (5), the canonical momentum conjugate to x^μ equals the momentum one-form of the particle:

$$p_\mu \equiv \frac{\partial L_2}{\partial(dx^\mu/d\tau)} = g_{\mu\nu}\frac{dx^\nu}{d\tau} \ . \tag{6}$$

The coincidence of the conjugate momentum with the momentum one-form encourages us to consider the Hamiltonian approach as an alternative to the geodesic equation. In the Hamiltonian approach, coordinates and conjugate momenta are treated on an equal footing and are varied independently during the extremization of the action. The Hamiltonian is given by a Legendre transformation of the Lagrangian,

$$H(p, x, \tau) \equiv p_\mu\frac{dx^\mu}{d\tau} - L(x, dx/d\tau, \tau) \tag{7}$$

where the coordinate velocity $dx^\mu/d\tau$ must be expressed in terms of the coordinates and momenta. For Lagrangian L_2 this is simple, with the result

$$H_2(p_\mu, x^\nu, \tau) = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu \ . \tag{8}$$

Notice the consistency of the spacetime tensor component notation in equations (6)-(8). The rules for placement of upper and lower indices automatically imply that the conjugate momentum must be a one-form and that the Hamiltonian is a scalar.

The reader will notice that the Hamiltonian H_2 exactly equals the Lagrangian L_2 (eq. 5) when evaluated at a given point in phase space (p, x). However, in its meaning and use the Hamiltonian is very different from the Lagrangian. In the Hamiltonian approach, we treat the position and conjugate momentum on an equal footing. By requiring the action to be stationary under independent variations $\delta x^\mu(\tau)$ and $\delta p_\nu(\tau)$, we obtain Hamilton's equations in four-dimensional covariant tensor form:

$$\frac{dx^\mu}{d\tau} = \frac{\partial H_2}{\partial p_\mu} \ , \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H_2}{\partial x^\mu} \ . \tag{9}$$

Evaluating them using equation (8) yields the canonical equations of motion,

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu} p_\nu \ , \quad \frac{dp_\mu}{d\tau} = -\frac{1}{2}\frac{\partial g^{\kappa\lambda}}{\partial x^\mu} p_\kappa p_\lambda = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} g^{\kappa\alpha} g^{\beta\lambda} p_\kappa p_\lambda = g^{\beta\lambda} \Gamma^\kappa{}_{\mu\beta} p_\kappa p_\lambda \ . \tag{10}$$

These equations may be combined to give equation (4).

The canonical equations (9) imply $dH/d\tau = \partial H/\partial\tau$. Because H_2 is independent of the parameter τ, it is therefore conserved along the trajectory. Indeed, its value follows simply from the particle mass:

$$g^{\mu\nu} p_\mu p_\nu = -m^2 \ \rightarrow \ H_2(p, x) = -\frac{1}{2} m^2 \ . \tag{11}$$

It follows that solutions of Hamilton's equations (10) satisfy $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \propto d\tau^2$, hence τ must be an affine parameter.

At this point, it is worth explaining why we did not use the original, parameterization-invariant Largrangian of equation (1) as the basis of a Hamiltonian treatment. Because L_1 is homogeneous of first degree in the coordinate velocity, $(dx^\mu/d\tau)\partial L_1/\partial(dx^\mu/d\tau) = L_1$ and the Hamiltonian vanishes identically. This is a consequence of the parameterization invariance of equation (1). The parameterization-invariance was an extra symmetry not needed for the dynamics. With a non-zero Hamiltonian, the dynamics itself (through the conserved Hamiltonian) showed that the appropriate parameter is path length.

3 Separating Time and Space

The Hamiltonian formalism developed above is elegant and manifestly covariant, i.e. the results are tensor equations and therefore hold for any coordinates and any reference frame. However, the covariant formulation is inconvenient for practical use. For one thing, every test particle has its own affine parameter; there is no global invariant clock by which to synchronize a system of particles. Sometimes this is regarded, incorrectly, as a shortcoming of relativity. In fact, relativity allows us to parameterize the spatial position of any number of particles using the coordinate time $t = x^0$. (After all, time was invented precisely to label spacetime events with a timelike coordinate.) An observer would report the results of measurement of any number of particle trajectories as $x^i(t)$; there is no ambiguity nor any loss of generality as long as we specify the metric.

Our goal is to obtain a Hamiltonian on the six-dimensional phase space $\{p_i, x^j\}$ which yields the form of Hamilton's equations familiar from undergraduate mechanics:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \ , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \ . \tag{12}$$

However, unlike undergraduate mechanics, we require that these equations of motion be fully correct in general relativity. Their solutions must be consistent with solutions of equation (10). We might hope simply to eliminate τ as a parameter, replacing it with t, while retaining the spatial components of p_μ and x^ν for our phase space variables. But what is the Hamiltonian, and can we ensure relativistic covariance?

The answer comes from a third expression for the action, regarded now as a functional of the 6-dimensional phase space trajectory $\{p_i(t), x^j(t)\}$:

$$S_3[p_i(t), x^j(t)] = 2S_2 = \int p_\mu dx^\mu = \int \left(p_0 + p_i \frac{dx^i}{dt}\right) dt \ . \tag{13}$$

Note that S_3 is manifestly a spacetime scalar, but that we have separated time and space components of the momentum one-form. Our desire to have a global time parameter has forced this space-time split.

Equation (13) is highly suggestive if we recall the Legendre transformation $H = p_i dx^i/dt - L$ (written here for three spatial coordinates parameterized by t rather than four coordinates parameterized by τ). Inverting the transformation, we conclude that the factor in parentheses in equation (13) must be the Lagrangian so that $S_3 = \int L\,dt$, and therefore the Hamiltonian is $H = -p_0$.

This result is appealing: the Hamiltonian naturally works out to be (minus) the time component of the momentum one-form. It is suggestive that, in locally flat coordinates, $-p_0 = p^0$ is the energy. However, despite appearances, the Hamiltonian is *not* in general the proper energy. Our formalism works for arbitrary spacetime coordinates and is not restricted to flat coordinates or inertial frames. We only require that t be time-like so that it can parameterize timelike spacetime trajectories.

Equation (13) with $p_0 = -H$ is not useful until we write the Hamiltonian in terms of the phase space coordinates and time: $H = H(p_i, x^j, t)$. We could do this by writing $L = p_\mu dx^\mu/dt$ in terms of x^i and dx^i/dt, but it is simpler to write p_0 directly in terms of (p_i, x^j, t). How?

A hint is given by the fact that in abandoning the affine parameterization by τ, we don't obtain the normalization of the four-momentum (eq. 11) automatically. Therefore we must add it as a constraint to the action of equation (13). We wish to use the energy integral $H_2 = -\frac{1}{2}m^2$ to reduce the order of the system (eqs. 10). Solving this relation for $-p_0$ in terms of the other variables yields the Hamiltonian on our reduced (6-dimensional) phase space.

For this procedure to be valid, it has to be shown that extremizing S_3 with respect to all possible phase space trajectories $\{p_i(t), x^i(t)\}$ is equivalent to extremizing S_2 with respect to $\{x^i(\tau), t(\tau)\}$ for τ being an affine parameter. Equivalently, we must show that solutions of equations (9) are solutions of equations (9) and vice versa. A proof is presented in Section 4.2 below.

Before presenting the technicalities, we state the key result of these notes, the Hamiltonian on our six-dimensional phase space $\{p_i, x^j\}$, obtained by solving $H_2(p_i, p_0, x^j, t) = -\frac{1}{2}m^2$ for $p_0 = -H$:

$$H(p_i, x^j, t) = -p_0 = \frac{g^{0i}p_i}{g^{00}} + \left[\frac{(g^{ij}p_ip_j + m^2)}{-g^{00}} + \left(\frac{g^{0i}p_i}{g^{00}}\right)^2\right]^{1/2} . \qquad (14)$$

Note that here, as in the covariant case, the conjugate momenta are given by the (here, spatial) components of the momentum one-form. The inverse metric components $g^{\mu\nu}$ are, in general, functions of x^i and t. Equation (14) is exact; no approximation to the metric has been made. We only require that t be timelike, i.e. $g_{00} < 0$, in order to parameterize timelike geodesics.

The next section presents mathematical material that is optional for 8.962. However, it is recommended for those students prepared to explore differential geometry somewhat further. The application to Hamiltonian mechanics should help the student to better understand the mathematics of general relativity.

4 Hamiltonian mechanics and symplectic manifolds

The proof that the 8-dimensional phase space may be reduced to the six spatial dimensions while retaining a Hamiltonian description becomes straightforward in the context of symplectic differential geometry (see Section 4.2 below). Classical Hamiltonian mechanics is naturally expressed using differential forms and exterior calculus (Arnold 1989; see also Exercise 4.11 of Misner et al 1973). We present an elementary summary here, both to provide background for the proof to follow and to elucidate differential geometry through its use in another context. In fact, we are not ignoring general relativity but extending it; the Hamiltonian mechanics we develop is fully consistent with general relativity.

The material presented in this section is mathematically more advanced than Schutz (1985). Treatments may be found in Misner et al (1973, Chapter 4), Schutz (1980), Arnold (1989), and, briefly, in Appendix B of Wald (1984) and Carroll (1997).

We begin with the configuration space of a mechanical system of n degrees of freedom characterized by the generalized coordinates q^i (which may, for example, be the four spacetime coordinates of a single particle's worldline, or the three spatial coordinates only). The configuration space is a manifold V whose tangent space $TV_{\mathbf{q}}$ at each point $\mathbf{q}$ in the manifold is given by the set of all generalized velocity vectors $d\vec{q}/dt$ at $\mathbf{q}$. Note that t is any parameter for a curve $\mathbf{q}(t)$; we are not restricting ourselves to Newtonian mechanics with its absolute time.

The union of all tangent spaces at all points of the manifold is called the tangent bundle, denoted TV. The set TV has the structure of a manifold of dimension $2n$. There exists a differentiable function on TV, the Lagrangian, whose partial derivatives with respect to the velocity vector components defines the components of a one-form, the canonical momentum:

$$\tilde{p} \equiv \frac{\partial L}{\partial(d\vec{q}/dt)} \ . \tag{15}$$

To see that this is a one-form, we note that it is a linear function of a tangent vector: $\tilde{p}(d\vec{q}) = p_i dq^i$ is a scalar. At each point in the configuration space manifold, the set of all $\tilde{p}$ defines the cotangent space $T^*V_{\mathbf{q}}$. (The name cotangent is used to distinguish the dual space of one-forms from the space of vectors.)

The union of all cotangent spaces at all points of the manifold is called the cotangent bundle, T^*V. Like the tangent bundle, the cotangent bundle is a manifold of dimension $2n$. A point of T^*V is specified by the coordinates (p_i, q^j). The cotangent bundle is well known: it is phase space.

Having set up the phase space, we now discard the original configuration space V, its tangent vector space $TV_{\mathbf{q}}$ and the tangent bundle TV. To emphasize that the phase space is a manifold of dimension $2n$, we will denote it M^{2n} rather than by T^*V.

Being a manifold, the phase space has a tangent space of vectors. Each parameterized curve $\gamma(t)$ in phase space has, at each point in the manifold, a tangent vector $\vec{\xi}$ whose coordinate components are the $2n$ numbers $(dp_i/dt, dq^j/dt)$. The phase space also has one-forms, or linear functions of vectors. For example, the gradient of a scalar field $H(p_i, q^j)$ in phase space is a one-form. However, it will prove convenient to denote the gradient of a scalar using a new notation, the exterior derivative: $\mathbf{d}H \equiv \tilde{\nabla} H$. In the coordinate basis, $\mathbf{d}H$ has components $(\partial H/\partial p_i, \partial H/\partial q^j)$. In this section, forms will be denoted with boldface symbols.

One must be careful not to read too much into the positions of indices: $\partial H/\partial p_i$ and $\partial H/\partial q^i$ are both components of a one-form in phase space. They may also happen to be spacetime vectors and one-forms, respectively, but we are now working in phase space. In phase space, p_i and q^j have equal footing as coordinates. We will retain the placement of indices (i, j go from 1 to n) simply as a reminder that our momenta and position displacements may be derived from spacetime one-forms and vectors. This way we can arrive at physical equations of Hamiltonian dynamics that are tensor equations (hence valid for any coordinate system) in both spacetime and phase space.

As in spacetime, we define the basis one-forms by the gradient (here, the exterior derivative) of the coordinate fields: $\{\mathbf{d}p_i, \mathbf{d}q^j\}$. We can combine one-forms and vectors to produce higher-rank tensors through the operations of gradient and tensor product. It proves especially useful to define the antisymmetric tensor product, or wedge product. The wedge product of two one-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \equiv \boldsymbol{\alpha} \otimes \boldsymbol{\beta} - \boldsymbol{\beta} \otimes \boldsymbol{\alpha} \ . \tag{16}$$

The wedge product of two one-forms gives a 2-form, an antisymmetric $(0,2)$ tensor. The wedge product (tensor product with antisymmetrization) can be extended to produce p-forms with p less than or equal to the dimension of the manifold. A p-form is a fully antisymmetric, linear function of p vectors. Forms will be denoted by Greek letters.

Given a p-form $\boldsymbol{\alpha}$, we can obtain a $(p+1)$-form by exterior differentiation, $\mathbf{d}\boldsymbol{\alpha}$. Exterior differentiation consists of the gradient followed by antisymmetrization on all arguments. For p-form $\boldsymbol{\omega}^p$ and q-form $\boldsymbol{\omega}^q$, the exterior derivative obeys the relation

$$\mathbf{d}(\boldsymbol{\omega}^p \wedge \boldsymbol{\omega}^q) = \mathbf{d}\boldsymbol{\omega}^p \wedge \boldsymbol{\omega}^q + (-1)^p \boldsymbol{\omega}^p \wedge \mathbf{d}\boldsymbol{\omega}^q \ . \tag{17}$$

(Here p and q are integers having nothing to do with phase space coordinates.) Note that $\mathbf{dd}\boldsymbol{\omega} = 0$ for any form $\boldsymbol{\omega}$. Any form $\boldsymbol{\omega}$ for which $\mathbf{d}\boldsymbol{\omega} = 0$ is called a closed form.

Forms are most widely used to provide a definition of integration free from coordinates and the metric. Consider, for example, the line integral giving the work done by a force, $\int \vec{F} \cdot d\vec{x}$. If the force were a one-form $\boldsymbol{\theta}$ instead of a vector, and if $\vec{\xi}$ were the tangent vector to a path γ ($\vec{\xi} = d\vec{x}/dt$ where t parameterizes the path), we could write the work as $\int_\gamma \boldsymbol{\theta}(\vec{\xi})$ or $\int_\gamma \boldsymbol{\theta}$ for short. No coordinates are involved until we choose a coordinate basis, and no metric is required because we integrate a one-form instead of a vector with a dot product.

Similarly, a 2-form may be integrated over an orientable 2-dimensional surface. Integration is built up by adding together the results from many small patches of the surface. An infinitesimal patch may be taken to be the parallelogram defined by two tangent vectors, $\vec{\xi}$ and $\vec{\eta}$. The integral of the 2-form $\boldsymbol{\omega}$ over the surface σ is $\int_\sigma \boldsymbol{\omega}(\vec{\xi}, \vec{\eta})$ or $\int_\sigma \boldsymbol{\omega}$ for short.

The spacetime manifold received additional structure with the introduction of the metric, a $(0,2)$ tensor used to give the magnitude of a vector (and to distinguish timelike, spacelike and null vectors). A manifold with a positive-definite symmetric $(0,2)$ tensor defining magnitude is called a Riemannian manifold. When the eigenvalues of the metric have mixed signs (as in the case of spacetime), the manifold is called pseudo-Riemannian.

Phase space has no metric; there is no concept of distance between points of phase space. It has a special antisymmetric $(0,2)$ tensor instead, in other words a 2-form. We will call this fundamental form the **symplectic form** $\boldsymbol{\omega}$; Arnold (1989) gives it the cumbersome name "the form giving the symplectic structure." In terms of the coordinate basis one-forms $\mathbf{d}p_i$ and $\mathbf{d}q^j$, the symplectic form is

$$\boldsymbol{\omega} \equiv \mathbf{d}p_i \wedge \mathbf{d}q^i = \mathbf{d}p_1 \wedge \mathbf{d}q^1 + \cdots + \mathbf{d}p_n \wedge \mathbf{d}q^n \ . \tag{18}$$

Note the implied sum on paired upper and lower indices.

One of the uses of the metric is to map vectors to one-forms; the symplectic form fulfills the same role in phase space. Filling one slot of $\boldsymbol{\omega}$ with a vector yields a one-form, $\boldsymbol{\omega}(\cdot, \vec{\xi}\,)$. It is easy to show that this mapping is invertible by representing $\boldsymbol{\omega}$ in the coordinate basis and showing that it is an orthogonal matrix. Therefore, every one-form has a corresponding vector.

There is a particular one-form of special interest in phase space, $\mathbf{d}H$ where $H(p, q, t)$ is the Hamiltonian function. The corresponding vector is the phase space velocity, i.e. the tangent to the phase space trajectory:

$$\begin{aligned} \boldsymbol{\omega}(\cdot, \vec{\xi}\,) &= \mathbf{d}p_i(\cdot)\mathbf{d}q^i(\vec{\xi}\,) - \mathbf{d}q^i(\cdot)\mathbf{d}p_i(\vec{\xi}\,) = \frac{dq^i}{dt}\mathbf{d}p_i - \frac{dp_i}{dt}\mathbf{d}q^i \\ &= \mathbf{d}H(\cdot) = \frac{\partial H}{\partial p_i}\mathbf{d}p_i + \frac{\partial H}{\partial q^i}\mathbf{d}q^i \ . \end{aligned} \tag{19}$$

Equating terms, we see that Hamilton's equations are given concisely by $\boldsymbol{\omega}(\vec{\xi}\,) = \mathbf{d}H$.

Besides giving the antisymmetric relationship between coordinates and momenta apparent in Hamilton's equations, the symplectic form allows us to define canonical transformations of the coordinates and momenta. The phase space components (p_i, q^j) transform with a $2n \times 2n$ matrix Λ to $(p_{\bar{i}}, q^{\bar{j}})$. A canonical transformation is one that leaves the symplectic form invariant. In matrix notation, this implies $\Lambda^T \omega \Lambda = \omega$. Thus, canonical invariance of a Hamiltonian system is analogous to Lorentz invariance in special relativity, where the transformations obey $\Lambda^T \eta \Lambda = \eta$ where η is the Minkowski metric.

The standard results of Hamiltonian mechanics are elegantly derived and expressed using the language of symplectic differential geometry. For example, Arnold (1989, §38 and §44D) shows that transformation of phase space induced by Hamiltonian evolution is canonical. This implies that the phase space area (the integral of $\boldsymbol{\omega}$, a 2-form) is preserved by Hamiltonian evolution. It is easy to show that not only $\boldsymbol{\omega}$ but also $\boldsymbol{\omega}^2 \equiv \boldsymbol{\omega} \wedge \boldsymbol{\omega}$ is a canonical invariant, as is $\boldsymbol{\omega}^p \equiv \boldsymbol{\omega} \wedge \cdots \wedge \boldsymbol{\omega}$ with p factors of $\boldsymbol{\omega}$, for all $p \leq n$. (Antisymmetry limits the rank of a p-form to $p \leq n$.) Thus, phase space volume is preserved by Hamiltonian evolution (Liouville theorem).

4.1 Extended phase space

Inspired by relativity, we can absorb the time parameter into the phase space to obtain a manifold of $2n + 1$ dimensions, denoted M^{2n+1} and called extended phase space. As we will see, this extension allows a concise derivation of the extremal form of the action under Hamiltonian motion.

Before proceeding, we should emphasize that the results of the previous section are not limited to nonrelativistic systems. Indeed, they apply to the phase space (p_μ, x^ν) of a single particle in general relativity where the role of time is played by the affine parameter τ. The relativistic Hamilton's equations (9) follow immediately from equation (19). Nonetheless, if we wish to parameterize trajectories by coordinate time (as we must for a system of more than one particle), we must show the consistency of the space-time split apparent in equation (14). We can do this by re-uniting coordinates and time in M^{2n+1}.

In M^{2n}, the symplectic form $\mathbf{d}p_i \wedge \mathbf{d}q^i$ is the fundamental object. In M^{2n+1}, we must incorporate the one-form $\mathbf{d}t$. This is done with a new one-form, the **integral invariant of Poincaré-Cartan**:

$$\boldsymbol{\omega} \equiv p_i \mathbf{d}q^i - H(p_i, q^j, t)\mathbf{d}t \ . \tag{20}$$

(The reader must note from context whether $\boldsymbol{\omega}$ refers to this one-form or to the symplectic 2-form.) This one-form looks deceptively like the integrand of the action, or the Lagrangian. However, it is a differential form on the extended phase space, not a function. Once we integrate it over a curve γ in M^{2n+1}, however, we get the action:

$$S = \int_\gamma \boldsymbol{\omega} = \int_A^B \left[p_i dq^i - H(p_i, q^j, t) dt \right] \ . \tag{21}$$

The integration is taken from A to B in the extended phase space.

Now suppose we integrate $\boldsymbol{\omega}$ from A to B along two slightly different paths and take the difference to get a close loop integral. To evaluate this integral we can use Stokes' theorem. In the language of differential forms, Stokes' theorem is written (Misner et al 1973, Chapter 4, or Wald 1984, Appendix B)

$$\int_{\partial M} \boldsymbol{\omega} = \int_M \mathbf{d}\boldsymbol{\omega} \tag{22}$$

Here, M is a p-dimensional compact orientable manifold with boundary ∂M and $\boldsymbol{\omega}$ is a $(p-1)$-form; $\mathbf{d}\boldsymbol{\omega}$ is its exterior derivative, a p-form. Note that M can be a submanifold of a larger space, so that Stokes' theorem actually implies a whole set of relations including the familiar Gauss and Stokes laws of ordinary vector calculus.

Applying equation (22) to the difference of actions computed along two neighboring paths with (q^i, t) fixed at the endpoints and using equation (17), we get

$$\delta S = \int_\sigma \mathbf{d}\boldsymbol{\omega} = \int_\sigma \mathbf{d}p_i \wedge \mathbf{d}q^i - \mathbf{d}H \wedge \mathbf{d}t \ , \tag{23}$$

where σ denotes the surface area in the extended phase space bounded by the two paths from A to B. Note the emergence of the fundamental symplectic form on M^{2n}.

Now, let us express the integrand of equation (23) in the coordinate basis of one-forms in M^{2n+1}, evaluating one of the vector slots using the tangent vector $\vec{\xi}$ to one of the two curves from A to B. The result is similar to equation (19):

$$\mathbf{d}\boldsymbol{\omega}(\cdot,\vec{\xi}\,) = \left(\frac{dq^i}{dt} - \frac{\partial H}{\partial p_i}\right)\mathbf{d}p_i + \left(-\frac{dp_i}{dt} - \frac{\partial H}{\partial q^i}\right)\mathbf{d}q^i + \left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right)\mathbf{d}t\ . \tag{24}$$

The principal of least action states that $\delta S = 0$ for small variations about the true path, with (q^i, t) fixed at the end points. This will be true, for arbitrary small variations, if and only if $\mathbf{d}\boldsymbol{\omega}(\cdot,\vec{\xi}\,) = 0$ for the tangent vector along the extremal path. From equation (24), Hamilton's equations follow.

The solution of Hamilton's equations gives an extended phase-space trajectory with tangent vector $\vec{\xi}$ being an eigenvector of the 2-form $\mathbf{d}\boldsymbol{\omega}$ with zero eigenvalue. Arnold (1989) proves that, for any differentiable function H defined on M^{2n+1}, the two-form $\mathbf{d}\boldsymbol{\omega}$ has exactly one eigenvector with eigenvalue zero, $(\partial H/\partial p_i, -\partial H/\partial q^i, 1)$. This is a vector field in M^{2n+1} and it defines a set of integral curves (field lines, to which it is tangent) called the "vortex lines" of the one-form $\boldsymbol{\omega}$. The vortex lines are precisely the trajectories of Hamiltonian flow, i.e. the solutions of equations (12).

A bundle of vortex lines is called a vortex tube. From Stokes' theorem, the circulation of a vortex tube, defined as the integral of the Poincaré-Cartan integral invariant around a closed loop bounding the vortex tube, is an integral of motion. (This is why $\boldsymbol{\omega}$ is called an integral invariant.) If the bounding curves are taken to lie on hypersurfaces of constant time, it follows that $\oint p_i \mathbf{d}q^i$ is also an integral of motion. By Stokes' theorem, this implies that the fundamental form $\mathbf{d}p_i \wedge \mathbf{d}q^i$ is an integral invariant. Thus, Hamiltonian evolution is canonical and preserves phase space areas and volumes.

By adding t to our manifold we have partially unified coordinates and time. Can we go all the way to obtain a spacetime covariant formulation of Hamiltonian dynamics? For the case of single particle motion, the answer is clearly yes. If we write $H = -p_0$ and $t = q^0$, the integral invariant of Poincaré-Cartan takes the simple form $\boldsymbol{\omega} = p_\mu \mathbf{d}q^\mu$ where μ takes the range 0 to n. Now $\mathbf{d}\boldsymbol{\omega}$ looks like the symplectic form on M^{2n+2}, except that here p_0 is not a dynamical variable but rather a function on M^{2n+1}. However, we can promote it to the status of an independent variable by defining a new Hamiltonian $H'(p_\mu, q^\nu)$ on M^{2n+2} such that $H' =$ constant can be solved for p_0 to give $-p_0 = H(p_i, q^j, q^0 = t)$. A simple choice is $H' = p_0 + H$.

Having subsumed the parameter for trajectories into the phase space, we must introduce a new parameter, τ. Because $\partial H'/\partial\tau = 0$, the solution of Hamilton's equations in M^{2n+2} will ensure that H' is a constant of motion. This is exactly what happened with the relativistically covariant Hamiltonian H_2 in Section 2 (eqs. 8 and 11).

The reader may now ask, if the Hamiltonian is independent of time, is it possible to reduce the dimensionality of phase space by two? The answer is yes; the next section shows how.

4.2 Reduction of order

Hamilton's equations imply that when $\partial H/\partial t = 0$, H is an integral of motion. In this case, phase space trajectories in M^{2n} are confined to the $(2n-1)$-dimensional hypersurface $H =$ constant. This condition may be used to eliminate t and choose one of the coordinates to become a new "time" parameter, with a new Hamiltonian defined on the reduced phase space.

This procedure was used in Section 3 to reduce the relativistically covariant 8-dimensional phase space $\{p_\mu, x^\nu\}$ with Hamiltonian given by equation (8) to the 6-dimensional phase space $\{p_i, x^j\}$ with the Hamiltonian of equation (14). While this reduction is plausible, it remains to be proved that the reduced phase space is a symplectic manifold and that the new Hamiltonian is given by the momentum conjugate to the time coordinate. The proof is given here.

Starting from the conserved Hamiltonian $H(p,q) \equiv H(p_0, p_i, q^0, q^j) = h$ with $1 \le i, j \le n-1$, we assume that (in some region) this equation can be solved for the momentum coordinate p_0:

$$p_0 = -K(P_i, Q^j, T; h) \tag{25}$$

where $P_i = p_i$, $Q^i = q^i$, and $T = q^0$. Note that any of the coordinates may be eliminated, with its conjugate momentum becoming (minus) the new Hamiltonian. Thus, the reduction of order is compatible with relativistic covariance. However, it can be applied to any Hamiltonian system, relativistic or not.

Next we write the integral invariant of Poincaré-Cartan in terms of the new variables:

$$\boldsymbol{\omega} = p_0 \mathbf{d}q^0 + p_i \mathbf{d}q^i - H\mathbf{d}t = P_i \mathbf{d}Q^i - K\mathbf{d}T - \mathbf{d}(Ht) + t\mathbf{d}H \ . \tag{26}$$

Recall that this is a one-form defined on M^{2n+1}.

Now let γ be an integral curve of the canonical equations (12) lying on the $2n$-dimensional surface $H(p,q) = h$ in the $(2n+1)$-dimensional extended phase space $\{p, q, t\}$. Thus, γ is a vortex line of the two-form $p\mathbf{d}q - H\mathbf{d}t = p_0\mathbf{d}q^0 + p_i\mathbf{d}q^i - H\mathbf{d}t$. We project the extended phase space M^{2n+1} onto the phase space $M^{2n} = \{p, q\}$ by discarding the time parameter t. The surface $H = h$ projects onto a $(2n-1)$-dimensional manifold M^{2n-1} with coordinates $\{P_i, Q^j, T\}$. Discarding t, the integral curve γ projects onto a curve $\bar{\gamma}$ also in M^{2n-1}.

The coordinates $(P_i, Q^j, T) = (p_i, q^j, q^0)$ locally (and perhaps globally) cover the submanifold M^{2n-1} (the surface $H =$ constant in $M^{2n} = \{p, q\}$). We now show that M^{2n-1} is the extended phase space for a symplectic manifold with Hamiltonian K.

We do this by examining equation (26) while noting that the integral curve γ lies on the surface $H =$ constant. Clearly the last term in equation (26) vanishes on M^{2n-1}. Next, $\mathbf{d}(Ht)$ does not affect the vortex lines of $\boldsymbol{\omega}$ because $\mathbf{dd}(Ht) = 0$. (The variation of the action is invariant under the addition of a total derivative to the Lagrangian.) But the vortex lines of $P_i\mathbf{d}Q^i - K\mathbf{d}T$ satisfy Hamilton's equations (Sect. 4.1). Thus we have proven that reduction of order preserves Hamiltonian evolution.

The solution curves $\bar{\gamma}$ on M^{2n-1} are vortex lines of $p\mathbf{d}q = P\mathbf{d}Q - K\mathbf{d}T$. Thus, they are extremals of the integral $\int p\mathbf{d}q$. In other words, if the Hamiltonian function $H(q,p)$ in M^{2n+1} is independent of time, then the phase space trajectories satisfying Hamilton's equations are extremals of the integral $\int p\mathbf{d}q$ in the class of curves lying on M^{2n-1} with fixed endpoints of integration. The converse is also true (Arnold 1989): if $\partial H/\partial t = 0$, the extremals of the "reduced action"

$$\int_\gamma p\mathbf{d}q = \int_\gamma \frac{\partial L}{\partial(d\vec{q}/d\tau)}(\tau)\frac{d\vec{q}}{d\tau}\,d\tau \tag{27}$$

with fixed endpoints, $\delta q = 0$, are solutions of Hamilton's equations in M^{2n+1}. This is known as Maupertuis' principle of least action. Note that the principle can only be implemented if p_i is expressed as a function of q and $\dot{q}$ so that the integral is a functional of the configuration space trajectory. Also, because the time parameterization is arbitrary, Maupertuis' principle determines the shape of a trajectory but not the time (t does not appear in eq. 27); in order to determine the time we must use the energy integral.

These results justify the approach of Section 3. The spacetime trajectories are extremals of equation (13) as a consequence of $\partial H_2/\partial\tau = 0$ (eq. 8) and Maupertuis' principle. The order is reduced further by using $H_2 = -\frac{1}{2}m^2$ to solve for $-p_0$ as the new Hamiltonian $H(p_i, x^j, t)$, equation (14).

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Physics 8.962 Spring 1999

Measuring the Metric, and Curvature versus Acceleration

1 Introduction

These notes show how observers can set up a coordinate system and measure the spacetime geometry using clocks and lasers. The approach is similar to that of special relativity, but the reader must not be misled. Spacetime diagrams with rectilinear axes do not imply flat spacetime any more than flat maps imply a flat earth.

