

Phys 543

Homework Set 2

Problem 2.21

The electric potential is defined as

$$V(\vec{r}) = - \int_0^{\vec{r}} \vec{E} \cdot d\vec{r}$$

Our reference point is at infinity and we take $V(\infty) = 0$.

Let us first determine the electric field due to a uniformly charged sphere of radius R carrying the total charge q . Pick a spherical Gaussian sphere whose center coincides with the center of the solid sphere and that has a radius $r > R$, i.e., $Q_{\text{enclosed}} = q$

$$\oint \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} q$$

$$E 4\pi r^2 = \frac{1}{\epsilon_0} q$$

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

If we pick the Gaussian sphere with radius $r < R$, then

$$Q_{\text{enclosed}} = \frac{q}{4\pi R^3} \frac{4}{3}\pi r^3 = q \frac{r^3}{R^3}$$

$$\oint \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} q \frac{r^3}{R^3}$$

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r \hat{r}$$

For $r > R$:

$$V(r) - V(\infty) \xrightarrow{r \rightarrow \infty} = - \int_{\infty}^r \vec{E} \cdot d\vec{r} = - \int_{\infty}^r \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr'$$

$$= + \frac{1}{4\pi\epsilon_0} \frac{q}{r'} \left[r \right]_{\infty}^r = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

For $r < R$

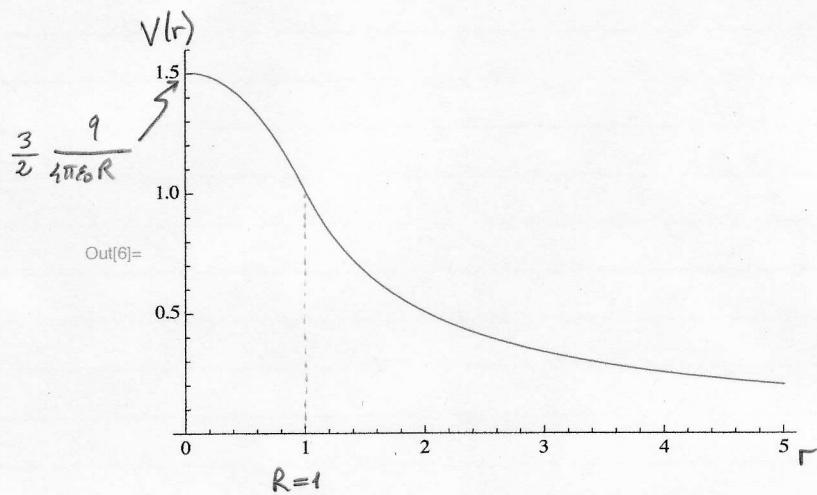
$$V(r) - V(R) = - \int_R^r \vec{E} \cdot d\vec{r} = - \int_R^r \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r dr$$

$$= - \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \frac{r^2}{2} \Big|_R^r = \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \frac{1}{2} (R^2 - r^2)$$

$$V(r) = V(R) + \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \frac{1}{2} (R^2 - r^2)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R} + \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \frac{1}{2} (R^2 - r^2)$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{2R} \left(3 - \frac{r^2}{R^2} \right)$$



$$\text{For } r > R, \quad \nabla V(r) = \hat{r} \frac{\partial}{\partial r} \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r} \right) = -\hat{r} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

$$\vec{E}(r) = -\nabla V(r) = \hat{r} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

$$\text{For } r < R, \quad \nabla V(r) = \hat{r} \frac{\partial}{\partial r} \left(\frac{q}{4\pi\epsilon_0} \frac{1}{2R} \left(3 - \frac{r^2}{R^2} \right) \right) = -\hat{r} \frac{q}{4\pi\epsilon_0} \frac{r}{R^3}$$

$$\vec{E}(r) = -\nabla V(r) = \hat{r} \frac{q}{4\pi\epsilon_0} \frac{r}{R^3}$$

Problem 2.22.

