Fermi's Two Golden Rules

Many authors have pointed out that Fermi's Golden Rules were first derived by Dirac and should therefore be attributed to Dirac.

Fermi knew this, of course. He wasn't taking credit for them, he was simply pointing out how important they are to the practice of quantum mechanics. In his only published reference to them---in the book on Nuclear Physics that his students compiled based on Fermi's course at the University of Chicago from January to June 1949---Fermi refers to Schiff's Quantum Mechanics book for their derivation.

Fermi's point was simply that the two Golden Rules are the most important tools for a practicing physicist. Fermi's impact as a teacher was legendary. His students honored him by referring to the two rules as Fermi's Golden Rules.

Fermi called them Golden Rule #1 and Golden Rule #2.

Curiously, modern authors seem unaware that Fermi had two Golden Rules.
They seem only to know about his second Golden Rule.

Fermi's First Golden Rule is the result of applying second-order time-dependent perturbation theory to quantum scattering and resonances.

Fermi's Second Golden Rule is the result of applying second-order time-dependent perturbation theory to absorption.

Why did Fermi refer to the second-order result as the First Rule? I would like to think that it was because Fermi considered scattering more important than absorption.

Quantum mechanics---Perturbation theory, first and second order (Fermi's golden rules).

George Nickel's "What Makes A Qualified Physicist?"

Editorial Note: Dr. Edward Teller, in teaching a quantum mechanics class at Berkeley in 1955 (in which the editor was in attendance), mentioned that this was actually Fermi's golden rule 2. Since Teller did not subscribe to golden rule 1, the class was not informed what rule 1 was.

From Modern Quantum Mechanics by J.J. Sakurai

Practicing the Golden Rule without a license is a common offense, although it tends to pale compared to the misuse of the uncertainty relations.

Herbert Kroemer

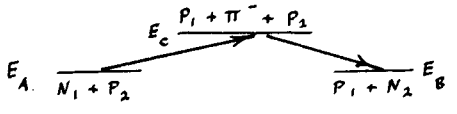
$$e^{-\kappa r}\left[\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \cdots\right]$$

with some directional terms.

Unfortunately it is impossible to find a solution for Schrödinger's equation when the potential diverges faster than $1/r^2$ at the origin. Where $1/r^3$ terms appear, the field must be arbitrarily cut off in a finite volume, but this makes it impossible to formulate the problem in a relativistically invariant way.

Because of these difficulties there are as yet no self-consistent results from meson theory.

In order to point out another important difficulty in meson theory, we must now discuss, exceedingly briefly, the quantum-mechanical formulation of the problem. As an example we shall take one of the "reactions" postulated on p. 134. We have illus-



$$\left.\begin{array}{ll} N_1 \rightarrow P_1 + \pi^- & (A \rightarrow C) \\ \pi^- + P_2 \rightarrow N_2 & (C \rightarrow B) \end{array}\right\}$$

trated in the sketch at left that the intermediate state (C) is energetically impossible for nucleons at rest, since it "costs" 145 Mev to create a π^-. In quantum mechanical perturbation theory, however, states with energies above or below that of the system are important as intermediate, or virtual, states. We shall make extensive use of intermediate states (for example in Ch. VIII to derive the Breit-Wigner formula). Since the mean life of the intermediate state is short ($\tau \sim \hbar/\Delta E$ by the uncertainty principle) there is no violation of conservation of energy.

The transition probability and energy perturbation can be calculated with the help of perturbation theory (ie., there is no better way known). Since the direct matrix element coupling the initial and final states is assumed to be zero, we use "Golden Rule #1" for the second order transition:

$$H'_{BA} = \sum_C \frac{H'_{BC}\, H'_{CA}}{E_A - E_C}$$

Now we can point out the difficulty. It turns out that only the first non-vanishing matrix element (in this example the second-order one) is finite, but that the higher order elements are sums that are not negligible- in fact they diverge. The divergences in the corresponding terms in the electromagnetic case can be removed relativistically by the recent advances in quantum mechanics, but the way out of the difficulty has not been found in meson theory.

Even if the divergences of the individual higher-order transitions could be removed there is another difficulty. Perturbation theory applied to the electromagnetic case gives an expansion of successive orders of the interaction Hamiltonian in powers of $(e^2/\hbar c) = 1/137$. This parameter is quite small, so that there is hope that the whole series will converge. But meson perturbation theory is an expansion in powers of $(g^2/\hbar c)$. This cannot be made smaller than about 1/5 if the theory is to give the right magnitude of nuclear forces. There is considerably less hope that the entire series will converge, even if the individual terms can be made finite.

The explanation of this phenomenon is based on the assumption that the transition $A + a \rightarrow B + b$ occurs through an intermediate state C:

$$A + a \longrightarrow C \longrightarrow B + b$$

State C is the "Compound nucleus". The idea of the compound nucleus is due to Bohr.*

The idea of how resonances in cross section result from this assumption can be obtained from the quantum mechanics of second order transitions. The probability of transition, per unit time, is given by "Golden Rule No. 1":**

$$\text{trans. prob./sec} = \frac{2\pi}{\hbar}\left|\frac{\mathcal{H}_{CA}\mathcal{H}_{BC}}{E_A - E_C}\right|^2 \times \begin{pmatrix}\text{energy}\\ \text{density of}\\ \text{states}\end{pmatrix} \qquad \text{VIII.19}$$

provided there are no direct transitions from A to B. The cross section is, from VIII.9,

$$\sigma_{A\rightarrow B} = \frac{1}{\pi\hbar^4}\left|\Omega\mathcal{H}\right|^2 \frac{p_b^2}{v_a v_b} \qquad \text{VIII.9'}$$

which becomes, analogously,

$$\sigma_{A\rightarrow B} = \frac{1}{\pi\hbar^4}\left|\frac{\mathcal{H}_{CA}\mathcal{H}_{BC}}{E_A - E_C}\right|^2 \frac{p_b^2}{v_a v_b} \qquad \text{VIII.20}$$

Near $E_A = E_C$, (resonance), σ is large. This formula gives infinite σ at the resonance energy, but the formula does not take into account the short lifetime of the compound state. A correct formula is derived in section F.

