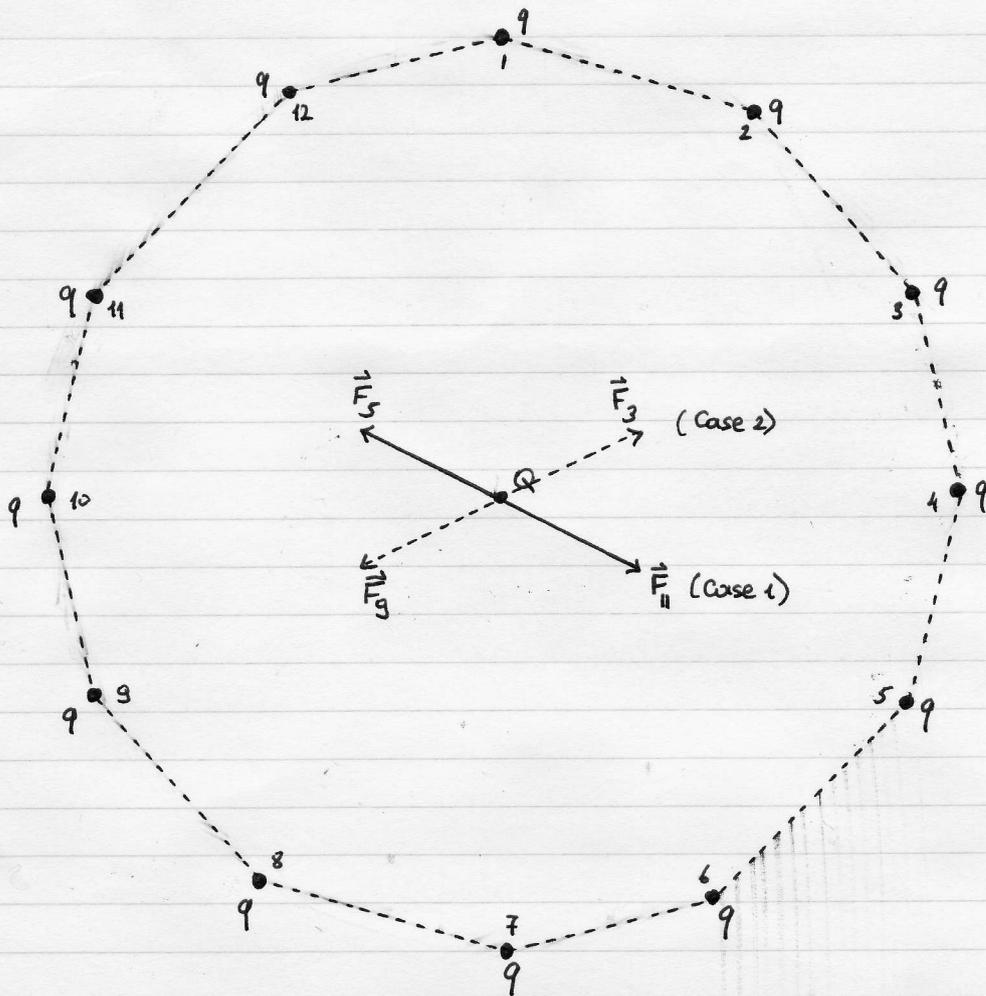


Phy 543

Homework Set 1

Problem 2.1.

a)



In the above figure I would like demonstrate two cases:

Case 1:  $q$  and  $Q$  have the same signs.

First note, that all charges ( $q$ 's) are at the same distance to  $Q$ , i.e. the magnitude of the force that they are applying is the same!

As I pointed out in the figure,  $\vec{F}_5$ , the force due to the 5th charge, is ~~in the~~ pointing in the opposite direction to  $\vec{F}_1$ , due to symmetry. It is easy to see that we have six such pairs each of which have net zero force on  $Q$ . Hence, the total net force is zero.

Case 2:  $q$  and  $Q$  have the opposite signs.

We have again a perfect cancellation of the force applied by symmetric pairs with the difference of the direction of the individual forces.

Remark: i) The above argument can be generalized to any  $2n$ -polygon.