Cartography provides an excellent starting point for understanding the metric. Terrestrial maps always provide a scale of the sort "One inch equals 1000 miles." If the map is of a sufficiently small region and is free from distortion, one scale will suffice. However, a projection of the entire sphere requires a scale that varies with location and even direction. The Mercator projection suggests that Greenland is larger than South America until one notices the scale difference. The simplest map projection, with latitude and longitude plotted as a Cartesian grid, has a scale that depends not only on position but also on direction. Close to the poles, one degree of latitude represents a far greater distance than one degree of longitude.

The map scale is the metric. The spacetime metric has the same meaning and use: it translates coordinate distances and times ("one inch on the map") to physical ("proper") distances and times.

The terrestrial example also helps us to understand how coordinate systems can be defined in practice on a curved manifold. Let us consider how coordinates are defined on the Earth. First pick one point and call it the north pole. The pole is chosen along the rotation axis. Now extend a family of geodesics from the north pole, called meridians of longitude. Label each meridian by its longitude ϕ. We choose the meridian going through Greenwich, England, and call it the"prime meridian," $\phi = 0$. Next, we define latitude λ as an affine parameter along each meridian of longitude, scaled to $\pi/2$ at the north pole and decreasing linearly to $-\pi/2$ at the point where the meridians intersect again (the south pole). With these definitions, the proper distance between the nearby points with coordinates (λ, ϕ) and $(\lambda + d\lambda, \phi + d\phi)$ is given by $ds^2 = R^2(d\lambda^2 + \cos^2 \lambda \, d\phi^2)$. In this way, every point on the sphere gets coordinates along with a scale which converts coordinate intervals to proper distances.

This example seems almost trivial. However, it faithfully illustrates the concepts involved in setting up a coordinate system and measuring the metric. In particular, coordinates are numbers assigned by obsevers who exchange information with each other. There is no conceptual need to have the idealized dense system of clocks and rods filling spacetime. Observe any major civil engineering project. The metric is measured by two surveyors with transits and tape measures or laser ranging devices. Physicists can do the same, in principle and in practice. These notes illustrate this through a simple thought experiment. The result will be a clearer understanding of the relation between curvature, gravity, and acceleration.

2 The metric in 1+1 spacetime

We study coordinate systems and the metric in the simplest nontrivial case, spacetime with one space dimension. This analysis leaves out the issue of orientation of spatial axes. It also greatly reduces the number of degrees of freedom in the metric. As a symmetric 2 matrix, the metric has three independent coefficients. Fixing two coordinates imposes two constraints, leaving one degree of freedom in the metric. This contrasts with the six metric degrees of freedom in a 3+1 spacetime. However, if one understands well the 1+1 example, it is straightforward (albeit more complicated) to generalize to 2+1 and 3+1 spacetime.

We will construct a coordinate system starting from one observer called A. Observer A may have any motion whatsoever relative to other objects, including acceleration. But neither spatial position nor velocity is meaningful for A before we introduce other observers or coordinates ("velocity relative to what?") although A's acceleration (relative to a local inertial frame!) is meaningful: A stands on a scale, reads the weight, and divides by rest mass. Observer A could be you or me, standing on the surface of the earth. It could equally well be an astronaut landing on the moon. It may be helpful in this example to think of the observers as being stationary with respect to a massive gravitating body (e.g. a black hole or neutron star). However, we are considering a completely general case, in which the spacetime may not be at all static. (That is, there may not be any Killing vectors whatsoever.)

We take observer A's worldine to define the t-axis: A has spatial coordinate $x_A \equiv 0$. A second observer, some finite (possibly large) distance away, is denoted B. Both A and B carry atomic clocks, lasers, mirrors and detectors.

Observer A decides to set the spacetime coordinates over all spacetime using the following procedure, illustrated in Figure 1. First, the reading of A's atomic clock gives

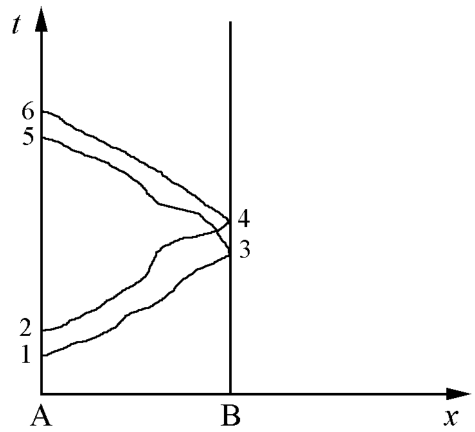

Figure 1: Setting up a coordinate system in curved spacetime. There are two timelike worldlines and two pairs of null geodesics. The appearance of flat coordinates is misleading; the metric varies from place to place.

the t-coordinate along the t-axis ($x = 0$). Then, A sends a pair of laser pulses to B, who reflects them back to A with a mirror. If the pulses do not return with the same time separation (measured by A) as they were sent, A deduces that B is moving and sends signals instructing B to adjust her velocity until $t_6 - t_5 = t_2 - t_1$. The two continually exchange signals to ensure that this condition is maintained. A then declares that B has a constant space coordinate (by definition), which is set to half the round-trip light-travel time, $x_B \equiv \frac{1}{2}(t_5 - t_1)$. A sends signals to inform B of her coordinate.

Having set the spatial coordinate, A now sends time signals to define the t-coordinate along B's worldline. A's laser encodes a signal from Event 1 in Figure 1, "This pulse was sent at $t = t_1$. Set your clock to $t_1 + x_B$." B receives the pulse at Event 3 and sets her clock. A sends a second pulse from Event 2 at $t_2 = t_1 + \Delta t$ which is received by B at Event 4. B compares the time difference quoted by A with the time elapsed on her atomic clock, the proper time $\Delta\tau_B$. To her surprise, $\Delta\tau_B \neq \Delta t$.

At first A and B are sure something went wrong; maybe B has begun to drift. But repeated exchange of laser pulses shows that this cannot be the explanation: the round-trip light-travel time is always the same. Next they speculate that the lasers may be traveling through a refractive medium whose index of refraction is changing with time. (A constant index of refraction wouldn't change the differential arrival time.) However, they reject this hypothesis when they find that B's atomic clock continually runs at a different rate than the timing signals sent by A, while the round-trip light-travel time

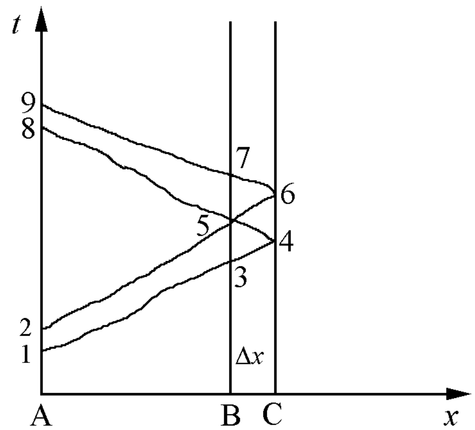

Figure 2: Testing for space curvature.

measured by A never changes. Moreover, laboratory analysis of the medium between them shows no evidence for any change.

Becoming suspicious, B decides to keep two clocks, an atomic clock measuring τ_B and another set to read the time sent by A, denoted t. The difference between the two grows increasingly large.

The observers next speculate that they may be in a non-inertial frame so that special relativity remains valid despite the apparent contradiction of clock differences ($g_{tt} \neq 1$) with no relative motion ($dx_B/dt = 0$). We will return to this speculation in Section 3. In any case, they decide to keep track of the conversion from coordinate time (sent by A) to proper time (measured by B) for nearby events on B's worldline by defining a metric coefficient:

$$g_{tt}(t, x_B) \equiv \lim_{\Delta t \to 0} - \left(\frac{\Delta \tau_B}{\Delta t} \right)^2 \ . \tag{1}$$

The observers now wonder whether measurements of spatial distances will yield a similar mystery. To test this, a third observer is brought to help in Figure 2. Observer C adjusts his velocity to be at rest relative to A. Just as for B, the definition of rest is that the round-trip light-travel time measured by A is constant, $t_8 - t_1 = t_9 - t_2 = 2x_C \equiv 2(x_B + \Delta x)$. Note that the coordinate distances are expressed entirely in terms of readings of A's clock. A sends timing signals to both B and C. Each of them sets one clock to read the time sent by A (corrected for the spatial coordinate distance x_B and x_C, respectively) and also keeps time by carrying an undisturbed atomic clock. The former is called coordinate time t while the latter is called proper time.

The coordinate time synchronization provided by A ensures that $t_2 - t_1 = t_5 - t_3 = t_6 - t_4 = t_7 - t_5 = t_9 - t_8 = 2\Delta x$. Note that the procedure used by A to set t and x relates the coordinates of events on the worldlines of B and C:

$$\begin{aligned} (t_4, x_4) &= (t_3, x_3) + (1,1)\Delta x\ , \quad (t_5, x_5) = (t_4, x_4) + (1,-1)\Delta x\ , \\ (t_6, x_6) &= (t_5, x_5) + (1,1)\Delta x\ , \quad (t_7, x_7) = (t_6, x_6) + (1,-1)\Delta x\ . \end{aligned} \tag{2}$$

Because they follow simply from the synchronization provided by A, these equations are exact; they do not require Δx to be small. However, by themselves they do not imply anything about the *physical* separations between the events. Testing this means measuring the metric.

To explore the metric, C checks his proper time and confirms B's observation that proper time differs from coordinate time. However, the metric coefficient he deduces, $g_{tt}(x_C, t)$, differs from B's. (The difference is first-order in Δx.)

The pair now wonder whether spatial coordinate intervals are similarly skewed relative to proper distance. They decide to measure the proper distance between them by using laser-ranging, the same way that A set their spatial coordinates in the first place. B sends a laser pulse at Event 3 which is reflected at Event 4 and received back at Event 5 in Figure 2. From this, she deduces the proper distance of C,

$$\Delta s = \frac{1}{2}(\tau_5 - \tau_3) \tag{3}$$

where τ_i is the reading of her atomic clock at event i. To her surprise, B finds that Δx does not measure proper distance, not even in the limit $\Delta x \to 0$. She defines another metric coefficient to convert coordinate distance to proper distance,

$$g_{xx} \equiv \lim_{\Delta x \to 0} \left(\frac{\Delta s}{\Delta x} \right)^2 \ . \tag{4}$$

The measurement of proper distance in equation (4) must be made at fixed t; otherwise the distance must be corrected for relative motion between B and C (should any exist). Fortunately, B can make this measurement at $t = t_4$ because that is when her laser pulse reaches C (see Fig. 2 and eqs. 2). Expanding $\tau_5 = \tau_B(t_4 + \Delta x)$ and $\tau_3 = \tau_B(t_4 - \Delta x)$ to first order in Δx using equations (1), (3), and (4), she finds

$$g_{xx}(x,t) = -g_{tt}(x,t)\ . \tag{5}$$

The observers repeat the experiment using Events 5, 6, and 7. They find that, while the metric may have changed, equation (5) still holds.

The observers are intrigued to find such a relation between the time and space parts of their metric, and they wonder whether this is a general phenomenon. Have they discovered a modification of special relativity, in which the Minkowski metric is simply multipled by a conformal factor, $g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$?

They decide to explore this question by measuring g_{tx}. A little thought shows that they cannot do this using pairs of events with either fixed x or fixed t. Fortunately, they have ideal pairs of events in the lightlike intervals between Events 3 and 4:

$$ds_{34}^2 \equiv \lim_{\Delta t, \Delta x \to 0} g_{tt}(t_4 - t_3)^2 + 2g_{tx}(t_4 - t_3)(x_4 - x_3) + g_{xx}(x_4 - x_3)^2 \ . \tag{6}$$

Using equations (2) and (5) and the condition $ds = 0$ for a light ray, they conclude

$$g_{tx} = 0 \ . \tag{7}$$

Their space and time coordinates are orthogonal but on account of equations (5) and (7) all time and space intervals are stretched by $\sqrt{g_{xx}}$.

Our observers now begin to wonder if they have discovered a modification of special relativity, or perhaps they are seeing special relativity in a non-inertial frame. However, we know better. Unless the Riemann tensor vanishes identically, the metric they have determined cannot be transformed everywhere to the Minkowski form. Instead, what they have found is simply a consequence of how A fixed the coordinates. Fixing two coordinates means imposing two gauge conditions on the metric. A defined coordinates so as to make the problem look as much as possible like special relativity (eqs. 2). Equations (5) and (7) are the corresponding gauge conditions.

It is a special feature of 1+1 spacetime that the metric can always be reduced to a conformally flat one, i.e.

$$ds^2 = \Omega^2(x)\eta_{\mu\nu}dx^\mu dx^\nu \tag{8}$$

for some function $\Omega(x^\mu)$ called the conformal factor. In two dimensions the Riemann tensor has only one independent component and the Weyl tensor vanishes identically. Advanced GR and differential geometry texts show that spacetimes with vanishing Weyl tensor are conformally flat.

Thus, A has simply managed to assign conformally flat coordinates. This isn't a coincidence; by defining coordinate times and distances using null geodesics, he forced the metric to be identical to Minkowski up to an overall factor that has no effect on null lines. Equivalently, in two dimensions the metric has one physical degree of freedom, which has been reduced to the conformal factor $\Omega \equiv \sqrt{g_{xx}} = \sqrt{-g_{tt}}$.

This does not mean that A would have had such luck in more than two dimensions. In n dimensions the Riemann tensor has $n^2(n^2-1)/12$ independent components (Wald p. 54) and for $n \geq 3$ the Ricci tensor has $n(n+1)/2$ independent components. For $n = 2$ and $n = 3$ the Weyl tensor vanishes identically and spacetime is conformally flat. Not so for $n > 3$.

It would take a lot of effort to describe a complete synchronization in 3+1 spacetime using clocks and lasers. However, even without doing this we can be confident that the metric will not be conformally flat except for special spacetimes for which the Weyl tensor vanishes. Incidentally, in the weak-field limit conformally flat spacetimes have no deflection of light (can you explain why?). The solar deflection of light rules out conformally flat spacetime theories including ones proposed by Nordstrom and Weyl.

It is an interesting exercise to show how to transform an arbitrary metric of a 1+1 spacetime to the conformally flat form. The simplest way is to compute the Ricci scalar. For the metric of equation (8), one finds

$$R = \Omega^{-2}(\partial_t^2 - \partial_x^2)\ln\Omega^2 \ . \tag{9}$$

Starting from a 1+1 metric in a different form, one can compute R everywhere in spacetime. Equation (9) is then a nonlinear wave equation for $\Omega(t,x)$ with source $R(t,x)$. It can be solved subject to initial Cauchy data on a spacelike hypersurface on which $\Omega = 1$, $\partial_t\Omega = \partial_x\Omega = 0$ (corresponding to locally flat coordinates).

We have exhausted the analysis of 1+1 spacetime. Our observers have discerned one possible contradiction with special relativity: clocks run at different rates in different places (and perhaps at different times). If equation (9) gives Ricci scalar $R = 0$ everywhere with $\Omega = \sqrt{-g_{tt}}$, then the spacetime is really flat and we must be seeing the effects of accelerated motion in special relativity. If $R \neq 0$, then the variation of clocks is an entirely new phenomenon, which we call gravitational redshift.

3 The metric for an accelerated observer

It is informative to examine the problem from another perspective by working out the metric that an arbitrarily accelerating observer in a flat spacetime would deduce using the synchronization procedure of Section 2. We can then more clearly distinguish the effects of curvature (gravity) and acceleration.

Figure 3 shows the situation prevailing in special relativity when observer A has an arbitrary timelike trajectory $x^\mu_A(\tau_A)$ where τ_A is the proper time measured by his atomic clock. While A's worldline is erratic, those of light signals are not, because here $t = x^0$ and $x = x^1$ are flat coordinates in Minkowski spacetime. Given an arbitrary worldline $x^\mu_A(\tau_A)$, how can we possibly find the worldines of observers at fixed coordinate displacement as in the preceding section?

The answer is the same as the answer to practically all questions of measurement in GR: use the metric! The metric of flat spacetime is the Minkowski metric, so the paths of laser pulses are very simple. We simply solve an algebra problem enforcing that Events 1 and 2 are separated by a null geodesic (a straight line in Minkowski spacetime) and likewise for Events 2 and 3, as shown in Figure 3. Notice that the lengths (i.e. coordinate differences) of the two null rays need not be the same.

The coordinates of Events 1 and 3 are simply the coordinates along A's worldine, while those for Event 2 are to be determined in terms of A's coordinates. As in Section 2, A defines the spatial coordinate of B to be twice the round-trip light-travel time. Thus, if event 0 has $x^0 = t_A(\tau_0)$, then Event 3 has $x^0 = t_A(\tau_0 + 2L)$. For convenience we will

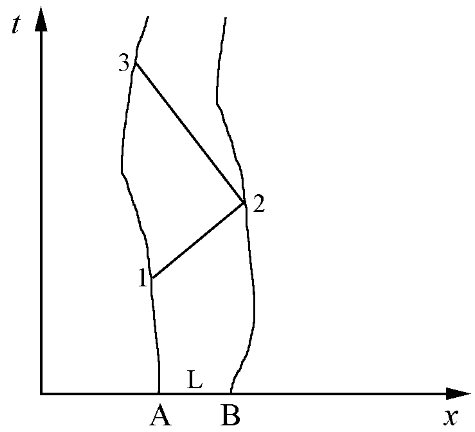

Figure 3: An accelerating observer sets up a coordinate system with an atomic clock, laser and detector.

set $\tau_0 \equiv \tau_A - L$. Then, according to the prescription of Section 2, A will assign to Event 2 the coordinates (τ_A, L). The coordinates in our flat Minkowksi spacetime are

$$\begin{aligned} \text{Event 1:} \quad x^0 &= t_A(\tau_A - L)\ , \quad x^1 = x_A(\tau_A - L)\ , \\ \text{Event 2:} \quad x^0 &= t(\tau_A, L)\ , \quad x^1 = x(\tau_A, L)\ , \\ \text{Event 3:} \quad x^0 &= t_A(\tau_A + L)\ , \quad x^1 = x_A(\tau_A + L)\ . \end{aligned} \tag{10}$$

Note that the argument τ_A for Event 2 is not an affine parameter along B's wordline; it is the clock time sent to B by A. A second argument L is given so that we can look at a family of worldlines with different L. A is setting up coordinates by finding the spacetime paths corresponding to the coordinate lines $L =$ constant and $\tau_A =$ constant. We are performing a coordinate transformation from (t, x) to (τ_A, L).

Requiring that Events 1 and 2 be joined by a null geodesic in flat spacetime gives the condition $x_2^\mu - x_1^\mu = (C_1, C_1)$ for some constant C_1. The same condition for Events 2 and 3 gives $x_3^\mu - x_2^\mu = (C_2, -C_2)$ (with a minus sign because the light ray travels toward decreasing x). These conditions give four equations for the four unknowns C_1, C_2, $t(\tau_A, L)$, and $x(\tau_A, L)$. Solving them gives the coordinate transformation between (τ_A, L) and the Minkowski coordinates:

$$\begin{aligned} t(\tau_A, L) &= \frac{1}{2}\left[t_A(\tau_A + L) + t_A(\tau_A - L) + x_A(\tau_A + L) - x_A(\tau_A - L)\right]\ , \\ x(\tau_A, L) &= \frac{1}{2}\left[x_A(\tau_A + L) + x_A(\tau_A - L) + t_A(\tau_A + L) - t_A(\tau_A - L)\right]\ . \end{aligned} \tag{11}$$

Note that these results are exact; they do not assume that L is small nor do they restrict A's worldline in any way except that it must be timelike. The student may easily evaluate C_1 and C_2 and show that they are not equal unless $x_A(\tau_A + L) = x_A(\tau_A - L)$.

Using equations (11), we may transform the Minkowski metric to get the metric in the coordinates A has set with his clock and laser, (τ_A, L):

$$ds^2 = -dt^2 + dx^2 = g_{tt} d\tau_A^2 + 2g_{tx} d\tau_A dL + g_{xx} dL^2 \ . \tag{12}$$

Substituting equations (11) gives the metric components in terms of A's four-velocity components,

$$-g_{tt} = g_{xx} = \left[V_A^t(\tau_A + L) + V_A^x(\tau_A + L)\right] \left[V_A^t(\tau_A - L) - V_A^x(\tau_A - L)\right] \ , \quad g_{tx} = 0 \ . \tag{13}$$

This is precisely in the form of equation (8), as it must be because of the way in which A coordinatized spacetime.

It is straightforward to work out the Riemann tensor from equation (13). Not surprisingly, it vanishes identically. Thus, an observer *can* tell, through measurements, whether he or she lives in a flat or nonflat spacetime. The metric is measurable.

Now that we have a general result, it is worth simplifying to the case of an observer with constant acceleration g_A in Minkowski spacetime. Problem 3 of Problem Set 1 showed that one can write the trajectory of such an observer (up to the addition of constants) as $x = g_A^{-1} \cosh g_A \tau_A$, $t = g_A^{-1} \sinh g_A \tau_A$. Equation (13) then gives

$$ds^2 = e^{2g_A L} \left(-d\tau_A^2 + dL^2\right) \ . \tag{14}$$

One word of caution is in order about the interpretation of equation (14). Our derivation assumed that the acceleration g_A is constant for observer A at $L = 0$. However, this does not mean that other observers (at fixed, nonzero L) have the same acceleration. To see this, we can differentiate equations (11) to derive the 4-velocity of observer B at (τ_A, L) and the relation between coordinate time τ_A and proper time along B's worldline, with the result

$$V_B^\mu(\tau_A, L) = (\cosh g_A \tau_A, \sinh g_A \tau_A) = (\cosh g_B \tau_B, \sinh g_B \tau_B) \ , \quad \frac{d\tau_B}{d\tau_A} = \frac{g_A}{g_B} = e^{gL} \ . \tag{15}$$

The four-acceleration of B follows from $a_B^\mu = dV_B^\mu / d\tau_B = e^{-gL} dV^\mu / d\tau_A$ and its magnitude is therefore $g_B = g_A e^{-gL}$. The proper acceleration varies with L precisely so that the proper distance between observers A and B, measured at constant τ_A, remains constant.

4 Gravity versus acceleration in 1+1 spacetime

Equation (14) gives one form of the metric for a flat spacetime as seen by an accelerating observer. There are many other forms, and it is worth noting some of them in order to gain some intuition about the effects of acceleration. For simplicity, we will restrict our discussion here to static spacetimes, i.e. metrics with $g_{0i} = 0$ and $\partial_t g_{\mu\nu} = 0$. In 1+1 spacetime this means the line element may be written

$$ds^2 = -e^{2\phi(x)}dt^2 + e^{-2\psi(x)}dx^2 \ . \tag{16}$$

(The metric may be further transformed to the conformally flat form, eq. 8, but we leave it in this form because of its similarity to the form often used in $3+1$ spacetime.)

Given the metric (16), we would like to know when the spacetime is flat. If it is flat, we would like the explicit coordinate transformation to Minkowski. Both of these are straightforward in 1+1 spacetime. (One might hope for them also to be straightforward in more dimensions, at least in principle, but the algebra rapidly increases.)

The definitive test for flatness is given by the Riemann tensor. Because the Weyl tensor vanishes in 1+1 spacetime, it is enough to examine the Ricci tensor. With equation (16), the Ricci tensor has nonvanishing components

$$R_{tt} = e^{\phi+\psi}\,\frac{d\tilde{g}}{dx} \ , \quad R_{xx} = -e^{-(\phi+\psi)}\frac{d\tilde{g}}{dx} \quad \text{where} \quad \tilde{g}(x) = e^{\phi}\,g(x) = e^{\phi+\psi}\,\frac{d\phi}{dx} \ . \tag{17}$$

The function $g(x)$ is the proper acceleration along the x-coordinate line, along which the tangent vector (4-velocity) is $V_x^\mu = e^{-\phi}(1,0)$. This follows from computing the 4-acceleration with equation (16) using the covariant prescription $a^\mu(x) = \nabla_V V^\mu = V_x^\nu \nabla_\nu V_x^\mu$. The magnitude of the acceleration is then $g(x) \equiv (g_{\mu\nu}a^\mu a^\nu)^{1/2}$, yielding $g(x) = e^\psi d\phi/dx$. The factor e^ψ converts $d\phi/dx$ to $g(x) = d\phi/dl$ where $dl = \sqrt{g_{xx}}\,dx$ measures proper distance.

A stationary observer, i.e. one who remains at fixed spatial coordinate x, feels a time-independent effective gravity $g(x)$. Nongravitational forces (e.g. a rocket, or the contact force from a surface holding the observer up) are required to maintain the observer at fixed x. The gravity field $g(x)$ can be measured very simply by releasing a test particle from rest and measuring its acceleration relative to the stationary observer. For example, we measure g on the Earth by dropping masses and measuring their acceleration in the lab frame.

We will see following equation (18) below why the function $\tilde{g}(x) = (d\tau/dt)g(x)$ rather than $g(x)$ determines curvature. For now, we simply note that equation (17) implies that spacetime curvature is given (for a static 1+1 metric) by the gradient of the gravitational redshift factor $\sqrt{-g_{tt}} = e^\phi$ rather than by the "gravity" (i.e. acceleration) gradient dg/dx.

In linearized gravitation, $g = \tilde{g}$ and so we deduced (in the notes *Gravitation in the Weak-Field Limit*) that a spatially uniform gravitational (gravitoelectric) field can be transformed away by making a global coordinate transformation to an accelerating frame. For strong fields, $g \neq \tilde{g}$ and it is no longer true that a uniform gravitoelectric field can be transformed away. Only if the gravitational redshift factor $e^{\phi(x)}$ varies linearly with proper distance, i.e. $\tilde{g} \equiv d(e^{\phi})/dl$ is a constant, is spacetime is flat, enabling one to transform coordinates so as to remove all evidence for acceleration. If, on the other hand, $d\tilde{g}/dx \neq 0$ — even if $dg/dx = 0$ — then the spacetime is not flat and no coordinate transformation can transform the metric to the Minkowski form.

Suppose we have a line element for which $\tilde{g}(x) =$ constant. We know that such a spacetime is flat, because the Ricci tensor (hence Riemann tensor, in 1+1 spacetime) vanishes everywhere. What is the coordinate transformation to Minkowski?

The transformation may be found by writing the metric as $g = \Lambda^T \eta \Lambda$ where $\Lambda^{\bar{\mu}}{}_{\nu} = \partial \bar{x}^{\bar{\mu}} / \partial x^{\nu}$ is the Jacobian matrix for the transformation $\bar{x}(x)$. (Note that here g is the matrix with entries $g_{\mu\nu}$ and not the gravitational acceleration!) By writing $\bar{t} = \bar{t}(t, x)$ and $\bar{x} = \bar{x}(t, x)$, substituting into $g = \Lambda^T \eta \Lambda$, using equation (16) and imposing the integrability conditions $\partial^2 \bar{t}/\partial t \partial x = \partial^2 \bar{t}/\partial x \partial t$ and $\partial^2 \bar{x}/\partial t \partial x = \partial^2 \bar{x}/\partial x \partial t$, one finds

$$\bar{t}(t, x) = \frac{1}{g} \sinh \tilde{g}\, t \ , \quad \bar{x}(t, x) = \frac{1}{g} \cosh \tilde{g}\, t \quad \text{if} \quad \frac{d\tilde{g}}{dx} = 0 \ , \tag{18}$$

up to the addition of irrelevant constants. We recognize this result as the trajectory in flat spacetime of a constantly accelerating observer.

Equation (18) is easy to understand in light of the discussion following equation (14). The proper time τ for the stationary observer at x is related to coordinate time t by $d\tau = \sqrt{-g_{tt}(x)}\, dt = e^{\phi} dt$. Thus, $g(x)\tau = e^{\phi}\, g\, t = \tilde{g}\, t$ or, in the notation of equation (15), $g_B \tau_B = g_A \tau_A$ (since τ_A was used there as the global t-coordinate). The condition $e^{\phi}\, g = \tilde{g}(x) =$ constant amounts to requiring that all observers be able to scale their gravitational accelerations to a common value for the observer at $\phi(x) = 0$, $\tilde{g}$. If they cannot (i.e. if $d\tilde{g}/dx \neq 0$), then the metric is not equivalent to Minkowski spacetime seen in the eyes of an accelerating observer.

With equations (16)–(18) in hand, we can write the metric of a flat spacetime in several new ways, with various spatial dependence for the acceleration of our coordinate observers:

$$\begin{aligned} ds^2 &= e^{2\tilde{g}\, x}(-dt^2 + dx^2) \ , \quad g(x) = \tilde{g}\, e^{-\tilde{g}\, x} && (19) \\ &= -\tilde{g}^2 (x - x_0)^2 dt^2 + dx^2 \ , \quad g(x) = \frac{1}{x - x_0} && (20) \\ &= -[2\tilde{g}(x - x_0)] dt^2 + [2\tilde{g}(x - x_0)]^{-1} dx^2 \ , \quad g(x) = \sqrt{\frac{\tilde{g}}{2(x - x_0)}} \ . && (21) \end{aligned}$$

The first form was already given above in equation (14). The second and third forms are peculiar in that there is a coordinate singularity at $x = x_0$; these coordinates only work for $x > x_0$. This singularity is very similar to the one occuring in the Schwarzschild line element. Using the experience we have obtained here, we will remove the Schwarzschild singularity at $r = 2GM$ by performing a coordinate transformation similar to those used here. The student may find it instructive to write down the coordinate transformations for these cases using equation (18) and drawing the (t, x) coordinate lines on top of the Minkowski coordinates $(\bar{t}, \bar{x})$. While the singularity at $x = x_0$ can be transformed away, it does signal the existence of an event horizon. Equation (20) is called Rindler spacetime. Its event horizon is discussed briefly in Schutz (p. 150) and in more detail by Wald (pp. 149–152).

Actually, equation (21) is closer to the Schwarzschild line element. Indeed, it becomes the r-t part of the Schwarzschild line element with the substitutions $x \to r$, $-2\tilde{g}x_0 \to 1$ and $\tilde{g} \to -GM/r^2$. These identifications show that the Schwarzschild spacetime differs from Minkowski in that the acceleration needed to remain stationary is radially directed and falls off as $e^{-\phi}\, r^{-2}$. We can understand many of its features through this identification of gravity and acceleration.

For completeness, I list three more useful forms for a flat spacetime line element:

$$\begin{aligned} ds^2 &= -dt^2 + \tilde{g}^2 (t - t_0)^2 dx^2 \ , \quad g(x) = 0 &\quad (22)\\ &= -dUdV &\quad (23)\\ &= -e^{v-u} dudv \ . &\quad (24) \end{aligned}$$

The first is similar to Rindler spacetime but with t and x exchanged. The result is suprising at first: the acceleration of a stationary observer vanishes. Equation (22) has the form of Gaussian normal or synchronous coordinates (Wald, p. 42). It represents the coordinate frame of a freely-falling observer. It is interesting to ask why, if the observer is freely-falling, the line element does not reduce to Minkowski despite the fact that this spacetime is flat. The answer is that different observers (i.e., worldlines of different x) are in uniform motion relative to one another. In other words, equation (22) is Minkowski spacetime in expanding coordinates. It is very similar to the Robertson-Walker spacetime, which reduces to it (short of two spatial dimensions) when the mass density is much less than the critical density.

Equations (23) and (24) are Minkowski spacetime in null (or light-cone) coordinates. For example, $U = \bar{t} - \bar{x}$, $V = \bar{t} + \bar{x}$. These coordinates are useful for studying horizons.

