

Let us compare W_{SO} with H_0 , which is of the order of e^2/R :

$$\frac{W_{SO}}{H_0} \simeq \frac{\frac{e^2 \hbar^2}{m_e^2 c^2 R^3}}{\frac{e^2}{R}} = \frac{\hbar^2}{m_e^2 c^2 R^2} \quad (\text{B-11})$$

R is of the order of the Bohr radius, $a_0 = \hbar^2/m_e e^2$. Consequently:

$$\frac{W_{SO}}{H_0} \simeq \frac{e^4}{\hbar^2 c^2} = \alpha^2 = \left(\frac{1}{137}\right)^2 \quad (\text{B-12})$$

γ . *The Darwin term W_D*

(i) *The physical origin*

In the Dirac equation, the interaction between the electron and the Coulomb field of the nucleus is "local"; it only depends on the value of the field at the electron position \mathbf{r} . However, the non-relativistic approximation (the series expansion in v/c) leads, for the two-component spinor which describes the electron state, to an equation in which the interaction between the electron and the field has become non-local. The electron is then affected by all the values taken on by the field in a domain centered at the point \mathbf{r} , and whose size is of the order of the Compton wave length $\hbar/m_e c$ of the electron. This is the origin of the correction represented by the Darwin term.

To understand this more precisely, assume that the potential energy of the electron, instead of being equal to $V(\mathbf{r})$, is given by an expression of the form:

$$\int d^3 \rho f(\boldsymbol{\rho}) V(\mathbf{r} + \boldsymbol{\rho}) \quad (\text{B-13})$$

where $f(\boldsymbol{\rho})$ is a function whose integral is equal to 1, which only depends on $|\boldsymbol{\rho}|$, and which takes on significant values only inside a volume of the order of $(\hbar/m_e c)^3$, centered at $\boldsymbol{\rho} = \mathbf{0}$.

If we neglect the variation of $V(\mathbf{r})$ over a distance of the order of $\hbar/m_e c$, we can replace $V(\mathbf{r} + \boldsymbol{\rho})$ by $V(\mathbf{r})$ in (B-13) and take $V(\mathbf{r})$ outside the integral, which is then equal to 1. (B-13) reduces, in this case, to $V(\mathbf{r})$.

A better approximation consists of replacing, in (B-13), $V(\mathbf{r} + \boldsymbol{\rho})$ by its Taylor series expansion in the neighborhood of $\boldsymbol{\rho} = \mathbf{0}$. The zeroth-order term gives $V(\mathbf{r})$. The first-order term is zero because of the spherical symmetry of $f(\boldsymbol{\rho})$. The second-order term involves the second derivatives of the potential energy $V(\mathbf{r})$ at the point \mathbf{r} and quadratic functions of the components of $\boldsymbol{\rho}$, weighted by $f(\boldsymbol{\rho})$ and integrated over $d^3 \rho$. This leads to a result of the order of

$$(\hbar/m_e c)^2 \Delta V(\mathbf{r})$$

It is therefore easy to accept the idea that this second-order term should be the Darwin term.

(ii) Order of magnitude

Replacing $V(R)$ by $-e^2/R$, we can write the Darwin term in the form :

$$-e^2 \frac{\hbar^2}{8m_e^2 c^2} \Delta \left(\frac{1}{R} \right) = \frac{\pi e^2 \hbar^2}{2m_e^2 c^2} \delta(\mathbf{R}) \quad (\text{B-14})$$

(we have used the expression for the Laplacian of $1/R$ given by formula (61) of appendix II).

When we take the mean value of (B-14) in an atomic state, we find a contribution equal to :

$$\frac{\pi e^2 \hbar^2}{2m_e^2 c^2} |\psi(\mathbf{0})|^2$$

where $\psi(\mathbf{0})$ is the value of the wave function at the origin. The Darwin term therefore affects only the s electrons, which are the only ones for which $\psi(\mathbf{0}) \neq 0$ (cf. chap. VII, § C-4-c). The order of magnitude of $|\psi(\mathbf{0})|^2$ can be obtained by taking the integral of the square of the modulus of the wave function over a volume of the order of a_0^3 (where a_0 is the Bohr radius) to be equal to 1. Thus we obtain :

$$|\psi(\mathbf{0})|^2 \simeq \frac{1}{a_0^3} = \frac{m_e^3 e^6}{\hbar^6} \quad (\text{B-15})$$

which gives the order of magnitude of the Darwin term :

$$W_D \simeq \frac{\pi e^2 \hbar^2}{2m_e^2 c^2} |\psi(\mathbf{0})|^2 \simeq m_e c^2 \frac{e^8}{\hbar^4 c^4} = m_e c^2 \alpha^4 \quad (\text{B-16})$$

