

FIGURE 9

The Zeeman diagram (for an arbitrary field) of the $1s$ ground state of the hydrogen atom: m_F remains a good quantum number for any value of the field. We obtain two straight lines, of opposite slopes, corresponding to the values $+1$ and -1 of m_F , as well as a hyperbola whose two branches are associated with the two $m_F = 0$ levels. Figures 5 and 7 give, respectively, the tangents at the origin and the asymptotes of the levels shown in this diagram.

We see that the energy depends only on n and J , and not on l .

If we make a limited expansion of formula (C-27) in powers of α , we obtain:

$$E_{n,J} = m_e c^2 - \frac{1}{2} m_e c^2 \alpha^2 \frac{1}{n^2} - \frac{m_e c^2}{2n^4} \left(\frac{n}{J + 1/2} - \frac{3}{4} \right) \alpha^4 + \dots \quad (\text{C-28})$$

The first term is the rest-mass-energy of the electron. The second term follows from the theory of chapter VII. The third term gives the correction to first order in W_f calculated in this chapter.

- (iv) Even in the absence of an external field and incident photons, a fluctuating electromagnetic field must be considered to exist in space (*cf.* complement K_v, § 3-d- δ). This phenomenon is related to the quantum mechanical nature of the electromagnetic field, which we have not taken into consideration here. The coupling of the atom with these fluctuations of the electromagnetic field removes the degeneracy between the $2s_{1/2}$ and $2p_{1/2}$ levels. The $2s_{1/2}$ level is raised with respect to the $2p_{1/2}$ level by a quantity called the "Lamb shift" which is of the order of 1 060 MHz (fig. 4, page 1231).

The theoretical and experimental study of this phenomenon, which was discovered in 1949, has been the object of a great deal of research, leading to the development of modern quantum electrodynamics.

D. THE HYPERFINE STRUCTURE OF THE $n = 1$ LEVEL

It would now seem logical to study the effect of W_{hf} inside the fine structure levels $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$, in order to see if the interactions related to the proton spin I cause a hyperfine structure to appear in each of these levels. However, since W_f does not remove the degeneracy of the ground state $1s$, it is simpler to study the effect of W_{hf} on this state. The results obtained in this special case can easily be generalized to the $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$ levels.

1. Statement of the problem

a. THE DEGENERACY OF THE $1s$ LEVEL

For the $1s$ level, there is no orbital degeneracy ($l = 0$). On the other hand, the S_z and I_z components of \mathbf{S} and \mathbf{I} can still take on two values: $m_S = \pm 1/2$ and $m_I = \pm 1/2$. The degeneracy of the $1s$ level is therefore equal to 4, and a possible basis in this level is given by the vectors:

$$\left\{ \left| n = 1; l = 0; m_L = 0; m_S = \pm \frac{1}{2}; m_I = \pm \frac{1}{2} \right\rangle \right\} \quad (\text{D-1})$$

b. THE $1s$ LEVEL HAS NO FINE STRUCTURE

We shall show that the W_f term does not remove the degeneracy of the $1s$ level.

The W_{mv} and W_D terms do not act on m_S and m_I , and are represented in the $1s$ subspace by multiples of the unit matrix. We find (*cf.* complement B_{XII}):

$$\langle W_{mv} \rangle_{1s} = -\frac{5}{8} m_e c^2 \alpha^4 \quad (\text{D-2})$$

$$\langle W_D \rangle_{1s} = \frac{1}{2} m_e c^2 \alpha^4 \quad (\text{D-3})$$

Finally, calculation of the matrix elements of the W_{SO} term involves the "angular" matrix elements $\langle l = 0, m_L = 0 | L_{x,y,z} | l = 0, m_L = 0 \rangle$, which are obviously zero ($l = 0$); therefore:

$$\langle W_{SO} \rangle_{1s} = 0 \quad (\text{D-4})$$

In conclusion, W_f merely shifts the $1s$ level as a whole by a quantity equal to:

$$\left(-\frac{5}{8} + \frac{1}{2} \right) m_e c^2 \alpha^4 = -\frac{1}{8} m_e c^2 \alpha^4 \quad (\text{D-5})$$

without splitting the level. This result could have been foreseen: since $l = 0$ and $s = 1/2$, J can take on only one value, $J = 1/2$, and the $1s$ level therefore gives rise to only one fine structure level, $1s_{1/2}$.