Having derived many results in $1 + 1$ spacetime, I close with the cautionary remark that in $2 + 1$ and $3 + 1$ spacetime, there are additional degrees of freedom in the metric that are quite unlike Newtonian gravity and cannot be removed (even locally) by transformation to a linearly accelerating frame. Nonetheless, it should be possible to extend the treatment of these notes to account for these effects — gravitomagnetism and gravitational radiation. As shown in the notes *Gravitation in the Weak-Field Limit*, a uniform gravitomagnetic field is equivalent to uniformly rotating coordinates. Gravitational radiation is different; there is no such thing as a spatially uniform gravitational wave. However, one can always choose coordinates so that gravitational radiation strain s_{ij} and its first derivatives vanish at a point.

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Physics 8.962 Spring 1999

Measuring the Metric, and Curvature versus Acceleration

1 Introduction

These notes show how observers can set up a coordinate system and measure the spacetime geometry using clocks and lasers. The approach is similar to that of special relativity, but the reader must not be misled. Spacetime diagrams with rectilinear axes do not imply flat spacetime any more than flat maps imply a flat earth.

Cartography provides an excellent starting point for understanding the metric. Terrestrial maps always provide a scale of the sort "One inch equals 1000 miles." If the map is of a sufficiently small region and is free from distortion, one scale will suffice. However, a projection of the entire sphere requires a scale that varies with location and even direction. The Mercator projection suggests that Greenland is larger than South America until one notices the scale difference. The simplest map projection, with latitude and longitude plotted as a Cartesian grid, has a scale that depends not only on position but also on direction. Close to the poles, one degree of latitude represents a far greater distance than one degree of longitude.

The map scale is the metric. The spacetime metric has the same meaning and use: it translates coordinate distances and times ("one inch on the map") to physical ("proper") distances and times.

The terrestrial example also helps us to understand how coordinate systems can be defined in practice on a curved manifold. Let us consider how coordinates are defined on the Earth. First pick one point and call it the north pole. The pole is chosen along the rotation axis. Now extend a family of geodesics from the north pole, called meridians of longitude. Label each meridian by its longitude ϕ. We choose the meridian going through Greenwich, England, and call it the"prime meridian," $\phi = 0$. Next, we define latitude λ as an affine parameter along each meridian of longitude, scaled to $\pi/2$ at the north pole and decreasing linearly to $-\pi/2$ at the point where the meridians intersect again (the south pole). With these definitions, the proper distance between the nearby points with coordinates (λ, ϕ) and $(\lambda + d\lambda, \phi + d\phi)$ is given by $ds^2 = R^2(d\lambda^2 + \cos^2\lambda\, d\phi^2)$. In this way, every point on the sphere gets coordinates along with a scale which converts coordinate intervals to proper distances.

This example seems almost trivial. However, it faithfully illustrates the concepts involved in setting up a coordinate system and measuring the metric. In particular, coordinates are numbers assigned by obsevers who exchange information with each other. There is no conceptual need to have the idealized dense system of clocks and rods filling spacetime. Observe any major civil engineering project. The metric is measured by two surveyors with transits and tape measures or laser ranging devices. Physicists can do the same, in principle and in practice. These notes illustrate this through a simple thought experiment. The result will be a clearer understanding of the relation between curvature, gravity, and acceleration.

2 The metric in 1+1 spacetime

We study coordinate systems and the metric in the simplest nontrivial case, spacetime with one space dimension. This analysis leaves out the issue of orientation of spatial axes. It also greatly reduces the number of degrees of freedom in the metric. As a symmetric 2 matrix, the metric has three independent coefficients. Fixing two coordinates imposes two constraints, leaving one degree of freedom in the metric. This contrasts with the six metric degrees of freedom in a 3+1 spacetime. However, if one understands well the 1+1 example, it is straightforward (albeit more complicated) to generalize to 2+1 and 3+1 spacetime.

We will construct a coordinate system starting from one observer called A. Observer A may have any motion whatsoever relative to other objects, including acceleration. But neither spatial position nor velocity is meaningful for A before we introduce other observers or coordinates ("velocity relative to what?") although A's acceleration (relative to a local inertial frame!) is meaningful: A stands on a scale, reads the weight, and divides by rest mass. Observer A could be you or me, standing on the surface of the earth. It could equally well be an astronaut landing on the moon. It may be helpful in this example to think of the observers as being stationary with respect to a massive gravitating body (e.g. a black hole or neutron star). However, we are considering a completely general case, in which the spacetime may not be at all static. (That is, there may not be any Killing vectors whatsoever.)

We take observer A's worldine to define the t-axis: A has spatial coordinate $x_A \equiv 0$. A second observer, some finite (possibly large) distance away, is denoted B. Both A and B carry atomic clocks, lasers, mirrors and detectors.

Observer A decides to set the spacetime coordinates over all spacetime using the following procedure, illustrated in Figure 1. First, the reading of A's atomic clock gives

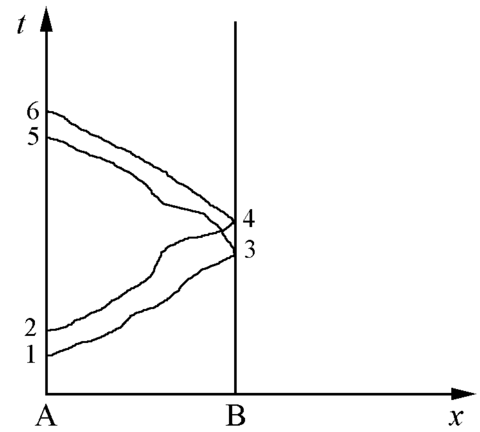

Figure 1: Setting up a coordinate system in curved spacetime. There are two timelike worldlines and two pairs of null geodesics. The appearance of flat coordinates is misleading; the metric varies from place to place.

the t-coordinate along the t-axis ($x = 0$). Then, A sends a pair of laser pulses to B, who reflects them back to A with a mirror. If the pulses do not return with the same time separation (measured by A) as they were sent, A deduces that B is moving and sends signals instructing B to adjust her velocity until $t_6 - t_5 = t_2 - t_1$. The two continually exchange signals to ensure that this condition is maintained. A then declares that B has a constant space coordinate (by definition), which is set to half the round-trip light-travel time, $x_B \equiv \frac{1}{2}(t_5 - t_1)$. A sends signals to inform B of her coordinate.

Having set the spatial coordinate, A now sends time signals to define the t-coordinate along B's worldline. A's laser encodes a signal from Event 1 in Figure 1, "This pulse was sent at $t = t_1$. Set your clock to $t_1 + x_B$." B receives the pulse at Event 3 and sets her clock. A sends a second pulse from Event 2 at $t_2 = t_1 + \Delta t$ which is received by B at Event 4. B compares the time difference quoted by A with the time elapsed on her atomic clock, the proper time $\Delta\tau_B$. To her surprise, $\Delta\tau_B \neq \Delta t$.

At first A and B are sure something went wrong; maybe B has begun to drift. But repeated exchange of laser pulses shows that this cannot be the explanation: the round-trip light-travel time is always the same. Next they speculate that the lasers may be traveling through a refractive medium whose index of refraction is changing with time. (A constant index of refraction wouldn't change the differential arrival time.) However, they reject this hypothesis when they find that B's atomic clock continually runs at a different rate than the timing signals sent by A, while the round-trip light-travel time

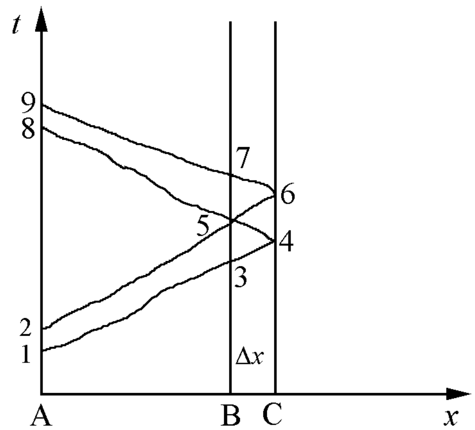

Figure 2: Testing for space curvature.

measured by A never changes. Moreover, laboratory analysis of the medium between them shows no evidence for any change.

Becoming suspicious, B decides to keep two clocks, an atomic clock measuring τ_B and another set to read the time sent by A, denoted t. The difference between the two grows increasingly large.

The observers next speculate that they may be in a non-inertial frame so that special relativity remains valid despite the apparent contradiction of clock differences ($g_{tt} \neq 1$) with no relative motion ($dx_B/dt = 0$). We will return to this speculation in Section 3. In any case, they decide to keep track of the conversion from coordinate time (sent by A) to proper time (measured by B) for nearby events on B's worldline by defining a metric coefficient:

$$g_{tt}(t, x_B) \equiv \lim_{\Delta t \to 0} -\left(\frac{\Delta\tau_B}{\Delta t}\right)^2 . \tag{1}$$

The observers now wonder whether measurements of spatial distances will yield a similar mystery. To test this, a third observer is brought to help in Figure 2. Observer C adjusts his velocity to be at rest relative to A. Just as for B, the definition of rest is that the round-trip light-travel time measured by A is constant, $t_8 - t_1 = t_9 - t_2 = 2x_C \equiv 2(x_B + \Delta x)$. Note that the coordinate distances are expressed entirely in terms of readings of A's clock. A sends timing signals to both B and C. Each of them sets one clock to read the time sent by A (corrected for the spatial coordinate distance x_B and x_C, respectively) and also keeps time by carrying an undisturbed atomic clock. The former is called coordinate time t while the latter is called proper time.

The coordinate time synchronization provided by A ensures that $t_2 - t_1 = t_5 - t_3 = t_6 - t_4 = t_7 - t_5 = t_9 - t_8 = 2\Delta x$. Note that the procedure used by A to set t and x relates the coordinates of events on the worldlines of B and C:

$$\begin{aligned}(t_4, x_4) &= (t_3, x_3) + (1, 1)\Delta x \ , \quad (t_5, x_5) = (t_4, x_4) + (1, -1)\Delta x \ , \\ (t_6, x_6) &= (t_5, x_5) + (1, 1)\Delta x \ , \quad (t_7, x_7) = (t_6, x_6) + (1, -1)\Delta x \ . \end{aligned} \tag{2}$$

Because they follow simply from the synchronization provided by A, these equations are exact; they do not require Δx to be small. However, by themselves they do not imply anything about the *physical* separations between the events. Testing this means measuring the metric.

To explore the metric, C checks his proper time and confirms B's observation that proper time differs from coordinate time. However, the metric coefficient he deduces, $g_{tt}(x_C, t)$, differs from B's. (The difference is first-order in Δx.)

The pair now wonder whether spatial coordinate intervals are similarly skewed relative to proper distance. They decide to measure the proper distance between them by using laser-ranging, the same way that A set their spatial coordinates in the first place. B sends a laser pulse at Event 3 which is reflected at Event 4 and received back at Event 5 in Figure 2. From this, she deduces the proper distance of C,

$$\Delta s = \frac{1}{2}(\tau_5 - \tau_3) \tag{3}$$

where τ_i is the reading of her atomic clock at event i. To her surprise, B finds that Δx does not measure proper distance, not even in the limit $\Delta x \to 0$. She defines another metric coefficient to convert coordinate distance to proper distance,

$$g_{xx} \equiv \lim_{\Delta x \to 0} \left(\frac{\Delta s}{\Delta x}\right)^2 \ . \tag{4}$$

The measurement of proper distance in equation (4) must be made at fixed t; otherwise the distance must be corrected for relative motion between B and C (should any exist). Fortunately, B can make this measurement at $t = t_4$ because that is when her laser pulse reaches C (see Fig. 2 and eqs. 2). Expanding $\tau_5 = \tau_B(t_4 + \Delta x)$ and $\tau_3 = \tau_B(t_4 - \Delta x)$ to first order in Δx using equations (1), (3), and (4), she finds

$$g_{xx}(x, t) = -g_{tt}(x, t) \ . \tag{5}$$

The observers repeat the experiment using Events 5, 6, and 7. They find that, while the metric may have changed, equation (5) still holds.

The observers are intrigued to find such a relation between the time and space parts of their metric, and they wonder whether this is a general phenomenon. Have they discovered a modification of special relativity, in which the Minkowski metric is simply multipled by a conformal factor, $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$?

They decide to explore this question by measuring g_{tx}. A little thought shows that they cannot do this using pairs of events with either fixed x or fixed t. Fortunately, they have ideal pairs of events in the lightlike intervals between Events 3 and 4:

$$ds_{34}^2 \equiv \lim_{\Delta t, \Delta x \to 0} g_{tt}(t_4 - t_3)^2 + 2g_{tx}(t_4 - t_3)(x_4 - x_3) + g_{xx}(x_4 - x_3)^2 \ . \tag{6}$$

Using equations (2) and (5) and the condition $ds = 0$ for a light ray, they conclude

$$g_{tx} = 0 \ . \tag{7}$$

Their space and time coordinates are orthogonal but on account of equations (5) and (7) all time and space intervals are stretched by $\sqrt{g_{xx}}$.

Our observers now begin to wonder if they have discovered a modification of special relativity, or perhaps they are seeing special relativity in a non-inertial frame. However, we know better. Unless the Riemann tensor vanishes identically, the metric they have determined cannot be transformed everywhere to the Minkowski form. Instead, what they have found is simply a consequence of how A fixed the coordinates. Fixing two coordinates means imposing two gauge conditions on the metric. A defined coordinates so as to make the problem look as much as possible like special relativity (eqs. 2). Equations (5) and (7) are the corresponding gauge conditions.

It is a special feature of 1+1 spacetime that the metric can always be reduced to a conformally flat one, i.e.

$$ds^2 = \Omega^2(x)\eta_{\mu\nu}dx^\mu dx^\nu \tag{8}$$

for some function $\Omega(x^\mu)$ called the conformal factor. In two dimensions the Riemann tensor has only one independent component and the Weyl tensor vanishes identically. Advanced GR and differential geometry texts show that spacetimes with vanishing Weyl tensor are conformally flat.

Thus, A has simply managed to assign conformally flat coordinates. This isn't a coincidence; by defining coordinate times and distances using null geodesics, he forced the metric to be identical to Minkowski up to an overall factor that has no effect on null lines. Equivalently, in two dimensions the metric has one physical degree of freedom, which has been reduced to the conformal factor $\Omega \equiv \sqrt{g_{xx}} = \sqrt{-g_{tt}}$.

This does not mean that A would have had such luck in more than two dimensions. In n dimensions the Riemann tensor has $n^2(n^2-1)/12$ independent components (Wald p. 54) and for $n \geq 3$ the Ricci tensor has $n(n+1)/2$ independent components. For $n = 2$ and $n = 3$ the Weyl tensor vanishes identically and spacetime is conformally flat. Not so for $n > 3$.

It would take a lot of effort to describe a complete synchronization in 3+1 spacetime using clocks and lasers. However, even without doing this we can be confident that the metric will not be conformally flat except for special spacetimes for which the Weyl tensor vanishes. Incidentally, in the weak-field limit conformally flat spacetimes have no deflection of light (can you explain why?). The solar deflection of light rules out conformally flat spacetime theories including ones proposed by Nordstrom and Weyl.

It is an interesting exercise to show how to transform an arbitrary metric of a 1+1 spacetime to the conformally flat form. The simplest way is to compute the Ricci scalar. For the metric of equation (8), one finds

$$R = \Omega^{-2}(\partial_t^2 - \partial_x^2)\ln\Omega^2 \ . \tag{9}$$

Starting from a 1+1 metric in a different form, one can compute R everywhere in spacetime. Equation (9) is then a nonlinear wave equation for $\Omega(t,x)$ with source $R(t,x)$. It can be solved subject to initial Cauchy data on a spacelike hypersurface on which $\Omega = 1$, $\partial_t\Omega = \partial_x\Omega = 0$ (corresponding to locally flat coordinates).

We have exhausted the analysis of 1+1 spacetime. Our observers have discerned one possible contradiction with special relativity: clocks run at different rates in different places (and perhaps at different times). If equation (9) gives Ricci scalar $R = 0$ everywhere with $\Omega = \sqrt{-g_{tt}}$, then the spacetime is really flat and we must be seeing the effects of accelerated motion in special relativity. If $R \neq 0$, then the variation of clocks is an entirely new phenomenon, which we call gravitational redshift.

3 The metric for an accelerated observer

It is informative to examine the problem from another perspective by working out the metric that an arbitrarily accelerating observer in a flat spacetime would deduce using the synchronization procedure of Section 2. We can then more clearly distinguish the effects of curvature (gravity) and acceleration.

Figure 3 shows the situation prevailing in special relativity when observer A has an arbitrary timelike trajectory $x_A^\mu(\tau_A)$ where τ_A is the proper time measured by his atomic clock. While A's worldline is erratic, those of light signals are not, because here $t = x^0$ and $x = x^1$ are flat coordinates in Minkowski spacetime. Given an arbitrary worldline $x_A^\mu(\tau_A)$, how can we possibly find the worldines of observers at fixed coordinate displacement as in the preceding section?

The answer is the same as the answer to practically all questions of measurement in GR: use the metric! The metric of flat spacetime is the Minkowski metric, so the paths of laser pulses are very simple. We simply solve an algebra problem enforcing that Events 1 and 2 are separated by a null geodesic (a straight line in Minkowski spacetime) and likewise for Events 2 and 3, as shown in Figure 3. Notice that the lengths (i.e. coordinate differences) of the two null rays need not be the same.

The coordinates of Events 1 and 3 are simply the coordinates along A's worldine, while those for Event 2 are to be determined in terms of A's coordinates. As in Section 2, A defines the spatial coordinate of B to be twice the round-trip light-travel time. Thus, if event 0 has $x^0 = t_A(\tau_0)$, then Event 3 has $x^0 = t_A(\tau_0 + 2L)$. For convenience we will

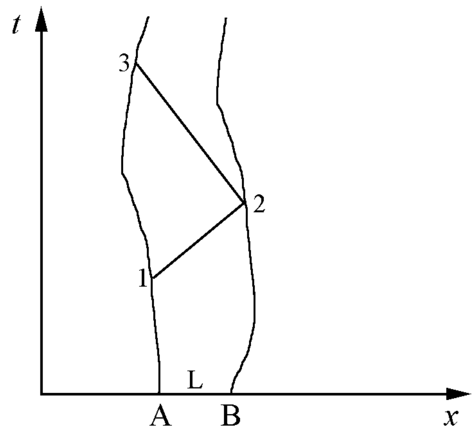

Figure 3: An accelerating observer sets up a coordinate system with an atomic clock, laser and detector.

set $\tau_0 \equiv \tau_A - L$. Then, according to the prescription of Section 2, A will assign to Event 2 the coordinates (τ_A, L). The coordinates in our flat Minkowksi spacetime are

$$\begin{aligned}
\text{Event 1:}\quad x^0 &= t_A(\tau_A - L)\ , \quad x^1 = x_A(\tau_A - L)\ ,\\
\text{Event 2:}\quad x^0 &= t(\tau_A, L)\ , \quad x^1 = x(\tau_A, L)\ ,\\
\text{Event 3:}\quad x^0 &= t_A(\tau_A + L)\ , \quad x^1 = x_A(\tau_A + L)\ .
\end{aligned} \tag{10}$$

Note that the argument τ_A for Event 2 is not an affine parameter along B's wordline; it is the clock time sent to B by A. A second argument L is given so that we can look at a family of worldlines with different L. A is setting up coordinates by finding the spacetime paths corresponding to the coordinate lines $L =$ constant and $\tau_A =$ constant. We are performing a coordinate transformation from (t, x) to (τ_A, L).

Requiring that Events 1 and 2 be joined by a null geodesic in flat spacetime gives the condition $x_2^\mu - x_1^\mu = (C_1, C_1)$ for some constant C_1. The same condition for Events 2 and 3 gives $x_3^\mu - x_2^\mu = (C_2, -C_2)$ (with a minus sign because the light ray travels toward decreasing x). These conditions give four equations for the four unknowns C_1, C_2, $t(\tau_A, L)$, and $x(\tau_A, L)$. Solving them gives the coordinate transformation between (τ_A, L) and the Minkowski coordinates:

$$\begin{aligned}
t(\tau_A, L) &= \frac{1}{2}\left[t_A(\tau_A + L) + t_A(\tau_A - L) + x_A(\tau_A + L) - x_A(\tau_A - L)\right]\ ,\\
x(\tau_A, L) &= \frac{1}{2}\left[x_A(\tau_A + L) + x_A(\tau_A - L) + t_A(\tau_A + L) - t_A(\tau_A - L)\right]\ .
\end{aligned} \tag{11}$$

Note that these results are exact; they do not assume that L is small nor do they restrict A's worldline in any way except that it must be timelike. The student may easily evaluate C_1 and C_2 and show that they are not equal unless $x_A(\tau_A + L) = x_A(\tau_A - L)$.

Using equations (11), we may transform the Minkowski metric to get the metric in the coordinates A has set with his clock and laser, (τ_A, L):

$$ds^2 = -dt^2 + dx^2 = g_{tt}d\tau_A^2 + 2g_{tx}d\tau_A dL + g_{xx}dL^2 \ . \tag{12}$$

Substituting equations (11) gives the metric components in terms of A's four-velocity components,

$$-g_{tt} = g_{xx} = \left[V_A^t(\tau_A + L) + V_A^x(\tau_A + L)\right]\left[V_A^t(\tau_A - L) - V_A^x(\tau_A - L)\right] \ , \quad g_{tx} = 0 \ . \tag{13}$$

This is precisely in the form of equation (8), as it must be because of the way in which A coordinatized spacetime.

It is straightforward to work out the Riemann tensor from equation (13). Not surprisingly, it vanishes identically. Thus, an observer *can* tell, through measurements, whether he or she lives in a flat or nonflat spacetime. The metric is measurable.

Now that we have a general result, it is worth simplifying to the case of an observer with constant acceleration g_A in Minkowski spacetime. Problem 3 of Problem Set 1 showed that one can write the trajectory of such an observer (up to the addition of constants) as $x = g_A^{-1}\cosh g_A\tau_A$, $t = g_A^{-1}\sinh g_A\tau_A$. Equation (13) then gives

$$ds^2 = e^{2g_A L}\left(-d\tau_A^2 + dL^2\right) \ . \tag{14}$$

One word of caution is in order about the interpretation of equation (14). Our derivation assumed that the acceleration g_A is constant for observer A at $L = 0$. However, this does not mean that other observers (at fixed, nonzero L) have the same acceleration. To see this, we can differentiate equations (11) to derive the 4-velocity of observer B at (τ_A, L) and the relation between coordinate time τ_A and proper time along B's worldline, with the result

$$V_B^\mu(\tau_A, L) = (\cosh g_A\tau_A, \sinh g_A\tau_A) = (\cosh g_B\tau_B, \sinh g_B\tau_B) \ , \quad \frac{d\tau_B}{d\tau_A} = \frac{g_A}{g_B} = e^{gL} \ . \tag{15}$$

The four-acceleration of B follows from $a_B^\mu = dV_B^\mu/d\tau_B = e^{-gL}dV^\mu/d\tau_A$ and its magnitude is therefore $g_B = g_A e^{-gL}$. The proper acceleration varies with L precisely so that the proper distance between observers A and B, measured at constant τ_A, remains constant.

4 Gravity versus acceleration in 1+1 spacetime

Equation (14) gives one form of the metric for a flat spacetime as seen by an accelerating observer. There are many other forms, and it is worth noting some of them in order to gain some intuition about the effects of acceleration. For simplicity, we will restrict our discussion here to static spacetimes, i.e. metrics with $g_{0i} = 0$ and $\partial_t g_{\mu\nu} = 0$. In 1+1 spacetime this means the line element may be written

$$ds^2 = -e^{2\phi(x)}dt^2 + e^{-2\psi(x)}dx^2 \ . \tag{16}$$

(The metric may be further transformed to the conformally flat form, eq. 8, but we leave it in this form because of its similarity to the form often used in $3+1$ spacetime.)

Given the metric (16), we would like to know when the spacetime is flat. If it is flat, we would like the explicit coordinate transformation to Minkowski. Both of these are straightforward in 1+1 spacetime. (One might hope for them also to be straightforward in more dimensions, at least in principle, but the algebra rapidly increases.)

The definitive test for flatness is given by the Riemann tensor. Because the Weyl tensor vanishes in 1+1 spacetime, it is enough to examine the Ricci tensor. With equation (16), the Ricci tensor has nonvanishing components

$$R_{tt} = e^{\phi+\psi}\,\frac{d\tilde{g}}{dx} \ , \quad R_{xx} = -e^{-(\phi+\psi)}\frac{d\tilde{g}}{dx} \quad \text{where} \quad \tilde{g}(x) = e^{\phi}\,g(x) = e^{\phi+\psi}\,\frac{d\phi}{dx} \ . \tag{17}$$

The function $g(x)$ is the proper acceleration along the x-coordinate line, along which the tangent vector (4-velocity) is $V^\mu_x = e^{-\phi}(1,0)$. This follows from computing the 4-acceleration with equation (16) using the covariant prescription $a^\mu(x) = \nabla_V V^\mu = V^\nu_x \nabla_\nu V^\mu_x$. The magnitude of the acceleration is then $g(x) \equiv (g_{\mu\nu}a^\mu a^\nu)^{1/2}$, yielding $g(x) = e^\psi d\phi/dx$. The factor e^ψ converts $d\phi/dx$ to $g(x) = d\phi/dl$ where $dl = \sqrt{g_{xx}}\,dx$ measures proper distance.

A stationary observer, i.e. one who remains at fixed spatial coordinate x, feels a time-independent effective gravity $g(x)$. Nongravitational forces (e.g. a rocket, or the contact force from a surface holding the observer up) are required to maintain the observer at fixed x. The gravity field $g(x)$ can be measured very simply by releasing a test particle from rest and measuring its acceleration relative to the stationary observer. For example, we measure g on the Earth by dropping masses and measuring their acceleration in the lab frame.

We will see following equation (18) below why the function $\tilde{g}(x) = (d\tau/dt)g(x)$ rather than $g(x)$ determines curvature. For now, we simply note that equation (17) implies that spacetime curvature is given (for a static 1+1 metric) by the gradient of the gravitational redshift factor $\sqrt{-g_{tt}} = e^\phi$ rather than by the "gravity" (i.e. acceleration) gradient dg/dx.

In linearized gravitation, $g = \tilde{g}$ and so we deduced (in the notes *Gravitation in the Weak-Field Limit*) that a spatially uniform gravitational (gravitoelectric) field can be transformed away by making a global coordinate transformation to an accelerating frame. For strong fields, $g \neq \tilde{g}$ and it is no longer true that a uniform gravitoelectric field can be transformed away. Only if the gravitational redshift factor $e^{\phi(x)}$ varies linearly with proper distance, i.e. $\tilde{g} \equiv d(e^{\phi})/dl$ is a constant, is spacetime is flat, enabling one to transform coordinates so as to remove all evidence for acceleration. If, on the other hand, $d\tilde{g}/dx \neq 0$ — even if $dg/dx = 0$ — then the spacetime is not flat and no coordinate transformation can transform the metric to the Minkowski form.

Suppose we have a line element for which $\tilde{g}(x) =$ constant. We know that such a spacetime is flat, because the Ricci tensor (hence Riemann tensor, in 1+1 spacetime) vanishes everywhere. What is the coordinate transformation to Minkowski?

The transformation may be found by writing the metric as $g = \Lambda^T \eta \Lambda$ where $\Lambda^{\bar{\mu}}{}_{\nu} = \partial \bar{x}^{\bar{\mu}}/\partial x^{\nu}$ is the Jacobian matrix for the transformation $\bar{x}(x)$. (Note that here g is the matrix with entries $g_{\mu\nu}$ and not the gravitational acceleration!) By writing $\bar{t} = \bar{t}(t,x)$ and $\bar{x} = \bar{x}(t,x)$, substituting into $g = \Lambda^T \eta \Lambda$, using equation (16) and imposing the integrability conditions $\partial^2 \bar{t}/\partial t \partial x = \partial^2 \bar{t}/\partial x \partial t$ and $\partial^2 \bar{x}/\partial t \partial x = \partial^2 \bar{x}/\partial x \partial t$, one finds

$$\bar{t}(t,x) = \frac{1}{g} \sinh \tilde{g}\, t \ , \quad \bar{x}(t,x) = \frac{1}{g} \cosh \tilde{g}\, t \quad \text{if} \quad \frac{d\tilde{g}}{dx} = 0 \ , \tag{18}$$

up to the addition of irrelevant constants. We recognize this result as the trajectory in flat spacetime of a constantly accelerating observer.

Equation (18) is easy to understand in light of the discussion following equation (14). The proper time τ for the stationary observer at x is related to coordinate time t by $d\tau = \sqrt{-g_{tt}(x)}\, dt = e^{\phi} dt$. Thus, $g(x)\tau = e^{\phi}\, g\, t = \tilde{g}\, t$ or, in the notation of equation (15), $g_B \tau_B = g_A \tau_A$ (since τ_A was used there as the global t-coordinate). The condition $e^{\phi}\, g = \tilde{g}(x) =$ constant amounts to requiring that all observers be able to scale their gravitational accelerations to a common value for the observer at $\phi(x) = 0$, $\tilde{g}$. If they cannot (i.e. if $d\tilde{g}/dx \neq 0$), then the metric is not equivalent to Minkowski spacetime seen in the eyes of an accelerating observer.

With equations (16)–(18) in hand, we can write the metric of a flat spacetime in several new ways, with various spatial dependence for the acceleration of our coordinate observers:

$$\begin{aligned} ds^2 &= e^{2\tilde{g}\,x}(-dt^2 + dx^2) \ , \quad g(x) = \tilde{g}\, e^{-\tilde{g}\,x} && (19) \\ &= -\tilde{g}^2(x - x_0)^2 dt^2 + dx^2 \ , \quad g(x) = \frac{1}{x - x_0} && (20) \\ &= -[2\tilde{g}(x - x_0)]dt^2 + [2\tilde{g}(x - x_0)]^{-1} dx^2 \ , \quad g(x) = \sqrt{\frac{\tilde{g}}{2(x - x_0)}} \ . && (21) \end{aligned}$$

The first form was already given above in equation (14). The second and third forms are peculiar in that there is a coordinate singularity at $x = x_0$; these coordinates only work for $x > x_0$. This singularity is very similar to the one occuring in the Schwarzschild line element. Using the experience we have obtained here, we will remove the Schwarzschild singularity at $r = 2GM$ by performing a coordinate transformation similar to those used here. The student may find it instructive to write down the coordinate transformations for these cases using equation (18) and drawing the (t, x) coordinate lines on top of the Minkowski coordinates $(\bar{t}, \bar{x})$. While the singularity at $x = x_0$ can be transformed away, it does signal the existence of an event horizon. Equation (20) is called Rindler spacetime. Its event horizon is discussed briefly in Schutz (p. 150) and in more detail by Wald (pp. 149–152).

Actually, equation (21) is closer to the Schwarzschild line element. Indeed, it becomes the r-t part of the Schwarzschild line element with the substitutions $x \to r$, $-2\tilde{g}x_0 \to 1$ and $\tilde{g} \to -GM/r^2$. These identifications show that the Schwarzschild spacetime differs from Minkowski in that the acceleration needed to remain stationary is radially directed and falls off as $e^{-\phi}\, r^{-2}$. We can understand many of its features through this identification of gravity and acceleration.