Since $H_0 \simeq m_e c^2 \alpha^2$, we again see that :

$$\frac{W_D}{H_0} \simeq \alpha^2 = \left(\frac{1}{137} \right)^2 \quad (\text{B-17})$$

Thus, all the fine structure terms are about 10^4 times smaller than the non-relativistic Hamiltonian of chapter VII.

2. Magnetic interactions related to proton spin: the hyperfine Hamiltonian

a. PROTON SPIN AND MAGNETIC MOMENT

Thus far, we have considered the proton to be a physical point of masse M_p and charge $q_p = -q$. Actually, the proton, like the electron, is a spin 1/2 particle. We shall denote by \mathbf{I} the corresponding spin observable.

With the spin \mathbf{I} of the proton is associated a magnetic moment \mathbf{M}_I . However, the gyromagnetic ratio is different from that of the electron :

$$\mathbf{M}_I = g_p \mu_n \mathbf{I} / \hbar \quad (\text{B-18})$$

where μ_n is the *nuclear Bohr magneton*:

$$\mu_n = \frac{q_p \hbar}{2M_p} \quad (\text{B-19})$$

and the factor g_p , for the proton, is equal to: $g_p \simeq 5.585$. Because of the presence of M_p (the proton mass) in the denominator of (B-19), μ_n is close to 2 000 times smaller than the Bohr magneton μ_B (recall that $\mu_B = q\hbar/2m_e$). Although the angular momenta of the proton and the electron are the same, nuclear magnetism, because of the mass difference, is much less important than electronic magnetism. The magnetic interactions due to the proton spin \mathbf{I} are therefore very weak.

b. THE MAGNETIC HYPERFINE HAMILTONIAN W_{hf}

The electron moves, therefore, not only in the electrostatic field of the proton, but also in the magnetic field created by \mathbf{M}_I . When we introduce the corresponding vector potential into the Schrödinger equation*, we find that we must add to the Hamiltonian (B-1) an additional series of terms for which the expression is (cf. complement A_{XII}):

$$W_{hf} = -\frac{\mu_0}{4\pi} \left\{ \frac{q}{m_e R^3} \mathbf{L} \cdot \mathbf{M}_I + \frac{1}{R^3} [3(\mathbf{M}_S \cdot \mathbf{n})(\mathbf{M}_I \cdot \mathbf{n}) - \mathbf{M}_S \cdot \mathbf{M}_I] + \frac{8\pi}{3} \mathbf{M}_S \cdot \mathbf{M}_I \delta(\mathbf{R}) \right\} \quad (\text{B-20})$$

\mathbf{M}_S is the spin magnetic moment of the electron, and \mathbf{n} is the unit vector of the straight line joining the proton to the electron (fig. 1).

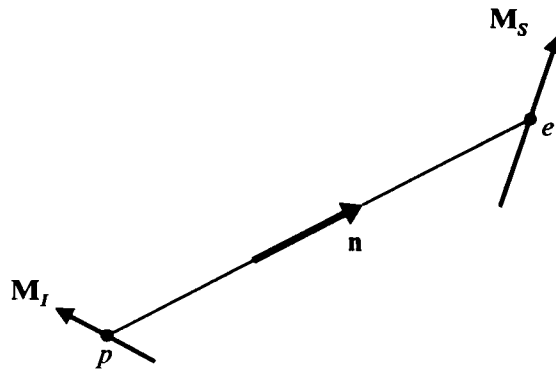


FIGURE 1

Relative disposition of the magnetic moments \mathbf{M}_I and \mathbf{M}_S of the proton and the electron; \mathbf{n} is the unit vector on the line joining the two particles.

We shall see that W_{hf} introduces energy shifts which are small compared to those created by W_f . This is why W_{hf} is called the “hyperfine structure Hamiltonian”.