Since the Hamiltonian W_f does not split the $1s$ level, we can now consider the effect of the W_{hf} term. To do so, we must first calculate the matrix which represents W_{hf} in the $1s$ level.

2. Matrix representation of W_{hf} in the $1s$ level

a. TERMS OTHER THAN THE CONTACT TERM

Let us show that the first two terms of W_{hf} [formula (B-20)] make no contribution.

Calculation of the contribution from the first term, $-\frac{\mu_0 q}{4\pi m_e R^3} \mathbf{L} \cdot \mathbf{M}_I$, leads to the "angular" matrix elements $\langle l = 0, m_L = 0 | \mathbf{L} | l = 0, m_L = 0 \rangle$, which are obviously zero ($l = 0$).

Similarly, it can be shown (*cf.* complement B_{XI}, § 3) that the matrix elements of the second term (the dipole-dipole interaction) are zero because of the spherical symmetry of the $1s$ state.

b. THE CONTACT TERM

The matrix elements of the last term of (B-20), that is, of the contact term, are of the form:

$$\langle n = 1; l = 0; m_L = 0; m'_S; m'_I | -\frac{2\mu_0}{3} \mathbf{M}_S \cdot \mathbf{M}_I \delta(\mathbf{R}) | n = 1; l = 0; m_L = 0; m_S; m_I \rangle \quad (\text{D-6})$$

If we go into the $\{ | \mathbf{r} \rangle \}$ representation, we can separate the orbital and spin parts of this matrix element and put it in the form:

$$\mathcal{A} \langle m'_S; m'_I | \mathbf{I} \cdot \mathbf{S} | m_S; m_I \rangle \quad (\text{D-7})$$

where \mathcal{A} is a number given by :

$$\begin{aligned} \mathcal{A} &= \frac{q^2}{3\epsilon_0 c^2} \frac{g_p}{m_e M_p} \langle n = 1; l = 0; m_L = 0 | \delta(\mathbf{R}) | n = 1; l = 0; m_L = 0 \rangle \\ &= \frac{q^2}{3\epsilon_0 c^2} \frac{g_p}{m_e M_p} \frac{1}{4\pi} |R_{10}(0)|^2 \\ &= \frac{4}{3} g_p \frac{m_e}{M_p} m_e c^2 \alpha^4 \left(1 + \frac{m_e}{M_p} \right)^{-3} \frac{1}{\hbar^2} \end{aligned} \quad (\text{D-8})$$

We have used the expressions relating M_S and M_I to \mathbf{S} and \mathbf{I} [cf. (B-18)], as well as the expression for the radial function $R_{10}(r)$ given in §C-4-c of chapter VII*.

The orbital variables have therefore completely disappeared, and we are left with a problem of two spin 1/2's, \mathbf{I} and \mathbf{S} , coupled by an interaction of the form :

$$\mathcal{A} \mathbf{I} \cdot \mathbf{S} \quad (\text{D-9})$$

where \mathcal{A} is a constant.

c. EIGENSTATES AND EIGENVALUES OF THE CONTACT TERM

To represent the operator $\mathcal{A} \mathbf{I} \cdot \mathbf{S}$, we have thus far considered only the basis :

$$\left\{ \left| s = \frac{1}{2}; I = \frac{1}{2}; m_S; m_I \right\rangle \right\} \quad (\text{D-10})$$

formed by the eigenvectors common to \mathbf{S}^2 , \mathbf{I}^2 , S_z , I_z . We can also, by introducing the total angular momentum**:

$$\mathbf{F} = \mathbf{S} + \mathbf{I} \quad (\text{D-11})$$

use the basis :

$$\left\{ \left| s = \frac{1}{2}; I = \frac{1}{2}; F; m_F \right\rangle \right\} \quad (\text{D-12})$$

formed by the eigenstates common to \mathbf{S}^2 , \mathbf{I}^2 , \mathbf{F}^2 , F_z . Since $s = I = 1/2$, F can take on only the two values $F = 0$ and $F = 1$. We can easily pass from one basis to the other by means of (B-22) and (B-23) of chapter X.

