Virtual probability current associated with the spin

Katsunori Mita
Department of Physics, St. Mary's College of Maryland, St. Mary's City, Maryland 20686

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A simple derivation of the spin probability current density from the expectation value of the spin operator is given. The properties of the spin probability current density are then examined in detail. We show that the spin probability current is solenoidal, virtual, and gives null contribution to the momentum of the particle. Expressions of the spin probability current density are derived for the Gaussian wave packet and the *s* states of the hydrogen atom.

I. INTRODUCTION

The spin current is a concept not often treated in textbooks of quantum mechanics, appearing in a very small number of texts. In the text by Landau and Lifshitz, the spin current density is derived, without mentioning its name, in an analysis of the current density for a charged particle moving in an external magnetic field.[1] In a more recent text, Greiner introduces the spin current density ad hoc, and without an example, drawing an analogy with the magnetization current density of classical electromagnetic theory.[2]

The lack of coverage is also reflected in this journal. We again find only a couple of papers on the spin current. Parker derived the hyperfine structure Hamiltonian for hydrogen by evaluating the magnetic field at the nucleus due to the electron's spin current density.[3] In an attempt to obtain a concrete physical picture of the spin, Ohanian used the spin current to argue that ''the spin may be regarded as an angular momentum generated by a circulating flow of energy in the wave field of the electron.''[4] His discussion is based on the momentum density of the Dirac field obtained from the symmetrized energy-momentum tensor. Though Ohanian's picture of the spin is intuitively appealing, it unfortunately goes beyond the level of undergraduate quantum mechanics, and is difficult to introduce in a classroom setting.

Except for Ohanian's paper, in all of the references previously cited, the spin current is introduced in conjunction with a magnetic field, whether the field is external or the electron's own. From this situation, one may acquire the impression that the spin current exists only in the context of the magnetic properties of the electron. Such is not the case.

In this article, we offer a straightforward derivation of the spin probability current, within the scope of nonrelativistic quantum mechanics, without relying on the magnetic properties of the electron (Sec. II). We then investigate the properties of the spin probability current (Sec. III). We will show that the spin probability current is solenoidal, virtual, and gives null contribution to the particle's momentum. Finally, expressions of spin probability current densities are derived for the Gaussian wave packet and the *s* states of the hydrogen atom, and their physical properties are examined (Sec. IV).

II. DERIVATION OF THE SPIN PROBABILITY CURRENT DENSITY

For a quantum particle of mass *m*, the expectation value of the orbital angular momentum operator **L** can be written in the form

$$\langle \mathbf{L} \rangle = m \int_{V_0} \mathbf{r} \times \mathbf{j} d^3 r, \tag{1}$$

where **j** is the probability current density defined by

$$\mathbf{j} = \frac{\hbar}{i2m} (\psi^* \nabla \psi - \psi \nabla \psi^*), \tag{2}$$

and V_0 denotes the entire space. The probability current density **j** satisfies the equation of continuity

$$\frac{\partial}{\partial t} (\psi^* \psi) + \nabla \cdot \mathbf{j} = 0, \tag{3}$$

expressing the local conservation of probability. Equation (1) expresses the orbital angular momentum of a quantum particle in terms of the circulating probability current. Equation (1) is derived in Appendix A.

For a particle with spin $\hbar/2$, we will rewrite the expectation value of the spin operator

$$\langle \mathbf{S} \rangle = \frac{\hbar}{2} \int_{V_0} \psi^\dagger \boldsymbol{\sigma} \psi d^3 r \tag{4}$$

in the same form as the orbital probability current, as expressed in Eq. (1). Following Ohanian, the idea is that the spin is another form of angular momentum due to another kind of circulating ''current.'' The nature of this current is investigated in Sec. III. In Eq. (4), ψ denotes a two-component spinor.