For completeness, I list three more useful forms for a flat spacetime line element:

$$\begin{aligned} ds^2 &= -dt^2 + \tilde{g}^2(t - t_0)^2 dx^2 \ , \quad g(x) = 0 && (22) \\ &= -dU dV && (23) \\ &= -e^{v-u} du dv \ . && (24) \end{aligned}$$

The first is similar to Rindler spacetime but with t and x exchanged. The result is suprising at first: the acceleration of a stationary observer vanishes. Equation (22) has the form of Gaussian normal or synchronous coordinates (Wald, p. 42). It represents the coordinate frame of a freely-falling observer. It is interesting to ask why, if the observer is freely-falling, the line element does not reduce to Minkowski despite the fact that this spacetime is flat. The answer is that different observers (i.e., worldlines of different x) are in uniform motion relative to one another. In other words, equation (22) is Minkowski spacetime in expanding coordinates. It is very similar to the Robertson-Walker spacetime, which reduces to it (short of two spatial dimensions) when the mass density is much less than the critical density.

Equations (23) and (24) are Minkowski spacetime in null (or light-cone) coordinates. For example, $U = \bar{t} - \bar{x}$, $V = \bar{t} + \bar{x}$. These coordinates are useful for studying horizons.

Having derived many results in $1 + 1$ spacetime, I close with the cautionary remark that in $2 + 1$ and $3 + 1$ spacetime, there are additional degrees of freedom in the metric that are quite unlike Newtonian gravity and cannot be removed (even locally) by transformation to a linearly accelerating frame. Nonetheless, it should be possible to extend the treatment of these notes to account for these effects — gravitomagnetism and gravitational radiation. As shown in the notes *Gravitation in the Weak-Field Limit*, a uniform gravitomagnetic field is equivalent to uniformly rotating coordinates. Gravitational radiation is different; there is no such thing as a spatially uniform gravitational wave. However, one can always choose coordinates so that gravitational radiation strain s_{ij} and its first derivatives vanish at a point.

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Physics 8.962 Spring 2002

Symmetry Transformations, the Einstein-Hilbert Action, and Gauge Invariance

1 Introduction

Action principles are widely used to express the laws of physics, including those of general relativity. For example, freely falling particles move along geodesics, or curves of extremal path length.

Symmetry transformations are changes in the coordinates or variables that leave the action invariant. It is well known that continuous symmetries generate conservation laws (Noether's Theorem). Conservation laws are of fundamental importance in physics and so it is valuable to investigate symmetries of the action.

It is useful to distinguish between two types of symmetries: **dynamical** symmetries corresponding to some inherent property of the matter or spacetime evolution (e.g. the metric components being independent of a coordinate, leading to a conserved momentum one-form component) and **nondynamical** symmetries arising because of the way in which we formulate the action. Dynamical symmetries constrain the solutions of the equations of motion while nondynamical symmetries give rise to mathematical identities. These notes will consider both.

An example of a nondynamical symmetry is the parameterization-invariance of the path length, the action for a free particle:

$$S[x^\mu(\tau)] = \int_{\tau_1}^{\tau_2} L_1\left(x^\mu(\tau), \dot{x}^\mu(\tau), \tau\right)\, d\tau = \int_{\tau_1}^{\tau_2} \left[g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{1/2} d\tau\ . \tag{1}$$

This action is invariant under arbitrary reparameterization $\tau \to \tau'(\tau)$, implying that any solution $x^\mu(\tau)$ of the variational problem $\delta S = 0$ immediately gives rise to other solutions

$y^\mu(\tau) = x^\mu(\tau'(\tau))$. Moreover, even if the action is not extremal with Lagrangian L_1 for some (non-geodesic) curve $x^\mu(\tau)$, it is still invariant under reparameterization of that curve.

There is another nondynamical symmetry of great importance in general relativity, coordinate-invariance. Being based on tensors, equations of motion in general relativity hold regardless of the coordinate system. However, when we write an *action* involving tensors, we must write the *components* of the tensors in some basis. This is because the calculus of variations works with functions, e.g. the components of tensors, treated as spacetime fields. Although the values of the fields are dependent on the coordinate system chosen, the action must be a scalar, and therefore invariant under coordinate transformations. This is true whether or not the action is extremized and therefore it is a nondynamical symmetry.

Nondynamical symmetries give rise to special laws called identities. They are distinct from conservation laws because they hold whether or not one has extremized the action.

The material in these notes is generally not presented in this form in the GR textbooks, although much of it can be found in Misner et al if you search well. Although these symmetry principles and methods are not needed for integrating the geodesic equation, they are invaluable in understanding the origin of the contracted Bianchi identities and stress-energy conservation in the action formulation of general relativity. More broadly, they are the cornerstone of gauge theories of physical fields including gravity.

Starting with the simple system of a single particle, we will advance to the Lagrangian formulation of general relativity as a classical field theory. We will discover that, in the field theory formulation, the contracted Bianchi identities arise from a non-dynamical symmetry while stress-energy conservation arises from a dynamical symmetry. Along the way, we will explore Killing vectors, diffeomorphisms and Lie derivatives, the stress-energy tensor, electromagnetism and charge conservation. We will discuss the role of continuous symmetries (gauge invariance and diffeomorphism invariance or general covariance) for a simple model of a relativistic fluid interacting with electromagnetism and gravity. Although this material goes beyond what is presented in lecture, it is not very advanced mathematically and it is recommended reading for students wishing to understand gauge symmetry and the parallels between gravity, electromagnetism, and other gauge theories.

2 Parameterization-Invariance of Geodesics

The parameterization-invariance of equation (1) may be considered in the broader context of Lagrangian systems. Consider a system with n degrees of freedom — the generalized coordinates q^i — with a parameter t giving the evolution of the trajectory in configuration space. (In eq. 1, q^i is denoted x^μ and t is τ.) We will drop the superscript on q^i when it is clear from the context.

Theorem: If the action $S[q(t)]$ is invariant under the infinitesimal transformation $t \to t + \epsilon(t)$ with $\epsilon = 0$ at the endpoints, then the Hamiltonian vanishes identically.

The proof is straightforward. Given a parameterized trajectory $q^i(t)$, we define a new parameterized trajectory $\bar{q}(t) = q(t+\epsilon)$. The action is

$$S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \ . \tag{2}$$

Linearizing $\bar{q}(t)$ for small ϵ,

$$\bar{q}(t) = q + \dot{q}\epsilon \ , \quad \frac{d\bar{q}}{dt} = \dot{q} + \frac{d}{dt}(\dot{q}\epsilon) \ .$$

The change in the action under the transformation $t \to t + \epsilon$ is, to first order in ϵ,

$$\begin{aligned} S[q(t+\epsilon)] - S[q(t)] &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial t}\epsilon + \frac{\partial L}{\partial q^i}\dot{q}^i\epsilon + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}(\dot{q}^i\epsilon) \right] dt \\ &= \int_{t_1}^{t_2} \left[\frac{dL}{dt}\epsilon + \left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i \right) \frac{d\epsilon}{dt} \right] dt \\ &= [L\epsilon]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \right) \frac{d\epsilon}{dt}\, dt \ . \end{aligned} \tag{3}$$

The boundary term vanishes because $\epsilon = 0$ at the endpoints. Parameterization-invariance means that the integral term must vanish for arbitrary $d\epsilon/dt$, implying

$$H \equiv \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L = 0 \ . \tag{4}$$

Nowhere did this derivation assume that the action is extremal or that $q^i(t)$ satisfy the Euler-Lagrange equations. Consequently, equation (4) is a nondynamical symmetry.

The reader may easily check that the Hamiltonian H_1 constructed from equation (1) vanishes identically. This symmetry does not mean that there is no Hamiltonian formulation for geodesic motion, only that the Lagrangian L_1 has non-dynamical degrees of freedom that must be eliminated before a Hamiltonian can be constructed. (A similar circumstance arises in non-Abelian quantum field theories, where the non-dynamical degrees of freedom are called Faddeev-Popov ghosts.) This can be done by replacing the parameter with one of the coordinates, reducing the number of degrees of freedom in the action by one. It can also be done by changing the Lagrangian to one that is no longer invariant under reparameterizations, e.g. $L_2 = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. In this case, $\partial L_2/\partial\tau = 0$ leads to a *dynamical* symmetry, $H_2 = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu =$ constant along trajectories which satisfy the equations of motion.

The identity $H_1 = 0$ is very different from the conservation law $H_2 =$ constant arising from a time-independent Lagrangian. The conservation law holds only for solutions of the equations of motion; by contrast, when the action is parameterization-invariant, $H_1 = 0$ holds for any trajectory. The nondynamical symmetry therefore does not constrain the motion.

3 Generalized Translational Symmetry

Continuing with the mechanical analogy of Lagrangian systems exemplified by equation (2), in this section we consider translations of the configuration space variables. If the Lagrangian is invariant under the translation $q^i(t) \to q^i(t) + a^i$ for constant a^i, then $p_i a^i$ is conserved along trajectories satisfying the Euler-Lagrange equations. This well-known example of translational invariance is the prototypical dynamical symmetry, and it follows directly from the Euler-Lagrange equations. In this section we generalize the concept of translational invariance by considering spatially-varying shifts and coordinate transformations that leave the action invariant. Along the way we will introduce several important new mathematical concepts.

In flat spacetime it is common to perform calculations in one reference frame with a fixed set of coordinates. In general relativity there are no preferred frames or coordinates, which can lead to confusion unless one is careful. The coordinates of a trajectory may change either because the trajectory has been shifted or because the underlying coordinate system has changed. The consequences of these alternatives are very different: under a coordinate transformation the Lagrangian is a scalar whose form and value are unchanged, while the Lagrangian can change when a trajectory is shifted. The Lagrangian is always taken to be a scalar in order to ensure local Lorentz invariance (no preferred frame of reference). In this section we will carefully sort out the effects of both shifting the trajectory and transforming the coordinates in order to identify the underlying symmetries. As we will see, conservation laws arise when shifting the trajectory is equivalent to a coordinate transformation.

We consider a general, relativistically covariant Lagrangian for a particle, which depends on the velocity, the metric, and possibly on additional fields:

$$S[x(\tau)] = \int_{\tau_1}^{\tau_2} L(g_{\mu\nu}, A_\mu, \dots, \dot{x}^\mu)\, d\tau \ . \tag{5}$$

Note that the coordinate-dependence occurs in the fields $g_{\mu\nu}(x)$ and $A_\mu(x)$. An example of such a Lagrangian is

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + q A_\mu \dot{x}^\mu \ . \tag{6}$$

The first piece is the quadratic Lagrangian L_2 that gives rise to the geodesic equation. The additional term gives rise to a non-gravitational force. The Euler-Lagrange equation for this Lagrangian is

$$\frac{D^2 x^\mu}{d\tau^2} = q F^\mu{}_\nu \frac{dx^\nu}{d\tau} \ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ . \tag{7}$$

We see that the non-gravitational force is the Lorentz force for a charge q, assuming that the units of the affine parameter τ are chosen so that $dx^\mu/d\tau$ is the 4-momentum (i.e. $md\tau$ is proper time for a particle of mass m). The one-form field $A_\mu(x)$ is the

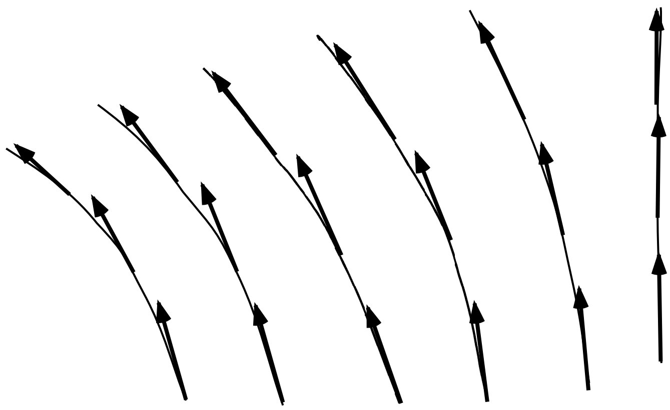

Figure 1: A vector field and its integral curves.

electromagnetic potential. We will retain the electromagnetic interaction term in the Lagrangian in the presentation that follows in order to illustrate more broadly the effects of symmetry.

Symmetry appears only when a system is changed. Because L is a scalar, coordinate transformations for a fixed trajectory change nothing and therefore reveal no symmetry. So let us try changing the trajectory itself. Keeping the coordinates (and therefore the metric and all other fields) fixed, we will shift the trajectory along the integral curves of some vector field $\xi^\mu(x)$. (Here $\vec{\xi}$ is any vector field.) As we will see, a vector field provides a one-to-one mapping of the manifold back to itself, providing a natural translation operator in curved spacetime.

Figure 1 shows a vector field and its integral curves $x^\mu(\lambda, \tau)$ where τ labels the curve and λ is a parameter along each curve. Any vector field $\vec{\xi}(x)$ has a unique set of integral curves whose tangent vector is $\partial x^\mu/\partial\lambda = \xi^\mu(x)$. If we think of $\vec{\xi}(x)$ as a fluid velocity field, then the integral curves are streamlines, i.e. the trajectories of fluid particles.

The integral curves of a vector field provide a continuous one-to-one mapping of the manifold back to itself, called a pushforward. (The mapping is one-to-one because the integral curves cannot intersect since the tangent is unique at each point.) Figure 2 illustrates the pushforward. This mapping associates each point on the curve $x^\mu(\tau)$ with a corresponding point on the curve $y^\mu(\tau)$. For example, the point P_0 $(\lambda = 0, \tau = 3)$ is mapped to another point P $(\lambda = 1, \tau = 3)$. The mapping $x \to y$ is obtained by

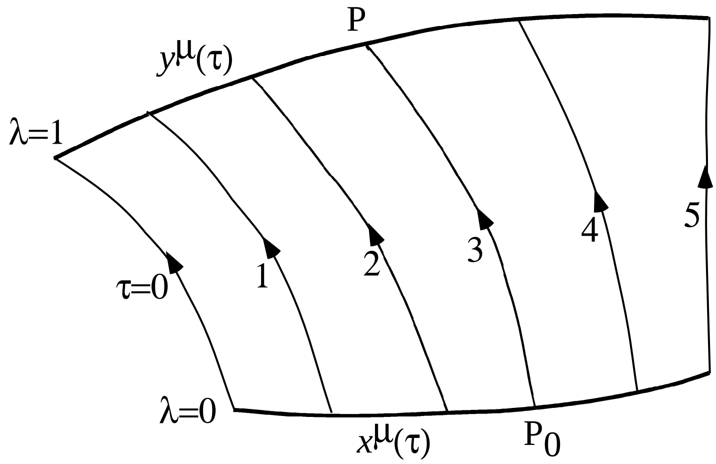

Figure 2: Using the integral curves of a vector field to shift a curve $x^\mu(\tau)$ to a new curve $y^\mu(\tau)$. The shift, known as a pushforward, defines a continuous one-to-one mapping of the space back to itself.

integrating along the vector field $\vec{\xi}(x)$:

$$\frac{\partial x^\mu}{\partial \lambda} = \xi^\mu(x) \ , \quad x^\mu(\lambda = 0, \tau) \equiv x^\mu(\tau) \ , \quad y^\mu(\tau) \equiv x^\mu(\lambda = 1, \tau) \ . \tag{8}$$

The shift amount $\lambda = 1$ is arbitrary; any shift along the integral curves constitutes a pushforward. The inverse mapping from $y \to x$ is called a pullback.

The pushforward generalizes the simple translations of flat spacetime. A finite translation is built up by a succession of infinitesimal shifts $y^\mu = x^\mu + \xi^\mu d\lambda$. Because the vector field $\vec{\xi}(x)$ is a tangent vector field, the shifted curves are guaranteed to reside in the manifold.

Applying an infinitesimal pushforward yields the action

$$S[x(\tau) + \xi(x(\tau))d\lambda] = \int_{\tau_1}^{\tau_2} L(g_{\mu\nu}(x + \xi d\lambda), A_\mu(x + \xi d\lambda), \dot{x}^\mu + \dot{\xi}^\mu d\lambda)\, d\tau \ . \tag{9}$$

This is similar to the usual variation $x^\mu \to x^\mu + \delta x^\mu$ used in deriving the Euler-Lagrange equations, except that ξ is a field defined everywhere in space (not just on the trajectory) and we do not require $\xi = 0$ at the endpoints. Our goal here is not to find a trajectory that makes the action stationary; rather it is to identify symmetries of the action that result in conservation laws.

We will ask whether applying a pushforward to one solution of the Euler-Lagrange equations leaves the action invariant. If so, there is a dynamical symmetry and we will obtain a conservation law. Note that our shifts are more general than the uniform translations and rotations considered in nonrelativistic mechanics and special relativity (here the shifts can vary arbitrarily from point to point, so long as the transformation has an inverse), so we expect to find more general conservation laws.

On the face of it, any pushforward changes the action:

$$S[x(\tau)+\xi(x(\tau))d\lambda] = S[x(\tau)] + d\lambda \int_{\tau_1}^{\tau_2} \left[\frac{\partial L}{\partial g_{\mu\nu}}(\partial_\alpha g_{\mu\nu})\xi^\alpha + \frac{\partial L}{\partial A_\mu}(\partial_\alpha A_\mu)\xi^\alpha + \frac{\partial L}{\partial \dot{x}^\mu}\frac{d\xi^\mu}{d\tau} \right] d\tau \ . \tag{10}$$

It is far from obvious that the term in brackets ever would vanish. However, we have one more tool to use at our disposal: coordinate transformations. Because the Lagrangian is a scalar, we are free to transform coordinates. In some circumstances the effect of the pushforward may be eliminated by an appropriate coordinate transformation, revealing a symmetry.

We consider transformations of the coordinates $x^\mu \to \bar{x}^\mu(x)$, where we assume this mapping is smooth and one-to-one so that $\partial\bar{x}^\mu/\partial x^\alpha$ is nonzero and nonsingular everywhere. A trajectory $x^\mu(\tau)$ in the old coordinates becomes $\bar{x}^\mu(x(\tau)) \equiv \bar{x}^\mu(\tau)$ in the new ones, where τ labels a fixed point on the trajectory independently of the coordinates.

The action depends on the metric tensor, one-form potential and velocity components, which under a coordinate transformation change to

$$g_{\bar{\mu}\bar{\nu}} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} \ , \quad A_{\bar{\mu}} = A_\alpha \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \ , \quad \frac{d\bar{x}^\mu}{d\tau} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} \ . \tag{11}$$

We have assumed that $\partial\bar{x}^\mu/\partial x^\alpha$ is invertible. Under coordinate transformations the action does not even change form (only the coordinate labels change), so coordinate transformations alone cannot generate any nondynamical symmetries. However, we will show below that coordinate invariance can generate *dynamical* symmetries which apply only to solutions of the Euler-Lagrange equations.

Under a pushforward, the trajectory $x^\mu(\tau)$ is shifted to a different trajectory with coordinates $y^\mu(\tau)$. After the pushforward, we transform the coordinates to $\bar{x}^\mu(y(\tau))$. Because the pushforward is a one-to-one mapping of the manifold to itself, we are free to choose our coordinate transformation so that $\bar{x} = x$, i.e. $\bar{x}^\mu(y(\tau)) \equiv \bar{x}^\mu(\tau) = x^\mu(\tau)$. In other words, we transform the coordinates so that the new coordinates of the new trajectory are the same as the old coordinates of the old trajectory. The pushforward changes the trajectory; the coordinate transformation covers our tracks.

The combination of pushforward and coordinate transformation is an example of a **diffeomorphism**. A diffeomorphism is a one-to-one mapping between the manifold and itself. In our case, the pushforward and transformation depend on one parameter λ and we have a one-parameter family of diffeomorphisms. After a diffeomorphism, the point P in Figure 2 has the same values of the transformed coordinates as the point P_0 has in the original coordinates: $\bar{x}^\mu(\lambda,\tau) = x^\mu(\tau)$.

Naively, it would seem that a diffeomorphism automatically leaves the action unchanged because the coordinates of the trajectory are unchanged. However, the Lagrangian depends not only on the coordinates of the trajectory; it also depends on tensor components that change according to equation (11). More work will be required before we can tell whether the action is invariant under a diffeomorphism. While a coordinate transformation by itself does not change the action, in general a diffeomorphism, because it involves a pushforward, does. A continuous symmetry occurs when a diffeomorphism does *not* change the action. This is the symmetry we will be studying.

The diffeomorphism is an important operation in general relativity. We therefore digress to consider the diffeomorphism in greater detail before returning to examine its effect on the action.

3.1 Infinitesimal Diffeomorphisms and Lie derivatives

In a diffeomorphism, we shift the point at which a tensor is evaluated by pushing it forward using a vector field and then we transform (pull back) the coordinates so that the shifted point has the same coordinate labels as the old point. Since a diffeomorphism maps a manifold back to itself, under a diffeomorphism a rank (m, n) tensor is mapped to another rank (m, n) tensor. This subsection asks how tensors change under diffeomorphisms.

The pushforward mapping may be symbolically denoted ϕ_λ (following Wald 1984, Appendix C). Thus, a diffeomorphism maps a tensor $\mathsf{T}(P_0)$ at point P_0 to a tensor $\bar{\mathsf{T}}(P) \equiv \phi_\lambda \mathsf{T}(P_0)$ such that the coordinate values are unchanged: $\bar{x}^\mu(P) = x^\mu(P_0)$. (See Fig. 2 for the roles of the points P_0 and P.) The diffeomorphism may be regarded as an active coordinate transformation: under a diffeomorphism the spatial point is changed but the coordinates are not.

We illustrate the diffeomorphism by applying it to the components of the one-form $\tilde{A} = A_\mu \tilde{e}^\mu$ in a coordinate basis:

$$\bar{A}_\mu(P_0) \equiv A_\alpha(P) \frac{\partial x^\alpha}{\partial \bar{x}^\mu}(P) \ , \quad \text{where} \ \ \bar{x}^\mu(P) = x^\mu(P_0) \ . \tag{12}$$

Starting with A_α at point P_0 with coordinates $x^\mu(P_0)$, we push the coordinates forward to point P, we evaluate A_α there, and then we transform the basis back to the coordinate basis at P with new coordinates $\bar{x}^\mu(P)$.

The diffeomorphism is a continuous, one-parameter family of mappings. Thus, a general diffeomorphism may be obtained from the infinitesimal diffeomorphism with pushforward $y^\mu = x^\mu + \xi^\mu d\lambda$. The corresponding coordinate transformation is (to first order in $d\lambda$)

$$\bar{x}^\mu = x^\mu - \xi^\mu d\lambda \tag{13}$$

so that $\bar{x}^\mu(P) = x^\mu(P_0)$. This yields (in the x^μ coordinate system)

$$\bar{A}_\mu(x) \equiv A_\alpha(x + \xi d\lambda)\frac{\partial x^\alpha}{\partial \bar{x}^\mu} = A_\mu(x) + [\xi^\alpha \partial_\alpha A_\mu(x) + A_\alpha(x)\partial_\mu \xi^\alpha]\, d\lambda + O(d\lambda)^2 \ . \quad (14)$$

We have inverted the Jacobian $\partial\bar{x}^\mu/\partial x^\alpha = \delta^\mu{}_\alpha - \partial_\alpha \xi^\mu d\lambda$ to first order in $d\lambda$, $\partial x^\alpha/\partial\bar{x}^\mu = \delta^\alpha{}_\mu + \partial_\mu \xi^\alpha d\lambda + O(d\lambda)^2$. In a similar manner, the infinitesimal diffeomorphism of the metric gives

$$\begin{aligned} \bar{g}_{\mu\nu}(x) &\equiv g_{\alpha\beta}(x + \xi d\lambda)\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} \\ &= g_{\mu\nu}(x) + [\xi^\alpha \partial_\alpha g_{\mu\nu}(x) + g_{\alpha\nu}(x)\partial_\mu \xi^\alpha + g_{\mu\alpha}(x)\partial_\nu \xi^\alpha]\, d\lambda + O(d\lambda)^2 \ . \end{aligned} \quad (15)$$

In general, the infinitesimal diffeomorphism $\bar{\mathsf{T}} \equiv \phi_{\Delta\lambda}\mathsf{T}$ changes the tensor by an amount first-order in $\Delta\lambda$ and linear in $\vec{\xi}$. This change allows us to define a linear operator called the **Lie derivative**:

$$\mathcal{L}_\xi \mathsf{T} \equiv \lim_{\Delta\lambda\to 0} \frac{\phi_{\Delta\lambda}\mathsf{T}(x) - \mathsf{T}(x)}{\Delta\lambda} \quad \text{with } \bar{x}^\mu(P) = x^\mu(P_0) = x^\mu(P) - \xi^\mu \Delta\lambda + O(\Delta\lambda)^2 \ . \quad (16)$$

The Lie derivatives of $A_\mu(x)$ and $g_{\mu\nu}(x)$ follow from equations (14)–(16):

$$\mathcal{L}_\xi A_\mu(x) = \xi^\alpha \partial_\alpha A_\mu + A_\alpha \partial_\mu \xi^\alpha \ , \quad \mathcal{L}_\xi g_{\mu\nu}(x) = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu \xi^\alpha + g_{\mu\alpha}\partial_\nu \xi^\alpha \ . \quad (17)$$

The first term of the Lie derivative, $\xi^\alpha\partial_\alpha$, corresponds to the pushforward, shifting a tensor to another point in the manifold. The remaining terms arise from the coordinate transformation back to the original coordinate values. As we will show in the next subsection, this combination of terms makes the Lie derivative a tensor in the tangent space at x^μ.

Under a diffeomorphism the transformed tensor components, regarded as functions of coordinates, are evaluated at exactly the same numerical values of the transformed coordinate fields (but a different point in spacetime!) as the original tensor components in the original coordinates. This point is fundamental to the diffeomorphism and therefore to the Lie derivative, and distinguishes the latter from a directional derivative. Thinking of the tensor components as a set of functions of coordinates, we are performing an active transformation: the tensor component functions are changed but they are evaluated at the original values of the coordinates. The Lie derivative generates an infinitesimal diffeomorphism. That is, under a diffeomorphism with pushforward $x^\mu \to x^\mu + \xi^\mu d\lambda$, any tensor T is transformed to $\mathsf{T} + \mathcal{L}_\xi \mathsf{T} d\lambda$.

The fact that the coordinate values do not change, while the tensor fields do, distinguishes the diffeomorphism from a simple coordinate transformation. An important implication is that, in integrals over spacetime volume, the volume element d^4x does not change under a diffeomorphism, while it does change under a coordinate transformation. By contrast, the volume element $\sqrt{-g}\, d^4x$ is invariant under a coordinate transformation but not under a diffeomorphism.

3.2 Properties of the Lie Derivative

The Lie derivative $\mathcal{L}_\xi$ is similar to the directional derivative operator ∇_ξ in its properties but not in its value, except for a scalar where $\mathcal{L}_\xi f = \nabla_\xi f = \xi^\mu \partial_\mu f$. The Lie derivative of a tensor is a tensor of the same rank. To show that it is a tensor, we rewrite the partial derivatives in equation (17) in terms of covariant derivatives in a coordinate basis using the Christoffel connection coefficients to obtain

$$\begin{aligned}\mathcal{L}_\xi A_\mu &= \xi^\alpha \nabla_\alpha A_\mu + A_\alpha \nabla_\mu \xi^\alpha + T^\alpha{}_{\mu\beta} A_\alpha \xi^\beta \ , \\ \mathcal{L}_\xi g_{\mu\nu} &= \xi^\alpha \nabla_\alpha g_{\mu\nu} + g_{\alpha\nu} \nabla_\mu \xi^\alpha + g_{\mu\alpha} \nabla_\nu \xi^\alpha + T^\alpha{}_{\mu\beta} g_{\alpha\nu} \xi^\beta + T^\alpha{}_{\nu\beta} g_{\mu\alpha} \xi^\beta \ ,\end{aligned} \tag{18}$$

where $T^\alpha{}_{\mu\beta}$ is the torsion tensor, defined by $T^\alpha{}_{\mu\beta} = \Gamma^\alpha{}_{\mu\beta} - \Gamma^\alpha{}_{\beta\mu}$ in a coordinate basis. The torsion vanishes by assumption in general relativity. Equations (18) show that $\mathcal{L}_\xi A_\mu$ and $\mathcal{L}_\xi g_{\mu\nu}$ are tensors.

The Lie derivative $\mathcal{L}_\xi$ differs from the directional derivative ∇_ξ in two ways. First, the Lie derivative requires no connection: equation (17) gave the Lie derivative solely in terms of partial derivatives of tensor components. [The derivatives of the metric should not be regarded here as arising from the connection; the Lie derivative of any rank $(0,2)$ tensor has the same form as $\mathcal{L}_\xi g_{\mu\nu}$ in eq. 17.] Second, the Lie derivative involves the derivatives of the vector field $\vec{\xi}$ while the covariant derivative does not. The Lie derivative trades partial derivatives of the metric (present in the connection for the covariant derivative) for partial derivatives of the vector field. The directional derivative tells how a fixed tensor field changes as one moves through it in direction $\vec{\xi}$. The Lie derivative tells how a tensor field changes as it is pushed forward along the integral curves of $\vec{\xi}$.

More understanding of the Lie derivative comes from examining the first-order change in a vector expanded in a coordinate basis under a displacement $\vec{\xi} d\lambda$:

$$d\vec{A} = \vec{A}(x + \xi d\lambda) - \vec{A}(x) = A^\mu(x + \xi d\lambda) \vec{e}_\mu(x + \xi d\lambda) - A^\mu(x) \vec{e}_\mu(x) \ . \tag{19}$$

The nature of the derivative depends on how we obtain $\vec{e}_\mu(x + \xi d\lambda)$ from $\vec{e}_\mu(x)$. For the directional derivative ∇_ξ, the basis vectors at different points are related by the connection:

$$\vec{e}_\mu(x + \xi\lambda) = \left(\delta^\beta{}_\mu + d\lambda\, \xi^\alpha \Gamma^\beta{}_{\mu\alpha}\right) \vec{e}_\beta(x) \quad \text{for } \nabla_\xi \ . \tag{20}$$

For the Lie derivative $\mathcal{L}_\xi$, the basis vector is mapped back to the starting point with

$$\vec{e}_\mu(x + \xi d\lambda) = \frac{\partial \bar{x}^\beta}{\partial x^\mu} \vec{e}_\beta(x) = \left(\delta^\beta{}_\mu - d\lambda\, \partial_\mu \xi^\beta\right) \vec{e}_\beta(x) \quad \text{for } \mathcal{L}_\xi \ . \tag{21}$$

Similarly, the basis one-form is mapped using

$$\tilde{e}^\mu(x + \xi d\lambda) = \frac{\partial x^\mu}{\partial \bar{x}^\beta} \tilde{e}^\beta(x) = \left(\delta^\mu{}_\beta + d\lambda\, \partial_\beta \xi^\mu\right) \tilde{e}^\beta(x) \quad \text{for } \mathcal{L}_\xi \ . \tag{22}$$

These mappings ensure that $d\vec{A}/d\lambda = \mathcal{L}_\xi \vec{A}$ is a tangent vector on the manifold.

The Lie derivative of any tensor may be obtained using the following rules: (1) The Lie derivative of a scalar field is the directional derivative, $\mathcal{L}_\xi f = \xi^\alpha \partial_\alpha f = \nabla_\xi f$. (2) The Lie derivative obeys the Liebnitz rule, $\mathcal{L}_\xi(TU) = (\mathcal{L}_\xi T)U + T(\mathcal{L}_\xi U)$, where T and U may be tensors of any rank, with a tensor product or contraction between them. The Lie derivative commutes with contractions. (3) The Lie derivatives of the basis vectors are $\mathcal{L}_\xi \vec{e}_\mu = -\vec{e}_\alpha \partial_\mu \xi^\alpha$. (4) The Lie derivatives of the basis one-forms are $\mathcal{L}_\xi \tilde{e}^\mu = \tilde{e}^\alpha \partial_\alpha \xi^\mu$.