The life-time of the compound state is long enough for the nucleus C to "forget" how it was formed,*** and this results in a basic simplification in the interpretation.

From the Heisenberg relation $\Delta t \Delta E \gtrsim \hbar$, the lifetime of the compound nucleus and the uncertainty Γ in its energy are related by

$$\Gamma \gtrsim \frac{\hbar}{\text{lifetime}} \qquad \text{VIII.21}$$

The reasons why the compound nucleus has a lifetime greater than zero are the following:

1) For charged particle decay, the barrier factor (VIII.12) reduces the rate of decay.

2) Decay by γ radiation is very slow compared to the times in which the nucleus changes its organization: the lifetime against γ emission is $\sim 10^{-13} - 10^{-14}$ sec. The characteristic time of the nucleus, i.e., the time for a nucleon to cross the nucleus, is $\sim$(size)/(velocity) $\approx 10^{-13}/10^{9}$, or about 10^{-22} sec.

3) A particularly important reason is the tendency toward equipartition of energy in the nucleus. The excess energy due to the absorption of the bombarding particle is distributed among all the nucleons. It is rare that there is a fluctuation in which a large fraction of the excess energy is on one nucleon.

4) Selection rules forbid some modes of decay.

---

* Bohr, Nature 137 344 (1936)

** Schiff, p. 196, eq. (29.20)

*** Discussed in Peierl's review article in Reports on the Progress in Physics VIII (1940), Phys. Soc. of London, 1941.

state m and the final state k, the second need not. It is not difficult to see that the second bracket term arises from the 1 in the numerator of (29.9), which in turn comes from the initial condition at zero time. This initial condition means that the perturbation is turned on suddenly; thus the mathematical formulation suggests that the second-order transitions that do not conserve energy are caused by the sudden turning on of the perturbation. This is in agreement with Eqs. (29.8) and (29.17), which show that a perturbation that has nonzero frequency Fourier components can give up energy to or absorb energy from the system that it perturbs. In the case we are now considering, these Fourier components are not marked enough to produce in first order a transition probability that is proportional to the time, but they do in second order.

In most practical problems, the sudden turning on of the perturbation is introduced as a mathematical artifice that simplifies the calculation. Actually, in such cases, the perturbation either is always present, or is turned on very slowly, and we are concerned with transitions that conserve energy between initial and final states. Problems that can be treated by the sudden approximation (see end of Sec. 31) are an exception; there energy need not be conserved. Throughout this section and the next, we assume that only transitions that conserve the energy actually occur ($\omega_{km} \cong 0$).

Suppose now that the perturbation produces no transitions in first order; this means that there are no states n that conserve energy ($\omega_{nm} \cong 0$) for which the matrix element $H'_{nm} \neq 0$. Since $\omega_{km} \cong 0$, this means also that $H'_{nm} = 0$ whenever $\omega_{kn} \cong 0$. In this case, the second term in the bracket of (29.19) is never appreciable. The calculation of w is carried through as before, except that $a_k^{(2)}$ replaces $a_k^{(1)}$; thus (29.12) can be used if the matrix element H'_{km} is replaced by the second-order matrix element

$$\mathsf{S} \frac{H'_{kn}H'_{nm}}{E_m - E_n} \tag{29.20}$$

Effect of First-order Transitions. In the event that transitions can occur in first order, but they are not to the state in which we are interested, we can proceed as follows. It is still true that the second term in the bracket of (29.19) is negligible for states n that have energies appreciably different from E_k (or E_m), since then ω_{kn} is large. However, there may now be states n for which E_n, E_m, and E_k are all close together and neither H'_{kn} nor H'_{nm} is zero. The second bracket term cannot be ignored, for without it the summation or integration over n would have a singularity when ω_{nm} is zero. It is not difficult to see that for any value of ω_{km} (zero or otherwise), the entire bracket is proportional to ω_{nm} (which is equal to $\omega_{km} - \omega_{kn}$) when ω_{nm} is small; this cancels out the ω_{nm}

transition = number of transitions per unit time = w is given by "Golden Rule No. 2": *

$$\boxed{w = \frac{2\pi}{\hbar} |\mathcal{H}|^2 \frac{dn}{dE}} \qquad \text{VIII.2}$$

where $\mathcal{H}$ is the matrix element of the perturbation causing the transition, and dn/dE = energy density of final states, counting each degenerate state separately.

$|\mathcal{H}|^2$ may be the same for all energetically possible final states; more often it depends on the state. (For instance, $|\mathcal{H}|^2$ may depend on the direction of emission.) Then $|\mathcal{H}|^2$ in the formula is a suitable average over the possible final states.****

$dn/dE = \infty$ for a continuum of states. But in that case $|\mathcal{H}| \longrightarrow 0$, so that the expression $|\mathcal{H}|^2\, dn/dE$ has the indeterminate form $0 \times \infty$. This difficulty is removed by limiting space to a box of volume Ω. $|\mathcal{H}|$ is then small but finite and dn/dE large but finite. Ω drops out of the result. The number of final states equals the number of states of the emitted particle. This is because a change in momentum of one particle compels a change in momentum of the other, by conservation of linear and angular momentum of the system.