To carry this out, first observe the vector identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B}), \tag{5}$$

and let $\mathbf{A} = \mathbf{r}$ and $\mathbf{B} = \psi^\dagger \boldsymbol{\sigma} \psi$ in Eq. (5). Then we obtain

$$\psi^+ \boldsymbol{\sigma} \psi = \tfrac{1}{2} \mathbf{r} \times [\nabla \times (\psi^\dagger \boldsymbol{\sigma} \psi)] - \tfrac{1}{2} \nabla [\mathbf{r} \cdot (\psi^\dagger \boldsymbol{\sigma} \psi)] + \frac{1}{2} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} [x_i (\psi^+ \boldsymbol{\sigma} \psi)]. \tag{6}$$

Integrating Eq. (6) over the entire space, we have

$$\langle \mathbf{S}\rangle=\frac{\hbar}{4}\int_{V_0}\mathbf{r}\times[\nabla\times(\psi^\dagger\boldsymbol{\sigma}\psi)]d^3r-\frac{\hbar}{4}\int_{V_0}\nabla[\mathbf{r}\cdot(\psi^\dagger\boldsymbol{\sigma}\psi)]d^3r+\frac{\hbar}{4}\sum_{i=1}^{3}\int_{V_0}\frac{\partial}{\partial x_i}[x_i(\psi^\dagger\boldsymbol{\sigma}\psi)]d^3r. \tag{7}$$

The second integral on the right-hand side of Eq. (7) is rewritten as

$$\int_{V_0}\nabla[\mathbf{r}\cdot(\psi^\dagger\boldsymbol{\sigma}\psi)]d^3r=\int_{S_0}\mathbf{r}\cdot(\psi^\dagger\boldsymbol{\sigma}\psi)d^3r, \tag{8}$$

where S_0 denotes the surface at infinity, and an alternate form of Gauss's theorem[5] is used in Eq. (8). This integral vanishes if $\psi^\dagger\boldsymbol{\sigma}\psi\to 0$ faster than $1/r$ as $r\to\infty$. Similarly, the integrals in the third term of Eq. (7) are all vanishing under the same condition.

Thus we may write

$$\langle \mathbf{S}\rangle=m\int_{V_0}\mathbf{r}\times\mathbf{j}_S d^3r, \tag{9}$$

where

$$\mathbf{j}_S=\nabla\times\mathbf{V}_S, \tag{10}$$

with

$$\mathbf{V}_S=\frac{\hbar}{4m}\psi^\dagger\boldsymbol{\sigma}\psi. \tag{11}$$

Comparing Eqs. (1) and (9), we see that $\mathbf{j}_S$ may be regarded as a form of probability current density, giving rise to the spin of the particle. For this reason, we will refer to $\mathbf{j}_S$ as the spin probability current density, in contrast to the orbital probability current density appearing in Eq. (1). Then $\mathbf{V}_S$ in Eq. (11) is recognized as the vector potential of the spin probability current density $\mathbf{j}_S$.

The spin probability current density given in Eqs. (10) and (11) differs by a constant factor from the definition of the spin current density given by Greiner or Parker. These authors define the spin current as the electric current that gives rise to the correct magnetic moment of the electron including the g factor of 2. In this paper, we define $\mathbf{j}_S$ as the virtual probability current density that gives rise to the spin angular momentum of the electron. If we assume that the electric current density is given by the electron charge times $\mathbf{j}_S$ and calculate the magnetic moment of the electron, we obtain the quantity without the g factor. Hence the proper gyromagnetic ratio of the particle cannot be obtained in the context of our analysis based on nonrelativistic quantum mechanics.[6] It seems that some sort of additional physical mechanism, whether it is relativity or otherwise,[7] is necessary to obtain the proper gyromagnetic ratio of the electron.

III. PROPERTIES OF THE SPIN PROBABILITY CURRENT DENSITY

We next investigate the properties of the spin probability current density derived in Sec. II. First, since the divergence of a curl is always vanishing, we have

$$\nabla\cdot\mathbf{j}_S=0, \tag{12}$$

and $\mathbf{j}_S$ is intrinsically solenoidal. In view of the equation of continuity given by Eq. (3), this implies that the probability density is never altered by $\mathbf{j}_S$. This is in fact the case if $\mathbf{j}_S$ is an effective probability current density without a transport of probability density.