* The factor $(1 + m_e/M_p)^{-3}$ in (D-8) arises from the fact that it is the reduced mass μ which enters into $R_{10}(0)$. It so happens that, for the contact term, it is correct to take the nuclear finite mass effect into account in this way.

** The total angular momentum is actually $\mathbf{F} = \mathbf{L} + \mathbf{S} + \mathbf{I}$, that is, $\mathbf{F} = \mathbf{J} + \mathbf{I}$. However, for the ground state, the orbital angular momentum is zero, so \mathbf{F} reduces to (D-11).

The $\{ |F, m_F\rangle \}$ basis is better adapted than the $\{ |m_S, m_I\rangle \}$ basis to the study of the operator $\mathcal{A} \mathbf{I} \cdot \mathbf{S}$, as this operator is represented in the $\{ |F, m_F\rangle \}$ basis by a diagonal matrix (for the sake of simplicity, we do not explicitly write $s = 1/2$ and $I = 1/2$). This is true, since we obtain, from (D-11):

$$\mathcal{A} \mathbf{I} \cdot \mathbf{S} = \frac{\mathcal{A}}{2} (\mathbf{F}^2 - \mathbf{I}^2 - \mathbf{S}^2) \quad (\text{D-13})$$

It follows that the states $|F, m_F\rangle$ are eigenstates of $\mathcal{A} \mathbf{I} \cdot \mathbf{S}$:

$$\mathcal{A} \mathbf{I} \cdot \mathbf{S} |F, m_F\rangle = \frac{\mathcal{A} \hbar^2}{2} [F(F+1) - I(I+1) - S(S+1)] |F, m_F\rangle \quad (\text{D-14})$$

We see from (D-14) that the eigenvalues depend only on F , and not on m_F . They are equal to:

$$\frac{\mathcal{A} \hbar^2}{2} \left[2 - \frac{3}{4} - \frac{3}{4} \right] = \frac{\mathcal{A} \hbar^2}{4} \quad (\text{D-15})$$

for $F = 1$, and:

$$\frac{\mathcal{A} \hbar^2}{2} \left[0 - \frac{3}{4} - \frac{3}{4} \right] = -\frac{3\mathcal{A} \hbar^2}{4} \quad (\text{D-16})$$

for $F = 0$.

The four-fold degeneracy of the $1s$ level is therefore partially removed by W_{hf} . We obtain a three-fold degenerate $F = 1$ level and a non-degenerate $F = 0$ level. The $(2F + 1)$ -fold degeneracy of the $F = 1$ level is essential and is related to the invariance of W_{hf} under a rotation of the total system.

3. The hyperfine structure of the $1s$ level

a. POSITIONS OF THE LEVELS

Under the effect of W_f , the energy of the $1s$ level is lowered by a quantity $m_e c^2 \alpha^4 / 8$ with respect to the value $-\mu c^2 \alpha^2 / 2$ calculated in chapter VII. W_{hf} then splits the $1s_{1/2}$ level into two hyperfine levels, separated by an energy $\mathcal{A} \hbar^2$ (fig. 3). $\mathcal{A} \hbar^2$ is often called the "hyperfine structure of the ground state".

COMMENT :

It could be found, similarly, that W_{hf} splits each of the fine structure levels $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$ into a series of hyperfine levels, corresponding to all the values of F separated by one unit and included between $J + I$ and $|J - I|$. For the $2s_{1/2}$ and $2p_{1/2}$ levels, we have $J = 1/2$. Therefore, F takes on the two values $F = 1$ and $F = 0$. For the $2p_{3/2}$ level, $J = 3/2$, and, consequently, we have $F = 2$ and $F = 1$ (cf. fig. 4).