It was shown in Chapter IV, p. 76 that the number of states available to a free particle, "b", with momentum between p and p + dp, confined to a box of volume Ω, is

$$dn = \frac{4\pi p_b^2\, dp_b\, \Omega}{(2\pi\hbar)^3} \qquad \text{VIII.3}$$

This must be multiplied by the multiplicity in the final state** caused by spin orientation, which is given by the factor $(2I_b+1)\times(2I_B+1)$, where I_b is the spin of the emitted particle and I_B the spin of the nucleus. If b is a photon, $(2I_b+1)$ is put equal to two.***

$$dE = v_b\, dp_b \quad \text{(true relativistically)} \qquad \text{VIII.4}$$

where p_b and v_b are the momentum and velocity in the center of mass frame of reference of the final (B+b) state. Since "B" is usually massive compared with "b", p_b and v_b can usually be measured in the laboratory frame. Combining these two equations:

$$\frac{dn}{dE} = \frac{4\pi p_b^2\, \Omega}{(2\pi\hbar)^3 v_b}(2I_b+1)(2I_B+1) \qquad \text{VIII.5}$$

From this and VIII.2 we get

$$\text{No. transitions per unit time} = \frac{1}{\pi\hbar^4}\frac{p_b^2}{v_b}\Omega|\mathcal{H}|^2(2I_b+1)(2I_B+1) \qquad \text{VIII.6}$$

The following equation is essentially a definition of the cross-section $\sigma_{A\to B}$ per A nucleus:

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* Derived in Schiff, Quantum Mechanics, p. 193. ("Golden Rule No. 1" is on page 148 of this text).

** This is discussed in greater detail in section C, this chapter.

*** This point is discussed by Bethe and Placzek, Phys.Rev. 51 450, Appendix, p. 483. Multiplicity is caused by the two possible independent polarizations.

**** See page 214 for more complete discussion.

to note that conservation of energy, suitably modified by the uncertainty principle, is an automatic consequence of the calculation and does not have to be inserted as a separate assumption.

Transition Probability. In order to obtain an explicit expression for w, it is convenient to assume that the system is contained in a large cubical box of dimensions L that has periodic boundary conditions at its walls (Sec. 10). Then the eigenfunctions u_n form a discrete set and can be normalized to unity in the volume L^3. We now consider a particular group of final states k that have nearly the same energy as the initial state m and for which the matrix element H'_{km} of the perturbation is a slowly varying function of k. We define a density of final states $\rho(k)$ such that $\rho(k)dE_k$ is the number of such states in the energy range dE_k, and assume that $\rho(k)$ is also a slowly varying function of k.

The transition probability per unit time to one or another of this group of states can then be written

$$w = t^{-1} \sum_k |a_k^{(1)}(t)|^2 = t^{-1} \int |a_k^{(1)}(t)|^2 \rho(k) dE_k \tag{29.10}$$

when the box L is large enough so that the summation over k can be replaced by the integration over E_k. Since H'_{km} and $\rho(k)$ are slowly varying and most of the contribution to the integral comes from a narrow range of energy about $E_k = E_m$, they can be taken outside of the integral, and (29.10) can be rewritten as

$$w = \frac{1}{t} \frac{4|H'_{km}|^2}{\hbar} \rho(k) \int_{-\infty}^{\infty} \frac{\sin^2 \frac{1}{2}\omega_{km} t}{\omega_{km}^2} d\omega_{km} \tag{29.11}$$

where the index k now refers to a typical one of the group of states having about the energy E_m. The integral in (29.11) is $\frac{1}{2}t \int_{-\infty}^{\infty} x^{-2} \sin^2 x dx = \frac{1}{2}\pi t$, so that we finally obtain

$$w = \frac{2\pi}{\hbar} \rho(k) |H'_{km}|^2 \tag{29.12}$$

which is independent of t, as expected.

There may be several different groups of final states k, all of which have about the energy E_m but for which the perturbation matrix elements H'_{km} and the densities of states $\rho(k)$, while nearly constant within a group, differ from one group to another. Then (29.12) gives the transitions per unit time to a particular group; similar expressions of the same form give the rates of transition to other groups.

Scattering Cross Section. As a first application of Eq. (29.12), we calculate w when the initial and final states are free-particle momentum

Nuclear Physics

A Course Given by ENRICO FERMI
at the University of Chicago. Notes Compiled by
Jay Orear, A. H. Rosenfeld, and R. A. Schluter

Revised Edition

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PREFACE

This material is a reproduction, with some amplification, of our notes on lectures in Physics 262-3: Nuclear Physics, given by Enrico Fermi, Jan.-June 1949. The course covered a large number of topics, both experimental and theoretical.

The lectures presupposed a familiarity with physics generally acquired by a student who has completed one course in quantum mechanics (this to include a discussion of the Pauli spin operators and of perturbation theory, both time-independent and time-dependent). We shall make some use of elementary concepts of such topics as statistical mechanics and electrodynamics, but we give references, and a reader could probably pick up the necessary ideas as he goes along; or he could omit a few sections.

Dr. Fermi has not read this material; he is not responsible for errors. We have made some attempt to confine the classroom presentation to the text proper, putting much of our amplifications in footnotes, appendices, and in the solutions to the problems. Most of the problems were assigned in class, but the solutions are not due to Dr. Fermi.

The literature references in the text apply to the list on page 239. At the end of the book there is also a summary of the notation and a list of pertinent constants, values, and relationships.

We would very much appreciate your calling errors to our attention; we would like to hear any suggestions and comments that you may have.

May we thank warmly all those who have helped us to prepare these notes.

Jay Orear
A.H. Rosenfeld
R.A. Schluter

January, 1950

This second printing of these notes differs from the first in that corrections and minor revisions have been made on approximately 70 pages in the first nine chapters, and major revisions have been made in the chapter on cosmic rays. We are grateful to the many people who have given suggestions and corrections; in particular, we are indebted to Prof. Marcel Schein for his suggestions and generous aid in revision of Chapter X.

JO, AHR, RAS

September 1950

An attempt to bring this second printing of the revised edition up to date has been made by adding new footnotes and two pages (237,238) of recent developments. Corrections and minor revisions have been made on approximately 40 pages.

JO, AHR, RAS

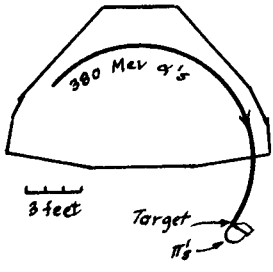
CHAPTER VII. MESONS

A. PROPERTIES KNOWN FROM EXPERIMENT

In this section we shall discuss briefly some of the facts known about mesons, and summarize them in a table; except for one of the problems, however, we shall not discuss the experiments behind the facts*.