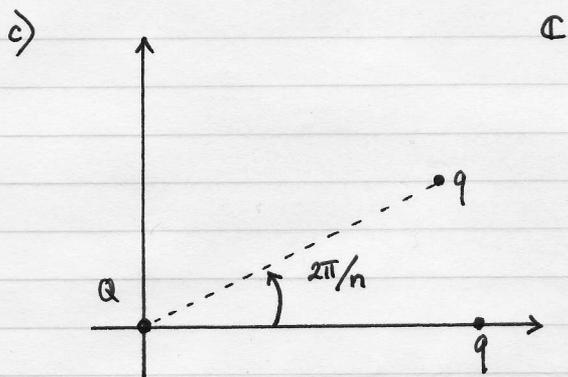
ii) We made use of the principle of superposition.

b) If we remove the 7th charge, the force due to the 1st charge won't be cancelled:

$$|\vec{F}| = \frac{1}{4\pi\epsilon_0} \frac{|Qq|}{r^2} \quad r: \text{the distance of } Q \text{ to } q\text{'s.}$$

$Qq > 0$ , the force points to the 7th charge

$Qq < 0$ , the force points to the 1st charge



In this part, I am going to present an argument which doesn't refer to any specific number of corners, like 13, but it is general for any  $n$ ! Imagine the charges are located in complex plane rather than  $\mathbb{R}^2$ !

For the sake of argument, without losing generality, assume  $Qq < 0$ !

The force vectors can be then identified (I mean their directions, since they all have the same magnitude) with the roots of unity:

$$\left(e^{2\pi i/n}\right)^k \equiv \omega_k^k \quad k=0,1,\dots,n-1$$

The net force reads

$$\sum_{k=0}^{n-1} \omega_k^k = 1 + \omega + \omega^2 + \dots + \omega^{n-1}$$

$$= \frac{1 - \omega^n}{1 - \omega} = 0 \quad \text{by the definition of } \omega!$$

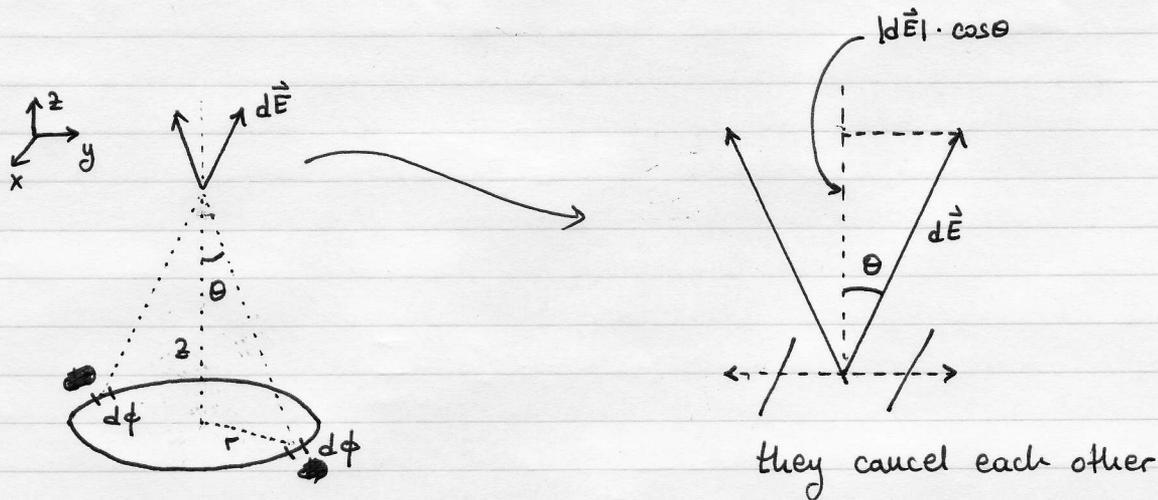
The argument applies for any  $n > 1$ !

d) It is obvious from the above argument that the net force will not vanish if we remove one of the charges.

Let me remove the charge corresponding to  $k=0$ !

$$\omega + \omega^2 + \dots + \omega^{n-1} = -1 \quad \text{hence } |\vec{F}| = \frac{1}{4\pi\epsilon_0} \frac{|Qq|}{r^2} \quad \text{and pointing to } -1 \text{ for } (qQ) < 0!$$

Problem 2.5.



The net field is in the z-direction

$$2 dE_z = 2 \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2+z^2} \cos\theta = \frac{2}{4\pi\epsilon_0} \frac{\rho d\phi}{r^2+z^2} \cos\theta \quad \rho = \text{charge density}$$

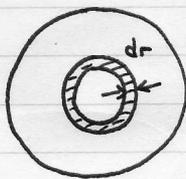
$$\cos\theta = \frac{z}{\sqrt{r^2+z^2}}$$

$$E_z = \int_0^\pi 2 dE_z = \frac{1}{4\pi\epsilon_0} \frac{z}{(r^2+z^2)^{3/2}} \rho \int_0^\pi 2 d\phi = \frac{1}{4\pi\epsilon_0} \frac{z}{(r^2+z^2)^{3/2}} \underbrace{(2\pi\rho)}_Q$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{z}{(r^2+z^2)^{3/2}} \hat{z}$$

Problem 2.6.