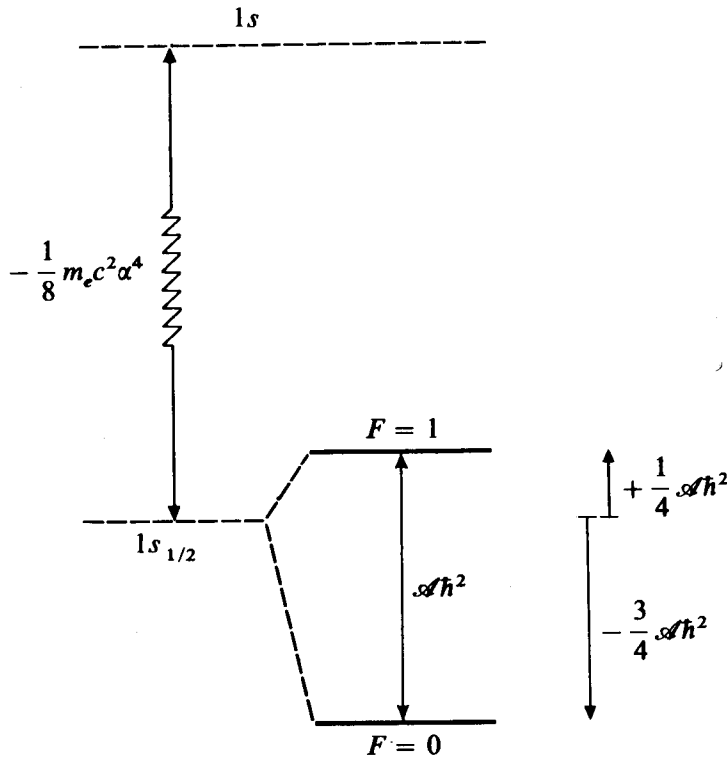


FIGURE 3

The hyperfine structure of the $n = 1$ level of the hydrogen atom. Under the effect of W_f , the $n = 1$ level undergoes a global shift equal to

$$-\frac{1}{8} m_e c^2 \alpha^4;$$

J can take on only one value, $J = 1/2$. When the hyperfine coupling W_{hf} is taken into account, the $1s_{1/2}$ level splits into two hyperfine levels, $F = 1$ and $F = 0$. The hyperfine transition

$$F = 1 \leftrightarrow F = 0$$

(the 21 cm line studied in radioastronomy) has a frequency which is known experimentally to twelve significant figures (thanks to the hydrogen maser).

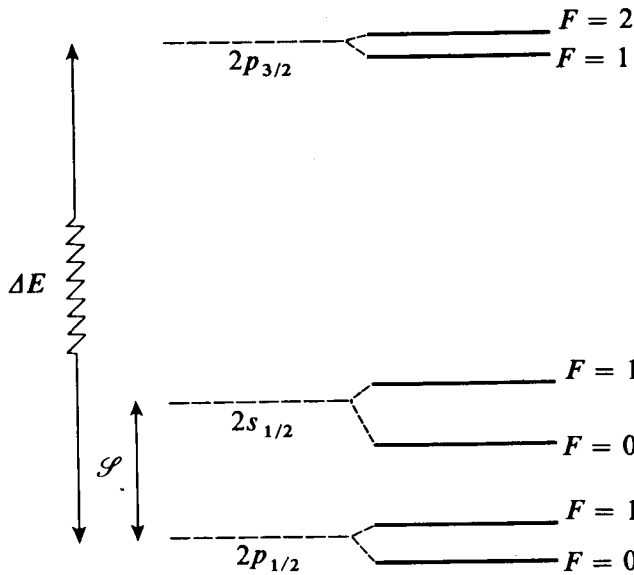


FIGURE 4

The hyperfine structure of the $n = 2$ level of the hydrogen atom. The separation \mathcal{S} between the two levels $2s_{1/2}$ and $2p_{1/2}$ is the Lamb shift, which is about ten times smaller than the fine structure splitting ΔE separating the two levels $2p_{1/2}$ and $2p_{3/2}$ ($\mathcal{S} \approx 1\,057.8$ MHz; $\Delta E \approx 10\,969.1$ MHz). When the hyperfine coupling W_{hf} is taken into account, each level splits into two hyperfine sublevels (the corresponding value of the quantum number F is indicated on the right-hand side of the figure). The hyperfine splittings are equal to 23.7 MHz for the $2p_{3/2}$ level, 177.56 MHz for the $2s_{1/2}$ level and 59.19 MHz for the $2p_{1/2}$ level (for the sake of clarity, the figure is not drawn to scale).