By mesons we mean unstable particles of mass greater than that of the electron, less than that of the nucleon. The only ones directly observable so far have either a positive or negative fundamental charge.

Mesons were postulated by Yukawa in 1935, and soon thereafter μ-mesons (they will be called "μ's" or muons from here on) were observed as secondary particles in cosmic radiation***. In 1948 π-mesons (π's or pions) were created artificially by bombarding various targets in the Berkeley cyclotron**. During 1949-50, overwhelming evidence has been found for the existence of a neutral pion π^0. This is discussed further on p. 237



Production of $\pi^{\pm}$'s in a cyclotron.

So far only two sorts of mesons, π and μ, have been identified beyond all doubt, but there are rumors of others.

The names ρ and σ are also used in the literature. This is because the various kinds of meson tracks observed were classified phenomenologically by Powell and his associates according to what was observed at the end of the tracks. This nomenclature is confusing because the number of different kinds of mesons turned out to be less than the number of categories chosen, so that identical mesons may be called by different names.

A ρ meson is one which is observed to stop in the emulsion without producing any observable product. This is a rather time-dependent definition, since more sensitive films are currently being developed. Thus previously unobservable singly-charged relativistic particles (particles travelling at "minimum ionization" -- see Fig. II.4, p. 33) may now be detected.

A σ meson (σ for "star-producing") denotes a meson which produces a nuclear disintegration at the end of its track.

*For nice discussions see "Mesons Old and New" by Keller, Am. Jour. Phys. 17, 356 (Sept. 1949) and a 10-page article by Snyder, Nucleonics 5, 42 (July '49). See also Occhialini and Powell, "Nuclear Physics in Photographs" (1947); and all the references on p. 239 of this book.

**Gardner and Lattes, Phys. Rev. 74, 1236 ('48), Science 107, 270 ('48); Burfening and Lattes, Phys. Rev. 75, 382 ('49).

***Neddermeyer and Anderson, Phys. Rev. 51, 884 ('37), Street and Stevenson, Phys. Rev. 51, 1005 ('37).

The π-Meson (π for "primary" -- for a summary of its properties, see TABLE VII.I):

1. Charged Pions.

The mean life, $\tau \sim 10^{-8}$ sec*, given in the table, applies in the c-m system of the π. Observed in the laboratory system, this time appears dilated by a factor of $\gamma = (1-\beta^2)^{-1/2} = W/M_\pi c^2$. Therefore a $\pi^\pm$ formed with an energy of several Bev during a collision of a high energy cosmic ray particle and a nucleus could travel, at a speed approaching that of light, many meters before it decays. In this case it will probably decay at high energy (before it slows down), into a high-energy μ and a neutrino (?).

If a $\pi^\pm$ slows down before decaying (or is formed at low energy) then as it slows down to about 10 Mev, its rate of ionization increases slowly to about five times minimum ionization, at which point it becomes visible even in the older nuclear films. The last 10 Mev of its path is about 2500 microns long.

A π^+, which is repelled by nuclei, simply comes to rest and decays. But a slow π^- is attracted****and frequently absorbed by a nucleus, giving up its rest energy and probably boiling off several nucleons. These two sorts of tracks are illustrated in FIG. VII.1.

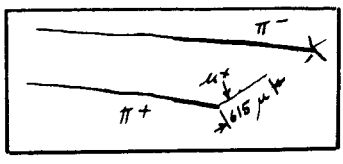


FIG. VII.1 Tracks of π Mesons in Emulsion.

When not captured by a nucleon, a π decays as follows:

$$\pi^\pm \xrightarrow{10^{-8}\ \text{sec}} \mu^\pm + \nu'$$

where ν' is thought to be a neutrino (we shall refer to it as such). As illustrated in problem 1, p. 138, $M_{\nu'}c^2$ is known to be < 15 Mev.

2. Neutral Pions -- see p. 237.

The μ-Meson (again, see TABLE VII.I for mass, etc.):

If the decay reaction mentioned just above takes place while the π is at rest, the μ has a kinetic energy of 4.1 Mev and travels almost exactly 615 microns in Ilford emulsions. Of course, most of the μ's in cosmic radiation are formed when π's decay at high energy, so their range is >> 615 microns.**

On most film the end of the path looks blank, but with cloud chambers, g-m tubes, or minimum ionization film, it has been determined that, when there are no heavy nuclei around, one of the products of the μ-disintegration is an electron which may have one of several energies and is thought to have a continuous spectrum from 9 to 55 Mev***. No other particles have been detected during the reaction, so that the most logical guess is

$$\mu^\pm \xrightarrow{2.15\ \mu\text{sec}} e^\pm + 2\nu \quad (\text{i.e. at least } 2\nu) \qquad \text{in vacuo}.$$

The electron is so light compared to the μ that, on the average, we can think of the energy as being essentially divided equally among the three particles, all extremely relativistic.

*Richardson, Phys. Rev. 74, 1720 ('48)

**An energy spectrum of cosmic ray μ's is given in FIG. X.5, p.220.

***Steinberger, Phys. Rev. 75, 1136 ('49) and Leighton, Anderson, Seriff, Phys. Rev. 75, 1432 ('49): Current data is inadequate for differentiation between several discreet energies (as for α's) and a continuum.

****See the discussion at the top of p. 133.

The qualification "in vacuo" concerning the mean life of u's is needed because, in matter, a positive meson is repelled by nuclei, but a negative meson may fall into a stable Bohr orbit just as an electron does***. The orbits have radius and energy

$$r = \frac{(n\hbar)^2}{M Z e^2} \qquad E = -\frac{M(Ze^2)^2}{2(n\hbar)^2}$$

where M is the reduced mass. The μ orbit is smaller than the corresponding electronic orbit by a factor of 216, and the binding energy is increased by the same factor. For heavy elements, the smallest orbit is only slightly larger than the nucleus itself, so that the μ spends a large fraction of its time inside the nucleus. If the μ interacted strongly with the nucleons (as a π does) it would be immediately captured by the nucleus, but we find that the interaction is very weak. Ticho * gives a curve showing that τ_μ-drops from 2.15 μsec for Z = 1 to 0.7 μsec for Z = 16 , where the capture probability has started to compete seriously with the natural decay. Remember that these mean lives apply to the c-m system of the meson.