* Since the hyperfine interactions are very small corrective terms, they can be found using the non-relativistic Schrödinger equation.

c. INTERPRETATION OF THE VARIOUS TERMS OF W_{hf}

The first term of W_{hf} represents the interaction of the nuclear magnetic moment M_I with the magnetic field $(\mu_0/4\pi)q\mathbf{L}/m_e r^3$ created at the proton by the rotation of the electronic charge.

The second term represents the dipole-dipole interaction between the electronic and nuclear magnetic moments: the interaction of the magnetic moment of the electron spin with the magnetic field created by M_I (*cf.* complement B_{XI}) or vice versa.

Finally, the last term, also called Fermi's "contact term", arises from the singularity at $r = 0$ of the field created by the magnetic moment of the proton. In reality, the proton is not a point. It can be shown (*cf.* complement A_{XII}) that the magnetic field inside the proton does not have the same form as the one created outside by M_I (and which enters into the dipole-dipole interaction). The contact term describes the interaction of the magnetic moment of the electron spin with the magnetic field inside the proton (the "delta" function expresses the fact that this contact term exists, as its name indicates, only when the wave functions of the electron and proton overlap).

d. ORDERS OF MAGNITUDE

It can easily be shown that the order of magnitude of the first two terms of W_{hf} is:

$$\frac{q^2 \hbar^2}{m_e M_p R^3} \frac{\mu_0}{4\pi} = \frac{e^2 \hbar^2}{m_e M_p c^2} \frac{1}{R^3} \quad (\text{B-21})$$

By using (B-10), we see that these terms are about 2 000 times smaller than W_{SO} .

As for the last term of (B-20), it is also 2 000 times smaller than the Darwin term, which also contains a $\delta(\mathbf{R})$ function.

C. THE FINE STRUCTURE OF THE $n = 2$ LEVEL

1. Statement of the problem

a. DEGENERACY OF THE $n = 2$ LEVEL

We saw in chapter VII that the energy of the hydrogen atom depends only on the quantum number n . The $2s$ ($n = 2, l = 0$) and $2p$ ($n = 2, l = 1$) states therefore have the same energy, equal to:

$$-\frac{E_I}{4} = -\frac{1}{8} \mu c^2 \alpha^2$$

If the spins are ignored, the $2s$ subshell is composed of a single state, and the $2p$ subshell of three distinct states which differ by their eigenvalue $m_L \hbar$ of the component L_z of the orbital angular momentum \mathbf{L} ($m_L = 1, 0, -1$). Because of the

existence of electron and proton spins, the degeneracy of the $n = 2$ level is higher than the value calculated in chapter VII. The components S_z and I_z of the two spins can each take on two values : $m_s = \pm 1/2$, $m_I = \pm 1/2$. One possible orthonormal basis in the $n = 2$ level is then :

$$\left\{ \left| n = 2; l = 0; m_L = 0; m_s = \pm \frac{1}{2}; m_I = \pm \frac{1}{2} \right\rangle \right\} \quad (\text{C-1})$$

($2s$ subshell, of dimension 4)

$$\left\{ \left| n = 2; l = 1; m_L = -1, 0, +1; m_s = \pm \frac{1}{2}; m_I = \pm \frac{1}{2} \right\rangle \right\} \quad (\text{C-2})$$

($2p$ subshell, of dimension 12).

The $n = 2$ shell then has a total degeneracy equal to 16.

According to the results of chapter XI (§C), in order to calculate the effect of a perturbation W on the $n = 2$ level, it is necessary to diagonalize the 16×16 matrix representing the restriction of W to this level. The eigenvalues of this matrix are the first order corrections to the energy, and the corresponding eigenstates are the eigenstates of the Hamiltonian to zeroth order.

b. THE PERTURBATION HAMILTONIAN

In all of this section, we shall assume that no external field is applied to the atom. The difference W between the exact Hamiltonian H and the Hamiltonian H_0 of chapter VII (§C) contains fine structure terms, indicated in §B-1 above :

$$W_f = W_{mv} + W_{SO} + W_D \quad (\text{C-3})$$

and hyperfine structure terms W_{hf} , introduced in § B-2. We thus have :

$$W = W_f + W_{hf} \quad (\text{C-4})$$

Since W_f is close to 2 000 times larger than W_{hf} (cf. §B-2-d), we must obviously begin by studying the effect of W_f , before considering that of W_{hf} , on the $n = 2$ level. We shall see that the $n = 16$ degeneracy of this level is partially removed by W_f . The structure which appears in this way is called the "fine structure".