We can use the result of the previous problem. Let me imagine the disk is made of thin rings



$$dq = 2\pi r dr \sigma$$

↑ surface charge density

$$d\vec{E}(r) = \frac{dq}{4\pi\epsilon_0} \frac{z}{(r^2+z^2)^{3/2}} \hat{z}$$

Note that  $d\vec{E}(r)$  is a function of  $r$ !

$$\vec{E} = \int_0^R \frac{2\pi r dr \sigma}{4\pi\epsilon_0} \frac{z}{(r^2+z^2)^{3/2}} \hat{z}$$

$u = r^2+z^2 \quad du = 2r dr$

$$= \int_{z^2}^{R^2+z^2} \frac{\sigma du}{2\epsilon_0} \frac{z \hat{z}}{u^{3/2}} = \hat{z} \frac{\sigma z}{2\epsilon_0} (-2) \frac{1}{\sqrt{u}} \Big|_{z^2}^{R^2+z^2}$$

$$= \hat{z} \frac{\sigma z}{2\epsilon_0} \left( \frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right) = \hat{z} \frac{\sigma}{2\epsilon_0} \left( 1 - \frac{z}{\sqrt{R^2+z^2}} \right)$$

$$R \rightarrow \infty \Rightarrow \frac{z}{\sqrt{R^2+z^2}} \rightarrow 0$$

$$\vec{E} = \hat{z} \frac{\sigma}{2\epsilon_0}$$

which makes sense since this is the limit when the disk becomes an infinite plane!

$$z \gg R \Rightarrow \frac{R}{z} \ll 1 \quad \frac{z}{\sqrt{R^2+z^2}} = \frac{z}{z\sqrt{1+\frac{R^2}{z^2}}} = \frac{1}{\sqrt{1+\frac{R^2}{z^2}}} \approx 1 - \frac{1}{2} \frac{R^2}{z^2} + \mathcal{O}\left(\frac{R^4}{z^4}\right)$$

$$\vec{E} \approx \hat{z} \frac{\sigma}{2\epsilon_0} \left( 1 - 1 + \frac{1}{2} \frac{R^2}{z^2} \right) = \hat{z} \frac{\sigma R^2}{4\epsilon_0} \frac{1}{z^2} = \hat{z} \frac{\pi R^2 \sigma}{4\pi\epsilon_0} \frac{1}{z^2}$$

$$= \hat{z} \frac{Q}{4\pi\epsilon_0} \frac{1}{z^2} \quad \text{which also makes perfect sense since the disc will look like a point charge from infinite distance!}$$

Problem 2.12.

Apply Gauss' law:  $\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} Q_{\text{enclosed}}$

Take a Gaussian sphere (inside the charge sphere) of radius  $r < R$ !

$$Q_{\text{enclosed}} = \rho \frac{4}{3}\pi r^3 = \frac{Q}{\frac{4}{3}\pi R^3} \frac{4}{3}\pi r^3 = Q \frac{r^3}{R^3}$$

$$\oint \vec{E} \cdot d\vec{S} = E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} Q \frac{r^3}{R^3}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r \hat{r} = \frac{1}{3\epsilon_0} \rho r \hat{r}$$

Problem 2.15.

i)  $E=0$  since  $Q_{\text{enclosed}}=0$  for  $r < a$

ii) Take a Gaussian surface of radius  $r$  such that  $a < r < b$

$$\oint \vec{E} \cdot d\vec{S} = E 4\pi r^2 = \frac{1}{\epsilon_0} \int_a^r \rho(r') dV = \frac{1}{\epsilon_0} \int_a^r \frac{k}{r'} \underbrace{dr' r'^2 4\pi}_{\text{inf. volume element for sphere}}$$
$$= \frac{4\pi k}{\epsilon_0} (r-a)$$

$$\vec{E}(r) = \frac{k}{\epsilon_0} \frac{r-a}{r^2} \hat{r}$$

iii) Set  $r=b$  in the above integration:

$$\vec{E}(r) = \frac{k}{\epsilon_0} \frac{b-a}{r^2} \hat{r}$$