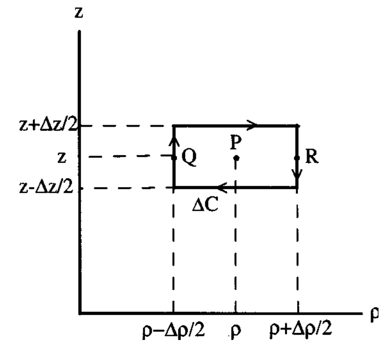

Fig. 1. Infinitesimal rectangle ΔC on a meridian plane.

We now show that the spin probability current is effective, or virtual, and has nothing to do with the motion of a particle. For this purpose, let the wave function of the particle with spin be in the form

$$\psi_\pm(\mathbf{r},t)=g(\mathbf{r},t)\chi_\pm, \tag{13}$$

where $\chi_\pm$ are eigenspinors of the spin operator S_z given by

$$\chi_+=\begin{pmatrix}1\\0\end{pmatrix},\quad \chi_-=\begin{pmatrix}0\\1\end{pmatrix}. \tag{14}$$

Then the vector potential of the spin current becomes

$$\mathbf{V}_{S\pm}=\frac{\hbar}{4m}|g|^2\chi_\pm^\dagger\boldsymbol{\sigma}\chi_\pm. \tag{15}$$

Since

$$\chi_\pm^\dagger\boldsymbol{\sigma}\chi_\pm=\pm\mathbf{k}, \tag{16}$$

where $\mathbf{k}$ is the unit vector along the z axis, and

$$|g|^2=\psi_\pm^\dagger\psi_\pm, \tag{17}$$

we obtain

$$\mathbf{V}_{S\pm}=\pm\mathbf{k}\frac{\lambda_\pm}{m}, \tag{18}$$

where

$$\lambda_\pm=\frac{\hbar}{4}\psi_\pm^\dagger\psi_\pm. \tag{19}$$

In order to examine the meaning of the spin probability current, consider an infinitesimal rectangle ΔC, as shown in Fig. 1, on a meridian plane surrounding a point P of the coordinates (ρ,φ,z), where $\rho=\sqrt{x^2+y^2}$ and $\varphi=\tan^{-1}(y/x)$. We will consider the spin-up current $\mathbf{j}_{S+}$ only. The extension of the analysis to the spin-down case is trivial, the only difference being the reversed direction of the current. The flux of the spin-up current $\mathbf{j}_{S+}$ through the infinitesimal surface $\Delta S=\Delta\rho\Delta z$ is given by

$$\int_{\Delta S}\mathbf{j}_{S+}\cdot d\mathbf{a}=\oint_{\Delta C}\mathbf{V}_{S+}\cdot d\mathbf{r}=\frac{1}{m}[(\lambda_+)_Q-(\lambda_+)_R]\Delta z, \tag{20}$$

where Eqs. (10), (18), and the Stokes's theorem are used to obtain Eq. (20). From Eq. (20), it follows that

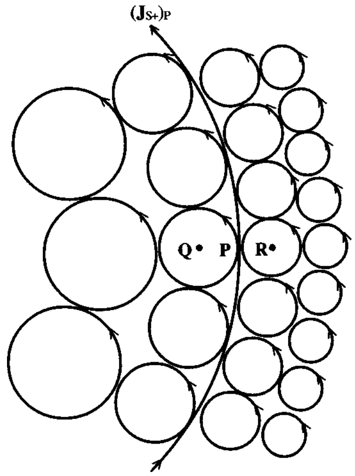

Fig. 2. Spin probability current density $\mathbf{j}_{S+}$ induced by the inhomogeneity of the angular momentum density λ_+.

$$(j_{S+})_P = \frac{1}{m} \lim_{\Delta\rho\to 0} \frac{(\lambda_+)_Q - (\lambda_+)_R}{\Delta\rho}. \tag{21}$$

Alternatively, by evaluating the curl of Eq. (18), we have

$$\mathbf{j}_{S+} = \frac{1}{m} \boldsymbol{\varphi}_0 \left(-\frac{\partial \lambda_+}{\partial \rho} \right), \tag{22}$$

where $\boldsymbol{\varphi}_0$ is the unit vector along the increasing azimuth angle φ. Equations (21) and (22) are equivalent to each other. Note that λ_+ is the angular momentum density proportional to the probability density $\psi_+^\dagger \psi_+$, and $\mathbf{V}_{S+}$ is proportional to λ_+ directed in the positive z direction everywhere. It appears that, at a given time, the particle ''spins'' at each point in space at a different rate proportional to the probability density, pointing in the positive z direction. And according to Eq. (21) or Eq. (22), the spin probability current is induced by the imbalance of the angular momentum density from one point to another along the radial direction. Figure 2 gives the pictorial representation of this situation in the vicinity of the point P. In this figure, the radius of each circle is proportional to the magnitude of the angular momentum density.