W_{hf} may then remove the remaining degeneracy of the fine structure levels and cause a "hyperfine structure" to appear inside each of these levels.

In this section (§C), we shall confine ourselves to the study of the fine structure of the $n = 1$ level. The calculations can easily be generalized to other levels.

2. Matrix representation of the fine-structure Hamiltonian W_f inside the $n = 2$ level

a. GENERAL PROPERTIES

The properties of W_f , as we shall see, enable us to show that the 16×16 matrix which represents it in the $n = 2$ level can be broken down into a series of square submatrices of smaller dimensions. This will considerably simplify the determination of the eigenvalues and eigenvectors of this matrix.

$\alpha.$ W_f does not act on the spin variables of the proton

We see from (B-1) that the fine structure terms do not depend on \mathbf{I} . It follows that the proton spin can be ignored in the study of the fine structure (afterwards, we multiply by 2 all the degrees of degeneracy obtained). The dimension of the matrix to be diagonalized therefore falls from 16 to 8.

$\beta.$ W_f does not connect the $2s$ and $2p$ subshells

Let us first prove that L^2 commutes with W_f . The operator L^2 commutes with the various components of \mathbf{L} , with R (L^2 acts only on the angular variables), with \mathbf{P}^2 [cf. formula (A-16) of chapter VII], and with \mathbf{S} (L^2 does not act on the spin variables). L^2 therefore commutes with W_{mv} (which is proportional to \mathbf{P}^4), with W_{SO} (which depends only on $R, \mathbf{L}, \mathbf{S}$), and with W_D (which depends only on R).

The $2s$ and $2p$ states are eigenstates of L^2 with different eigenvalues (0 and $2\hbar^2$). Therefore, W_f , which commutes with L^2 , has no matrix elements between a $2s$ state and a $2p$ state. The 8×8 matrix representing W_f inside the $n = 2$ level can be broken down, consequently, into a 2×2 matrix relative to the $2s$ state and a 6×6 matrix relative to the $2p$ state :

$$(W_f)_{n=2} = \begin{array}{c} \begin{array}{cc} & \begin{array}{c} 2s \\ 2p \end{array} \\ \begin{array}{c} 2s \\ 2p \end{array} & \begin{array}{|c|c|} \hline \text{diagonal} & 0 \\ \hline 0 & \text{diagonal} \\ \hline \end{array} \end{array} \end{array}$$

COMMENT :

The preceding property can also be considered to be a consequence of the fact that W_f is even. Under a reflection, \mathbf{R} changes to $-\mathbf{R}$ ($R = |\mathbf{R}|$ remains unchanged), \mathbf{P} , to $-\mathbf{P}$, \mathbf{L} to \mathbf{L} , and \mathbf{S} to \mathbf{S} . It is then easy to see that W_f remains invariant. W_f therefore has no matrix elements between the $2s$ and $2p$ states, which are of opposite parity (cf. complement F_{II}).

b. MATRIX REPRESENTATION OF W_f IN THE $2s$ SUBSHELL

The dimension 2 of the $2s$ subspace is the result of the two possible values $m_s = \pm 1/2$ of S_z (since we are ignoring I_z for the moment).

W_{mv} and W_D do not depend on S . The matrices which represent these two operators in the $2s$ subspace are therefore multiples of the unit matrix, with proportionality coefficients equal, respectively, to the purely orbital matrix elements:

$$\langle n = 2; l = 0; m_L = 0 | -\frac{\mathbf{P}^4}{8m_e^3c^2} | n = 2; l = 0; m_L = 0 \rangle$$

and:

$$\langle n = 2; l = 0; m_L = 0 | \frac{\hbar^2}{8m_e^2c^2} q \Delta V(R) | n = 2; l = 0; m_L = 0 \rangle$$