b. **IMPORTANCE OF THE HYPERFINE STRUCTURE OF THE 1s LEVEL**

The hyperfine structure of the ground state of the hydrogen atom is currently the physical quantity which is known experimentally to the highest number of significant figures. Expressed in Hz, it is equal to*:

$$\frac{\mathcal{A}\hbar}{2\pi} = 1\,420\,405\,751.768 \pm 0.001 \text{ Hz} \quad (\text{D-17})$$

Such a high degree of experimental accuracy was made possible by the development of the "hydrogen maser" in 1963. The principle of such a device is, very schematically, the following: hydrogen atoms, previously sorted (by a magnetic selection of the Stern-Gerlach type) so as to choose those in the upper hyperfine level $F = 1$, are stored in a glass cell (the arrangement is similar to the one shown in figure 6 of complement F_{IV}). This constitutes an amplifying medium for the hyperfine frequency $\frac{E(F = 1) - E(F = 0)}{h}$. If the cell is placed in a cavity tuned to the hyperfine frequency, and if the losses of the cavity are small enough for the gain to be greater than the losses, the system becomes unstable and can oscillate: we obtain an "atomic oscillator" (a maser). The frequency of the oscillator is very stable and of great spectral purity. Its measurement gives directly the value of the hyperfine splitting, expressed in Hz.

Note, finally, that hydrogen atoms in interstellar space are detected in radioastronomy by the radiation they emit spontaneously when they fall from the $F = 1$ hyperfine level to the $F = 0$ hyperfine level of the ground state (this transition corresponds to a wave length of 21 cm). Most of the information we possess about interstellar hydrogen clouds is supplied by the study of this 21 cm line.

E. **THE ZEEMAN EFFECT
OF THE 1s GROUND STATE HYPERFINE STRUCTURE**

1. **Statement of the problem**

a. **THE ZEEMAN HAMILTONIAN W_z**

We now assume the atom to be placed in a static uniform magnetic field \mathbf{B}_0 parallel to Oz . This field interacts with the various magnetic moments present in the atom: the orbital and spin magnetic moments of the electron, $\mathbf{M}_L = \frac{q}{2m_e} \mathbf{L}$ and $\mathbf{M}_S = \frac{q}{m_e} \mathbf{S}$, and the magnetic moment of the nucleus, $\mathbf{M}_I = -\frac{qg_p}{2M_p} \mathbf{I}$ [cf. expression (B-18)].

* The calculations presented in this chapter are obviously completely incapable of predicting all these significant figures. Moreover, even the most advanced theories cannot, at the present time, explain more than the first five or six figures of (D-17).

The Zeeman Hamiltonian W_Z which describes the interaction energy of the atom with the field \mathbf{B}_0 can then be written:

$$\begin{aligned} W_Z &= -\mathbf{B}_0 \cdot (\mathbf{M}_L + \mathbf{M}_S + \mathbf{M}_I) \\ &= \omega_0(L_z + 2S_z) + \omega_n I_z \end{aligned} \quad (\text{E-1})$$

where ω_0 (the Larmor angular frequency in the field \mathbf{B}_0) and ω_n are defined by:

$$\left\{ \begin{array}{l} \omega_0 = -\frac{q}{2m_e} B_0 \end{array} \right. \quad (\text{E-2})$$

$$\left\{ \begin{array}{l} \omega_n = \frac{q}{2M_p} g_p B_0 \end{array} \right. \quad (\text{E-3})$$

Since $M_p \gg m_e$, we clearly have:

$$|\omega_0| \gg |\omega_n| \quad (\text{E-4})$$

COMMENT:

Rigorously, W_Z contains another term, which is quadratic in B_0 (the diamagnetic term). This term does not act on the electronic and nuclear spin variables and merely shifts the $1s$ level as a whole, without modifying its Zeeman diagram, which we shall study later. Moreover, it is much smaller than (E-1). Recall that a detailed study of the effect of the diamagnetic term is presented in complement D_{VII}.

b. THE PERTURBATION "SEEN" BY THE $1s$ LEVEL

In this section, we propose to study the effect of W_Z on the $1s$ ground state of the hydrogen atom (the case of the $n = 2$ level is slightly more complicated since, in a zero magnetic field, this level possesses both a fine and a hyperfine structure, while the $n = 1$ level has only a hyperfine structure; the principle of the calculation is nevertheless the same). Even with the strongest magnetic fields that can be produced in the laboratory, W_Z is much smaller than the distance between the $1s$ level and the other levels; consequently, its effect can be treated by perturbation theory.