These rules ensure that the Lie derivative of a tensor is a tensor. Using them, the Lie derivative of any tensor may be obtained by expanding the tensor in a basis, e.g. for a rank $(1,2)$ tensor,

$$\begin{aligned} \mathcal{L}_\xi \mathsf{S} &= \mathcal{L}_\xi (S^\mu{}_{\nu\kappa} \vec{e}_\mu \otimes \tilde{e}^\nu \otimes \tilde{e}^\kappa) \equiv (\mathcal{L}_\xi S^\mu{}_{\nu\kappa})\, \vec{e}_\mu \otimes \tilde{e}^\nu \otimes \tilde{e}^\kappa \\ &= [\xi^\alpha \partial_\alpha S^\mu{}_{\nu\kappa} - S^\alpha{}_{\nu\kappa} \partial_\alpha \xi^\mu + S^\mu{}_{\alpha\kappa} \partial_\nu \xi^\alpha + S^\mu{}_{\nu\alpha} \partial_\kappa \xi^\alpha]\, \vec{e}_\mu \otimes \tilde{e}^\nu \otimes \tilde{e}^\kappa \ . \end{aligned} \tag{23}$$

The partial derivatives can be changed to covariant derivatives without change (with vanishing torsion, the connection coefficients so introduced will cancel each other), confirming that the Lie derivative of a tensor really is a tensor.

The Lie derivative of a vector field is an antisymmetric object known also as the commutator or Lie bracket:

$$\mathcal{L}_V \vec{U} = (V^\mu \partial_\mu U^\nu - U^\mu \partial_\mu V^\nu) \vec{e}_\nu \equiv [\vec{V}, \vec{U}] \ . \tag{24}$$

The commutator was introduced in the notes *Tensor Calculus, Part 2*, Section 2.2. With vanishing torsion, $[\vec{V}, \vec{U}] = \nabla_V \vec{U} - \nabla_U \vec{V}$. Using rule (4) of the Lie derivative given after equation (22), it follows at once that the commutator of any pair of coordinate basis vector fields vanishes: $[\vec{e}_\mu, \vec{e}_\nu] = 0$.

3.3 Diffeomorphism-invariance and Killing Vectors

Having defined and investigated the properties of diffeomorphisms and the Lie derivative, we return to the question posed at the beginning of Section 3: How can we tell when the action is translationally invariant? Equation (10) gives the change in the action under a generalized translation or pushforward by the vector field $\vec{\xi}$. However, it is not yet in a form that highlights the key role played by diffeomorphisms.

To uncover the diffeomorphism we must perform the infinitesimal coordinate transformation given by equation (13). To first order in $d\lambda$ this has no effect on the $d\lambda$ term already on the right-hand side of equation (10) but it does add a piece to the unperturbed action. Using equation (11) and the fact that the Lagrangian is a scalar, to $O(d\lambda)$ we obtain

$$\begin{aligned} S[x(\tau)] &= \int_{\tau_1}^{\tau_2} L(g_{\mu\nu}, A_\mu, \dot{x}^\mu)\, d\tau = \int_{\tau_1}^{\tau_2} L\left(g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}, A_\alpha \frac{\partial x^\alpha}{\partial \bar{x}^\mu}, \frac{dx^\alpha}{d\tau} \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\right) d\tau \\ &= S[x(\tau)] + d\lambda \int_{\tau_1}^{\tau_2} \left[\frac{\partial L}{\partial g_{\mu\nu}} (g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\alpha} \partial_\nu \xi^\alpha) + \frac{\partial L}{\partial A_\mu} (A_\alpha \partial_\mu \xi^\alpha) - \frac{\partial L}{\partial \dot{x}^\mu} \frac{d\xi^\mu}{d\tau} \right] d\tau \ . \end{aligned} \tag{25}$$

The integral multiplying $d\lambda$ always has the value zero for any trajectory $x^\mu(\tau)$ and vector field $\vec{\xi}$ because of the coordinate-invariance of the action. However, it is a special kind of zero because, when added to the pushforward term of equation (10), it gives a diffeomorphism:

$$S[x(\tau) + \xi(x(\tau))d\lambda] = S[x(\tau)] + d\lambda \int_{\tau_1}^{\tau_2} \left[\frac{\partial L}{\partial g_{\mu\nu}} \mathcal{L}_\xi g_{\mu\nu} + \frac{\partial L}{\partial A_\mu} \mathcal{L}_\xi A_\mu \right] d\tau \ . \tag{26}$$

If the action contains additional fields, under a diffeomorphism we obtain a Lie derivative term for each field.

Thus, we have answered the question of translation-invariance: the action is translationally invariant if and only if the Lie derivative of each tensor field appearing in the Lagrangian vanishes. The uniform translations of Newtonian mechanics are generalized to diffeomorphisms, which include translations, rotations, boosts, and any continuous, one-to-one mapping of the manifold back to itself.

In Newtonian mechanics, translation-invariance leads to a conserved momentum. What about diffeomorphism-invariance? Does it also lead to a conservation law?

Let us suppose that the original trajectory $x^\mu(\tau)$ satisfies the equations of motion before being pushed forward, i.e. the action, with Lagrangian $L(g_{\mu\nu}(x), A_\mu, \dot{x}^\mu)$, is stationary under first-order variations $x^\mu \to x^\mu + \delta x^\mu(x)$ with fixed endpoints $\delta x^\mu(\tau_1) = \delta x^\mu(\tau_2) = 0$. From equation (26) it follows that the action for the shifted trajectory is also stationary, if and only if $\mathcal{L}_\xi g_{\mu\nu} = 0$ and $\mathcal{L}_\xi A_\mu = 0$. (When the trajectory is varied $x^\mu \to x^\mu + \delta x^\mu$, cross-terms $\xi\delta x$ are regarded as being second-order and are ignored.)

If there exists a vector field $\vec{\xi}$ such that $\mathcal{L}_\xi g_{\mu\nu} = 0$ and $\mathcal{L}_\xi A_\mu = 0$, then we can shift solutions of the equations of motion along $\vec{\xi}(x(\tau))$ and generate new solutions. This is a new continuous symmetry called diffeomorphism-invariance, and it generalizes translational-invariance in Newtonian mechanics and special relativity. The result is a dynamical symmetry, which may be deduced by rewriting equation (26):

$$\begin{aligned}
\lim_{\Delta\lambda \to 0} \frac{S[x(\tau) + \xi(x(\tau))\Delta\lambda] - S[x(\tau)]}{\Delta\lambda} &= \int_{\tau_1}^{\tau_2} \left[\frac{\partial L}{\partial g_{\mu\nu}} \mathcal{L}_\xi g_{\mu\nu} + \frac{\partial L}{\partial A_\mu} \mathcal{L}_\xi A_\mu \right] d\tau \\
&= \int_{\tau_1}^{\tau_2} \left[\frac{\partial L}{\partial x^\alpha} \xi^\alpha + \frac{\partial L}{\partial \dot{x}^\mu} \frac{d\xi^\mu}{d\tau} \right] d\tau \\
&= \int_{\tau_1}^{\tau_2} \left[\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \xi^\mu + \frac{\partial L}{\partial \dot{x}^\mu} \frac{d\xi^\mu}{d\tau} \right] d\tau \\
&= \int_{\tau_1}^{\tau_2} \left[\frac{d}{d\tau} (p_\mu \xi^\mu) \right] d\tau \\
&= [p_\mu \xi^\mu]_{\tau_1}^{\tau_2} \ .
\end{aligned} \tag{27}$$

All of the steps are straightforward aside from the second line. To obtain this we first expanded the Lie derivatives using equation (17). The terms multiplying ξ^α were then combined to give $\partial L/\partial x^\alpha$ (regarding the Lagrangian as a function of x^μ and $\dot{x}^\mu$). For the terms multiplying the gradient $\partial_\mu \xi^\alpha$, we used $d\xi^\mu(x(\tau))/d\tau = \dot{x}^\alpha \partial_\alpha \xi^\mu$ combined with equation (6) to convert partial derivatives of L with respect to the fields $g_{\mu\nu}$ and A_μ to partial derivatives with respect to $\dot{x}^\mu$. (This conversion is dependent on the Lagrangian, of course, but works for any Lagrangian that is a function of $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ and $A_\mu \dot{x}^\mu$.) To obtain the third line we used the assumption that $x^\mu(\tau)$ is a solution of the Euler-Lagrange equations. To obtain the fourth line we used the definition of canonical momentum,

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} \ . \tag{28}$$

For the Lagrangian of equation (6), $p_\mu = g_{\mu\nu}\dot{x}^\nu + qA_\mu$ is not the mechanical momentum (the first term) but also includes a contribution from the electromagnetic field.

Nowhere in equation (27) did we assume that ξ^μ vanishes at the endpoints. The vector field $\vec{\xi}$ is not just a variation used to obtain equations of motion, nor is it a constant; it is an arbitrary small shift.

Theorem: If the Lagrangian is invariant under the diffeomorphism generated by a vector field $\vec{\xi}$, then $\tilde{p}(\vec{\xi}\,) = p_\mu \xi^\mu$ is conserved along curves that extremize the action, i.e. for trajectories obeying the equations of motion.

This result is a generalization of conservation of momentum. The vector field $\vec{\xi}$ may be thought of as the coordinate basis vector field for a cyclic coordinate, i.e. one that does not appear in the Lagrangian. In particular, if $\partial L/\partial x^\alpha = 0$ for a particular coordinate x^α (e.g. $\alpha = 0$), then L is invariant under the diffeomorphism generated by $\vec{e}_\alpha$ so that p_α is conserved.

When gravity is the only force acting on a particle, diffeomorphism-invariance has a purely geometric interpretation in terms of special vector fields known as Killing vectors. Using equation (18) for a manifold with a metric-compatible connection (implying $\nabla_\alpha g_{\mu\nu} = 0$) and vanishing torsion (both of these are true in general relativity), we find that diffeomorphism-invariance implies

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \ . \tag{29}$$

This equation is known as Killing's equation and its solutions are called Killing vector fields, or Killing vectors for short. Thus, our theorem may be restated as follows: If the spacetime has a Killing vector $\vec{\xi}(x)$, then $p_\mu \xi^\mu$ is conserved along any geodesic. A much shorter proof of this theorem follows from $\nabla_V(p_\mu \xi^\mu) = \xi^\mu \nabla_V p_\mu + p_\mu V^\nu \nabla_\nu \xi^\mu$. The first term vanishes by the geodesic equation, while the second term vanishes from Killing's equation with $p^\mu \propto V^\mu$. Despite being longer, however, the proof based on the Lie derivative is valuable because it highlights the role played by a continuous symmetry, diffeomorphism-invariance of the metric.

One is not free to choose Killing vectors; general spacetimes (i.e. ones lacking symmetry) do not have *any* solutions of Killing's equation. As shown in Appendix C.3 of

Wald (1984), a 4-dimensional spacetime has at most 10 Killing vectors. The Minkowski metric has the maximal number, corresponding to the Poincaré group of transformations: three rotations, three boosts, and four translations. Each Killing vector gives a conserved momentum.

The existence of a Killing vector represents a symmetry: the geometry of spacetime as represented by the metric is invariant as one moves in the $\vec{\xi}$-direction. Such a symmetry is known as an isometry. In the perturbation theory view of diffeomorphisms, isometries correspond to perturbations of the coordinates that leave the metric unchanged.

Any vector field can be chosen as one of the coordinate basis fields; the coordinate lines are the integral curves. In Figure 2, the integral curves were parameterized by λ, which becomes the coordinate whose corresponding basis vector is $\vec{e}_\lambda \equiv \vec{\xi}(x)$. For definiteness, let us call this coordinate $\lambda = x^0$. If $\vec{\xi} = \vec{e}_0$ is a Killing vector, then x^0 is a cyclic coordinate and the spacetime is stationary: $\partial_0 g_{\mu\nu} = 0$. In such spacetimes, and only in such spacetimes, p_0 is conserved along geodesics (aside from special cases like the Robertson-Walker spacetimes, where p_0 is conserved for massless but not massive particles because the spacetime is conformally stationary).

Another special feature of spacetimes with Killing vectors is that they have a conserved 4-vector energy-current $S^\nu = \xi_\mu T^{\mu\nu}$. Local stress-energy conservation $\nabla_\mu T^{\mu\nu} = 0$ then implies $\nabla_\nu S^\nu = 0$, which can be integrated over a volume to give the usual form of an integral conservation law. Conversely, spacetimes without Killing vectors do not have an tensor integral energy conservation law, except for spacetimes that are asymptotically flat at infinity. (However, all spacetimes have a conserved energy-momentum pseudotensor, as discussed in the notes *Stress-Energy Pseudotensors and Gravitational Radiation Power.*)

4 Einstein-Hilbert Action for the Metric

We have seen that the action principle is useful not only for concisely expressing the equations of motion; it also enables one to find identities and conservation laws from symmetries of the Lagrangian (invariance of the action under transformations). These methods apply not only to the trajectories of individual particles. They are readily generalized to spacetime fields such as the electromagnetic four-potential A_μ and, most significantly in GR, the metric $g_{\mu\nu}$ itself.

To understand how the action principle works for continuous fields, let us recall how it works for particles. The action is a functional of configuration-space trajectories. Given a set of functions $q^i(t)$, the action assigns a number, the integral of the Lagrangian over the parameter t. For continuous fields the configuration space is a Hilbert space, an infinite-dimensional space of functions. The single parameter t is replaced by the full set of spacetime coordinates. Variation of a configuration-space trajectory, $q^i(t) \to q^i(t) + \delta q^i(t)$, is generalized to variation of the field values at all points of spacetime, e.g.

$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$. In both cases, the Lagrangian is chosen so that the action is stationary for trajectories (or field configurations) that satisfy the desired equations of motion. The action principle concisely specifies those equations of motion and facilitates examination of symmetries and conservation laws.

In general relativity, the metric is the fundamental field characterizing the geometric and gravitational properties of spacetime, and so the action must be a functional of $g_{\mu\nu}(x)$. The standard action for the metric is the Hilbert action,

$$S_{\rm G}[g_{\mu\nu}(x)] = \int \frac{1}{16\pi G}\, g^{\mu\nu} R_{\mu\nu} \sqrt{-g}\, d^4x \ . \tag{30}$$

Here, $g = \det g_{\mu\nu}$ and $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ is the Ricci tensor. The factor $\sqrt{-g}$ makes the volume element invariant so that the action is a scalar (invariant under general coordinate transformations). The Einstein-Hilbert action was first shown by the mathematician David Hilbert to yield the Einstein field equations through a variational principle. Hilbert's paper was submitted five days before Einstein's paper presenting his celebrated field equations, although Hilbert did not have the correct field equations until later (for an interesting discussion of the historical issues see L. Corry et al., Science 278, 1270, 1997).

(The Einstein-Hilbert action is a scalar under general coordinate transformations. As we will show in the notes *Stress-Energy Pseudotensors and Gravitational Radiation Power*, it is possible to choose an action that, while not a scalar under general coordinate transformations, still yields the Einstein field equations. The action considered there differs from the Einstein-Hilbert action by a total derivative term. The only real invariance of the action that is required on physical grounds is local Lorentz invariance.)

In the particle actions considered previously, the Lagrangian depended on the generalized coordinates and their first derivatives with respect to the parameter τ. In a spacetime field theory, the single parameter τ is expanded to the four coordinates x^μ. If it is to be a scalar, the Lagrangian for the spacetime metric cannot depend on the first derivatives $\partial_\alpha g_{\mu\nu}$, because $\nabla_\alpha g_{\mu\nu} = 0$ and the first derivatives can all be transformed to zero at a point. Thus, unless one drops the requirement that the action be a scalar under general coordinate transformations, for gravity one is forced to go to second derivatives of the metric. The Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ is the simplest scalar that can be formed from the second derivatives of the metric. Amazingly, when the action for matter and all non-gravitational fields is added to the simplest possible scalar action for the metric, the least action principle yields the Einstein field equations.

To look for symmetries of the Einstein-Hilbert action, we consider its change under variation of the functions $g_{\mu\nu}(x)$ with fixed boundary hypersurfaces (the generalization of the fixed endpoints for an ordinary Lagrangian). It proves to be simpler to regard the inverse metric components $g^{\mu\nu}$ as the field variables. The action depends explicitly on $g^{\mu\nu}$ and the Christoffel connection coefficients, $\Gamma^{\alpha}{}_{\mu\nu}$, the latter appearing in the Ricci tensor in a coordinate basis:

$$R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}{}_{\mu\nu} - \partial_\mu \Gamma^{\alpha}{}_{\alpha\nu} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta} - \Gamma^{\alpha}{}_{\beta\mu}\Gamma^{\beta}{}_{\alpha\nu} \ . \tag{31}$$

Lengthy algebra shows that first-order variations of $g^{\mu\nu}$ produce the following changes in the quantities appearing in the Einstein-Hilbert action:

$$
\begin{aligned}
\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} = +\frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} \ , \\
\delta\Gamma^{\alpha}{}_{\mu\nu} &= -\frac{1}{2}\left[\nabla_\mu(g_{\nu\lambda}\delta g^{\alpha\lambda}) + \nabla_\nu(g_{\mu\lambda}\delta g^{\alpha\lambda}) - \nabla_\beta(g_{\mu\kappa}g_{\nu\lambda}g^{\alpha\beta}\delta g^{\kappa\lambda})\right] \ , \\
\delta R_{\mu\nu} &= \nabla_\alpha(\delta\Gamma^{\alpha}{}_{\mu\nu}) - \nabla_\mu(\delta\Gamma^{\alpha}{}_{\alpha\nu}) \ , \\
g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\mu\nabla_\nu\left(-\delta g^{\mu\nu} + g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}\right) \ , \\
\delta(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}) &= (G_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})\sqrt{-g} \ , \qquad (32)
\end{aligned}
$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. The covariant derivative ∇_μ appearing in these equations is taken with respect to the zeroth-order metric $g_{\mu\nu}$. Note that, while $\Gamma^{\alpha}{}_{\mu\nu}$ is not a tensor, $\delta\Gamma^{\alpha}{}_{\mu\nu}$ is. Note also that the variations we perform are not necessarily diffeomorphisms (that is, $\delta g_{\mu\nu}$ is not necessarily a Lie derivative), although diffeomorphisms are variations of just the type we are considering (i.e. variations of the tensor component fields for fixed values of their arguments). Equations (32) are straightforward to derive but take several pages of algebra.

Equations (32) give us the change in the gravitational action under variation of the metric:

$$
\begin{aligned}
\delta S_{\mathrm{G}} &\equiv S_{\mathrm{G}}[g^{\mu\nu} + \delta g^{\mu\nu}] - S_{\mathrm{G}}[g^{\mu\nu}] \\
&= \frac{1}{16\pi G}\int (G_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu v^\mu)\sqrt{-g}\, d^4x \ , \quad v^\mu \equiv \nabla_\nu(-\delta g^{\mu\nu} + g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}) \ . \qquad (33)
\end{aligned}
$$

Besides the desired Einstein tensor term, there is a divergence term arising from $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu v^\mu$ which can be integrated using the covariant Gauss' law. This term raises the question of what is fixed in the variation, and what the endpoints of the integration are.

In the action principle for particles (eq. 2), the endpoints of integration are fixed time values, t_1 and t_2. When we integrate over a four-dimensional volume, the endpoints correspond instead to three-dimensional hypersurfaces. The simplest case is when these are hypersurfaces of constant t, in which case the boundary terms are integrals over spatial volume.

In equation (33), the divergence term can be integrated to give the flux of v^μ through the bounding hypersurface. This term involves the derivatives of $\delta g^{\mu\nu}$ normal to the boundary (e.g. the time derivative of $\delta g^{\mu\nu}$, if the endpoints are constant-time hypersurfaces), and is therefore inconvenient because the usual variational principle sets $\delta g^{\mu\nu}$ but not its derivatives to zero at the endpoints. One may either revise the variational principle so that $g^{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$ are independently varied (the Palatini action), or one can add a boundary term to the Einstein-Hilbert action, involving a tensor called the extrinsic curvature, to cancel the $\nabla_\mu v^\mu$ term (Wald, Appendix E.1). In the following we will ignore this term, understanding that it can be eliminated by a more careful treatment.

(The Schrödinger action presented in the later notes *Stress-Energy Pseudotensors and Gravitational Radiation Power* eliminates the $\nabla_\mu v^\mu$ term.)

For convenience below, we introduce a new notation for the integrand of a functional variation, the functional derivative $\delta S/\delta\psi$, defined by

$$\delta S[\psi] \equiv \int \left(\frac{\delta S}{\delta \psi}\right) \delta\psi \sqrt{-g}\, d^4x \ . \tag{34}$$

Here, ψ is any tensor field, e.g. $g^{\mu\nu}$. The functional derivative is strictly defined only when there are no surface terms arising from the variation. Neglecting the surface term in equation (33), we see that $\delta S_{\rm G}/\delta g^{\mu\nu} = (16\pi G)^{-1} G_{\mu\nu}$.

4.1 Stress-Energy Tensor and Einstein Equations

To see how the Einstein equations arise from an action principle, we must add to $S_{\rm G}$ the action for matter, the source of spacetime curvature. Here, "matter" refers to all particles and fields excluding gravity, and specifically includes all the quarks, leptons and gauge bosons in the world (excluding gravitons). At the classical level, one could include electromagnetism and perhaps a simplified model of a fluid. The total action would become a functional of the metric and matter fields. Independent variation of each field yields the equations of motion for that field. Because the metric implicitly appears in the Lagrangian for matter, matter terms will appear in the equation of motion for the metric. This section shows how this all works out for the simplest model for matter, a classical sum of massive particles.

Starting from equation (1), we sum the actions for a discrete set of particles, weighting each by its mass:

$$S_{\rm M} = \sum_a \int -m_a \left(-g_{00} - 2g_{0i}\dot{x}_a^i - g_{ij}\dot{x}_a^i \dot{x}_a^j\right)^{1/2} dt \ . \tag{35}$$

The subscript a labels each particle. We avoid the problem of having no global proper time by parameterizing each particle's trajectory by the coordinate time. Variation of each trajectory, $x_a^i(t) \to x_a^i(t) + \delta x_a^i(t)$ for particle a with $\Delta S_{\rm M} = 0$, yields the geodesic equations of motion.

Now we wish to obtain the equations of motion for the metric itself, which we do by combining the gravitational and matter actions and varying the metric. After a little algebra, equation (33) gives the variation of $S_{\rm G}$; we must add to it the variation of $S_{\rm M}$. Equation (35) gives

$$\delta S_{\rm M} = \int dt \sum_a \frac{1}{2} m_a \frac{V_a^\mu V_a^\nu}{V_a^0} \delta g_{\mu\nu}(x_a^i(t), t) = \int dt \sum_a -\frac{1}{2} m_a \frac{V_{a\mu} V_{a\nu}}{V_a^0} \delta g^{\mu\nu}(x_a^i(t), t) \ . \tag{36}$$

Variation of the metric naturally gives the normalized 4-velocity for each particle, $V_a^\mu = dx^\mu/d\tau_a$ with $V_{a\mu}V_a^\mu = -1$, with a correction factor $1/V_a^0 = d\tau_a/dt$. Now, if we are to combine equations (33) and (36), we must modify the latter to get an integral over 4-volume. This is easily done by inserting a Dirac delta function. The result is

$$\delta S_{\mathrm{M}} = -\int \left[\frac{1}{2}\sum_a \frac{m_a}{\sqrt{-g}}\frac{V_{a\mu}V_{a\nu}}{V_a^0}\delta^3(x^i - x_a^i(t))\right] \delta g^{\mu\nu}(x)\sqrt{-g}\, d^4x \ . \tag{37}$$

The term in brackets may be rewritten in covariant form by inserting an integral over affine parameter with a delta function to cancel it, $\int d\tau_a\, \delta(t - t(\tau_a))(dt/d\tau_a)$. Noting that $V_a^0 = dt/d\tau_a$, we get

$$\delta S_{\mathrm{M}} = -\int \frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu}(x)\sqrt{-g}\, d^4x = +\int \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}(x)\sqrt{-g}\, d^4x \ , \tag{38}$$

where the functional differentiation has naturally produced the stress-energy tensor for a gas of particles,

$$T^{\mu\nu} = 2\frac{\delta S_{\mathrm{M}}}{\delta g_{\mu\nu}} = \sum_a \int d\tau_a \frac{\delta^4(x - x(\tau_a))}{\sqrt{-g}}\, m_a V_a^\mu V_a^\nu \ . \tag{39}$$

Aside from the factor $\sqrt{-g}$ needed to correct the Dirac delta function for non-flat coordinates (because $\sqrt{-g}\, d^4x$ is the invariant volume element), equation (39) agrees exactly with the stress-energy tensor worked out in the 8.962 notes *Number-Flux Vector and Stress-Energy Tensor*.

Equation (38) is a general result, and we take it as the definition of the stress-energy tensor for matter (cf. Appendix E.1 of Wald). Thus, given any action S_{M} for particles or fields (matter), we can vary the coordinates or fields to get the equations of motion and vary the metric to get the stress-energy tensor,

$$T^{\mu\nu} \equiv 2\frac{\delta S_{\mathrm{M}}}{\delta g_{\mu\nu}} \ . \tag{40}$$

Taking the action to be the sum of S_{G} and S_{M}, requiring it to be stationary with respect to variations $\delta g^{\mu\nu}$, now gives the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{41}$$

The pre-factor $(16\pi G)^{-1}$ on S_{G} was chosen to get the correct coefficient in this equation. The matter action is conventionally normalized so that it yields the stress-energy tensor as in equation (38).

4.2 Diffeomorphism Invariance of the Einstein-Hilbert Action

We return to the variation of the Einstein-Hilbert action, equation (33) without the surface term, and consider diffeomorphisms $\delta g^{\mu\nu} = \mathcal{L}_\xi g^{\mu\nu}$:

$$16\pi G\, \delta S_{\mathrm{G}} = \int G_{\mu\nu}(\mathcal{L}_\xi g^{\mu\nu})\sqrt{-g}\, d^4x = -2\int G^{\mu\nu}(\nabla_\mu \xi_\nu)\sqrt{-g}\, d^4x \ . \tag{42}$$

Here, $\vec{\xi}$ is *not* a Killing vector; it is an arbitrary small coordinate displacement. The Lie derivative $\mathcal{L}_\xi g^{\mu\nu}$ has been rewritten in terms of $-\mathcal{L}_\xi g_{\mu\nu}$ using $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$. Note that diffeomorphisms are a class of field variations that correspond to mapping the manifold back to itself. Under a diffeomorphism, the integrand of the Einstein-Hilbert action is varied, including the $\sqrt{-g}$ factor. However, as discussed at the end of §3.1, the volume element d^4x is fixed under a diffeomorphism even though it does change under coordinate transformations. The reason for this is apparent in equation (16): under a diffeomorphism, the coordinate values do not change. The pushforward cancels the transformation. If we simply performed either a passive coordinate transformation or pushforward alone, d^4x would not be invariant. Under a diffeomorphism the variation $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$ is a tensor on the "unperturbed background" spacetime with metric $g_{\mu\nu}$.

We now show that any scalar integral is invariant under a diffeomorphism that vanishes at the endpoints of integration. Consider the integrand of any action integral, $\Psi\sqrt{-g}$, where Ψ is any scalar constructed out of the tensor fields of the problem; e.g. $\Psi = R/(16\pi G)$ for the Hilbert action. From the first of equations (32) and the Lie derivative of the metric,

$$\mathcal{L}_\xi\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu} = (\nabla_\alpha\xi^\alpha)\sqrt{-g}\ . \tag{43}$$

Using the fact that the Lie derivative of a scalar is the directional derivative, we obtain

$$\delta S = \int \mathcal{L}_\xi(\Psi\sqrt{-g})\, d^4x = \int (\xi^\mu\nabla_\mu\Psi + \Psi\nabla_\mu\xi^\mu)\sqrt{-g}\, d^4x = \int \Psi\xi^\mu\, d^3\Sigma_\mu\ . \tag{44}$$

We have used the covariant form of Gauss' law, for which $d^3\Sigma_\mu$ is the covariant hypersurface area element for the oriented boundary of the integrated 4-volume. Physically it represents the difference between the spatial volume integrals at the endpoints of integration in time.

For variations with $\xi^\mu = 0$ on the boundaries, $\delta S = 0$. The reason for this is simple: diffeomorphism corresponds exactly to reparameterizing the manifold by shifting and relabeling the coordinates. Just as the action of equation (1) is invariant under arbitrary reparameterization of the path length with fixed endpoints, a spacetime field action is invariant under reparameterization of the coordinates (with no shift on the boundaries). The diffeomorphism differs from a standard coordinate transformation in that the variation is made so that d^4x is invariant rather than $\sqrt{-g}\, d^4x$, but the result is the same: scalar actions are diffeomorphism-invariant.

In considering diffeomorphisms, we do not assume that $g^{\mu\nu}$ extremizes the action. Thus, using $\delta S_{\mathrm{G}} = 0$ under diffeomorphisms, we will get an identity rather than a conservation law.

Integrating equation (42) by parts using Gauss's law gives

$$8\pi G\, \delta S_{\mathrm{G}} = -\int G^{\mu\nu}\xi_\nu\, d^3\Sigma_\mu + \int \xi_\nu\nabla_\mu G^{\mu\nu}\sqrt{-g}\, d^4x\ . \tag{45}$$

Under reparameterization, the boundary integral vanishes and $\delta S_{\rm G} = 0$ from above, but ξ_ν is arbitrary in the 4-volume integral. Therefore, diffeomorphism-invariance implies

$$\nabla_\mu G^{\mu\nu} = 0 \ . \tag{46}$$

Equation (46) is the famous contracted Bianchi identity. Mathematically, it is an identity akin to equation (4). It may also be regarded as a geometric property of the Riemann tensor arising from the full Bianchi identities,

$$\nabla_\sigma R^\alpha{}_{\beta\mu\nu} + \nabla_\mu R^\alpha{}_{\beta\nu\sigma} + \nabla_\nu R^\alpha{}_{\beta\sigma\mu} = 0 \ . \tag{47}$$

Contracting on α and μ, then multiplying by $g^{\sigma\beta}$ and contracting again gives equation (46). One can also explicitly verify equation (46) using equation (31), noting that $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ and $R^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta}$. Wald gives a shorter and more sophisticated proof in his Section 3.2; an even shorter proof can be given using differential forms (Misner et al chapter 15). Our proof, based on diffeomorphism-invariance, is just as rigorous although quite different in spirit from these geometric approaches.

The next step is to inquire whether diffeomorphism-invariance can be used to obtain true conservation laws and not just offer elegant derivations of identities. Before answering this question, we digress to explore an analogous symmetry in electromagnetism.

4.3 Gauge Invariance in Electromagnetism

Maxwell's equations can be obtained from an action principle by adding two more terms to the total action. In SI units these are

$$S_{\rm EM}[A_\mu, g^{\mu\nu}] = \int -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \sqrt{-g}\, d^4x \ , \quad S_{\rm I}[A_\mu] = \int A_\mu J^\mu \sqrt{-g}\, d^4x \ , \tag{48}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Note that $g^{\mu\nu}$ is present in $S_{\rm EM}$ implicitly through raising indices of $F_{\mu\nu}$, and that the connection coefficients occurring in $\nabla_\mu A_\nu$ are cancelled in $F_{\mu\nu}$. Electromagnetism adds two pieces to the action, $S_{\rm EM}$ for the free field A_μ and $S_{\rm I}$ for its interaction with a source, the 4-current density J^μ. Previously we considered $S_I = \int q A_\mu \dot{x}^\mu \, d\tau$ for a single particle; now we couple the electromagnetic field to the current density produced by many particles.