TABLE VII.I summarizes the material discussed in this section:

	Electron Masses	Mc^2	Probable Spin	MEAN Life in Vacuo	Interaction with Nuclei
$\pi^\pm$	276 ± 6	136 Mev	0 or 1	$\sim 10^{-8}$ sec	Strong, → exchange forces
$\mu^\pm$	210 ± 4	107 Mev	$\frac{1}{2}$	2.15 usec	Weak, → exchange forces
π^o	≈276 - 6 see p.237	135 Mev	0	$<10^{-13}$ sec	Strong, → ordinary forces

	Path Length in Emulsion	Decay Products
$\pi^\pm$	Non-relativistic ~2500μ. (see text)	π^- usually → star in film $\pi^+ \longrightarrow \mu^+$ (4.1 Mev) + ν
$\mu^\pm$	615 μ (μ ≡ micron)	$e^\pm$ (<55 Mev) + 2ν (?)
π^o	not observable	2 photons

TABLE VII.I Mesons.**

B. MESON THEORY

From electrostatics we know that two particles attract or repel one another according to Coulomb's law. For a classical treatment we say that this force arises from the potential field $\phi = e/r$ of one of the particles. However if we wish to take into account the corpuscular nature of light, we can describe this interaction by saying that one particle"emits" a photon which is subsequently absorbed by the other.

Analogously, the interaction of two nucleons can be partially interpreted by the picture of one nucleon "emitting" a quantum which is promptly absorbed by the second nucleon. These quanta are called mesons, and we shall call them π-mesons in this discussion. The reason for this nomenclature is that we know experimentally that nucleons interact

*Phys. Rev. 74, 1337 ('48)
**This footnote has been expanded and put on p. 237
***Fermi and Teller, Phys. Rev. 72, 399 ('47), J.A. Wheeler, Rev. Mod.Phys. 21, 133 ('49)

more strongly with π's than with μ's. If we are going to attribute nuclear forces to one sort of meson, we might as well call it a π.

If the π is uncharged then we can write the "reaction"

$$N_1 \rightarrow N_1' + \pi^0, \qquad \pi^0 + P_2 \rightarrow P_2'$$

The charge of the individual nucleons (they may be similar or different) undergoes no change during the "reaction", and it turns out that nuclear forces arising from π^0's are of the non-exchange, or ordinary type. On the other hand, if the π is charged we have either

$$N_1 \rightarrow P_1 + \pi^-, \qquad \pi^- + P_2 \rightarrow N_2 ;$$

or

$$P_1 \rightarrow N_1 + \pi^+, \qquad \pi^+ + N_2 \rightarrow P_2 .$$

In this case it turns out that the $\pi^{\pm}$ produces an exchange force.

Since there is evidence that nuclear forces are a mixture of both exchange and ordinary forces, an acceptable theory will probably have to involve both charged and neutral mesons.

Yukawa introduced the meson in 1935, and found that he had to assign it a rest mass of 100-200 m in order to fit the experimental data on the range of nuclear forces.

Fields whose quanta have zero rest mass are long-range; those with quanta of finite mass decrease exponentially. We can illustrate this statement classically as follows:

The potential field of a single electric charge fixed at the origin, $\rho = e\,\delta(\underline{r})$ VII.1′
obeys Laplace's equation,

$$\nabla^2\phi - \frac{1}{c^2}\ddot{\phi} = -4\pi e\,\delta(\underline{r}) \qquad \text{VII.2′}$$

and

$$\phi_{static} = \frac{e}{r}. \qquad \text{VII.3′}$$

If there is a second charge e at a distance r, the interaction energy

$$U = e\phi = e^2/r. \qquad \text{VII.4′}$$

A scalar neutral meson field generated by a nucleon of strength g, at the origin $\rho = g\,\delta(\underline{r})$ VII.1
obeys the Klein-Gordon equation*

$$\nabla^2\phi - K^2\phi - \frac{1}{c^2}\ddot{\phi} = -4\pi g\,\delta(\underline{r}) \qquad \text{VII.2}$$

and

$$\phi_{static} = g\frac{e^{-Kr}}{r} \qquad \text{VII.3}$$

---

*The Klein-Gordon equation may be obtained directly by substituting the operators

$$E = -\frac{\hbar}{i}\frac{\partial}{\partial t}, \qquad \underline{p} = \frac{\hbar}{i}\underline{\nabla}$$

into the equation for total relativistic energy

$$W^2 = M^2c^4 + p^2c^2$$

where M is the rest mass and p the momentum of the meson.

$$-\hbar^2\frac{\partial^2}{\partial t^2} - M^2c^4 + \hbar^2\nabla^2c^2 = 0$$

If we place a second nucleon at $\underline{r}$, it may be shown that**

$$U = -g\phi = -g^2 \frac{e^{-\kappa r}}{r} \qquad \text{VII.4}$$

The "range" of ϕ is $1/\kappa = \lambda\!\!\!^{-}_{\text{compton}} \frac{m}{M} = 3.86 \times 10^{-11} \frac{m}{M}$ cm.***

ϕ is the field variable (or one of its components) and must not be confused with the Schrödinger wave function. In the electromagnetic case, for example, the field variable may be a 4-vector (the electromagnetic 4-potential) or two 3-vectors ($\underline{\mathcal{E}}$ and $\underline{\mathcal{H}}$), depending upon one's point of view. In the simplest case (the first that one would try for a meson field) ϕ is simply a scalar or a pseudo-scalar. A scalar does not change sign on inversion of space; a pseudo-scalar does. When ϕ is a $\left\{\begin{matrix}\text{scalar}\\ \text{pseudo-scalar}\end{matrix}\right\}$ then the non-homogeneous right-hand side of VII.2 is also a $\left\{\begin{matrix}\text{scalar}\\ \text{pseudo-scalar}\end{matrix}\right\}$.