The physical situation described above is very similar to how the magnetization current is induced by the nonuniformity of the magnetization.[8] Just as the magnetization current is an effective electric current without an electric charge transport, the spin probability current is an effective, or virtual, probability current without a probability density transport. Since there is no transport of the probability density associated with the spin probability current, there is no motion of the particle from one point of space to another.

In other words, while the convection probability current density $\mathbf{j}$ gives the particle momentum in that

$$\langle \mathbf{P} \rangle = m \int_{V_0} \mathbf{j} d^3 r, \tag{23}$$

the spin probability current density yields the null contribution to the momentum of the particle. This point is seen directly from

$$m \int_{V_0} \mathbf{j}_S d^3 r = m \int_{V_0} \boldsymbol{\nabla} \times \mathbf{V}_S d^3 r = m \int_{S_0} d\mathbf{a} \times \mathbf{V}_S = 0, \tag{24}$$

under the condition described after Eq. (8).

The spin probability current is a virtual current which does not give rise to the particle's momentum. From this, it is natural to assume that $\mathbf{j}_S$ does not contribute to the particle's kinetic energy, either. Though it is difficult to examine this point in the context of the nonrelativistic quantum mechanics without making a conjecture, we mention that Belinfante showed some time ago that the spin energy-momentum tensor of arbitrary fields does not give any contribution to the energy of the particle.[9]

IV. EXAMPLES: GAUSSIAN WAVE PACKET AND *S* STATES OF THE HYDROGEN ATOM

We now present examples of spin probability currents and show how this concept offers an intuitive view of the spin of a particle. To illustrate the spin probability current in its simplest form, it is a good idea to eliminate the orbital probability current. Thus we first consider the motion of a particle with spin, say an electron, along a straight line represented by a three-dimensional, Gaussian wave packet. If the electron is moving along the positive z axis with a velocity $v_0 = \hbar k_0 / m_e$, with its spin directed parallel or antiparallel to the z axis, the wave packet may be expressed as

$$\psi_\pm(\mathbf{r},t) = \frac{1}{\sqrt{(\epsilon\sqrt{\pi})^3}} e^{-\xi^2/2\epsilon^2} e^{i(k_0 z - \omega_0 t\mu + \beta)} \chi_\pm, \tag{25}$$

where $\omega_0 = \hbar k_0^2 / 2m$,

$$\xi^2 = x^2 + y^2 + (z - v_0 t)^2, \tag{26}$$

and

$$\beta = \frac{\xi^2}{2\epsilon^2} \frac{t}{\tau} - \frac{3}{2} \tan^{-1}\left(\frac{t}{\tau}\right). \tag{27}$$

The spread of the Gaussian distribution ϵ at time $\mathbf{t}$ is given by

$$\epsilon = \epsilon_0 \sqrt{1 + \frac{t^2}{\tau^2}}, \tag{28}$$

where ϵ_0 denotes the spread at time $t=0$, and the time constant τ is given by

$$\tau = \frac{m_e \epsilon_0^2}{\hbar}. \tag{29}$$

Substituting Eq. (25) into Eqs. (18) and (19), we obtain the vector potential

$$\mathbf{V}_{S\pm} = \pm \mathbf{k} \frac{\hbar}{4 m_e (\epsilon\sqrt{\pi})^3} e^{-\xi^2/\epsilon^2}, \tag{30}$$

with the corresponding spin current density

$$\mathbf{j}_{S\pm} = \pm \boldsymbol{\varphi}_0 \frac{\hbar}{2 m_e (\sqrt{\pi})^3 \epsilon^5} \rho e^{-\xi^2/\epsilon^2}, \tag{31}$$

due to Eq. (22). Figure 3 shows the spin current of the spin-up wave packet at time $t=0$ on the xy plane with $z=0$. The circulating current forms a vortex and the center of the vortex is located at (0,0,0) at time $t=0$. As time elapses, the center of the vortex moves along the z axis with a constant velocity of v_0 as the size of the vortex increases according to Eqs. (28) and (29). Though the size of the vortex