The effect of a magnetic field on an atomic energy level is called the "Zeeman effect". When B_0 is plotted on the x -axis and the energies of the various sublevels it creates are plotted on the y -axis, a *Zeeman diagram* is obtained.

If B_0 is sufficiently strong, the Zeeman Hamiltonian W_Z can be of the same order of magnitude as the hyperfine Hamiltonian W_{hf} *, or even larger. On the other hand, if B_0 is very weak, $W_Z \ll W_{hf}$. Therefore, in general it is not possible to establish the relative importance of W_Z and W_{hf} . To obtain the energies of the various sublevels, $(W_Z + W_{hf})$ must be diagonalized inside the $n = 1$ level.

* Recall that W_f shifts the $1s$ level as a whole; it therefore also shifts the Zeeman diagram as a whole.

We showed in § D-2 that the restriction of W_{hf} to the $n = 1$ level could be put in the form $\mathcal{A} \mathbf{I} \cdot \mathbf{S}$. Using expression (E-1) for W_z , we see that we must also calculate matrix elements of the form:

$$\langle n = 1; l = 0; m_L = 0; m'_S; m'_I | \omega_0(L_z + 2S_z) + \omega_n I_z | n = 1; l = 0; m_L = 0; m_S; m_I \rangle \quad (\text{E-5})$$

The contribution of $\omega_0 L_z$ is zero, since l and m_L are zero. Since $2\omega_0 S_z + \omega_n I_z$ acts only on the spin variables, we can, for this term, separate the orbital part of the matrix element:

$$\langle n = 1; l = 0; m_L = 0 | n = 1; l = 0; m_L = 0 \rangle = 1 \quad (\text{E-6})$$

from the spin part.

In conclusion, therefore, we must, ignoring the quantum numbers n, l, m_L , diagonalize the operator:

$$\mathcal{A} \mathbf{I} \cdot \mathbf{S} + 2\omega_0 S_z + \omega_n I_z \quad (\text{E-7})$$

which acts only on the spin degrees of freedom. To do so, we can use either the $\{ | m_S, m_I \rangle \}$ basis or the $\{ | F, m_F \rangle \}$ basis.

According to (E-4), the last term of (E-7) is much smaller than the second one. To simplify the discussion, we shall neglect the term $\omega_n I_z$ from now on (it would be possible, however, to take it into account*). The perturbation "seen" by the $1s$ level can therefore be written, finally:

$$\mathcal{A} \mathbf{I} \cdot \mathbf{S} + 2\omega_0 S_z \quad (\text{E-8})$$

c. DIFFERENT DOMAINS OF FIELD STRENGTH

By varying B_0 , we can continuously modify the magnitude of the Zeeman term $2\omega_0 S_z$. We shall consider three different field strengths, determined by the respective orders of magnitude of the hyperfine term and the Zeeman term:

- (i) $\hbar\omega_0 \ll \mathcal{A}\hbar^2$: weak fields
- (ii) $\hbar\omega_0 \gg \mathcal{A}\hbar^2$: strong fields
- (iii) $\hbar\omega_0 \simeq \mathcal{A}\hbar^2$: intermediate fields

We shall later see that it is possible to diagonalize operator (E-8) exactly. However, in order to give a particularly simple example of perturbation theory, we shall use a slightly different method in cases (i) and (ii). In case (i), we shall treat $2\omega_0 S_z$ like a perturbation with respect to $\mathcal{A} \mathbf{I} \cdot \mathbf{S}$. On the other hand, in case (ii), we shall treat $\mathcal{A} \mathbf{I} \cdot \mathbf{S}$ like a perturbation with respect to $2\omega_0 S_z$. The exact diagonalization of the set of two operators, indispensable in case (iii), will allow us to check the preceding results.