The potential of VII.4 serves only as an example and could not adequately explain nuclear forces, since it is not spin dependent. Attempts have been made to employ more complicated interactions and to introduce vector and tensor fields. The various couplings all give fields of the general form

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Footnote continued from p. 134:
Now introduce a function $\phi(\underline{r},t)$ which has here the significance of a potential and which we shall call the field variable

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{M^2c^2}{\hbar^2}\right)\phi = 0$$

We shall call

$$\frac{\hbar}{Mc} = \kappa^{-1} = \lambda\!\!\!^{-}_{\text{compton}} \frac{m}{M}$$

and then get VII.2, and if M = 0 we get VII.2', for $\underline{r} \neq 0$.

Since the nuclear velocity is low, the main features of the problem show up in the time-independent equation

$$(\nabla^2 - \kappa^2)\phi = -4\pi g\, \delta(\underline{r}) \qquad \text{VII.5}$$

Let $\phi = \frac{u}{r}$; then, for $\underline{r} \neq 0$,

$$\left(\frac{d^2}{dr^2} - \kappa^2\right)u = 0$$

$$\phi = \frac{u}{r} = \text{const}\frac{e^{\pm \kappa r}}{r}$$

The constant is easily shown to be g by integrating both sides of VII.5 over a small region including the origin, and then equating the results. In this small region $\exp(\kappa r) \to 1$, so we have complete analogy with the electrostatic case. We discard the positive exponential case to get a localized field.

**The analogy is between the meson field ϕ, and the components of the electromagnetic 4-potential. The ϕ_{em} of VII.3' is only a factor in the 4th component $i\phi$ of this 4-vector. Therefore there is a difference of a factor of $i^2 = -1$ in the signs of the potential energies VII.4' and VII.4.

*** $3.86 \times 10^{-11}\ (m/M_{\pi^\pm}) = 1.40 \times 10^{-13}$ cm.

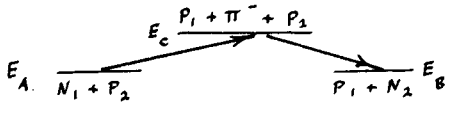
$$e^{-\kappa r}\left[\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \cdots\right]$$

with some directional terms.

Unfortunately it is impossible to find a solution for Schrödinger's equation when the potential diverges faster than $1/r^2$ at the origin. Where $1/r^3$ terms appear, the field must be arbitrarily cut off in a finite volume, but this makes it impossible to formulate the problem in a relativistically invariant way.

Because of these difficulties there are as yet no self-consistent results from meson theory.

In order to point out another important difficulty in meson theory, we must now discuss, exceedingly briefly, the quantum-mechanical formulation of the problem. As an example we shall take one of the "reactions" postulated on p. 134. We have illus-



$$\left.\begin{array}{ll} N_1 \rightarrow P_1 + \pi^- & (A \rightarrow C) \\ \pi^- + P_2 \rightarrow N_2 & (C \rightarrow B) \end{array}\right\}$$

trated in the sketch at left that the intermediate state (C) is energetically impossible for nucleons at rest, since it "costs" 145 Mev to create a π^-. In quantum mechanical perturbation theory, however, states with energies above or below that of the system are important as intermediate, or virtual, states. We shall make extensive use of intermediate states (for example in Ch. VIII to derive the Breit-Wigner formula). Since the mean life of the intermediate state is short ($\tau \sim \hbar/\Delta E$ by the uncertainty principle) there is no violation of conservation of energy.

The transition probability and energy perturbation can be calculated with the help of perturbation theory (ie., there is no better way known). Since the direct matrix element coupling the initial and final states is assumed to be zero, we use "Golden Rule #1" for the second order transition:

$$H'_{BA} = \sum_C \frac{H'_{BC}\, H'_{CA}}{E_A - E_C}$$

Now we can point out the difficulty. It turns out that only the first non-vanishing matrix element (in this example the second-order one) is finite, but that the higher order elements are sums that are not negligible- in fact they diverge. The divergences in the corresponding terms in the electromagnetic case can be removed relativistically by the recent advances in quantum mechanics, but the way out of the difficulty has not been found in meson theory.

Even if the divergences of the individual higher-order transitions could be removed there is another difficulty. Perturbation theory applied to the electromagnetic case gives an expansion of successive orders of the interaction Hamiltonian in powers of $(e^2/\hbar c) = 1/137$. This parameter is quite small, so that there is hope that the whole series will converge. But meson perturbation theory is an expansion in powers of $(g^2/\hbar c)$. This cannot be made smaller than about 1/5 if the theory is to give the right magnitude of nuclear forces. There is considerably less hope that the entire series will converge, even if the individual terms can be made finite.

Meson Theory and Beta Decay. By writing in sequence the reactions

$$N \rightarrow P + (\text{Meson})^-; \qquad (\text{Meson})^- \rightarrow e^- + n\nu \quad [n = 1, 2, 3 \ldots]$$

or $P \rightarrow N + (\text{Meson})^+$, etc.

Yukawa hoped to explain β-decay. Now that it is known that there are two sorts of mesons (maybe more), only one of which decays into an electron

$$\mu^\pm \rightarrow e^\pm + 2\nu$$

it is difficult to reconcile β-decay with the known meson mean lives in a quantitative way.

Summary A great deal of attention has been given to meson theories, from which has come relatively little quantitative results. Qualitatively, however the theory is valuable. Thus physicists predicted the creation of mesons during high-energy collisions before mesons had ever been observed. Meson theory was of considerable weight in the decision to build the large synchro-cyclotrons. Another example of the qualitative application of meson theory is the discussion in Ch. I (p. 14) where we obtain a numerically wrong but qualitatively useful value for the magnetic moment of the deuteron by assuming that part of the time

$$P \rightarrow N + \pi^+$$
$$N \rightarrow P + \pi^-$$

The formalism of meson theory may be greatly modified or abandoned, but the fundamental ideas are likely to survive.