The action principle says that the action $S_{\rm EM} + S_{\rm I}$ should be stationary with respect to variations δA_μ that vanish on the boundary. Applying this action principle (left as a homework exercise for the student) yields the equations of motion

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu \ . \tag{49}$$

In the language of these notes, the other pair of Maxwell equations, $\nabla_{[\alpha} F_{\mu\nu]} = 0$, arises from a non-dynamical symmetry, the invariance of $S_{\rm EM}[A_\mu]$ under a gauge transformation

$A_\mu \to A_\mu + \nabla_\mu \Phi$. (Expressed using differential forms, $\mathbf{dF} = 0$ because $\mathbf{F} = \mathbf{dA}$ is a closed 2-form. A gauge transformation adds to $\mathbf{F}$ the term $\mathbf{dd}\Phi$, which vanishes for the same reason. See the 8.962 notes *Hamiltonian Dynamics of Particle Motion*.) The source-free Maxwell equations are simple identities in that $\nabla_{[\alpha}F_{\mu\nu]} = 0$ for *any* differentiable A_μ, whether or not it extremizes any action.

If we require the complete action to be gauge-invariant, a new conservation law appears, charge conservation. Under a gauge transformation, the interaction term changes by

$$\begin{aligned} \delta S_{\rm I} &\equiv S_{\rm I}[A_\mu + \nabla_\mu \Phi] - S_{\rm I}[A_\mu] = \int J^\mu (\nabla_\mu \Phi) \sqrt{-g}\, d^4x \\ &= \int \Phi J^\mu \, d^3\Sigma_\mu - \int \Phi (\nabla_\mu J^\mu) \sqrt{-g}\, d^4x \ . \end{aligned} \tag{50}$$

For gauge transformations that vanish on the boundary, gauge-invariance is equivalent to conservation of charge, $\nabla_\mu J^\mu = 0$. This is an example of Noether's theorem: a continuous symmetry generates a conserved current. Gauge invariance is a dynamical symmetry because the action is extremized if and only if J^μ obeys the equations of motion for whatever charges produce the current. (There will be other action terms, such as eq. 35, to give the charges' equations of motion.) Adding a gauge transformation to a solution of the Maxwell equations yields another solution. All solutions necessarily conserve total charge.

Taking a broad view, physicists regard gauge-invariance as a fundamental symmetry of nature, from which charge conservation follows. A similar phenomenon occurs with the gravitational equivalent of gauge invariance, as we discuss next.

4.4 Energy-Momentum Conservation from Gauge Invariance

The example of electromagnetism sheds light on diffeomorphism-invariance in general relativity. We have already seen that every piece of the action is automatically diffeomorphism-invariant because of parameterization-invariance. However, we wish to single out gravity — specifically, the metric $g_{\mu\nu}$ — to impose a symmetry requirement akin to electromagnetic gauge-invariance.

We do this by defining a gauge transformation of the metric as an infinitesimal diffeomorphism,

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{51}$$

where $\xi^\mu = 0$ on the boundary of our volume. (If the manifold is compact, it has a natural boundary; otherwise we integrate over a compact subvolume. See Appendix A of Wald for mathematical rigor.) Gauge-invariance (diffeomorphism-invariance) of the Einstein-Hilbert action leads to a mathematical identity, the twice-contracted Bianchi identity, equation (46). The rest of the action, including all particles and fields, must also be diffeomorphism-invariant. In particular, this means that the matter action must be invariant under the gauge transformation of equation (51). Using equation (38), this requirement leads to a conservation law:

$$\delta S_{\mathrm{M}} = \int T^{\mu\nu}(\nabla_\mu \xi_\nu)\sqrt{-g}\, d^4x = -\int \xi_\nu (\nabla_\mu T^{\mu\nu})\sqrt{-g}\, d^4x = 0 \quad \Rightarrow \quad \nabla_\mu T^{\mu\nu} = 0 \ . \tag{52}$$

In general relativity, total stress-energy conservation is a consequence of gauge-invariance as defined by equation (51). Local energy-momentum conservation therefore follows as an application of Noether's theorem (a continuous symmetry of the action leads to a conserved current) just as electromagnetic gauge invariance implies charge conservation.

There is a further analogy with electromagnetism. Physical observables in general relativity must be gauge-invariant. If we wish to try to deduce physics from the metric or other tensors, we will have to work with gauge-invariant quantities or impose gauge conditions to fix the coordinates and remove the gauge freedom. This issue will arise later in the study of gravitational radiation.

5 An Example of Gauge Invariance and Diffeomorphism Invariance: The Ginzburg-Landau Model

The discussion of gauge invariance in the preceding section is incomplete (although fully correct) because under a diffeomorphism all fields change, not only the metric. Similarly, the matter fields for charged particles also change under an electromagnetic gauge transformation and under the more complicated symmetry transformations of non-Abelian gauge symmetries such as those present in the theories of the electroweak and strong interactions. In order to give a more complete picture of the role of gauge symmetries in both electromagnetism and gravity, we present here the classical field theory for the simplest charged field, a complex scalar field $\phi(x)$ representing spinless particles of charge q and mass m. Although there are no fundamental particles with spin 0 and nonzero electric charge, this example is very important in physics as it describes the effective field theory for superconductivity developed by Ginzburg and Landau.

The Ginzburg-Landau model illustrates the essential features of gauge symmetry arising in the standard model of particle physics and its classical extension to gravity. At the classical level, the Ginzburg-Landau model describes a charged fluid, e.g. a fluid of Cooper pairs (the electron pairs that are responsible for superconductivity). Here we couple the charged fluid to gravity as well as to the electromagnetic field.

The Ginzburg-Landau action is (with a sign difference in the kinetic term compared with quantum field theory textbooks because of our choice of metric signature)

$$S_{\mathrm{GL}}[\phi, A_\mu, g^{\mu\nu}] = \int \left[-\frac{1}{2} g^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) + \frac{1}{2}\mu^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2 \right] \sqrt{-g}\, d^4x \ , \tag{53}$$

where ϕ^* is the complex conjugate of ϕ and

$$D_\mu \equiv \nabla_\mu - iqA_\mu(x) \tag{54}$$

is called the **gauge covariant derivative**. The electromagnetic one-form potential appears so that the action is automatically gauge-invariant. Under an electromagnetic gauge transformation, both the electromagnetic potential and the scalar field change, as follows:

$$A_\mu(x) \to A_\mu(x) + \nabla_\mu\Phi(x) \ , \quad \phi(x) \to e^{iq\Phi(x)}\phi(x) \ , \quad D_\mu\phi \to e^{iq\Phi(x)}D_\mu\phi \ , \tag{55}$$

where $\Phi(x)$ is any real scalar field. We see that $(D^\mu\phi)^*(D_\nu\phi)$ and the Ginzburg-Landau action are gauge-invariant. Thus, an electromagnetic gauge transformation corresponds to an independent change of phase at each point in spacetime, or a local $U(1)$ symmetry.

The gauge covariant derivative automatically couples our charged scalar field to the electromagnetic field so that no explicit interaction term is needed, unlike in equation (48). The first term in the Ginzburg-Landau action is a "kinetic" part that is quadratic in the derivatives of the field. The remaining parts are "potential" terms. The quartic term with coefficient $\lambda/4$ represents the effect of self-interactions that lead to a phenomenon called spontaneous symmetry breaking. Although spontaneous symmetry breaking is of major importance in modern physics, and is an essential feature of the Ginzburg-Landau model, it has no effect on our discussion of symmetries and conservation laws so we ignore it in the following.

The appearance of A_μ in the gauge covariant derivative is reminiscent of the appearance of the connection $\Gamma^\mu_{\alpha\beta}$ in the covariant derivative of general relativity. However, the gravitational connection is absent for derivatives of scalar fields. We will not discuss the field theory of charged vector fields (which represent spin-1 particles in non-Abelian theories) or spinors (spin-1/2 particles).

A complete model includes the actions for gravity and the electromagnetic field in addition to $S_{\rm GL}$: $S[\phi, A_\mu, g^{\mu\nu}] = S_{\rm GL}[\phi, A_\mu, g^{\mu\nu}] + S_{\rm EM}[A_\mu, g^{\mu\nu}] + S_{\rm G}[g^{\mu\nu}]$. According to the action principle, the classical equations of motion follow by requiring the total action to be stationary with respect to small independent variations of $(\phi, A_\mu, g^{\mu\nu})$ at each point in spacetime. Varying the action yields

$$\begin{aligned}
\frac{\delta S}{\delta\phi} &= g^{\mu\nu}D_\mu D_\nu\phi + \left(\mu^2 - \lambda\phi^*\phi\right)\phi \ , \\
\frac{\delta S}{\delta A_\mu} &= -\frac{1}{4\pi}\nabla_\nu F^{\mu\nu} + J^\mu_{\rm GL} \ , \\
\frac{\delta S}{\delta g^{\mu\nu}} &= \frac{1}{16\pi G}G_{\mu\nu} - \frac{1}{2}T^{\rm EM}_{\mu\nu} - \frac{1}{2}T^{\rm GL}_{\mu\nu} \ ,
\end{aligned} \tag{56}$$

where the current and stress-energy tensor of the charged fluid are

$$J^{\rm GL}_\mu \equiv \frac{iq}{2}\left[\phi(D_\mu\phi)^* - \phi^*(D_\mu\phi)\right] \ ,$$

$$T^{\rm GL}_{\mu\nu} \;\equiv\; (D_\mu\phi)^*(D_\nu\phi) + \left[-\frac{1}{2}g^{\alpha\beta}(D_\alpha\phi)^*(D_\beta\phi) + \frac{1}{2}\mu^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2\right] g_{\mu\nu} \; . \quad (57)$$

The expression for the current density is very similar to the probability current density in nonrelativistic quantum mechanics. The expression for the stress-energy tensor seems strange, so let us examine the energy density in locally Minkowski coordinates (where $g_{\mu\nu} = \eta_{\mu\nu}$):

$$\rho_{\rm GL} = T^{\rm GL}_{00} = \frac{1}{2}\left|D_0\phi\right|^2 + \frac{1}{2}\left|D_i\phi\right|^2 - \frac{1}{2}\mu^2\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2 \; . \quad (58)$$

Aside from the electromagnetic contribution to the gauge covariant derivatives and the potential terms involving $\phi^*\phi$, this looks just like the energy density of a field of relativistic harmonic oscillators. (The potential energy is minimized for $|\phi| = \mu/\sqrt{\lambda}$. This is a circle in the complex ϕ plane, leading to spontaneous symmetry breaking as the field acquires a phase. Those with a knowledge of field theory will recognize two modes for small excitations: a massive mode with mass $\sqrt{2}\mu$ and a massless Goldstone mode corresponding to the field circulating along the circle of minima.)

The equations of motion follow immediately from setting the functional derivatives to zero. The equations of motion for $g^{\mu\nu}$ and A_μ are familiar from before; they are simply the Einstein and Maxwell equations with source including the current and stress-energy of the charged fluid. The equation of motion for ϕ is a nonlinear relativistic wave equation. If $A_\mu = 0$, $\mu^2 = -m^2$, $\lambda = 0$, and $g_{\mu\nu} = \eta_{\mu\nu}$ then it reduces to the Klein-Gordon equation, $(\partial_t^2 - \partial^2 + m^2)\phi = 0$ where $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$ is the spatial Laplacian. Our equation of motion for ϕ generalizes the Klein-Gordon equation to include the effects of gravity (through $g^{\mu\nu}$), electromagnetism (through A_μ), and self-interactions (through $\lambda\phi^*\phi$).

Now we can ask about the consequences of gauge invariance. First, the Ginzburg-Landau current and stress-energy tensor are gauge-invariant, as is easily verified using equations (55) and (57). The action is explicitly gauge-invariant. Using equations (56), we can ask about the effect of an infinitesimal gauge transformation, for which $\delta\phi = iq\Phi(x)\phi$, $\delta A_\mu = \nabla_\mu\Phi$, and $\delta g^{\mu\nu} = 0$. The change in the action is

$$\begin{aligned} \delta S &= \int \left[\frac{\delta S}{\delta\phi}(iq\Phi\phi) + \frac{\delta S}{\delta A_\mu}(\nabla_\mu\Phi)\right]\sqrt{-g}\,d^4x \\ &= \int \left[iq\phi\frac{\delta S}{\delta\phi} - \nabla_\mu\left(\frac{\delta S}{\delta A_\mu}\right)\right]\Phi(x)\sqrt{-g}\,d^4x \; , \end{aligned} \quad (59)$$

where we have integrated by parts and dropped a surface term assuming that $\Phi(x)$ vanishes on the boundary. Now, requiring $\delta S = 0$ under a gauge transformation for the total action adds nothing new because we already required $\delta S/\delta\phi = 0$ and $\delta S/\delta A_\mu = 0$. However, we have constructed each piece of the action ($S_{\rm GL}$, $S_{\rm EM}$ and $S_{\rm G}$) to be gauge-invariant. This gives:

$$\begin{aligned}
\delta S_{\rm GL} = 0 \qquad &\Rightarrow \quad iq\phi\frac{\delta S}{\delta\phi} - \nabla_\mu J^\mu_{\rm GL} = 0\ , \\
\delta S_{\rm EM} = 0 \qquad &\Rightarrow \quad -\frac{1}{4\pi}\nabla_\mu\nabla_\nu F^{\mu\nu} = 0\ .
\end{aligned} \tag{60}$$

For $S_{\rm GL}$, gauge invariance implies charge conservation provided that the field ϕ obeys the equation of motion $\delta S/\delta\phi = 0$. For $S_{\rm EM}$, gauge invariance gives a trivial identity because $F^{\mu\nu}$ is antisymmetric.

Similar results occur for diffeomorphism invariance, the gravitational counterpart of gauge invariance. Under an infinitesimal diffeomorphism, $\delta\phi = \mathcal{L}_\xi\phi$, $\delta A_\mu = \mathcal{L}_\xi A_\mu$, and $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$. The change in the action is

$$\begin{aligned}
\delta S &= \int\left[\frac{\delta S}{\delta\phi}\mathcal{L}_\xi\phi + \frac{\delta S}{\delta A_\mu}\mathcal{L}_\xi A_\mu + \frac{\delta S}{\delta g_{\mu\nu}}\mathcal{L}_\xi g_{\mu\nu}\right]\sqrt{-g}\,d^4x \\
&= \int\left[\frac{\delta S}{\delta\phi}\xi^\mu\nabla_\mu\phi + \left(-\frac{1}{4\pi}\nabla_\nu F^{\mu\nu} + J^\mu\right)\mathcal{L}_\xi A_\mu + \right. \\
&\qquad\qquad \left. + \left(-\frac{1}{8\pi G}G^{\mu\nu} + T^{\mu\nu}\right)\nabla_\mu\xi_\nu\right]\sqrt{-g}\,d^4x\ ,
\end{aligned} \tag{61}$$

where $J^\mu = J^\mu_{\rm GL}$ and $T^{\mu\nu} = T^{\mu\nu}_{\rm GL} + T^{\mu\nu}_{\rm EM}$. As above, requiring that the total action be diffeomorphism-invariant adds nothing new. However, we have constructed each piece of the action to be diffeomorphism-invariant, i.e. a scalar under general coordinate transformations. Applying diffeomorphism-invariance to $S_{\rm GL}$ gives a subset of the terms in equation (61),

$$\begin{aligned}
0 &= \int\left[\frac{\delta S}{\delta\phi}\xi^\mu\nabla_\mu\phi + J^\mu\left(\xi^\alpha\nabla_\alpha A_\mu + A_\alpha\nabla_\mu\xi^\alpha\right) + T^{\mu\nu}_{\rm GL}\nabla_\mu\xi_\nu\right]\sqrt{-g}\,d^4x \\
&= \int\left[-\frac{\delta S}{\delta\phi}\nabla_\mu\phi + J^\alpha\nabla_\mu A_\alpha - \nabla_\alpha\left(J^\alpha A_\mu\right) - \nabla^\nu T^{\rm GL}_{\mu\nu}\right]\xi^\mu(x)\sqrt{-g}\,d^4x \\
&= \int\left[-\frac{\delta S}{\delta\phi}\nabla_\mu\phi - (\nabla_\alpha J^\alpha)A_\mu + J^\alpha F_{\mu\alpha} - \nabla^\nu T^{\rm GL}_{\mu\nu}\right]\xi^\mu(x)\sqrt{-g}\,d^4x\ ,
\end{aligned} \tag{62}$$

where we have discarded surface integrals in the second line assuming that $\xi^\mu(x) = 0$ on the boundary.

Equation (62) gives a nice result. First, as always, our continuous symmetry (here, diffeomorphism-invariance) only gives physical results for solutions of the equations of motion. Thus, $\delta S/\delta\phi = \nabla_\alpha J^\alpha = 0$ can be dropped without further consideration. The remaining terms individually need not vanish from the equations of motion. From this we conclude

$$\nabla_\nu T^{\mu\nu}_{\rm GL} = F^{\mu\nu}J^{\rm GL}_\nu\ . \tag{63}$$

This has a simple interpretation: the work done by the electromagnetic field transfers energy-momentum to the charged fluid. Recall that the Lorentz force on a single charge with 4-velocity V^μ is $qF^{\mu\nu}V_\nu$ and that 4-force is the rate of change of 4-momentum. The current qV^μ for a single charge becomes the current density J^μ of a continuous fluid. Thus, equation (63) gives energy conservation for the charged fluid, including the transfer of energy to and from the electromagnetic field.

The reader can show that requiring $\delta S_{\rm EM} = 0$ under an infinitesimal diffeomorphism proceeds in a very similar fashion to equation (62) and yields the result

$$\nabla_\mu T^{\mu\nu}_{\rm EM} = -F^{\mu\nu} J^{\rm GL}_\nu \ . \tag{64}$$

This result gives the energy-momentum transfer from the viewpoint of the electromagnetic field: work done by the field on the fluid removes energy from the field. Combining equations (63) and (64) gives conservation of total stress-energy, $\nabla_\mu T^{\mu\nu} = 0$.

Finally, because $S_{\rm G}$ depends only on $g^{\mu\nu}$ and not on the other fields, diffeomorphism invariance yields the results already obtained in equations (45) and (46).

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Physics 8.962 Spring 1999

Gravitation in the Weak-Field Limit

1 Introduction

In special relativity, electromagnetism is described by a one-form field $A_\mu(x)$ in flat spacetime. Similarly, in the weak-field limit gravitation is described by a symmetric tensor field $h_{\mu\nu}(x)$ in flat spacetime. Pursuing the analogy can lead us to many insights about GR. These notes detail linearized GR, discussing particle motion via Hamiltonian dynamics, the gravitational field equations, the transverse gauge (giving the closest thing to an inertial frame in GR), gauge transformations, motion in accelerated and rotating frames, Mach's principle, and more.

Linear theory is also useful for most practical computations in general relativity. Linear theory suffices for nearly all experimental applications of general relativity performed to date, including the solar system tests (light deflection, perihelion precession, and Shapiro time delay measurements), gravitational lensing, and gravitational wave detection. The Hulse-Taylor binary pulsar offers some tests of gravity beyond lineaer theory (Taylor et al 1992), as do (in principle) cosmological tests of space curvature.

Some of this material is found in Thorne et al (1986) and some in Bertschinger (1996) but much of it is new. The notation differs slightly from chapter 4 of my Les Houches lectures (Bertschinger 1996); in particular, ϕ and ψ are swapped there, and h_{ij} in those notes is denoted s_{ij} here (eq. 11 below).

Throughout this set of notes, the Minkowski metric $\eta_{\mu\nu}$ is used to raise and lower indices. In this set of notes we refer to gravity as a field in flat spacetime as opposed to the manifestation of curvature in spacetime. With one important exception, this pretense can be made to work in the weak-field limit (although it breaks down for strong gravitational fields). As we will see, gravitational radiation can only be understood properly as a traveling wave of space curvature.

2 Particle Motion and Gauge Dependence

We begin by studying an analogue of general relativity, the motion of a charged particle. The covariant action for a particle of mass m and charge q has two terms: one for the free particle and another for its coupling to electromagnetism:

$$S[x^\mu(\tau)] = \int -m\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} d\tau + \int qA_\mu \frac{dx^\mu}{d\tau}\, d\tau \ . \tag{1}$$

Varying the trajectory and requiring it to be stationary, with τ being an affine parameter such that $V^\mu = dx^\mu/d\tau$ is normalized $\eta_{\mu\nu}V^\mu V^\nu = -1$, yields the equation of motion

$$\frac{dV^\mu}{d\tau} = \frac{q}{m}F^\mu{}_\nu V^\nu \ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2}$$

Regarding gravity as a weak (linearized) field on flat spacetime, the action for a particle of mass m also has two terms, one for the free particle and another for its coupling to gravity:

$$S[x^\mu(\tau)] = \int -m\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} d\tau + \int \frac{m}{2}\, h_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{-1/2} d\tau \ . \tag{3}$$

This result comes from using the free-field action with metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and linearizing in the small quantities $h_{\mu\nu}$. Note that for this to be valid, two requirements must be satisfied: First, the curvature scales given by the eigenvalues of the Ricci tensor (which have units of inverse length squared) must be large compared with the length scales under consideration (e.g. one must be far from the Schwarzschild radius of any black holes). Second, the coordinates must be nearly orthonormal. One cannot, for example, use spherical coordinates; Cartesian coordinates are required. (While this second condition can be relaxed, it makes the analysis much simpler. If the first condition holds, then coordinates can always be found such that the second condition holds also.)

Requiring the gravitational action to be stationary yields the equation of motion

$$\frac{dV^\mu}{d\tau} = -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right)V^\alpha V^\beta = -\Gamma^\mu{}_{\alpha\beta}V^\alpha V^\beta \ . \tag{4}$$

The object multiplying the 4-velocities on the right-hand side is just the linearized Christoffel connection (with $\eta^{\mu\nu}$ rather than $g^{\mu\nu}$ used to raise indices).

Equations (2) and (4) are very similar, as are the actions from which they were derived. Both $F_{\mu\nu}$ and $\Gamma^\mu{}_{\alpha\beta}$ are tensors under Lorentz transformations. This fact ensures that equations (2) and (4) hold in any Lorentz frame. Thus, in the weak field limit it is straightforward to analyze arbitrary relativistic motions of the sources and test particles, as long as all the components of the Lorentz-transformed field, $h_{\bar\mu\bar\nu} = \Lambda^\mu{}_{\bar\mu}\Lambda^\nu{}_{\bar\nu}h_{\mu\nu}$ are small compared with unity (otherwise the linear theory assumption breaks down). A simple example of the Lorentz transformation of weak gravitational fields was given in problem 1 of Problem Set 6.

From these considerations one might conclude that regarding linearized gravity as a field in flat spacetime with gravitational field strength tensor ${\Gamma^\mu}_{\alpha\beta}$ presents no difficulties. However, there is a very important difference: the electromagnetic force law is gauge-invariant while the gravitational one is not.

The electromagnetic field strength tensor, hence equation (2), is invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Phi(x) \ . \tag{5}$$

The Christoffel connection is, however, *not* invariant under the gravitational gauge transformation

$$h_{\mu\nu}(x) \to h_{\mu\nu} + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \ . \tag{6}$$

(Note that in both special relativity and linearized GR, $\nabla_\mu = \partial_\mu$.) While $F_{\mu\nu}$ is a tensor under general coordinate transformations, ${\Gamma^\mu}_{\alpha\beta}$ is not. Because the gravitational gauge transformation is simply an infinitesimal coordinate transformation, our putative gravitational field strength tensor is not gauge-invariant. While the form of equation (4) is unchanged by Lorentz transformations, it is not preserved by arbitrary coordinate transformations.

Try to imagine the Lorentz force law if the electromagnetic fields were not gauge-invariant. We would be unable to get a well-defined prediction for the motion of a particle.

The situation in gravity is less bleak because we recognize that the gauge transformation is equivalent to shifting the coordinates, $x^\mu \to x^\mu - \xi^\mu(x)$. If the coordinates are deformed, fictitious forces (like the Coriolis force) are introduced by the change in the Christoffel symbols. But while this perspective is natural in general relativity, it doesn't help one trying to obtain trajectories in the weak-field limit.

Can one ignore the gauge-dependence of ${\Gamma^\mu}_{\alpha\beta}$ by simply regarding $h_{\mu\nu}(x)$ as a given field? Yes, up to a point. However, as we will see later, the gauge-dependence rears its ugly head when one tries to solve the linearized field equations for $h_{\mu\nu}$. The Einstein equations contain extra degrees of freedom arising from the fact that a gauge-transformation of any solution is also a solution. Gravitational fields can mimic fictitious forces. In the full theory of GR this is no problem in principle, because gravity itself is a fictitious force — gravitational deflection arises from the use of curvilinear coordinates. (Of course, in a curved manifold we have no choice!)

Regardless of how we interpret gravity, in practice we must eliminate the gauge freedom somehow. There are two ways to do this: one may form gauge-invariant quantities (e.g. the electromagnetic field strength tensor) or impose gauge conditions that fix the potentials $h_{\mu\nu}$.

It happens that while the Christoffel connection is not gauge-invariant, in linearized gravity (but not in general) the Riemann tensor is gauge-invariant. Thus one way to form gauge-invariant quantities is to replace equation (4) by the geodesic deviation equation,

$$\frac{d^2(\Delta x)^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\nu} V^\alpha V^\beta (\Delta x)^\nu \tag{7}$$

where $(\Delta x)^\nu$ is the infinitesimal separation vector between a pair of geodesics. While this tells us all about the local environment of a freely-falling observer, it fails to tell us where the observer goes. In most applications we need to know the trajectories. Thus we will have to find other strategies for coping with the gauge problem.

3 Hamiltonian Formulation and Gravitomagnetism

Some aid in solving the gauge problem comes if we abandon manifest covariance and use $t = x^0$ to parameterize trajectories instead of the proper time $d\tau$. This yields the added benefit of highlighting the similarities between linearized gravity and electromagnetism. In particular, it illustrates the phenomenon of gravitomagnetism.

Changing the parameterization in equation (1) from τ to t and performing a Legendre transformation gives the Hamiltonian

$$H(x^i, \pi_i, t) = \left(p^2 + m^2\right)^{1/2} + q\phi \ , \quad p^i \equiv \pi_i - qA_i \ , \quad \phi \equiv -A_0 \ . \tag{8}$$

Here we denote the conjugate momentum by π^i to distinguish it from the mechanical momentum p^i. (Note that p^i and π_i are the components of 3-vectors in Euclidean space, so that their indices may be raised or lowered without change.) It is very important to treat the Hamiltonian as a function of the conjugate momentum and not the mechanical momentum, because only in this way do Hamilton's equations give the correct equations of motion:

$$\frac{dx^i}{dt} = \frac{p^i}{E} \equiv v^i \ , \quad \frac{d\pi_i}{dt} = q\left(-\partial_i\phi + v^j\partial_i A_j\right) \ , \quad E \equiv \sqrt{p^2+m^2} = \frac{m}{\sqrt{1-v^2}} \ . \tag{9}$$

Combining these gives the familiar form of the Lorentz force law,

$$\frac{dp^i}{dt} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)_i \ , \quad \underline{E} \equiv -\underline{\nabla}\phi - \partial_t \underline{A} \ , \quad \underline{B} = \underline{\nabla} \times \underline{A} \tag{10}$$

where underscores denote standard 3-vectors in Euclidean space. The dependence of the fields on the potentials ensures that the equation of motion is still invariant under the gauge transformation $\phi \to \phi - \partial_t\Phi$, $\underline{A} \to \underline{A} + \underline{\nabla}\Phi$.

Now we repeat these steps for gravity, starting from equation (3). For convenience, we first decompose $h_{\mu\nu}$ as

$$h_{00} = -2\phi \ , \quad h_{0i} = w_i \ , \quad h_{ij} = -2\psi\delta_{ij} + 2s_{ij} \ , \quad \text{where} \ \ s^j{}_j = \delta^{ij}s_{ij} = 0. \tag{11}$$

The ten degrees of freedom in $h_{\mu\nu}$ are incorporated into two scalars under spatial rotations (ϕ and ψ), one 3-vector, and one symmetric 2-index tensor, the traceless strain s_{ij}. Notice that w_i and s_{ij} generalize the weak-field metric used previously in 8.962.

To first order in $h_{\mu\nu}$, the Hamiltonian may now be written

$$\begin{aligned} &H(x^i,\pi_i,t) = (1+\phi)E \ , \quad E \equiv (\delta^{ij}p_ip_j + m^2)^{1/2} \ , \\ &p_i \equiv (1+\psi)\pi_i - (\delta^{ij}\pi_i\pi_j + m^2)^{1/2}w_i - s^j{}_i\pi_j \ . \end{aligned} \tag{12}$$

Here, π_i is the conjugate momentum while p_i and E are the proper 3-momentum and energy measured by an observer at fixed x^i, just as they are in equation (8). To prove this, we construct an orthonormal basis for such an observer:

$$\vec{e}_{\hat{0}} = \frac{1}{\sqrt{-g_{00}}}\,\vec{e}_0 = (1-\phi)\vec{e}_0 \ , \quad \vec{e}_{\hat{i}} = \vec{e}_i + g_{0i}\vec{e}_0 - \frac{1}{2}h^j{}_i\vec{e}_j = (1+\psi)\vec{e}_i + w_i\vec{e}_0 - s^j{}_i\vec{e}_j \ . \tag{13}$$

This basis is constructed by first setting $\vec{e}_{\hat{0}} \parallel \vec{e}_0$ and normalizing it with $\vec{e}_{\hat{0}}\cdot\vec{e}_{\hat{0}} = -1$. Next, $\vec{e}_{\hat{i}}$ is required to be orthogonal to $\vec{e}_{\hat{0}}$, giving the $g_{0i}\vec{e}_0$ term (to first order in the metric perturbations). Requiring $\vec{e}_{\hat{i}}\cdot\vec{e}_{\hat{j}} = \delta_{ij}$ gives the remaining term. Now, using the results from the notes *Hamiltonian Dynamics of Particle Motion*, the spacetime momentum one-form is $\tilde{P} = -H\tilde{e}^0 + \pi_i\tilde{e}^i$. Setting $E = -\tilde{P}(\vec{e}_{\hat{0}})$ and $p_i = \tilde{P}(\vec{e}_{\hat{i}})$ gives the desired results (to first order in the metric perturbations).

Equation (12) has the simple Newtonian interpretation that the Hamiltonian is the sum of E, the kinetic plus rest mass energy, and $E\phi$, the gravitational potential energy. This result is remarkably similar to equation (8), with just two differences. In place of charge q, the gravitational coupling is through the energy E. Gravitation also has a rank $(0,2)$ spatial tensor h_{ij} in addition to spatial scalar and vector potentials.

Although the gravitational potentials represent physical metric perturbations, having obtained the Hamiltonian we can forget about this for the moment in order to gain intuition about weak-field gravity by applying our understanding of analogous electromagnetic phenomena.

Hamilton's equations applied to equation (12) give

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H}{\partial \pi_i} = (1+\phi+\psi)v^j\left(\delta_{ij} - v_iw_j - s_{ij}\right) \ , \quad v^i \equiv \frac{p^i}{E} \ , \\ \frac{d\pi_i}{dt} &= -\frac{\partial H}{\partial x^i} = E\left[-\partial_i\phi + v^j\partial_iw_j - (\partial_i\psi)v^2 + (\partial_is_{jk})v^jv^k\right] \end{aligned} \tag{14}$$

Let us compare our result with equation (9). The equation for dx^i/dt is more complicated than the corresponding equation for electromagnetism because of the more complicated momentum-dependence of the gravitational "charge" E. Alternatively, one may adopt the curved spacetime perspective and note that dx^i and dt are coordinate differentials and not proper distances or times, so that the coordinate velocity dx^i/dt must be corrected to give the proper 3-velocity v^i measured by an observer at fixed x^i in an orthonormal frame, with the same result. The Newtonian and curved spacetime interpretations are consistent.