Let us first find the electric field due to an infinitely long straight wire



$$\oint \vec{E} \cdot d\vec{\sigma} = \frac{1}{\epsilon_0} Q_{\text{enclosed}}$$

$$E \cdot 2\pi s = \frac{1}{\epsilon_0} \lambda L$$

$$E(s) = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s}$$

Since the wire has infinite extent, we should set $V=0$ at a finite distance from the wire, call it a !

$$V(s) - V(a) = - \int_a^s \vec{E} \cdot d\vec{\ell}$$

$$= - \int_a^s \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s'} ds'$$

$$= - \frac{1}{2\pi\epsilon_0} \lambda \ln s \Big|_a^s = - \frac{1}{2\pi\epsilon_0} \lambda \ln \left(\frac{s}{a} \right)$$

$$\nabla V(s) = \hat{s} \frac{\partial}{\partial s} \left(- \frac{1}{2\pi\epsilon_0} \lambda \ln \left(\frac{s}{a} \right) \right) = - \hat{s} \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} = - \vec{E}$$

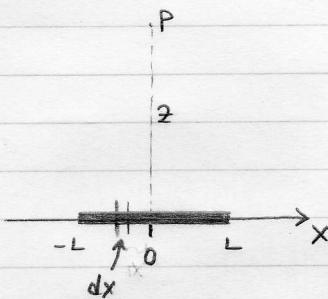
Problem 2.25

Fig a) For two point charges we need to use Eq. 2.27:

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{z^2 + (\frac{d}{2})^2}}$$

Fig b) For a line charge

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(r')}{|\vec{r} - \vec{r}'|} dr'$$



$$\vec{r} = z \hat{z} \quad \vec{r}' = x \hat{x}$$

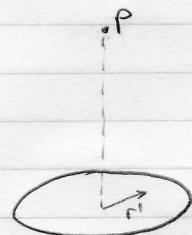
$$|\vec{r} - \vec{r}'|^2 = (z \hat{z} - x \hat{x})(z \hat{z} - x \hat{x}) = z^2 + x^2$$

$$|\vec{r} - \vec{r}'| = \sqrt{z^2 + x^2}$$

$$V(P) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda dx}{\sqrt{z^2 + x^2}} = \frac{\lambda}{4\pi\epsilon_0} \ln(x + \sqrt{z^2 + x^2}) \Big|_{-L}^L$$

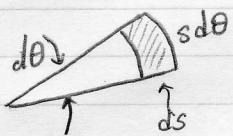
$$= \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{L + \sqrt{L^2 + z^2}}{-L + \sqrt{L^2 + z^2}} \right]$$

Fig c) Uniformly charged disc



$$\vec{r}' = s\hat{s} \quad \vec{r} = z\hat{z}$$

$$|\vec{r} - \vec{r}'|^2 = (\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}') = (z\hat{z} - s\hat{s}) \cdot (z\hat{z} - s\hat{s}) \\ = z^2 + s^2$$



$$|\vec{r} - \vec{r}'| = \sqrt{z^2 + s^2}$$

$$dA = s ds d\theta$$

$$V(P) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(r')}{|\vec{r} - \vec{r}'|} dA'$$

$$= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma s ds d\theta}{\sqrt{z^2 + s^2}}$$

$$= \frac{\sigma}{4\epsilon_0} \int_0^R \frac{s ds}{\sqrt{z^2 + s^2}} \quad z^2 + s^2 = u \quad du = 2s ds$$

$$= \frac{\sigma}{4\epsilon_0} \int_{z^2}^{z^2 + R^2} \frac{du}{\sqrt{u}} = \frac{\sigma}{4\epsilon_0} 2 \sqrt{u} \Big|_{z^2}^{z^2 + R^2}$$

$$= \frac{\sigma}{2\epsilon_0} \left(\sqrt{z^2 + R^2} - z \right)$$

Let us compute the electric fields due to the charge distributions depicted in the figures using

$$\vec{E} = -\nabla V$$

Fig a) $\vec{E} = -\nabla \left(\frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{z^2 + (\frac{d}{2})^2}} \right)$

$$= -\hat{z} \frac{2q}{4\pi\epsilon_0} \frac{\partial}{\partial z} \frac{1}{\sqrt{z^2 + (\frac{d}{2})^2}}$$

$$= \hat{z} \frac{2q}{4\pi\epsilon_0} \frac{1}{z^2} \frac{2z}{(z^2 + (\frac{d}{2})^2)^{3/2}}$$

$$= \hat{z} \frac{1}{4\pi\epsilon_0} \frac{2qz}{(z^2 + (\frac{d}{2})^2)^{3/2}}$$

Fig b) $\vec{E} = -\nabla \left(\frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{L + \sqrt{L^2 + z^2}}{-L + \sqrt{L^2 + z^2}} \right] \right)$

$$= -\hat{z} \frac{\lambda}{4\pi\epsilon_0} \left\{ \frac{1}{L + \sqrt{L^2 + z^2}} \frac{1}{\sqrt{L^2 + z^2}} - \frac{1}{-L + \sqrt{L^2 + z^2}} \frac{1}{\sqrt{L^2 + z^2}} \right\}$$