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CHAPTER VIII NUCLEAR REACTIONS

A. Notation

The nuclear reaction $A + a \longrightarrow B + p + Q$ is symbolized by

$$A(a,p)B$$

Particles are symbolized by: a alpha, p proton, d deuteron, γ gamma ray, and f for fission.

Q is (+) for an "exothermic" reaction, (-) for "endothermic".

The threshold is the minimum energy of the bombarding particle in order for the reaction to occur. Threshold is measured in the laboratory system, and therefore is not necessarily equal in magnitude to Q. If Q is positive, the threshold is, in principle, 0. If Q is negative, and if the bombarded particle A is approximately at rest, then (see Ch. I, page 5)

$$\text{Threshold energy} = (-Q) \times \frac{\text{Mass of incident particle}}{\text{Reduced mass of system}}$$

$$= (-Q) \times \frac{M_a + M_A}{M_A} \qquad \text{VIII.1}$$

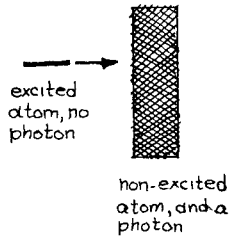
for the reaction symbolized above.

B. General Features of Cross-sections for Nuclear Reactions.

The following considerations apply to cross-sections for nuclear reactions in the absence of resonances. Resonance phenomena are discussed in section D.

Consider the transition $A + a \longrightarrow B + b + Q$, where the nucleus "A" and the particle "a" become the nucleus "B" and particle "b" Both the initial and final states of the system consist of a pair of unbound particles; therefore the transitions is to one of a continuous distribution of states. The initial state also has a continuous range of possible energies, but the experiment itself specifies a particular initial energy.

There are similar situations in atomic physics. For example, in emission of a photon by an excited atom, the transition is from a single state to one of a continuum of states: Conservation of energy selects the final state.



Another atomic example is the non-radiative or Auger transition. An excited atom may have two possible modes of decay. In addition to photon emission, the atom may decay by emission of an electron. Suppose, for example, the excitation corresponds to one missing electron in the K shell. The energy made available when an electron falls into this hole may be greater than the ionization energy, in which case an electron may be emitted from the atom. Again the final system consists of two unbound particles having a continuous range of possible energies.

Returning to the nuclear reaction $A + a \longrightarrow B + b$, we use a general principle of quantum mechanics to derive some essentially statistical results on the variation of the cross-section.

From quantum mechanics, the probability per unit time of

transition = number of transitions per unit time = w is given by "Golden Rule No. 2": *

$$\boxed{w = \frac{2\pi}{\hbar} |\mathcal{H}|^2 \frac{dn}{dE}} \qquad \text{VIII.2}$$

where $\mathcal{H}$ is the matrix element of the perturbation causing the transition, and dn/dE = energy density of final states, counting each degenerate state separately.

$|\mathcal{H}|^2$ may be the same for all energetically possible final states; more often it depends on the state. (For instance, $|\mathcal{H}|^2$ may depend on the direction of emission.) Then $|\mathcal{H}|^2$ in the formula is a suitable average over the possible final states.****

$dn/dE = \infty$ for a continuum of states. But in that case $|\mathcal{H}| \longrightarrow 0$, so that the expression $|\mathcal{H}|^2\, dn/dE$ has the indeterminate form $0 \times \infty$. This difficulty is removed by limiting space to a box of volume Ω. $|\mathcal{H}|$ is then small but finite and dn/dE large but finite. Ω drops out of the result. The number of final states equals the number of states of the emitted particle. This is because a change in momentum of one particle compels a change in momentum of the other, by conservation of linear and angular momentum of the system.

It was shown in Chapter IV, p. 76 that the number of states available to a free particle, "b", with momentum between p and p + dp, confined to a box of volume Ω, is

$$dn = \frac{4\pi p_b^2\, dp_b\, \Omega}{(2\pi\hbar)^3} \qquad \text{VIII.3}$$

This must be multiplied by the multiplicity in the final state** caused by spin orientation, which is given by the factor $(2I_b+1)\times(2I_B+1)$, where I_b is the spin of the emitted particle and I_B the spin of the nucleus. If b is a photon, $(2I_b+1)$ is put equal to two.***

$$dE = v_b\, dp_b \quad \text{(true relativistically)} \qquad \text{VIII.4}$$

where p_b and v_b are the momentum and velocity in the center of mass frame of reference of the final (B+b) state. Since "B" is usually massive compared with "b", p_b and v_b can usually be measured in the laboratory frame. Combining these two equations:

$$\frac{dn}{dE} = \frac{4\pi p_b^2\, \Omega}{(2\pi\hbar)^3 v_b}(2I_b+1)(2I_B+1) \qquad \text{VIII.5}$$

From this and VIII.2 we get

$$\text{No. transitions per unit time} = \frac{1}{\pi\hbar^4}\frac{p_b^2}{v_b}\Omega|\mathcal{H}|^2(2I_b+1)(2I_B+1) \qquad \text{VIII.6}$$

The following equation is essentially a definition of the cross-section $\sigma_{A\to B}$ per A nucleus:

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* Derived in Schiff, Quantum Mechanics, p. 193. ("Golden Rule No. 1" is on page 148 of this text).

** This is discussed in greater detail in section C, this chapter.

*** This point is discussed by Bethe and Placzek, Phys.Rev. 51 450, Appendix, p. 483. Multiplicity is caused by the two possible independent polarizations.