Equations (14) may be combined to give the weak-field gravitational force law

$$
\begin{aligned}
\frac{dp^i}{dt} &= E\left(\underline{g} + \underline{v} \times \underline{H}\right)_i + \frac{1}{2}E\left[-v^j\partial_t h_{ij} + v^j v^k(\partial_i h_{jk} - \partial_k h_{ij})\right] \ , \\
\underline{g} &\equiv -\underline{\nabla}\phi - \partial_t \underline{w} \ , \quad \underline{H} = \underline{\nabla} \times \underline{w} \ . \qquad (15)
\end{aligned}
$$

As before, underscores denote standard 3-vectors in Euclidean space. (The terms involving h_{ij} may be expanded by substituting $h_{ij} = -2\psi\delta_{ij} + 2s_{ij}$. No simplification results, so they are left in a more compact form above.) This equation, the gravitational counterpart of the Lorentz force law, is exact for linearized GR (though it is not valid for strong gravitational fields). Combined with the first of equations (14), it is equivalent to the geodesic equation for timelike or null geodesics in a weakly perturbed Minkowski spacetime.

Equation (15) is remarkably similar to the Lorentz force law. It reveals electric-type forces (present for particles at rest) and magnetic-type forces (force perpendicular to velocity). In addition there are velocity-dependent forces arising from the tensor potentials, i.e. from the spatial curvature terms in the metric. The Newtonian limit is obvious when $v \ll 1$. But equation (15) is correct also for relativistic particles and for relativistically moving gravitational sources, as long as the fields are weak, i.e. $|h_{\mu\nu}| \ll 1$.

It is straightforward to check that equation (15) is invariant under a gauge transformation generated by shifting the time coordinate, equation (6) with $\xi^0 = \Phi$ and $\xi^i = 0$. However, the force law is not invariant under gauge (coordinate) transformations generated by ξ^i. Thus, the Hamiltonian formulation has not solved the gauge problem, although it has isolated it. As a result, it has provided important insight into the nature of relativistic gravitation.

The fields $g_i = -\partial_i\phi - \partial_t w_i$ and $H^i = \epsilon^{ijk}\partial_j w_k$ are called the gravitoelectric and gravitomagnetic fields, respectively. (Here, ϵ^{ijk} is the fully antisymmetric three-dimensional Levi-Civita symbol, with $\epsilon^{123} = +1$.) They are invariant under the gauge transformation generated by $\xi^0 = \Phi$ and therefore are not sensitive to how one chooses hypersurfaces of constant time, although they do depend on the parameterization of spatial coordinates within these hypersurfaces. Once those coordinates are fixed, the gravitoelectric and gravitomagnetic fields have a clear meaning given by equation (15). Noting that $\underline{p} = E\underline{v}$, these fields contribute to the acceleration $d\underline{v}/dt = \underline{g} + \underline{v} \times \underline{H}$.

There are four distinct gravitational phenomena present in equation (15). They are

- The quasi-Newtonian gravitational field $\underline{g}$.

- The gravitomagnetic field $\underline{H}$, which is responsible for Lense-Thirring precession and the dragging of inertial frames.

- The scalar part of h_{ij}, i.e. $h_{ij} = -2\psi\delta_{ij}$, which (for $\psi = \phi$) doubles the deflection of light by the sun compared with the simple Newtonian calculation.

- The transverse-traceless part of h_{ij}, or gravitational radiation, described by the transverse-traceless strain matrix s_{ij}.

The rest of these notes will explore these phenomena in greater detail.

4 Field Equations

Greater understanding of the physics of weak-field gravitation comes from examining the Einstein equations and comparing them with the Maxwell equations. This will allow us to solve the gauge problem and thereby to explore the phenomena mentioned above with confidence that we are not being misled by coordinate artifacts.

Starting from equation (11), we obtain the linearized Christoffel symbols

$$\begin{aligned}
&\Gamma^0{}_{00} = \partial_t\phi \ , \quad \Gamma^0{}_{i0} = \partial_i\phi \ , \quad \Gamma^0{}_{ij} = -\partial_{(i}w_{j)} + \partial_t(s_{ij} - \delta_{ij}\psi) \ , \\
&\Gamma^i{}_{00} = \partial_i\phi + \partial_t w_i \ , \quad \Gamma^j{}_{i0} = \partial_{[i}w_{j]} + \partial_t(s_{ij} - \delta_{ij}\psi) \ , \\
&\Gamma^k{}_{ij} = \delta_{ij}\partial_k\psi - 2\delta_{k(i}\partial_{j)}\psi - \partial_k s_{ij} + 2\partial_{(i}s_{j)k} \ .
\end{aligned} \tag{16}$$

(Notice that the Kronecker delta is used to raise and lower spatial components.) The Ricci tensor has components

$$\begin{aligned}
R_{00} &= \partial^2\phi + \partial_t(\partial_i w^i) + 3\partial_t^2\psi \ , \\
R_{0i} &= -\frac{1}{2}\partial^2 w_i + \frac{1}{2}\partial_i(\partial_j w^j) + 2\partial_t\partial_i\psi + \partial_t\partial_j s^j{}_i \ , \\
R_{ij} &= -\partial_i\partial_j(\phi - \psi) - \partial_t\partial_{(i}w_{j)} + (\partial_t^2 - \partial^2)(s_{ij} - \psi\delta_{ij}) + 2\partial_k\partial_{(i}s_{j)}{}^k
\end{aligned} \tag{17}$$

where $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$. The Einstein tensor components are

$$\begin{aligned}
G_{00} &= 2\partial^2\psi + \partial_i\partial_j s^{ij} \ , \\
G_{0i} &= -\frac{1}{2}\partial^2 w_i + \frac{1}{2}\partial_i(\partial_j w^j) + 2\partial_t\partial_i\psi + \partial_t\partial_j s^j{}_i \ , \\
G_{ij} &= (\delta_{ij}\partial^2 - \partial_i\partial_j)(\phi - \psi) + \partial_t\left[\delta_{ij}(\partial_k w^k) - \partial_{(i}w_{j)}\right] + 2\delta_{ij}(\partial_t^2\psi) \\
&\quad + (\partial_t^2 - \partial^2)s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}(\partial_k\partial_l s^{kl}) \ .
\end{aligned} \tag{18}$$

It is fascinating that the time-time part of the Einstein tensor contains only the spatial parts of the metric, and $h_{00} = -2\phi$ appears only in G_{ij}. Although the equation of motion for nonrelativistic particles in the Newtonian limit is dependent only on h_{00} (through $\Gamma^i{}_{00}$), the Newtonian gravitational field equation (the Poisson equation) is sensitive only to h_{ij}! I do not know if this is a merely a coincidence; it is not true for the fully nonlinear Einstein equations.

It is also fascinating that G_{00} contains no time derivatives and G_{0i} contains only first time derivatives. If the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ are to provide evolution equations for the metric, we would have expected a total of ten independent second-order in time equations, one for each component of $g_{\mu\nu}$. (After all, typical mechanical systems have, from the Euler-Lagrange equations, second-order time evolution equations for each generalized coordinate.) What is going on?

A clue comes from similar behavior of the Maxwell equations:

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu \ , \quad \partial_{[\kappa} F_{\mu\nu]} = 0 \ . \tag{19}$$

The substitution $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ automatically satisfies the source-free Maxwell equations and gives

$$\partial^2 A^0 + \partial_t(\partial_j A^j) = -4\pi J^0 \ , \quad \partial_t(\partial_i A^0 + \partial_t A^i) + \partial_i(\partial_j A^j) - \partial^2 A^i = 4\pi J^i \ , \tag{20}$$

where once again $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$. Only the spatial parts of the Maxwell equations provide second-order time evolution equations. Does this mean that A^i evolves dynamically but A^0 does not?

The answer to that question is clearly no, because A_μ is gauge-dependent and one can easily choose a gauge in which $\partial_\nu F^{0\nu}$ contains second time derivatives of A^0. (The Lorentz gauge, with $\partial_\mu A^\mu = 0$, is a well-known example.)

However, there is a sense in which the time part of the Maxwell equations (the first of eqs. 20) is redundant and therefore need not provide an equation of motion for the field. As the reader may easily verify, the time derivative of this equation, when subtracted from the spatial divergence of the spatial equations (the second of eqs. 20), enforces charge conservation, $\partial_\mu J^\mu = 0$. (We are working in flat spacetime so there is no need for the covariant derivative symbol.) This is another way of expressing the statement that gauge-invariance implies charge conservation. We are perfectly at liberty to choose a gauge such that $\partial_j A^j = 0$ (the Coulomb or transverse gauge), in which case only A^i need be solved for by integrating a time evolution equation. Coulomb's law, $\partial^2 A^0 = -4\pi J^0$, may be regarded as a constraint equation to ensure conservation of charge.

Similarly, general relativity has a conservation law following from gauge-invariance: $\partial_\mu T^{\mu\nu} = 0$. Now there are four conserved quantities, the energy and momentum. (In the weak-field limit, but not in general, $T^{\mu\nu}$ can be integrated over volume to obtain a globally conserved energy and momentum.) The reader can easily verify the redundancy in equations (18): $\partial_t G^{00} + \partial_i G^{0i} = 0$, $\partial_t G^{0i} + \partial_j G^{ij} = 0$. Thus, if the matter evolves so as to conserve stress-energy $T^{\mu\nu}$, then the G_{00} and G_{0i} Einstein equations are redundant. They are present in order to enforce stress-energy conservation. In the literature they are known as the (linearized) Arnowitt-Deser-Misner (ADM) energy and momentum constraints (Arnowitt et al. 1962).

The Ricci and Einstein tensors are invariant (in linearized theory) under gauge transformations (eq. 6). This follows from the fact that a gauge transformation is a diffeomorphism and changes each tensor by the addition of a Lie derivative term: $R_{\mu\nu} \to R_{\mu\nu} + \mathcal{L}_{\vec{\xi}} R_{\mu\nu}$ and similarly for $G_{\mu\nu}$. The Lie derivative is first-order in both the shift vector $\vec{\xi}$ and the Ricci tensor, and therefore vanishes in linear theory. Put another way, because the Ricci tensor vanishes for the flat background spacetime, its Lie derivative vanishes.

Although $R_{\mu\nu}$ and $G_{\mu\nu}$ are gauge-invariant, their particular forms in equations (17) and (18) are not, because of the appearance of the metric perturbations $(\phi, w_i, \psi, s_{ij})$. Part of this dependence, in G_{0i} and G_{ij}, can be eliminated by using the gravitoelectromagnetic fields, giving

$$\begin{aligned} G_{0i} &= \frac{1}{2}(\underline{\nabla} \times \underline{H})_i + 2\partial_t\partial_i\psi + \partial_t\partial_j s^j{}_i \ , \\ G_{ij} &= \partial_{(i}g_{j)} - \delta_{ij}(\partial_k g^k) + (\partial_i\partial_j - \delta_{ij}\partial^2)\psi + 2\delta_{ij}(\partial_t^2\psi) \\ &\quad + (\partial_t^2 - \partial^2)s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}(\partial_k\partial_l s^{kl}) \ . \end{aligned} \tag{21}$$

Note that the potentials ϕ and w_i (from h_{00} and h_{0i}) enter into both the equations of motion and the Einstein equations only through the fields g_i and H_i, giving strong support to the interpretation of $\underline{g}$ and $\underline{H}$ as physical fields for linearized GR. But what of ψ and s_{ij}? We explore this question in the next section.

5 Gauge-fixing: Transverse Gauge

Up to this point, we have imposed no gauge conditions at all on the metric tensor potentials. However, we have four coordinate variations at our disposal. Under the gauge transformation (6), the potentials change by

$$\delta\phi = \partial_t\xi^0 \ , \quad \delta w_i = -\partial_i\xi^0 + \partial_t\xi^i \ , \quad \delta\psi = -\frac{1}{3}\partial_i\xi^i \ , \quad \delta s_{ij} = \partial_{(i}\xi_{j)} - \frac{1}{3}\delta_{ij}(\partial_k\xi^k) \ . \tag{22}$$

Examing equations (21), it is clear that substantial simplification would result if could choose a gauge such that

$$\partial_j s^j{}_i = 0 \ . \tag{23}$$

Indeed, this is possible, by gauge-transforming any s_{ij} which does not obey this condition using the spatial shift vector ξ^i obtained by solving $\partial_j(s^j{}_i + \delta s^j{}_i) = 0$, or

$$\partial^2\xi^i + \frac{1}{3}\partial_i(\partial_j\xi^j) = -2\partial_j s^j{}_i \ . \tag{24}$$

This is an elliptic equation which may be solved by decomposing ξ^i into longitudinal (curl-free) and transverse (divergence-free) parts. Solutions to this equation always exist; indeed, suitable boundary conditions must be specified in order to yield a unique solution. In Section 8 we will discuss the physical meaning of the extra solutions.

Equation (23) is called the transverse-traceless gauge condition. It is widely used when studying gravitational radiation, but we will see that it is also useful for other applications.

Similarly, although we have hidden the vector potential w_i in the gravitoelectromagnetic fields, the gauge may be fixed by requiring it to be transverse:

$$\partial_i w^i = 0 \ . \tag{25}$$

(The equations of motion depend only on $\underline{\nabla} \times \underline{w}$, so we expect to lose no physics by setting the longitudinal part to zero.) To convert a coordinate system that does not satisfy equation (25) to one that does, one solves the following elliptic equation for ξ^0:

$$\partial^2 \xi^0 - \partial_t(\partial_i \xi^i) = \partial_i w^i \ . \tag{26}$$

Once again, this equation (in combination with eq. 24 for ξ^i) may have multiple solutions depending on boundary conditions. (For given ξ^i, this is simply a Poisson equation for ξ^0.)

The combination of gauge conditions given by equations (23) and (25) imposes four conditions on the coordinates. They generalize the Coulomb gauge conditions of electromagnetism, $\partial_i A^i = 0$. As a result, both w_i and the traceless part of h_{ij} (i.e., s_{ij}) are transverse. The gauge condition on s_{ij} is well-known and is almost always used in studies of gravitational radiation; it reduces the number of degrees of freedom of s_{ij} from five to two, corresponding to the two orthogonal polarizations of gravitational radiation. However, the metric is not fully constrained until a gauge condition is imposed on w_i as well. Equation (25) reduces the number of degrees of freedom of w_i from three to two. The total number of physical degrees of freedom is six: one each for the spatial scalar fields ϕ and ψ, two for the transverse vector field w_i, and two for the transverse-traceless tensor field s_{ij}.

Based on its similarity with the Coulomb gauge of electromagnetism, Bertschinger (1996) dubbed these gauge conditions the Poisson gauge. Here we will call them **transverse gauge**. In transverse gauge, the Einstein equations become

$$\begin{aligned}
G_{00} &= 2\partial^2\psi = 8\pi G T_{00} \ , \\
G_{0i} &= \frac{1}{2}(\underline{\nabla} \times \underline{H})_i + 2\partial_t\partial_i\psi = 8\pi G T_{0i} \ , \\
G_{ij} &= (\delta_{ij}\partial^2 - \partial_i\partial_j)(\phi - \psi) - \partial_t\partial_{(i}w_{j)} + 2\delta_{ij}(\partial_t^2\psi) + (\partial_t^2 - \partial^2)s_{ij} \\
&= \partial_{(i}g_{j)} - \delta_{ij}(\partial_k g^k) + (\partial_i\partial_j - \delta_{ij}\partial^2)\psi + 2\delta_{ij}(\partial_t^2\psi) + (\partial_t^2 - \partial^2)s_{ij} = 8\pi G T_{ij} \ .
\end{aligned} \tag{27}$$

The G_{00} equation is precisely the Newtonian Poisson equation, justifying the alternative name Poisson gauge.

6 Scalar, Vector and Tensor Components

Having reduced the number of degrees of freedom in the metric to six, let us now reexamine the statement made at the end of Section 3 that there are four distinct gravitational phenomena. They may be classified by the form of the metric variables as scalar (ϕ and ψ), vector (w_i) and tensor (s_{ij}). The scalar-vector-tensor decomposition was first performed by Lifshitz (1946) in the context of perturbations of a Robertson-Walker spacetime, but it works (at least) for perturbations of any spacetime (such as Minkowski) with sufficient symmetry (i.e. with sufficient number of Killing vector fields). See Section 4.2 of Bertschinger (1996) for the cosmological application.

The scalar-vector-tensor decomposition is based on decomposing both the metric and stress-energy tensor components into longitudinal and transverse parts. Three-vectors like $\underline{w}$ (regarded as a three-vector in Euclidean space) and T_{0i} are decomposed as follows:

$$w^i = w^i_{\parallel} + w^i_{\perp} \ , \quad \underline{\nabla} \times \underline{w}_{\parallel} = \vec{e}_i \epsilon^{ijk} \partial_j w_{k,\parallel} = 0 \ , \quad \underline{\nabla} \cdot \underline{w}_{\perp} = \delta^{ij} \partial_i w_{j,\perp} = 0 \ . \tag{28}$$

In the transverse gauge, $\underline{w}_{\parallel} = 0$ but we are retaining it here for purposes of illustration.

The terms "longitudinal" and "transverse" come from the Fourier transform representation. Because $\underline{w}_{\parallel} = \underline{\nabla} \Phi_w$ for some scalar field Φ_w, the Fourier transform of $\underline{w}_{\parallel}$ is parallel to the wavevector $\underline{k}$. Similarly, $\underline{w}_{\perp} = \underline{\nabla} \times \underline{A}_w$ for some vector field $\underline{A}_w$, hence its Fourier transform is perpendicular (i.e. transverse) to $\underline{k}$. A spatial constant vector may be regarded as being either longitudinal or transverse.

Jackson (1975, Section 6.5) gives explicit expressions for the longitudinal and transverse parts of a three-vector field in flat space:

$$\underline{w}_{\parallel} = -\frac{1}{4\pi} \underline{\nabla} \int \frac{\underline{\nabla}' \cdot \underline{w}(\underline{x}')}{|\underline{x} - \underline{x}'|} \, d^3x' \ , \quad \underline{w}_{\perp} = \frac{1}{4\pi} \underline{\nabla} \times \underline{\nabla} \times \int \frac{\underline{w}(\underline{x}')}{|\underline{x} - \underline{x}'|} \, d^3x' \ . \tag{29}$$

Note that this decomposition is nonlocal, i.e. the longitudinal and transverse parts carry information about the vector field everywhere. Thus, if $\underline{w}$ is nonzero only in a small region of space, its longitudinal and transverse parts will generally be nonzero everywhere. One cannot deduce causality by looking at $\underline{w}_{\parallel}$ or $\underline{w}_{\perp}$ alone.

Similarly, a symmetric two-index tensor may be decomposed into three parts depending as to whether its divergence is longitudinal, transverse, or zero:

$$h_{ij} = h_{ij,\parallel} + h_{ij,\perp} + h_{ij,\mathrm{T}} \ . \tag{30}$$

We will refer to these parts as longitudinal (or scalar), rotational (or solenoidal or vector) and transverse (or tensor) parts of h_{ij}. In the transverse gauge $h + ij = h_{ij,\mathrm{T}}$, but we retain the other parts here for purpose of illustration.

The longitudinal and rotational parts are defined in terms of a scalar field $h_{\parallel}(x)$ and a transverse vector field $\underline{h}_{\perp}(x)$ such that

$$h_{ij,\parallel} = \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \partial^2 \right) h_{\parallel} \ , \quad h_{ij,\perp} = \partial_{(i} h_{j),\perp} \ . \tag{31}$$

As stated above, the divergences of $h_{ij,\|}$ and $h_{ij,\perp}$ are longitudinal and transverse vectors, respectively, and the divergence of $h_{ij,\mathrm{T}}$ vanishes identically:

$$\delta^{jk}\partial_k h_{ij,\|} = \frac{2}{3}\partial_i(\partial^2 h_\|) \ , \quad \delta^{jk}\partial_k h_{ij,\perp} = \frac{1}{2}\partial^2 h_{i,\perp} \ , \quad \delta^{jk}\partial_k h_{ij,\mathrm{T}} = 0 \ . \tag{32}$$

Thus, the longitudinal part is obtainable from a scalar field, the rotational part is obtainable only from a (transverse) vector field, and the transverse part is obtainable only from a (transverse traceless) tensor field. The reader may find it a useful exercise to construct integral expressions for these parts, similar to equations (29).

The stress-energy tensor may be decomposed in a similar way. Doing this, the linearized Einstein equations (27) in transverse gauge give field equations for the physical fields (ψ, g_i, H_i, s_{ij}):

$$\begin{aligned}
&\partial^2\psi = 4\pi G T_{00} \ , \\
&\underline{\nabla}\cdot\underline{g} - 3\partial_t^2\psi = -4\pi G(T_{00} + T^i{}_i) \ , \\
&\underline{\nabla}\times\underline{H} = -16\pi G\underline{f}_\perp \ , \quad \underline{f} \equiv T^{0i}\vec{e}_i \ , \\
&(\partial_t^2 - \partial^2)s_{ij} = 8\pi G T_{ij,\mathrm{T}}
\end{aligned} \tag{33}$$

plus constraint equations to ensure $\partial_\mu T^{\mu\nu} = 0$:

$$\begin{aligned}
&\partial_t\underline{\nabla}\psi = -4\pi G\underline{f}_\| \ , \\
&\partial_{(i}g_{j)} - \frac{1}{3}\delta_{ij}(\partial_k g^k) + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2)\psi = 8\pi G(\Pi_{ij} - \Pi_{ij,\mathrm{T}}) \ , \\
&\text{where} \ \ \Pi_{ij} \equiv T_{ij} - \frac{1}{3}\delta_{ij}T^k{}_k \ .
\end{aligned} \tag{34}$$

Note that the third constraint equation may be further decomposed into longitudinal and rotational parts as follows:

$$(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2)(\psi - \phi) = 8\pi G\,\Pi_{ij,\|} \ , \quad -\partial_t\partial_{(i}w_{j)} = 8\pi G\,\Pi_{ij,\perp} \ . \tag{35}$$

Equations (33)–(35) may be regarded as the fundamental Einstein equations in linear theory. No approximations have been made in deriving them, aside from $|h_{\mu\nu}| \ll 1$.

7 Physical Content of the Einstein equations

Equations (33)–(35) are remarkable in bearing similarities to both Newtonian gravity and electrodynamics. They exhibit precisely the four physical features mentioned at the end of Section 3: the quasi-Newtonian gravitational field $\underline{g}$, the gravitomagnetic field $\underline{H}$, the spatial potential ψ, and the transverse-traceless strain s_{ij}.

To see the effects of these fields, let us rewrite the gravitational force law, equation (15), using equation (11) with the transverse gauge conditions (23) and (25):

$$\frac{d\underline{p}}{dt} = E\left[\underline{g} + \underline{v} \times \underline{H} + \underline{v}(\partial_t\psi) - v^2\underline{\nabla}_\perp\psi\right] + E\left[-v^j\partial_t s^i{}_j + \epsilon^{ijk}v_j v^l \Omega^k{}_l\right]\vec{e}_i \qquad (36)$$

where $\underline{\nabla}_\perp \equiv (\delta^{ij} - v^i v^j/v^2)\vec{e}_i\partial_j$ is the gradient perpendicular to $\underline{v}$ and

$$\Omega^{kl} \equiv \partial_m s_n{}^{(k}\epsilon^{l)mn} \qquad (37)$$

is the "curl" of the strain tensor s^{kl}. (We define the curl of a symmetric two-index tensor by this equation).

We can build intuition about each component of the gravitational field $(\underline{g}, \underline{H}, \psi, s_{ij})$ by comparing equations (33)–(36) with the corresponding equations of Newtonian gravitation and electrodynamics.

First, the gravitoelectric field $\underline{g}$ is similar to the static Newtonian gravitational field in its effects, but its field equation (33b) differs from the static Poisson equation. While the potential ψ obeys the Newtonian Poisson equation (33a), its time derivative enters the equations of motion for both $\underline{g}$ and for particle momenta. Why? Note first that we've regarded ϕ as the more natural generalization of the Newtonian potential because it gives the deflection for slowly-moving particles; the terms with ψ in equation (36) all vanish when $v^i = 0$. Under what conditions then do we have $\phi \neq \psi$ and why does equation (33b) differ from the Newtonian Poisson equation?

The answers lie in source motion and causality. If the sources are static (or their motion is negligible), $\partial_t\psi = 0$ from equation (33a). The first of equations (35) shows that if the shear stress is small (compared with T_{00}), then $\phi \approx \psi$ (up to solutions of $\partial_i\partial_j(\phi - \psi) = 0$). Small stresses imply slow motions, so we deduce that the gravitational effects are describable by static gravitational fields in the Newtonian limit. Thus, one cannot argue that the Einstein equations violate causality because ψ is the solution of a static elliptic equation. The gravitational effects on slowly moving particles come not from ψ but from $\underline{g}$, whose source depends on the $\partial_t^2\psi$ as well as on the pressure.

It is instructive to compare the field equations for $\underline{g}$ and $\underline{H}$ with the Maxwell equations for $\underline{E}$ and $\underline{B}$:

$$\begin{aligned} \underline{\nabla} \cdot \underline{E} = 4\pi\rho_c \ , \quad \underline{\nabla} \times \underline{E} + \partial_t\underline{B} = 0 \ , \\ \underline{\nabla} \cdot \underline{B} = 0 \ , \quad \underline{\nabla} \times \underline{B} - \partial_t\underline{E} = 4\pi\underline{J}_c \end{aligned} \qquad (38)$$

were $J^\mu = (\rho_c, \underline{J}_c)$ is the four-current density. By comparison, $\underline{g}$ and $\underline{H}$ obey

$$\begin{aligned} \underline{\nabla} \cdot \underline{g} - 3\partial_t^2\psi = -4\pi G(T_{00} + T^i{}_i) \ , \quad \underline{\nabla} \times \underline{g} + \partial_t\underline{H} = 0 \ , \\ \underline{\nabla} \cdot \underline{H} = 0 \ , \quad \underline{\nabla} \times \underline{H} = -16\pi G\underline{f}_\perp \ . \end{aligned} \qquad (39)$$

How do we interpret these?

Gauss' law for the gravitoelectric field differs from its electrostatic counterpart because of the time-dependence of ψ and the inclusion of spatial stress as as source. (The electromagnetic source, being a vector rather than a tensor, has no such possibility.) We already noted that the $\partial_t^2\psi$ term is needed to ensure causality. (The proof of this is somewhat detailed, requiring a transformation to the Lorentz gauge where all metric components obey wave equations.)

The source-free equations, of both electrodynamics and gravitodynamics ensure that magnetic field lines have no ends. Faraday's law of induction $\partial_t \underline{B} + \underline{\nabla} \times \underline{E} = 0$ (and its gravitational counterpart) ensures $\underline{\nabla} \cdot \underline{B} = 0$ persists when the current sources evolve in Ampere's law.

So far gravitation and electrodynamics appear similar. However, Ampere's law reveals a fundamental difference between the two theories. *There is no gravitational displacement current.* The gravitomagnetic field does not obey a causal evolution equation — it is determined by the instantaneous energy current. Moreover, it is not the whole current $\underline{f} = T^{0i}\vec{e}_i$ that appears as its source but rather only the transverse current. (The longitudinal current would be incompatible with the transverse field $\underline{\nabla} \times \underline{H}$.)

Recall that the Maxwell equations enforce charge conservation through the time derivative of Gauss' law combined with the divergence of Ampere's law. Gravitation is completely different: $\partial_\mu T^{\mu 0} = 0$ is enforced by equations (33a) and (34a), which are not even present in our gravitational "Maxwell" equations. So gravitation doesn't need a displacement current to enforce energy conservation. However, the displacement current plays another fundamental role in electromagnetism, which was recognized by Maxwell before there was any experimental evidence for this term: it leads to wave equations for the electromagnetic fields.

The conclusion is inescapable — $\underline{g}$ and $\underline{H}$ do not obey causal wave equations. This does not mean GR violates causality, because one must include the effects of ψ and s_{ij} on any particle motion (eq. 36). This is left as an extended exercise for the reader. However, it is worth noting that one cannot simply deduce causality from the fact that s_{ij} evolves according to a causal wave equation (eq. 33d). The source for s_{ij} is the transverse-traceless stress, which extends over all space even if $T^{\mu\nu} = 0$ outside a finite region. (This gives rise to "near-field" contributions from gravitational radiation sources similar to the near-field electromagnetic fields of radiating charges.)

So far we have discussed the physics of $\underline{g}$ and $\underline{H}$ in detail but there are some aspects of the spatial metric perturbation fields ψ and s_{ij} remaining to be discussed. Starting with equation (36), we see that ψ plays two roles. The first was discussed in the notes *Hamiltonian Dynamics of Particle Motion*: ψ doubles the deflection of light (or any particle with $v = 1$). Its effect on the proper 3-momentum is to produce a transverse force $-Ev^2\underline{\nabla}_\perp\psi$. However, a time-varying potential also changes the proper energy of a particle through the longitudinal force $E\underline{v}(\partial_t\psi) = \underline{p}\partial_t\psi$. This effect is not the same as a time-varying gravitational (or electric) field; the Lorentz force law contains no such term as $\underline{p}\partial_t\phi$. It is purely a relativistic effect arising from the tensorial nature of gravity.

Finally, the best-known relativistic phenomenon of gravity is gravitational radiation, described (in transverse gauge) by the transverse-traceless potential s_{ij}. One could deduce the whole set of linearized Einstein equations by starting from the premise that gravitational radiation should be represented by a traceless two-index tensor (physically representing a spin-two field) and, because static gravitational fields are long-ranged, the graviton must be massless hence gravitational radiation must be transverse. (These statements will not be proven; doing so requires some background in field theory.) All the other gravitational fields may be regarded as auxiliary potentials needed to enforce gauge-invariance (local stress-energy conservation). In a similar way, Maxwell's equations may be built up starting from the premise that the transverse vector potential $\underline{A}_\perp$ obeys a wave equation with source given by the transverse current.

Gravitational radiation affects particle motion in three ways. The first two are apparent in equation (36). Noting that $v^l \Omega^k{}_l$ appears in the equation of motion the same way as the gravitomagetic field $\underline{H}$, we conclude that gravitational radiation contributes a force perpendicular to the velocity. However, that force is quadratic rather than linear in the velocity (for a given energy). Second, gravitational radiation contributes a term to the force that is linear in the velocity but dependent on the time derivative: $-v^j \partial_t s_{ij}$.

Both of these effects appear only in motion relative to the coordinate system. Because gravitational radiation produces no "force" on particles at rest in the coordinates, particles at rest remain at rest. The Christoffel symbol $\Gamma^i{}_{00}$ receives no contribution from h_{ij}.