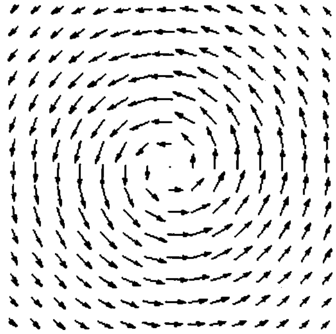

Fig. 3. Spin probability current of the spin-up wave packet at $t=0$ on the xy plane with $z=0$.

increases, the total angular momentum of the vortex must remain the same. To check this point, we evaluate

$$m_e \int_{V_0} \mathbf{r}\times\mathbf{j}_{S\pm} d^3r = \pm\mathbf{k}\frac{\hbar}{\epsilon^5\sqrt{\pi}}\int_0^\infty \rho^3 e^{-\rho^2/\epsilon^2} d\rho \times \int_{-\infty}^{\infty} e^{-(z-v_0t)^2/\epsilon^2} dz = \pm\mathbf{k}\frac{\hbar}{2}, \quad (32)$$

and the spin angular momentum is independent of time, though the integrand of the second integral in Eq. (32) contains time dependence.

As another example of the spin probability current, we look at the stationary states of the hydrogen atom. In particular, we examine the s states of the hydrogen atom with the spin of the electron included. For the s states, $l=0$ and the orbital probability currents are absent. But the spin probability currents do exist and we will see a novel property of $\mathbf{j}_S$ not possessed by the orbital probability currents. This property is a consequence of the fact that the expectation value $\langle s^2\rangle$ is constant for any state, unlike the case of $\langle L^2\rangle$.

For the s state with the principal quantum number n of the hydrogen atom, the wave functions for the spin-up and spin-down electron are given by

$$\psi_{ns\pm}(r) = \frac{1}{\sqrt{4\pi}} R_{n,0}(r)\chi_\pm, \quad (33)$$

where $r=\sqrt{x^2+y^2+z^2}$. We will use the spherical polar coordinates (r,θ,φ) in the following discussion of the hydrogen atom. The radial part of the hydrogen atom wave function is given by

$$R_{n,l}(r) = \sqrt{\frac{4(n-l-1)!}{n^3a_0^3 n(n+l)!}}\left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}(2r/na_0)e^{-r/na_0}, \quad (34)$$

where $L_p^q(\xi)$ are the associated Laguerre polynomials and a_0 is the Bohr radius. In particular, $R_{n,0}(r)$ appearing in Eq. (33) is given by

$$R_{n,0}(r) = -\frac{1}{\sqrt{n^3a_0}} e^{-r/na_0}\frac{d}{dr}[L_n(2r/na_0)], \quad (35)$$

where $L_p(\xi)$ are the Laguerre polynomials.

Substituting Eq. (33) into Eq. (11), we obtain the vector potential

$$(\mathbf{V}_S)_{ns\pm} = \pm\mathbf{k}\frac{\hbar}{16\pi m}[R_{n,l}(r)]^2, \quad (36)$$

where m here denotes the reduced mass of the system. Since the proton mass is much larger than the electron mass, we may approximate m as the electron mass. Taking the curl of $(\mathbf{V}_S)_{ns\pm}$, we have

$$(\mathbf{j}_S)_{ns\pm} = \mp\boldsymbol{\varphi}_0\frac{\hbar}{16\pi m}\sin\theta\frac{d}{dr}[R_{n,0}(r)]^2. \quad (37)$$

Equation (37) gives the spin probability current density for all s states of arbitrary n.