2. The weak-field Zeeman effect

The eigenstates and eigenvalues of $\mathcal{A} \mathbf{I} \cdot \mathbf{S}$ have already been determined (§ D-2). We therefore obtain two different levels : the three-fold degenerate level,

$$\{ | F = 1; m_F = -1, 0, +1 \rangle \},$$

* This is what we do in complement C_{XII}, in which we study the hydrogen-like systems (muonium, positronium) for which it is not possible to neglect the magnetic moment of one of the two particles.

of energy $\mathcal{A}\hbar^2/4$, and the non-degenerate level, $\{|F=0; m_F=0\rangle\}$, of energy $-3\mathcal{A}\hbar^2/4$. Since we are treating $2\omega_0 S_z$ like a perturbation with respect to $\mathcal{A}\mathbf{I}\cdot\mathbf{S}$, we must now separately diagonalize the two matrices representing $2\omega_0 S_z$ in the two levels, $F=1$ and $F=0$, corresponding to two distinct eigenvalues of $\mathcal{A}\mathbf{I}\cdot\mathbf{S}$.

a. **MATRIX REPRESENTATION OF S_z IN THE $\{|F, m_F\rangle\}$ BASIS**

Since we shall need it later, we shall begin by writing the matrix which represents S_z in the $\{|F, m_F\rangle\}$ basis (for the problem which concerns us here, it would suffice to write the two submatrices corresponding to the $F=1$ and $F=0$ subspaces).

By using formulas (B-22) and (B-23) of chapter X, we easily obtain:

$$\left\{ \begin{array}{l} S_z |F=1; m_F=1\rangle = \frac{\hbar}{2} |F=1; m_F=1\rangle \\ S_z |F=1; m_F=0\rangle = \frac{\hbar}{2} |F=0; m_F=0\rangle \\ S_z |F=1; m_F=-1\rangle = -\frac{\hbar}{2} |F=1; m_F=-1\rangle \\ S_z |F=0; m_F=0\rangle = \frac{\hbar}{2} |F=1; m_F=0\rangle \end{array} \right. \quad (\text{E-9})$$

which gives the following expression for the matrix representing S_z in the $\{|F, m_F\rangle\}$ basis (the basis vectors are arranged in the order $|1, 1\rangle, |1, 0\rangle, |1, -1\rangle, |0, 0\rangle$):

$$(S_z) = \frac{\hbar}{2} \times \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array} \quad (\text{E-10})$$

COMMENT:

It is instructive to compare the preceding matrix with the one which represents F_z in the same basis:

$$(F_z) = \hbar \times \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \quad (\text{E-11})$$

We see, first of all, that the two matrices are not proportional: the (F_z) matrix is diagonal, while the (S_z) one is not.

However, if we confine ourselves to the restrictions of the two matrices in the $F = 1$ subspace [limited by the darker line in expressions (E-10) and (E-11)], we see that they are proportional. Denoting by P_1 the projector onto the $F = 1$ subspace (*cf.* complement B_{II}), we have:

$$P_1 S_z P_1 = \frac{1}{2} P_1 F_z P_1 \quad (\text{E-12})$$

It would be simple to show that the same relation exists between S_x and F_x on the one hand, and S_y and F_y , on the other.

We have thus found a special case of the Wigner-Eckart theorem (complement D_x), according to which, in a given eigensubspace of the total angular momentum, all the matrices which represent vector operators are proportional. It is clear from this example that this proportionality exists only for the restrictions of operators to a given eigensubspace of the total angular momentum, and not for the operators themselves.

Moreover, the proportionality coefficient $1/2$ which appears in (E-12) can be obtained immediately from the projection theorem. According to formula (30) of complement E_x, this coefficient is equal to:

$$\frac{\langle \mathbf{S} \cdot \mathbf{F} \rangle_{F=1}}{\langle \mathbf{F}^2 \rangle_{F=1}} = \frac{F(F+1) + s(s+1) - I(I+1)}{2F(F+1)} \quad (\text{E-13})$$

Since $s = I = 1/2$, (E-13) is indeed equal to $1/2$.

b. WEAK-FIELD EIGENSTATES AND EIGENVALUES

According to the results of §a, the matrix which represents $2\omega_0 S_z$ in the $F = 1$ level can be written:

$\hbar\omega_0$	0	0
0	0	0
0	0	$-\hbar\omega_0$

(E-14)

In the $F = 0$ level, this matrix reduces to a number, equal to 0.