Since we know the eigenfunctions of H_0 , the calculation of these matrix elements presents no theoretical difficulty. We find (*cf.* complement B_{XII}):

$$\langle W_{mv} \rangle_{2s} = -\frac{13}{128} m_e c^2 \alpha^4 \quad (\text{C-5})$$

$$\langle W_D \rangle_{2s} = \frac{1}{16} m_e c^2 \alpha^4 \quad (\text{C-6})$$

Finally, calculation of the matrix elements of W_{SO} involves "angular" matrix elements of the form $\langle l = 0, m_L = 0 | L_{x,y,z} | l = 0, m_L = 0 \rangle$, which are zero because of the value $l = 0$ of the quantum number l . Therefore:

$$\langle W_{SO} \rangle_{2s} = 0 \quad (\text{C-7})$$

Thus, under the effect of the fine structure terms, the $2s$ subshell is shifted as a whole with respect to the position calculated in chapter VII by an amount equal to $-5m_e c^2 \alpha^4 / 128$.

c. MATRIX REPRESENTATION OF W_f IN THE $2p$ SUBSHELL $\alpha.$ W_{mv} and W_D terms

The W_{mv} and W_D terms commute with the various components of \mathbf{L} , since \mathbf{L} acts only on the angular variables and commutes with R and \mathbf{P}^2 (which depends on these variables only through \mathbf{L}^2 ; *cf.* chapter VII). \mathbf{L} therefore commutes with W_{mv} and W_D . Consequently, W_{mv} and W_D are scalar operators with respect to the orbital variables (*cf.* complement B_{VI}, §5-b). Since W_{mv} and W_D do not act on the spin variables, it follows that the matrices which represent W_{mv} and W_D inside the $2p$ subspace are multiples of the unit matrix. The calculation of the proportionality coefficient is given in complement B_{XII} and leads to:

$$\langle W_{mv} \rangle_{2p} = -\frac{7}{384} m_e c^2 \alpha^4 \quad (\text{C-8})$$

$$\langle W_D \rangle_{2p} = 0 \quad (\text{C-9})$$

The result (C-9) is due to the fact that W_D is proportional to $\delta(\mathbf{R})$ and can therefore have a non-zero mean value only in an s state (for $l \geq 1$, the wave function is zero at the origin).

$\beta.$ W_{SO} term

We must calculate the various matrix elements:

$$\langle n = 2; l = 1; s = \frac{1}{2}; m'_L; m'_S | \xi(R) \mathbf{L} \cdot \mathbf{S} | n = 2; l = 1; s = \frac{1}{2}; m_L; m_S \rangle \quad (\text{C-10})$$

with:

$$\xi(R) = \frac{e^2}{2m_e^2 c^2} \frac{1}{R^3} \quad (\text{C-11})$$

If we use the $\{ | \mathbf{r} \rangle \}$ representation, we can separate the radial part of matrix element (C-10) from the angular and spin parts. Thus we obtain:

$$\xi_{2p} \langle l = 1; s = \frac{1}{2}; m'_L; m'_S | \mathbf{L} \cdot \mathbf{S} | l = 1; s = \frac{1}{2}; m_L; m_S \rangle \quad (\text{C-12})$$

where ξ_{2p} is a number, equal to the radial integral:

$$\xi_{2p} = \frac{e^2}{2m_e^2 c^2} \int_0^\infty \frac{1}{r^3} |R_{21}(r)|^2 r^2 dr \quad (\text{C-13})$$

Since we know the radial function $R_{21}(r)$ of the $2p$ state, we can calculate ξ_{2p} . We find (*cf.* complement B_{XII}):

$$\xi_{2p} = \frac{1}{48\hbar^2} m_e c^2 \alpha^4 \quad (\text{C-14})$$

The radial variables have therefore disappeared. According to (C-12), the problem is reduced to the diagonalization of the operator $\xi_{2p} \mathbf{L} \cdot \mathbf{S}$, which acts only on the angular and spin variables.

To represent the operator $\xi_{2p} \mathbf{L} \cdot \mathbf{S}$ by a matrix, several different bases can be chosen :

– first of all, the basis:

$$\left\{ | l = 1; s = \frac{1}{2}; m_L; m_S \rangle \right\} \quad (\text{C-15})$$

which we have used thus far and which is constructed from common eigenstates of $\mathbf{L}^2, \mathbf{S}^2, L_z, S_z$;

– or, introducing the total angular momentum:

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \quad (\text{C-16})$$

the basis:

$$\left\{ | l = 1; s = \frac{1}{2}; J; m_J \rangle \right\} \quad (\text{C-17})$$

constructed from the eigenstates common to L^2 , S^2 , J^2 , J_z . According to the results of chapter X, since $l = 1$ and $s = 1/2$, J can take on two values : $J = 1 + 1/2 = 3/2$ and $J = 1 - 1/2 = 1/2$. Furthermore, we know how to go from one basis to the other, thanks to the Clebsch-Gordan coefficients [formulas (36) of complement A_x].