For $n=1$, the spin probability current density is given by

$$(\mathbf{j}_S)_{1s\pm} = \pm\boldsymbol{\varphi}_0\frac{\hbar}{2\pi m a_0^4}e^{-2r/a_0}\sin\theta. \quad (38)$$

The angular momentum due to $(\mathbf{j}_S)_{1s+}$ is evaluated as

$$m\int_{V_0}\mathbf{r}\times(\mathbf{j}_S)_{1s+}d^3r = \mathbf{k}\int_0^\infty(\Lambda_S)_{1s+}dr = \mathbf{k}\frac{\hbar}{2}, \quad (39)$$

where

$$(\Lambda_S)_{1s+} = \frac{4\hbar}{3a_0^4}r^3e^{-2r/a_0} \quad (40)$$

is the angular momentum of the spherical shell of unit thickness located at the radius r from the origin. The spin probability current density in Eq. (38) flows in a manner similar to that of the Gaussian wave packet at a given time.

While the flow of the spin probability current for the $1s$ state is similar to that of the wave packet, $\mathbf{j}_S$ for the $2s$ state is another story. The contrast between the orbital probability current and the spin probability current becomes striking for $n=2$. For the $2s$ state, the spin probability current density is given by

$$(\mathbf{j}_S)_{2s\pm} = \boldsymbol{\varphi}_0\frac{\hbar}{16\pi m a_0^4}\left(1-\frac{r}{2a_0}\right)\left(1-\frac{r}{4a_0}\right)e^{-r/a_0}\sin\theta. \quad (41)$$

Figure 4(a) depicts the cross section of $(\mathbf{j}_S)_{2s+}$ on the yz plane. In this figure, axes are in units of the Bohr radius. The white region means that the current flows into the paper and the black region, out of the paper. Very little or no current flows through the gray area. As we see in Fig. 4(a), an intriguing point is that the current changes its direction at $r=2a_0$ and again at $r=4a_0$. It seems that this is the mechanism that maintains the total angular momentum at a constant value of $\hbar/2$ as the arm of the probability distribution increases for higher principal quantum number n. From Eq. (B2) in Appendix B, it is clear that the orbital probability current does not change its direction for a particular set of quantum numbers $\{n,l,m_l\}$.

The angular momentum carried by $(\mathbf{j}_S)_{2s+}$ is calculated as

$$m\int_{V_0}\mathbf{r}\times(\mathbf{j}_S)_{2s+}d^3r = \mathbf{k}\int_0^\infty(\Lambda_S)_{2s+}dr = \mathbf{k}\frac{\hbar}{2}, \quad (42)$$

where

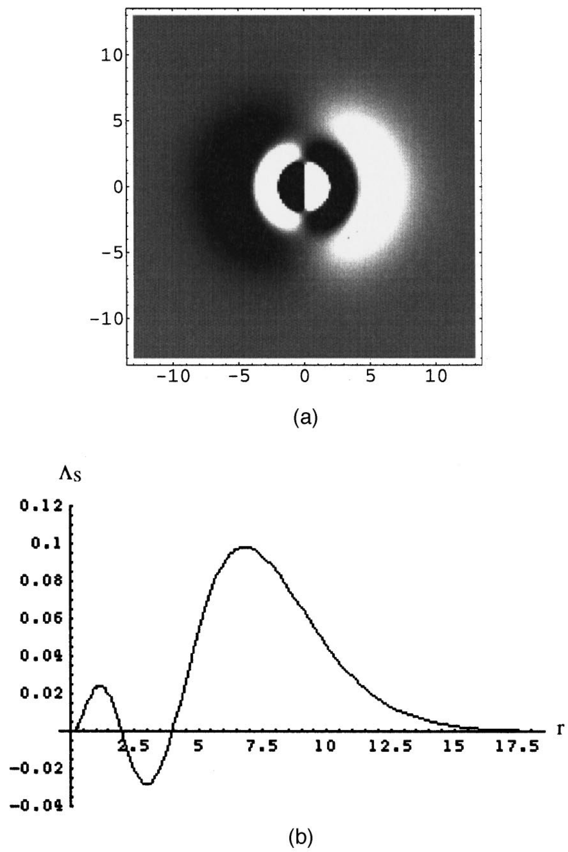

Fig. 4. Spin-up 2s state of the hydrogen atom: (a) cross-sectional view of the spin probability current density on the yz plane; (b) angular momentum density of the spherical shell.