Since these two matrices are diagonal, we can immediately find the weak-field eigenstates (to zeroth order in ω_0) and the eigenvalues (to first order in ω_0):

<i>Eigenstates</i>	<i>Eigenvalues</i>
$ F = 1; m_F = 1\rangle$	$\longleftrightarrow \frac{\mathcal{A}\hbar^2}{4} + \hbar\omega_0$
$ F = 1; m_F = 0\rangle$	$\longleftrightarrow \frac{\mathcal{A}\hbar^2}{4} + 0$
$ F = 1; m_F = -1\rangle$	$\longleftrightarrow \frac{\mathcal{A}\hbar^2}{4} - \hbar\omega_0$
$ F = 0; m_F = 0\rangle$	$\longleftrightarrow -3\frac{\mathcal{A}\hbar^2}{4} + 0$

(E-15)

In figure 5, we have plotted $\hbar\omega_0$ on the x -axis and the energies of the four Zeeman sublevels on the y -axis (Zeeman diagram). In a zero field, we have the two hyperfine levels, $F = 1$ and $F = 0$. When the field B_0 is turned on, the $|F = 0, m_F = 0\rangle$ sublevel, which is not degenerate, starts horizontally; as for the $F = 1$ level, its three-fold degeneracy is completely removed: three equidistant sublevels are obtained, varying linearly with $\hbar\omega_0$, with slopes of $+1$, 0 , -1 respectively.

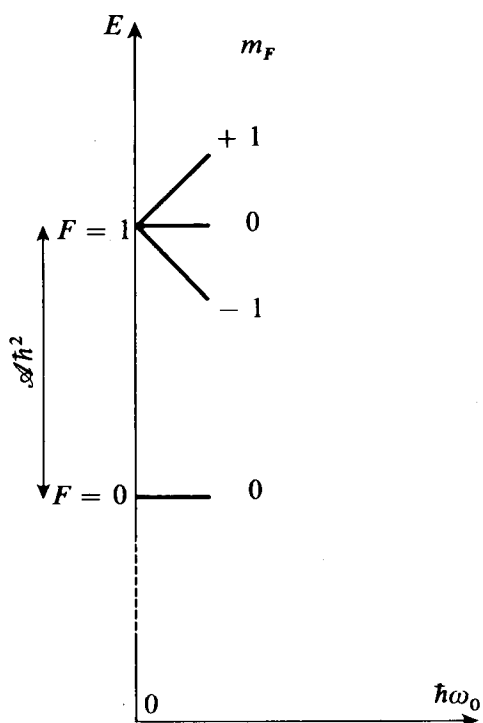


FIGURE 5

The weak-field Zeeman diagram of the $1s$ ground state of the hydrogen atom. The hyperfine $F = 1$ level splits into three equidistant levels, each of which corresponds to a well-defined value of the quantum number m_F . The $F = 0$ level does not undergo any shift to first order in ω_0 .

The preceding treatment is valid as long as the difference $\hbar\omega_0$ between two adjacent Zeeman sublevels of the $F = 1$ level remains much smaller than the zero-field difference between the $F = 1$ and $F = 0$ levels (the hyperfine structure).

COMMENT:

The Wigner-Eckart theorem, mentioned above, makes it possible to show that, in a given level F of the total angular momentum, the Zeeman Hamiltonian $\omega_0(L_z + 2S_z)$ is represented by a matrix proportional to F_z . Thus, we can write, denoting the projector onto the F level by P_F :

$$P_F[\omega_0(L_z + 2S_z)]P_F = g_F\omega_0 P_F F_z P_F \quad (\text{E-16})$$

g_F is called the *Landé factor* of the F state. In the case which concerns us here, $g_{F=1} = 1$.

c. THE BOHR FREQUENCIES INVOLVED IN THE EVOLUTION OF $\langle F \rangle$ AND $\langle S \rangle$. COMPARISON WITH THE VECTOR MODEL OF THE ATOM

In this section, we shall determine the different Bohr frequencies which appear in the evolution of $\langle \mathbf{F} \rangle$ and $\langle \mathbf{S} \rangle$, and show that certain aspects of the results obtained recall those found by using the vector model of the atom (*cf.* complement F_X).