$$= -\hat{z} \frac{\lambda}{4\pi\epsilon_0} \frac{z}{\sqrt{L^2 + z^2}} \left\{ \frac{1}{L + \sqrt{L^2 + z^2}} - \frac{1}{-L + \sqrt{L^2 + z^2}} \right\}$$

$$= -\hat{z} \frac{\lambda}{4\pi\epsilon_0} \frac{z}{\sqrt{L^2 + z^2}} \left\{ \frac{-L + \sqrt{L^2 + z^2} - L - \sqrt{L^2 + z^2}}{(L^2 + z^2) - z^2} \right\}$$

$$= \hat{z} \frac{2L\lambda}{4\pi\epsilon_0} \frac{1}{z\sqrt{L^2 + z^2}}$$

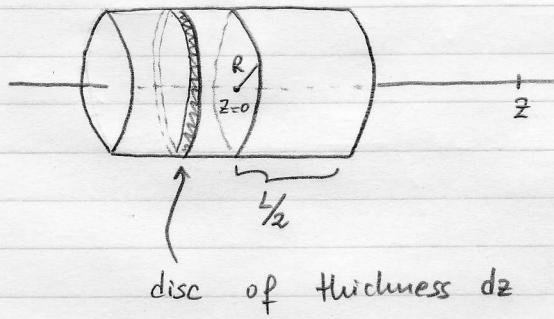
$$\text{Fig c)} \quad \vec{E} = -\nabla \left(\frac{\sigma}{2\epsilon_0} (\sqrt{z^2 + R^2} - z) \right)$$

$$= -\hat{z} \frac{\sigma}{2\epsilon_0} \left(\frac{1}{\sqrt{z^2 + R^2}} - 1 \right)$$

$$= \hat{z} \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right)$$

If the right-hand charge in Fig(a) is $-q$, then $V=0$, which suggests $\vec{E} = -\nabla V = 0$, in contradiction with the answer to Prob. 2.2b. The point is that we only know V on the z axis, and from this we cannot compute $E_x = -\partial V / \partial x$ or $E_y = -\partial V / \partial y$. That was OK with $+q$, because we knew from symmetry that $E_x = E_y = 0$. But now \vec{E} points in the x -direction, so knowing V on the z -axis is insufficient to determine \vec{E} .

Problem 2.27



$$dq = \sigma \pi R^2 dz' \Rightarrow d\sigma = \frac{dq}{\pi R^2} = \sigma dz'$$

From part c of the previous question we know

$$V(z) = \frac{\sigma}{2\epsilon_0} \left(\sqrt{(z-z')^2 + R^2} - (z-z') \right)$$

$$dV = \frac{d\sigma}{2\epsilon_0} \left(\sqrt{(z-z')^2 + R^2} - (z-z') \right)$$

$$= \frac{\sigma dz'}{2\epsilon_0} \left(\sqrt{(z-z')^2 + R^2} - (z-z') \right)$$

$$V(z) = \int dV = \frac{\sigma}{2\epsilon_0} \int_{-L/2}^{L/2} dz' \left[\sqrt{(z-z')^2 + R^2} - (z-z') \right]$$

let $z-z' = u$
 $du = -dz'$

$$= -\frac{\sigma}{2\epsilon_0} \int_{2+L/2}^{2-L/2} du \left(\sqrt{u^2 + R^2} - u \right)$$

$$= \frac{\sigma}{2\epsilon_0} \int_{2-L/2}^{2+L/2} du \left(\sqrt{u^2 + R^2} - u \right)$$

$$= \frac{\frac{q}{2}}{2\epsilon_0} \left\{ \frac{1}{2} u \sqrt{R^2 + u^2} + \frac{1}{2} R^2 \ln(u + \sqrt{R^2 + u^2}) - \frac{u^2}{2} \right\} \Big|_{z-L/2}^{z+L/2}$$

$$= \frac{q}{4\epsilon_0} \left\{ (z + \frac{L}{2}) \sqrt{R^2 + (z + \frac{L}{2})^2} - (z - \frac{L}{2}) \sqrt{R^2 + (z - \frac{L}{2})^2} \right. \\ \left. + R^2 \ln \left[\frac{z + \frac{L}{2} + \sqrt{R^2 + (z + \frac{L}{2})^2}}{z - \frac{L}{2} + \sqrt{R^2 + (z - \frac{L}{2})^2}} \right] - \underbrace{(z + \frac{L}{2})^2 + (z - \frac{L}{2})^2}_{-2zL} \right\}$$