$$(\Lambda_S)_{2s+}=\frac{\hbar}{6a_0^4}r^3\left(1-\frac{r}{2a_0}\right)\left(1-\frac{r}{4a_0}\right)e^{-r/a_0}. \tag{43}$$

Figure 4(b) shows the plot of $(\Lambda_S)_{2s+}$. As shown in Appendix B, for the orbital probability current, the angular momentum of the spherical shell of unit thickness becomes maximum for the radius of the Bohr orbit, $r=4a_0$ in this case. But for the spin probability current, $(\Lambda_S)_{2s+}=0$ for $r=4a_0$. Also, the angular momentum carried by the ''positive current'' centered around $r=4a_0$ is almost canceled by the ''negative current'' centered around $r=3a_0$, and the dominant contribution to the spin angular momentum stems from the next ''positive current'' centered around $r=7a_0$.

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APPENDIX A: DERIVATION OF EQ. (1)

In this appendix, we derive Eq. (1). Lévy-Leblond used this equation to show a case of the correspondence principle for a particle in a spherically symmetric potential.[10] The equation also appears in the textbook by Cohen-Tannoudji, Diu, and Laloë.[11] None of them, however, derived Eq. (1) or stated the conditions under which this equation holds valid. For this reason, we include the derivation of this equation.

The expectation value of the orbital angular momentum operator $\mathbf{L}$ is given by

$$\langle\mathbf{L}\rangle=\int_{V_0}\psi^*\mathbf{L}\psi d^3r, \tag{A1}$$

where V_0 denotes the entire space. The integrand in Eq. (A1) can be rewritten in the form

$$\psi^*\mathbf{L}\psi=m\mathbf{r}\times\mathbf{j}+\frac{\hbar}{i2}\mathbf{r}\times\boldsymbol{\nabla}(\psi^*\psi), \tag{A2}$$

where $\mathbf{j}$ is the probability current density defined in Eq. (2). Then $\langle\mathbf{L}\rangle$ is given as

$$\langle\mathbf{L}\rangle=m\int_{V_0}\mathbf{r}\times\mathbf{j}d^3r+\frac{\hbar}{i2}\int_{V_0}\mathbf{r}\times\boldsymbol{\nabla}(\psi^*\psi)d^3r. \tag{A3}$$

The integral in the second term of Eq. (A3) is rewritten as

$$\int_{V_0}\mathbf{r}\times\boldsymbol{\nabla}(\psi^*\psi)d^3r=-\int_{V_0}\boldsymbol{\nabla}\times(\mathbf{r}\psi^*\psi)d^3r$$

$$=-\int_{S_0}\psi^*\psi\mathbf{r}\times d\mathbf{a}, \tag{A4}$$

where S_0 is the surface at infinity, and an alternate form of Gauss's theorem is used to obtain the third expression in Eq. (A4). Now the integral vanishes under one of the following two conditions: first, when the quantum system in question possesses spherical symmetry so that $d\mathbf{a}=\mathbf{r}_0da$, where $\mathbf{r}_0$ denotes the unit vector in the radial direction. This may be in fact the case if the probability current is to be circulating; second, when the probability density $\psi^*\psi$ decreases faster than $1/r$ as $r\rightarrow\infty$. In many cases where an orbital probability current exists, the two conditions above may be simultaneously satisfied.

APPENDIX B: ORBITAL PROBABILITY CURRENT DENSITIES FOR THE HYDROGEN ATOM

In this appendix, we examine properties of the orbital probability current density for the hydrogen atom so that these properties may be contrasted to those of the spin probability current destiny.