We shall now show that the second basis (C-17) is better adapted than the first one to the problem which interests us here, since $\xi_{2p} \mathbf{L} \cdot \mathbf{S}$ is diagonal in the basis (C-17). To see this, we square both sides of (C-16). We find (\mathbf{L} and \mathbf{S} commute):

$$\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2 \mathbf{L} \cdot \mathbf{S} \quad (\text{C-18})$$

which gives :

$$\xi_{2p} \mathbf{L} \cdot \mathbf{S} = \frac{1}{2} \xi_{2p} (\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2) \quad (\text{C-19})$$

Each of the basis vectors (C-17) is an eigenstate of L^2 , S^2 , J^2 ; we thus have :

$$\begin{aligned} \xi_{2p} \mathbf{L} \cdot \mathbf{S} |l = 1; s = \frac{1}{2}; J; m_J \rangle \\ = \frac{1}{2} \xi_{2p} \hbar^2 \left[J(J + 1) - 2 - \frac{3}{4} \right] |l = 1; s = \frac{1}{2}; J; m_J \rangle \end{aligned} \quad (\text{C-20})$$

We see from (C-20) that the eigenvalues of $\xi_{2p} \mathbf{L} \cdot \mathbf{S}$ depend only on J and not on m_J ; they are equal to:

$$\frac{1}{2} \xi_{2p} \left[\frac{3}{4} - 2 - \frac{3}{4} \right] \hbar^2 = - \xi_{2p} \hbar^2 = - \frac{1}{48} m_e c^2 \alpha^4 \quad (\text{C-21})$$

for $J = 1/2$, and:

$$\frac{1}{2} \xi_{2p} \left[\frac{15}{4} - 2 - \frac{3}{4} \right] \hbar^2 = + \frac{1}{2} \xi_{2p} \hbar^2 = \frac{1}{96} m_e c^2 \alpha^4 \quad (\text{C-22})$$

for $J = 3/2$.

The six-fold degeneracy of the $2p$ level is therefore partially removed by W_{SO} . We obtain a four-fold degenerate level corresponding to $J = 3/2$, and a two-fold degenerate level corresponding to $J = 1/2$. The $(2J + 1)$ -fold degeneracy of each J state is an essential degeneracy related to the rotation invariance of W_f .

COMMENTS:

- (i) In the $2s$ subspace ($l = 0$, $s = 1/2$), J can take on only one value, $J = 0 + 1/2 = 1/2$.
- (ii) In the $2p$ subspace, W_{mv} and W_D are represented by multiples of the unit matrix. This property remains valid in any basis since the unit matrix is invariant under a change of basis. The choice of basis (C-17), required by the W_{SO} term, is therefore also adapted to the W_{mv} and W_D terms.

3. Results: the fine structure of the $n = 2$ level

a. SPECTROSCOPIC NOTATION

In addition to the quantum numbers n , l (and s), the preceding discussion introduced the quantum number J on which the energy correction due to the spin-orbit coupling term depends.

For the $2s$ level, $J = 1/2$; for the $2p$ level, $J = 1/2$ or $J = 3/2$. The level associated with a set of values, n , l , J is generally denoted by adding an index J to the symbol representing the (n, l) subshell in spectroscopic notation (*cf.* chap. VII, § C-4-b):

$$n l_J \quad (C-23)$$

where l stands for the letter s for $l = 0$, p for $l = 1$, d for $l = 2$, f for $l = 3$... Thus, the $n = 2$ level of the hydrogen atom gives rise to the $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$ levels.

b. POSITIONS OF THE $2s_{1/2}$, $2p_{1/2}$ AND $2p_{3/2}$ LEVELS

By regrouping the results of § 2, we can now calculate the positions of the $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$ levels with respect to the "unperturbed" energy of the $n = 2$ level calculated in chapter VII and equal to $-\mu c^2 \alpha^2 / 8$.