Does this mean that gravitational radiation has no effect on static particles? No — it means instead that gravitational radiation cannot be understood as a force in flat spacetime; it is fundamentally a wave of space curvature. One cannot deduce its effects from the coordinates alone; one must also use the metric. The proper spatial separation between two events (e.g. points on two particle worldlines) with small coordinate separation $\Delta x^i = (\Delta x) n^i$ is $(g_{ij}\Delta x^i \Delta x^j)^{1/2} = (\Delta x)(1 + s_{ij} n^i n^j)$. (Note that Schutz and most other references used $h_{ij} = \frac{1}{2} s_{ij}$.) We see that s_{ij} is the true strain — the change in distance divided by distance due to a passing gravitational wave. This strain effect, and not the velocity-dependent forces appearing in equation (36), is what is being sought by LIGO and other gravitational radiation detectors. The velocity-dependent forces do make a potentially detectable signature in the cosmic microwave background anisotropy, however, which provides a way to search for very long wavelength gravitational radiation.

8 Residual Gauge Freedom: Accelerating, Rotating, and Inertial Frames

Before concluding our discussion of linear theory, it is worthwhile examining equations (24) and (26) to deduce the gauge freedom remaining after we impose the transverse gauge conditions (23) and (25). Doing so will help to clarify the differences between gravity, acceleration, and rotation.

The gauge conditions are unaffected by linear transformations of the spatial coordinates, which are homogeneous solutions of equations (24) and (26):

$$\xi^0 = a^0(t) + b_i(t)x^i \ , \quad \xi^i = a^i(t) + \left[c_{(ij)} + c_{[ij]}(t)\right] x^j \ , \quad c_{(ij)} = \text{constant} \ . \tag{40}$$

Equations (24) and (26) also have quadratic solutions in the spatial coordinates, but these are excluded because gauge transformations require that the coordinate transformation $x^\mu \to y^\mu = x^\mu - \xi^\mu$ be one-to-one and invertible. (The symmetric tensor c_{ij} must be constant because otherwise ξ^0 would have a contribution $\frac{1}{2}\dot{c}_{ij}x^i x^j$.)

The various terms in equation (40) have straightforward physical interpretations: $a^0(t)$ represents a global redefinition of the time coordinate $t \to t - a^0(t)$, $b_i(t)$ is a velocity which tilts the t-axis as in an infinitesimal Lorentz transformation ($t' = t - vx$), da^i/dt is the other half of the Lorentz transformation (e.g. $x' = x - vt$), $c_{(ij)}$ represents a static stretching of the spatial coordinates, and $c_{[ij]}$ is a spatial rotation of the coordinates about the axis $\epsilon^{ijk}c_{jk}$.

Notice that the class of coordinate transformations allowed under a gauge transformation is broader than the Lorentz transformations of special relativity. Transformations to accelerating (d^2a^i/dt^2) and rotating ($dc_{[ij]}/dt$) frames occur naturally because the formulation of general relativity is covariant. That is, the equations of motion have the same form in any coordinate system. (However, the assumption $|h_{\mu\nu}| \ll 1$ greatly limits the coordinates allowed in linear theory.)

Using equations (22) and (40), the changes in the fields are

$$\delta \underline{g} = -\underline{\ddot{a}} + \underline{\dot{\omega}} \times \underline{r} \ , \quad \delta \underline{H} = -2\,\underline{\omega} \ , \quad \delta\psi = -\frac{1}{3}c^k{}_k \ , \quad \delta s_{ij} = c_{(ij)} - \frac{1}{3}\delta_{ij}c^k{}_k \tag{41}$$

where $\underline{r} \equiv x^i\vec{e}_i$ is the "radius vector" (which has the same meaning here as in special relativity) and the angular velocity ω^i is defined through

$$\frac{dc_{[ij]}}{dt} \equiv \epsilon_{ijk}\omega^k \ . \tag{42}$$

The spatial curvature force terms in equation (36) are invariant because the residual gauge freedom of transverse gauge in equation (41) allows only for constant spatial deformations (i.e., time-independent δh_{ij}). Gravitational radiation is necessarily time-dependent, so it is completely fixed by the transverse-traceless gauge condition equation (23). The spatial curvature potential ψ is arbitrary up to the addition of a constant. Thus, only the gravitoelectric and gravitomagnetic fields have physically relevant gauge freedom after the imposition of the transverse gauge conditions.

Note that equations (41) leave the Einstein equations (33)–(34) and (39) invariant. The Riemann, Ricci and Einstein tensors are gauge-invariant for a weakly perturbed Minkowski spacetime.

However, the gravitational force equation (36) is not gauge-invariant. Under the gauge transformation of equations (40) and (41) it acquires additional terms:

$$\delta\left(\frac{d\underline{p}}{dt}\right) = E\left(\delta\underline{g} + \underline{v} \times \delta\underline{H}\right) = E\left(-\ddot{\underline{a}} + \dot{\underline{\omega}} \times \underline{r} + 2\underline{\omega} \times \underline{v}\right) \ . \tag{43}$$

The reader will recognize these terms as exactly the fictitious forces arising from acceleration and rotation relative to an inertial frame. The famous Coriolis acceleration is $2\underline{\omega} \times \underline{v}$. (The centrifugal force term is absent because it is quadratic in the angular velocity and it vanishes in linear theory.)

The Weak Equivalence Principle is explicit in equation (43): acceleration is equivalent to a uniform gravitational (gravitoelectric) field $\underline{g}$. Moreover, we have also discovered that rotation is equivalent to a uniform gravitomagnetic field $\underline{H}$. Uniform fields are special because they can be transformed away while remaining in transverse gauge.

The observant reader may have noticed the word "inertial" used above and wondered about its meaning and relevance here. Doesn't GR single out no preferred frames? That is absolutely correct; GR distinguishes no preferred frames. However, *we* singled out a class of frames (i.e. coordinates) by imposing the transverse gauge conditions (23) and (25). **Transverse gauge provides the relativistic notion of inertial frames.** This is not just one frame but a class of frames because equation (36) is invariant under (small constant velocity) Lorentz transformations: b_i and da^i/dt are absent from equation (43). Thus, the Galilean-invariance of Newton's laws is extended to the Lorentz-invariance of the relativistic force law in transverse gauge. However, the gravitational force now includes magnetic and other terms not present in Newton's laws.

Although the gravitational force equation is not *invariant*, it is *covariant*. Fictitious forces are automatically incorporated into existing terms ($\underline{g}$ and $\underline{H}$); the *form* of equation (36) is invariant even though the *values* of each term are not. This points out a profound fact of gravity in general relativity: **nothing in the equations of motion distinguishes gravity from a fictitious force**.

Indeed, the curved spacetime perspective regards gravitation entirely as a fictitious force. Nonetheless, we can, by imposing the transverse gauge (or other gauge) conditions, make our own separation between physical and fictitious forces. (Here I must note the caveat that transverse gauge has not been extended to strong gravitational fields so I don't know whether all the conclusions obtained here are restricted to weak gravitational fields.) Uniform gravitoelectric or gravitomagnetic fields can always be transformed away, hence they may be regarded as being due to acceleration or rotation rather than gravity. Spatially varying gravitoelectric and gravitomagnetic fields cannot be transformed away. They can only be caused by the stress-energy tensor and they are not coordinate artifacts.

This separation between gravity and fictitious forces is somewhat unnatural in GR (and it requires a tremendous amount of preparation!), but it is helpful for building intuition by relating GR to Newtonian physics.

This discussion also sheds light on GR's connection to Mach's principle, which states that inertial frames are determined by the rest frame of distant matter in the universe (the "fixed stars"). This is not strictly true in GR. Non-rotating inertial frames would be ones in which $\underline{H} = 0$ everywhere. Locally, any nonzero $\underline{H}$ can be transformed away by a suitable rotation, but the rotation rates may be different at different places in which case there can exist no coordinate system in which $\underline{H} = 0$ everywhere. (In this case the coordinate lines would quickly tangle, cross and become unusable.) Transverse energy currents, due for example to rotating masses, produce gravitomagnetic fields that cannot be transformed away. (See the gravitational Ampere's law in eqs. 39.) However, the gravitomagnetic fields may be very small, in which case there do exist special frames in which $\underline{H} \approx 0$ and there are no Coriolis terms in the force law.

We happen to live in a universe with small transverse energy currents: the distant matter is not rotating. (Sensitive limits are placed by the isotropy of the cosmic microwave background radiation.) Thus, due good fortune, Mach was partly correct. However, were he to stand close to a rapidly rotating black hole, and remain fixed relative to the distant stars, he would get dizzy from the gravitomagnetic field. (He would literally feel like he was spinning.) Thus, Mach's principle is not built into GR but rather is a consequence of the fact that we live in a non-rotating (or very slowly rotating) universe.

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Physics 8.962 Spring 1999

Gravitational Lensing from Hamiltonian Dynamics

1 Introduction

The deflection of light by massive bodies is an old problem having few pedagogical treatments. The full machinery of general relativity seems like a sledge hammer when applied to weak gravitational fields. On the other hand, photons are relativistic particles and their propagation over cosmological distances demands more than Newtonian dynamics. In fact, for weak gravitational fields or for small perturbations of a simple cosmological model, it is possible to discuss gravitational lensing in a weak-field limit similar to Newtonian dynamics, albeit with light being deflected twice as much by gravity as a nonrelativistic particle.

The most common formalism for deriving the equations of gravitational lensing is based on Fermat's principle: light follows paths that minimize the time of arrival (Scheider et al. 1992). As we will show, light is deflected by weak static gravitational fields as though it travels in a medium with variable index of refraction $n = 1 - 2\phi$ where ϕ is the dimensionless gravitational potential.

With the framework of Hamiltonian dynamics given in the notes *Hamiltonian Dynamics of Particle Motion*, here we present a synopsis of the theory of gravitational lensing. The Hamiltonian formulation begins with general relativity and makes clear the approximations which are made at each step. It allows us to derive Fermat's least time principle in a weak gravitational field and to calculate the relative time delay when lensing produces multiple images. It is easily applied to lensing in cosmology, including a correct treatment of the inhomogeneity along the line of sight, by taking advantage of the standard formalism for perturbed cosmological models.

Portions of these notes are based on a chapter in the PhD thesis of Barkana (1997).

2 Hamiltonian Dynamics of Light

Starting from the notes *Hamiltonian Dynamics of Particle Motion* (Bertschinger 1999), we recall that geodesic motion of a particle of mass m in a metric $g_{\mu\nu}$ is equivalent to Hamiltonian motion in $3+1$ spacetime with Hamiltonian

$$H(p_i, x^j, t) = -p_0 = \frac{g^{0i}p_i}{g^{00}} + \left[\frac{(g^{ij}p_ip_j + m^2)}{-g^{00}} + \left(\frac{g^{0i}p_i}{g^{00}}\right)^2\right]^{1/2} . \tag{1}$$

This Hamiltonian is obtained by solving $g^{\mu\nu}p_\mu p_\nu = -m^2$ for p_0. The spacetime coordinates $x^\mu = (t, x^i)$ are arbitrary aside from the requirement that $g_{00} < 0$ so that t is timelike and is therefore a good parameter for timelike and null curves. The canonical momenta are the spatial components of the 4-momentum one-form p_μ. The inverse metric components $g^{\mu\nu}$ are, in general, functions of x^i and t. With this Hamiltonian, the exact spacetime geodesics are given by the solutions of Hamilton's equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} . \tag{2}$$

Our next step is to determine the Hamiltonian for the problem at hand, which requires specifying a metric. Because we haven't yet derived the Einstein field equations, all we can do is to pick an ad hoc metric. In order to obtain useful results, we will choose a physical metric representing a realistic cosmological model, an expanding Big Bang cosmology (a Robertson-Walker spacetime) superposed with small-amplitude spacetime curvature fluctuations arising from spatial variations in the matter density. For now, the reader will have to accept the exact form of the metric without proof.

The line element for our metric is

$$ds^2 = a^2(t)\left[-(1+2\phi)dt^2 + (1-2\phi)\gamma_{ij}dx^idx^j\right] . \tag{3}$$

In the literature, t is called "conformal" time and x^i are "comoving" spatial coordinates. The cosmic expansion scale factor is $a(t)$ and is related to the redshift of light emitted at time t by $a(t) = 1/(1+z)$. To get the non-cosmological limit (weak gravitational fields in Minkowski spacetime), one simply sets $a = 1$. The Newtonian gravitational potential $\phi(x^i, t)$ obeys (to a good approximation) the Poisson equation. (In cosmology, the source for ϕ is not ρ but rather $\rho - \bar\rho$ where $\bar\rho$ is the mean mass density; we will show this in more detail later in the course.) We assume $|\phi| \ll 1$ which is consistent with cosmological observations implying $\phi \sim 10^{-5}$.

In equation (3) we write $\gamma_{ij}(x^k)$ as the 3-metric of spatial hypersurfaces in the unperturbed Robertson-Walker space. For a flat space (the most popular model with theorists, and consistent with observations to date), we could adopt Cartesian coordinates for which $\gamma_{ij} = \delta_{ij}$. However, to allow for easy generalization to nonflat spaces as well as non-Cartesian coordinates in flat space we shall leave γ_{ij} unspecified for the moment.

Substituting the metric implied by equation (3) into equation (1) with $m = 0$ yields the Hamiltonian for a photon:

$$H(p_i, x^j, t) = p(1 + 2\phi) \ , \quad p \equiv \left(\gamma^{ij} p_i p_j\right)^{1/2} \ . \tag{4}$$

We have neglected all terms of higher order than linear in ϕ. Not surprisingly, in a perturbed spacetime the Hamiltonian equals the momentum plus a small correction for gravity. However, it differs from the proper energy measured by a stationary observer, $E = -V^\mu p_\mu$, because the 4-velocity of such an observer is $V^\mu = (a(1-\phi), 0, 0, 0)$ (since $g_{\mu\nu} V^\mu V^\nu = -1$) so that $E = a^{-1} p(1 + \phi)$. The latter expression is easy to understand because a^{-1} converts comoving to proper energy (the cosmological redshift) and in the Newtonian limit ϕ is the gravitational energy per unit mass (energy).

Why is the Hamiltonian not equal to the energy? The answer is because it is conjugate to the time coordinate t which does not measure proper time. The job of the Hamiltonian is to provide the equations of motion and not to equal the energy. The factor of 2 in equation (4) is important — it is responsible for the fact that light is deflected twice as much as nonrelativistic particles in a gravitational field.

To first order in ϕ, Hamilton's equations applied to equation (4) yield

$$\frac{dx^i}{dt} = n^i(1 + 2\phi) \ , \quad \frac{dp_i}{dt} = -2p\nabla_i\phi + \gamma^k{}_{ij} p_k n^j (1 + 2\phi) \ , \quad n^i \equiv \frac{\gamma^{ij} p_j}{p} \ . \tag{5}$$

We will drop terms $O(\phi^2)$ throughout. We have defined a unit three-vector n^i in the photon's direction of motion (normalized so that $\gamma_{ij} n^i n^j = 1$). The symbol $\gamma^k{}_{ij} = \frac{1}{2}\gamma^{kl}(\partial_i \gamma_{jl} + \partial_j \gamma_{il} - \partial_l \gamma_{ij})$ is a connection coefficient for the spatial metric that vanishes if we are in flat space and use Cartesian coordinates. Beware that ∇_i is the covariant derivative with respect to the 3-metric γ_{ij} and not the covariant derivative with respect to $\gamma_{\mu\nu}$, although there is no difference for a spatial scalar field: $\nabla_i \phi = \partial_i \phi$.

Note that the cosmological expansion factor has dropped out of equations (5). These equations are identical to what would be obtained for the deflection of light in a perturbed Minkowski spacetime. The reason for this is that the metric of equation (3) differs from the non-cosmological one solely by the factor $a^2(t)$ multiplying every term. This is called a conformal factor because it leaves angles invariant. In particular, it leaves null cones invariant, and therefore is absent from the equations of motion for massless particles.

In the following sections we shall represent three-vectors (and two-vectors) in the 3-space with metric γ_{ij} using arrows above the symbol. To lowest order in ϕ, we may interpret these formulae as giving the deflection of light in an unperturbed spacetime due to gravitational forces, just as in Newtonian mechanics. The difference is that our results are fully consistent with general relativity.

3 Fermat's Principle

When $\partial_t \phi = 0$, the Hamiltonian (eq. 4) is conserved along phase space trajectories and the equations of motion follow from an alternative variational principle, Maupertuis' principle (Bertschinger 1999). Maupertuis' principle states that if $\partial H(p_i, q^j, t)/\partial t = 0$, then the solution trajectories of the full Hamiltonian evolution are given by extrema of the reduced action $\int p_i \, dq^i$ with fixed endpoints. This occurs because

$$\int p_i \, dq^i - H \, dt = \int p_i \, dq^i - d(Ht) + t \, dH \ . \tag{6}$$

The Ht term, being a total derivative, vanishes for variations with fixed endpoints. The $t \, dH$ term vanishes for trajectories that satisfy energy conservation, and we already know (from the Hamilton's equations of the full action) that only such trajectories need be considered when $\partial H/\partial t = 0$. Thus, the condition $\delta \int p_i \, dq^i = 0$, when supplemented by conservation of H, is equivalent to the original action principle.

Expressing p_i in terms of dx^i/dt using Hamilton's equations (5) in the full phase space for the Hamiltonian of equation (4), the reduced action becomes

$$p_i \, dx^i = p \gamma_{ij} n^j \, dx^i = H(1 - 2\phi) \gamma_{ij} n^i \, dx^j = H \, dt \ . \tag{7}$$

Using $H = \text{constant} \equiv h$, Mauptertuis' principle yields Fermat's principle of least time,

$$\delta \int dt = \delta \int [1 - 2\phi(x)] \left(\gamma_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right)^{1/2} ds = 0 \tag{8}$$

for light paths parameterized by s. We leave it as an exercise for the reader to show, using the Euler-Lagrange equations, that if s measures path length, equation (8) yields equations (5) exactly (to lowest order in ϕ) when $\partial_t \phi = 0$. In comparing with equation (5), one must be careful to note that there the trajectory is parameterized by $dt = (1 - 2\phi) ds$ so that $\vec{n} = d\vec{x}/ds$ is a unit vector.

Thus, for a static potential ϕ (even in a non-static cosmological model with expansion factor $a(t)$), light travels along paths that minimize travel time but not path length (as measured by the spatial metric γ_{ij}). The null geodesics behave as though traveling through a medium with index of refraction $1 - 2\phi$. To minimize travel time, light rays will tend to avoid regions of negative ϕ; therefore light will be deflected around massive bodies.

Fermat's principle is exact for gravitational lensing only with static potentials. In most astrophysical applications, the potentials are sufficiently relaxed so that $\partial_t \phi$ may be neglected relative to $n^i \nabla_i \phi$ and Fermat's principle still applies. The one notable exception is microlensing, where the lensing is caused by stars (or other condensed objects) moving across the line of sight. In this case, one may still apply Fermat's principle after boosting to the rest frame of the lens.

4 Reduction to the Image Plane

In equation (8), the action is invariant under an arbitrary change of parameter, $s \to s'(s)$ with $ds'/ds > 0$. This is not a physical symmetry of the dynamics, and as a consequence we may eliminate a degree of freedom by using one of the coordinates to parameterize the trajectories. A similar procedure was used to eliminate t in going from equation (6) to equation (8). Here, as there, the Lagrangian is independent of the time parameter, enabling a reduction of order. However, for reasons that will soon become clear, this reduction cannot be done using the reduced action (Maupertuis' principle) but instead follows from reparameterization of the Lagrangian.

To clarify the steps, we start with

$$L_3(x^i, dx^j/ds) = [1 - 2\phi(x)] \left(\gamma_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right)^{1/2} \tag{9}$$

for the Lagrangian in the three-dimensional configuration space (eq. 8). Because the Lagrangian does not depend explicitly on s, the Hamiltonian is conserved and we may attempt to reduce the order as in the previous section. The first step is to construct the Hamiltonian. Under a Legendre transformation, $L_3 \to H_3(p_i, x^j, s) = p_i(dx^i/ds) - L_3$ where $p_i = \partial L_3/\partial(dx^i/ds)$ is the momentum conjugate to x^i. But we quickly run into trouble: as the reader may easily show, H_3 vanishes identically.

What causes this horror? The answer is that L_3 is homogeneous of first degree in the coordinate velocity dx^i/ds, which is equivalent to the statement that the action of equation (8) is invariant under reparameterization. Physically, the Hamiltonian vanishes because of the extra symmetry of the Lagrangian, which is unrelated to the dynamics. The physical Hamiltonian should include only the physical degrees of freedom, so we must eliminate the reparameterization-invariance if we are to use Hamiltonian methods.

This is done very simply by rewriting the action (eq. 8) using one of the coordinates as the parameter. The radial distance from the observer is a good choice: for small deflections of rays traveling nearly in the radial direction toward the observer, r will be single-valued along a trajectory.

To fix the parameterization we must write the spatial line element in a Robertson-Walker space in terms of r and two angular coordinates:

$$dl^2 \equiv \gamma_{ij} dx^i dx^j = dr^2 + R^2(r)\gamma_{ab}(\xi) d\xi^a d\xi^b \ . \tag{10}$$

Here $1 \le a, b \le 2$ and γ_{ab} is the metric of a unit 2-sphere. The coordinates ξ^a are angles and are dimensionless. Note that r measures radial distance ($\gamma_{rr} = 1$) and $R(r)$ measures angular distance. We will not give the exact form of $R(r)$ here except to note that for a flat space, $R(r) = r$. In the standard spherical coordinates, $\gamma_{\theta\theta} = 1$ and $\gamma_{\phi\phi} = \sin^2\theta$. We will leave the coordinates in the sphere arbitrary for the moment, and use γ_{ab} and its inverse γ^{ab} to lower and raise indices of two-vectors and one-forms in the sphere.

Our action, equation (8), is the total elapsed light-travel time t (using our original spacetime coordinates, eq. 3). The reparameterization means that now we express the action as a functional of the two-dimensional trajectory $\xi^a(r)$:

$$t[\xi^a(r)] = \int_0^{r_S} [1 - 2\phi(\xi, r)] \left[1 + R^2(r)\gamma_{ab}\frac{dx^a}{dr}\frac{dx^b}{dr}\right]^{1/2} dr \ . \tag{11}$$

This action is to be varied subject $\delta\xi^a = 0$ at $r = 0$ (the observer) and $r = r_S$ (the source).

In writing equation (11), we have neglected $\partial\phi/\partial t$ and we have neglected terms $O(\phi^2)$ (weak-field approximation). As we will see, the angular term inside the Lagrangian is small when the potential is small, and therefore we can expand the square root, dropping all but the lowest-order terms. To the same order of approximation, we may neglect the curvature of the unit sphere, and set $\gamma_{ab} = \delta_{ab}$. (We can always orient spherical coordinates so that $\gamma_{ab} = \delta_{ab}$ plus second-order corrections in ξ.) These approximations together constitute the small-angle approximation. In practice it is well satisfied; observed angular deflections of astrophysical lenses are much less than 10^{-3}.

With the weak-field and small-angle approximations, the action becomes

$$t[\xi^a(r)] = r_S + \int_0^{r_S} L_2 \, dr \ , \quad L_2\left(\xi^a, \frac{d\xi^b}{dr}, r\right) = \frac{1}{2}\, R^2(r)\delta_{ab}\frac{d\xi^a}{dr}\frac{d\xi^b}{dr} - 2\phi(\xi^a, r) \ . \tag{12}$$

Note that the Lagrangian now depends on the "time" parameter, so we have eliminated the parameterization-invariance.

To get a Hamiltonian system, we make the Legendre transformation of the Lagrangian L_2. The conjugate momentum is $p_a = R^2(r)\delta_{ab}d\xi^b/dr$. The Hamiltonian becomes

$$H(p_a, \xi^b, r) = \frac{p^2}{2R^2(r)} + 2\phi(\vec{\xi}, r) \ . \tag{13}$$

On account of the small-angle approximation, $\vec{p}$ and $\vec{\xi}$ are two-dimensional vectors in Euclidean space ($p^2 \equiv \delta^{ab}p_a p_b$). Noting that r plays the role of time, this Hamiltonian represents two-dimensional motion with a time-varying mass $R^2(r)$ and a time-dependent potential 2ϕ.

With the Hamiltonian of equation (13), Hamilton's equations give

$$\frac{d\vec{\xi}}{dr} = \frac{\vec{p}}{R^2(r)} \ , \quad \frac{d\vec{p}}{dr} = -2\frac{\partial\phi}{\partial\vec{\xi}} \ . \tag{14}$$

These equations and the action may be integrated subject to the "initial" conditions $\xi = \xi_0$, $\vec{p} = 0$ and $t = t_0$ at the observer, $r = 0$:

$$
\begin{aligned}
\vec{\xi}(r) &= \vec{\xi}_0 - \frac{2}{R(r)} \int_0^r \frac{R(r-r')}{R(r')} \frac{\partial \phi}{\partial \vec{\xi}}(\vec{\xi}(r'), r')\, dr' \\
\vec{p}(r) &= -2 \int_0^r \frac{\partial \phi}{\partial \vec{\xi}}(\vec{\xi}(r'), r')\, dr' \\
t(r) &= t_0 - r - \int_0^r \left[\frac{p^2(r')}{R^2(r')} - 2\phi(\vec{\xi}(r'), r') \right] dr' \; .
\end{aligned}
\tag{15}
$$

Note that here t is the coordinate time along the past light cone; the elapsed time (the action) is $t_0 - t$. The two terms in the time delay integral arise from geometric path length (the p^2 term) and gravity. Half of the gravitational potential part comes from the slowing down of clocks in a gravitational field (gravitational redshift) and the other half comes from the extra proper distance caused by the gravitational distortion of space.

Equations (15) provide only a formal solution, since ϕ is evaluated on the unknown path $\vec{\xi}(r')$. The reader may verify the solution by inserting into equations (14). One needs the following identity for the angular distance in a Robertson-Walker space, which we present without proof:

$$
\frac{\partial}{\partial r} \left[\frac{R(r-r')}{R(r)R(r')} \right] = \frac{1}{R^2(r)} \; . \tag{16}
$$

It is easy to verify this for the flat case $R(r) = r$.

When the potential varies with time, we cannot use Fermat's principle or the further reduction achieved in this section. Instead, one has to integrate the original equations of motion (5). It can be shown (Barkana 1997) that, under the small-angle approximation, these equations also have the formal solution given by equation (15), with the single change that ϕ also becomes a function of t and that t must be evaluated along the trajectory: $\phi(\vec{\xi}(r'), r', t(r'))$. Thus, we obtain the physical result that the potential is to be evaluated along the backward light cone.

5 Astrophysical Gravitational Lensing

The astrophysical application of gravitational lensing is based on the following considerations. Given an observed image position $\vec{\xi}_0$, we wish to deduce the source position $\vec{\xi}_S \equiv \vec{\xi}(r_S)$ using equation (15) to relate $\vec{\xi}(r_S)$ to $\vec{\xi}_0$. The result is a mapping from the image plane $\vec{\xi}_0$ to the source plane $\vec{\xi}_S$). This mapping is called the lens equation.

By integrating the deflection $\vec{\xi}_S - \vec{\xi}_0$ for a given distribution of mass (hence potential) along the line of sight from the observer, and for a given cosmological model (hence angular distance $R(r)$), one can compute the source plane positions for the observed images.

In practice, we wish to solve the inverse problem, namely to deduce properties of the mass and spatial geometry along the line of sight from observed lens systems. How can this be done if we know only the image positions but not the source positions?

There are several methods that can be used to deduce astrophysical information from gravitational lenses (Blandford and Narayan 1992). First, the lens mapping $\vec{\xi}_S(\vec{\xi}_0)$ can become multivalued so that a given source produces multiple images. In this case, the images provide constraints on lensing potential and geometry because all the ray paths must coincide in the source plane. This method can strongly constrain the mass of a lens, especially when the symmetry is high so that an Einstein ring or arc is produced.

Another method uses information from $t(r)$. If the source is time-varying and produces multiple images, then each image must undergo the same time variation, offset by the $t - t_0 + r$ integral in equation (15). Because this method involves measurement of a physical length scale (the time delay between images, multiplied by the speed of light), it offers the prospect of measuring cosmological distances in physical units, from which one can determine the Hubble constant. This is a favorite technique with MIT astrophysicists.

Another way to get a timescale occurs if the lens moves across the line of sight, in the phenomenon called microlensing. Gravitational lensing magnifies the image according to the determinant of the (inverse) magnification matrix $\frac{\partial \xi_S^a}{\partial \xi_0^b}$. If the angular position of the lens is close to ξ_S so that the rays pass close to the lens, the magnification can be substantial (e.g. a factor of ten). A lens moving transverse to the line of sight will therefore cause a systematic increase, then decrease, of the total flux from a source. From a statistical analysis of the event rates, magnifications and durations, it is possible to deduce some of the properties of a class of lensing objects, such as dim stars (or stellar remnants) in the halo surrounding our galaxy (more colorfully known as MACHOs for "MAssive Compact Halo Objects").

A fourth method, called weak lensing, uses statistical information about image distortions for the case where the deflections are not large enough to produce multiple images, but are large enough to produce detectable distortion. This method can provide statistical information about the lensing potential. It is a favorite method for trying to deduce the spectrum of dark matter density fluctuations.

There are many other applications of gravitational lensing. The study and observation of gravitationl lenses is one of the major areas of current research in astronomy.

6 Thin Lens Approximation

Our derivation of the lens equations (15) made the following, well-justified approximations: the spacetime is a weakly perturbed Roberston-Walker model with small-amplitude curvature fluctuations ($\phi^2 \ll 1$), the perturbing mass distribution is slowlyevolving ($\partial_t \phi$ neglected), and the angular deflections are small ($|\vec{\xi}_S - \vec{\xi}_0| \sim \phi < 10^{-3}$).

Nearly all calculations of lensing are made with an additional approximation, the thin-lens approximation. This approximation supposes that the image deflection occurs in a small range of distance δr about $r = r_L$. In this case, the first of equations (15) gives the thin lens equation

$$\vec{\xi}_S = \vec{\xi}_0 - \frac{R_{LS}}{R_S}\,\vec{\gamma}(\vec{\xi}_0, R_L) \ , \quad \vec{\gamma}(\xi, R) = 2\int \frac{\partial \phi}{\partial \vec{\xi}}(\vec{\xi}, r')\,\frac{dr'}{R} \ , \tag{17}$$

where $R_S \equiv R(r_S)$, $R_L \equiv R(r_L)$ and $R_{LS} \equiv R(r_S - r_L)$. The deflection angle $\vec{\gamma} = -2\int \vec{g}\,dr$ where $\vec{g} = -\vec{\nabla}_\perp \phi = -(1/R)\partial\phi/\partial\vec{\xi}$ is the Newtonian gravity vector (up to factors of a from the cosmology).

Let us estimate the deflection angle γ for a source directly behind a Newtonian point mass with $g = GM/r^2$ (here r is the proper distance from the point mass to a point on the light ray). The impact parameter in the thin-lens approximation is $b = \xi_0 R_L$. Because the deflection is small, the path is nearly a straight line past the lens, and the integral of g along the path gives, crudely, $2bg(b) = 2GM/b = 2GM/(\xi_0 R_L)$. (The factor of two is chosen so that this is, in fact, the exact result of a careful calculation.) With the source lying directly behind the lens, $\xi_S = 0$.

Substituting this deflection into the thin lens equation (17) gives

$$0 = \xi_0 - \frac{R_{LS}}{R_L R_S}\,\frac{4GM}{\xi_0} \ . \tag{18}$$

Vectors are suppressed because this lens equation holds at all positions around a ring of radius $\xi_0 = |\vec{\xi}_0|$ in the image plane. An image directly behind a point mass produces an Einstein ring. Solving for ξ_0 gives the Einstein ring radius:

$$\xi_0 = \left(\frac{4GMR_{LS}}{R_L R_S c^2}\right)^{1/2} \ . \tag{19}$$

References

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