The wave function for the stationary states of the hydrogen atom without electron spin is given by

$$\psi_{n,l,m_l}(\mathbf{r})=R_{n,l}(r)Y_l^{m_l}(\theta,\varphi), \tag{B1}$$

where $R_{n,l}(r)$ is the radial part of the wave function given in Eq. (34) and $Y_p^q(\theta,\varphi)$ is the spherical harmonic. Substituting Eq. (B1) into Eq. (2), we obtain

$$\mathbf{j}_{n,l,n_l}=\boldsymbol{\varphi}_0\frac{\hbar}{m}\frac{m_l}{r\sin\theta}|\psi_{n,l,m_l}|^2. \tag{B2}$$

The angular momentum due to this probability current density is evaluated as

$$m\int_{V_0}\mathbf{r}\times\mathbf{j}_{n,l,m_l}d^3r=\mathbf{k}\int_0^\infty\Lambda_{n,l,m_l}(r)dr=\mathbf{k}m_l\hbar, \tag{B3}$$

where

$$\Lambda_{n,l,m_l}(r)=m_l\hbar r^2[R_{n,l}(r)]^2. \tag{B4}$$

The quantity $\Lambda_{n,l,m_l} dr$ is the angular momentum of the spherical shell of thickness *dr* located at the radius *r* from the origin. To find $r_{\max}$ for which Λ_{n,l,m_l} becomes maximum, we let

$$\frac{d\Lambda_{n,l,m_l}}{dr}=0, \tag{B5}$$

which gives

$$R_{n,l}(\xi)+\xi\frac{d}{d\xi}R_{n,l}(\xi)=0, \tag{B6}$$

where

$$\xi=\frac{2r}{na_0}. \tag{B7}$$

Owing to

$$\xi\frac{d}{d\xi}[L_{n-l-1}^{2l+1}(\xi)]=(n-l-1)L_{n-l-1}^{2l+1}(\xi), \tag{B8}$$

the solution to Eq. (B6) is given by $\xi=2n$ or

$$r_{\max}=n^2a_0. \tag{B9}$$

Hence Λ_{n,l,m_l} assumes the maximum value for the radius of the Bohr orbit.

[1] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1965), 2nd ed., pp. 435–437.

[2] W. Greiner, *Quantum Mechanics* (Springer-Verlag, Berlin, 1989), pp. 241–242.

[3] G. W. Parker, ''Spin current density and the hyperfine interaction in hydrogen,'' Am. J. Phys. **52** (1), 36–39 (1984).

[4] H. C. Ohanian, ''What is spin?,'' Am. J. Phys. **54** (6), 500–505 (1986). Also see H. C. Ohanian, *Principles of Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1990), p. 244.

[5] G. Arfken, *Mathematical Methods for Physicists* (Academic, Orlando, 1985), 3rd ed., p. 59.

[6] In standard, nonrelativistic quantum mechanics, the *g* factor is inserted ''by hand.'' See, for example, R. Shankar, *Principles of Quantum Mechanics* (Plenum, New York, 1980), p. 400.

[7] J. M. Levy-Leblond showed that the correct *g* factor for the electron can be obtained through the linearization of the Schrödinger equation. See J. M. Lévy-Leblond, ''Nonrelativistic particles and wave equations,'' Commun. Math. Phys. **6**, 286–311 (1967). See also R. J. Gould, ''The intrinsic magnetic moment of elementary particles,'' Am. J. Phys. **64**, 597–601 (1996).

[8] For an excellent discussion of magnetization currents, see R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA, 1964), Vol. II, Chap. 36.

[9] F. J. Belinfante, ''On the spin angular momentum of mesons,'' Physica (Amsterdam) **6**, 887–898 (1939).

[10] J. M. Lévy-Leblond, ''The total probability current and the quantum period,'' Am. J. Phys. **55** (2), 146–149 (1987).

[11] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley, New York, 1977), Vol. I, p. 826.

NIGHTMARES

I can remember Bertrand Russell telling me of a horrible dream. He was in the top floor of the University Library, about A.D. 2100. A library assistant was going round the shelves carrying an enormous bucket, taking down book after book, glancing at them, restoring them to the shelves or dumping them into the bucket. At last he came to three large volumes which Russell could recognize as the last surviving copy of *Principia mathematica*. He took down one of the volumes, turned over a few pages, seemed puzzled for a moment by the curious symbolism, closed the volume, balanced it in his hand and hesitated. ...

G. H. Hardy, *A Mathematician's Apology* (Cambridge University Press, 1969; reprint of 1940 edition), p. 83.