$$\vec{E} = -\nabla V$$

$$= -\hat{z} \frac{q}{4\epsilon_0} \left\{ \sqrt{R^2 + (z + \frac{L}{2})^2} - (z + \frac{L}{2}) \cancel{\frac{1}{z} \frac{q(2 + \frac{L}{2})}{\sqrt{R^2 + (z + \frac{L}{2})^2}}} \right. \\ \left. - \sqrt{R^2 + (z - \frac{L}{2})^2} - (z - \frac{L}{2}) \cancel{\frac{1}{z} \frac{q(2 - \frac{L}{2})}{\sqrt{R^2 + (z - \frac{L}{2})^2}}} \right. \\ \left. + R^2 \left(\frac{1 + \cancel{\frac{1}{z} \frac{q(2 + \frac{L}{2})}{\sqrt{R^2 + (z + \frac{L}{2})^2}}}}{z + \frac{L}{2} + \sqrt{R^2 + (z + \frac{L}{2})^2}} - \frac{1 + \cancel{\frac{1}{z} \frac{q(2 - \frac{L}{2})}{\sqrt{R^2 + (z - \frac{L}{2})^2}}}}{z - \frac{L}{2} + \sqrt{R^2 + (z - \frac{L}{2})^2}} \right) - 2L \right\} \\ \underbrace{\frac{1}{\sqrt{R^2 + (z + \frac{L}{2})^2}} - \frac{1}{\sqrt{R^2 + (z - \frac{L}{2})^2}}}$$

$$\vec{E} = \frac{q}{2\epsilon_0} \left[L - \sqrt{R^2 + (z + \frac{L}{2})^2} + \sqrt{R^2 + (z - \frac{L}{2})^2} \right] \hat{z}$$

Problem 2.28.

Let us choose the point P along the z-axis, looked at $x=y=0$ and $z!$

$$\vec{r}' = r \sin\theta \cos\phi \hat{x} + r \sin\theta \sin\phi \hat{y} + r \cos\theta \hat{z}$$

$$\vec{r} = z \hat{z}$$

$$|\vec{r} - \vec{r}'|^2 = (\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}')$$

$$= (r \sin\theta \cos\phi \hat{x} + r \sin\theta \sin\phi \hat{y} + (r \cos\theta - z) \hat{z}) \cdot$$

$$(r \sin\theta \cos\phi \hat{x} + r \sin\theta \sin\phi \hat{y} + (r \cos\theta - z) \hat{z})$$

$$= r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + (r \cos\theta - z)^2$$

$$= r^2 \sin^2\theta + r^2 \cos^2\theta + z^2 - 2rz \cos\theta$$

$$= r^2 + z^2 - 2rz \cos\theta$$

$$V(\vec{r}) = \frac{1}{4\pi \epsilon_0} \int \frac{q}{(r^2 + z^2 - 2rz \cos\theta)^{1/2}} r^2 \sin\theta dr d\theta d\phi$$

$$\int d\phi = 2\pi$$

$$\int_0^\pi \frac{1}{(r^2 + z^2 - 2rz \cos\theta)^{1/2}} \sin\theta d\theta \quad u = \cos\theta \quad du = -\sin\theta d\theta$$

$$\begin{aligned} &= \int_{-1}^1 \frac{1}{(r^2 + z^2 - 2rzu)^{1/2}} du = -\frac{1}{2rz} \left[\sqrt{r^2 + z^2 - 2rzu} \right]_{-1}^1 \\ &= \frac{1}{rz} \left(\sqrt{r^2 + z^2 + 2rz} - \sqrt{r^2 + z^2 - 2rz} \right) \end{aligned}$$

$$= \frac{1}{r^2} (r+z - |r-z|) \quad (z>0 \text{ without loss of generality})$$

$$= \begin{cases} 2/z, & \text{if } r \leq z \\ 2/r, & \text{if } r > z \end{cases}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} 2\pi \cdot 2 \left\{ \int_0^z \frac{1}{r^2} r^2 dr + \int_z^R \frac{1}{r^2} r^2 dr \right\} = \frac{2}{2\epsilon_0} \left(R^2 - \frac{z^2}{3} \right)$$

$$\mathcal{D} = \frac{9}{\frac{4}{3}\pi R^3}$$

$$V(r) = \frac{9}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right)$$