According to the results of § 2-b, the $2s_{1/2}$ level is lowered by a quantity equal to:

$$-\frac{5}{128} m_e c^2 \alpha^4 \quad (C-24)$$

According to the results of § 2-c, the $2p_{1/2}$ level is lowered by a quantity equal to:

$$\left(-\frac{7}{384} - \frac{1}{48}\right) m_e c^2 \alpha^4 = -\frac{5}{128} m_e c^2 \alpha^4 \quad (C-25)$$

Thus we see that the $2s_{1/2}$ and $2p_{1/2}$ levels have the same energy. According to the theory presented here, this degeneracy must be considered to be accidental, as opposed to the essential $(2J + 1)$ -fold degeneracy of each J level.

Finally, the $2p_{3/2}$ level is lowered by a quantity:

$$\left(-\frac{7}{384} + \frac{1}{96}\right) m_e c^2 \alpha^4 = -\frac{1}{128} m_e c^2 \alpha^4 \quad (C-26)$$

The preceding results are shown in figure 2.

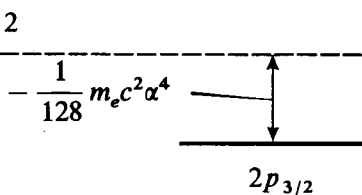
COMMENTS:

- (i) Only the spin-orbit coupling is responsible for the separation between the $2p_{1/2}$ and $2p_{3/2}$ levels, since W_{mv} and W_D shift the entire $2p$ level as a whole.

- (ii) The hydrogen atom can go from the $2p$ state to the $1s$ state by emitting a Lyman α photon ($\lambda = 1216 \text{ \AA}$). The material presented in this chapter shows that, because of the spin-orbit coupling, the Lyman α line actually contains two neighboring lines*, $2p_{1/2} \rightarrow 1s_{1/2}$ and $2p_{3/2} \rightarrow 1s_{1/2}$, separated by an energy difference equal to:

$$\frac{4}{128} m_e c^2 \alpha^4 = \frac{1}{32} m_e c^2 \alpha^4$$

When they are observed with a sufficient resolution, the lines of the hydrogen spectrum therefore present a "fine structure".



the hydrogen atom. Under the effect of the fine structure into three fine structure levels, written $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$. of the shifts, calculated to first order in W_f . The shifts are the result which remains valid, moreover, to all orders in W_f . um mechanical nature of the electromagnetic field, we find and $2p_{1/2}$ levels is removed (the Lamb shift; see figure 4).

els with the same J have the same energy. This result W_f : it remains valid to all orders. The exact solution the energy of a level characterized by the quantum

$$\left[1 + \sqrt{(J + 1/2)^2 - \alpha^2} \right]^{-2} \right]^{-\frac{1}{2}} \quad (\text{C-27})$$

$1/2$, so J can take on only the value $J = 1/2$. W_f therefore te, and there is only one fine structure level, the $1s_{1/2}$ level. is the only one for which l is necessarily zero. This is why = 2 level.

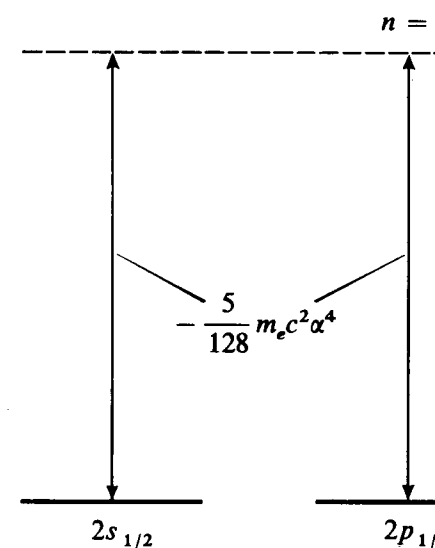


FIGURE 2

Fine structure of the $n = 2$ level of Hamiltonian W_f , the $n = 2$ level splits. We have indicated the algebraic values same for the $2s_{1/2}$ and $2p_{1/2}$ levels (a When we take into account the quant that the degeneracy between the $2s_{1/2}$

- (iii) We see in figure 2 that the two levels is not merely true to first order in of the Dirac equation gives, for numbers n, l, s, J , the value:

$$E_{n,J} = m_e c^2 \left[1 + \alpha^2 \left(n - J - \frac{1}{2} + \dots \right) \right]$$

* In the ground state, $l = 0$ and $s = 1/2$, the Dirac equation does not remove the degeneracy of the $1s$ state. This is a special case, since the ground state we have chosen here to study the excited $n = 2$ level.