**** See page 214 for more complete discussion.

$$\text{No. transitions/sec per "A" nucleus} = n_a \times v_{\text{"a" rel. to "A"}} \times \sigma_{A\to B} \qquad \text{VIII.7}$$

where A and B refer to the (A+a) and (B+b) states respectively, and n_a is the density of particles "a". Take n_a to be $1/\Omega$ cm^{-3} (one particle in the volume). Then

$$\frac{1}{\Omega} \times v_{\text{"a" rel. to "A"}} \times \sigma_{A\to B} = \frac{1}{\pi\hbar^4} \frac{p_b^2}{v_b} \Omega |\mathcal{H}|^2 (2I_b+1)(2I_B+1) \qquad \text{VIII.8}$$

Since nucleus "A" is often massive compared to "a", $v_{\text{"a" rel. to "A"}}$ is often nearly equal to v_a in the center of mass frame. In any case, these two velocity magnitudes are related by a constant factor. Writing $v_{\text{"a" rel. to "A"}} = v_a$,

$$\boxed{\sigma_{A\to B} = \frac{1}{\pi\hbar^4} |\Omega\mathcal{H}|^2 \frac{p_b^2}{v_a v_b} (2I_b+1)(2I_B+1)} \qquad \text{VIII.9}$$

In general, $\mathcal{H}$ is unknown. It has the form $\int d\tau \psi^*_{final} U \psi_{initial}$ where U is the interaction energy. If the wave functions used to compute $\mathcal{H}$ are normalized in volume Ω, Ω disappears from the expression $|\Omega\mathcal{H}|$ in VIII.9. This is seen as follows: Let Ψ have the form, at large distances, N exp(ikz). Then $\int |\psi|^2 d\tau = N^2\Omega$ Setting $N^2\Omega = 1$, we get $N = 1/\sqrt{\Omega}$

If $\Psi_{initial}$ and Ψ_{final} now mean the un-normalized plane wave functions, the matrix element factor in VIII.9 becomes

$$\Omega\mathcal{H} = \int d\tau \Psi^*_{final} U \Psi_{inital} \qquad \text{VIII.10}$$

(This may be looked upon as taking $\Omega = 1$.) Henceforth we use $\mathcal{H}$ for $\Omega\mathcal{H}$. In order to show the meaning of this expression, we write it as

$$|\mathcal{H}| = \bar{U} \times \text{Volume of nucleus} \times \overline{|\Psi_{initial}\Psi_{final}|} \qquad \text{VIII.11}$$

where $\overline{|\Psi_{in.}\Psi_{fin.}|}$ is a suitable average of the product of the wave functions over the volume of the nucleus. U, and hence the integrand, is zero outside the nucleus. $\bar{U}$ = average interaction energy $\approx$ depth of potential well. For our purposes here the important feature of VIII.11 is its dependence on the charge of the participating particles. If "a", say, is positively charged, its wave function will be reduced in <u>amplitude</u> at the nucleus by the barrier factor exp($-G_a/2$), where, by III.3, p. 58.

$$\frac{G_a}{2} = \sqrt{\frac{2M_a}{\hbar^2}} \int \sqrt{U_a - E_a}\, dr \longrightarrow \approx \frac{\pi Z_A g_a e^2}{\hbar v_a} \quad \text{for } \underline{\text{high}} \text{ barriers} \qquad \text{VIII.12}$$

U_a denotes the charge of "a" times the Coulomb potential of "A". Physically this factor represents Coulomb repulsion. The wave function of an outgoing particle at the nucleus is also reduced by such a barrier factor. The result for the <u>squared</u> matrix element is:

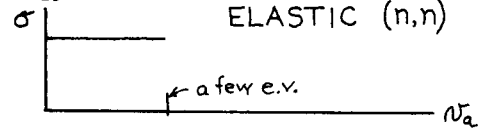
For neutral particles: $|\mathcal{H}|^2 \propto (\bar{U} \times \text{Vol. of nucleus})^2$ VIII.13
For + charged particles: $|\mathcal{H}|^2 \propto (\bar{U} \times \text{Vol.})^2 \times \exp(-G_a - G_b)$

(emission of negatively charged particles (electrons)is treated in Ch. IV)

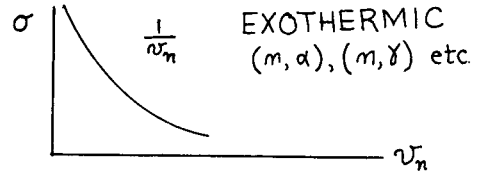
For endothermic reactions there is a threshold energy for the bombarding particle. For exothermic reactions in which the energy liberated is much larger than the energy of the bombarding particle, there are two simplifications in equation VIII.9: 1) the barrier factor $\exp(-G_b)$ for the outgoing particle is almost constant because it is a function of energy of the emitted particle "b", which is almost constant; 2) p_b and v_b are almost constant and therefore the statistical weight factor in VIII.9, $p_b^2/v_a v_b$, is proportional to $1/v_a$.

These results are now applied to specific cases to deduce the general features of the σ vs. energy and σ vs. velocity curves.

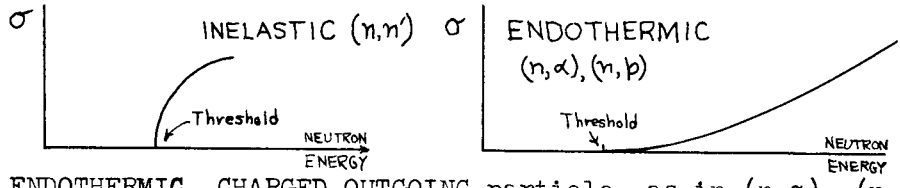
1) ELASTIC (n,n) (both particles uncharged)
$v_a = v_b$, therefore $p_b^2/v_a v_b = (M_{neut})^2$, a constant
At low energy $|\mathcal{H}|$ is approximately constant, therefore
$\sigma \approx$ constant at low energy.



2) EXOTHERMIC, low energy UNCHARGED bombarding particle, as in (n,α), (n,p), (n,γ), (n,f). Q is usually ~ Mev. while neutron energy is ~ e.v., therefore $v_b \approx$ constant. Therefore $p_b^2/v_a v_b \approx 1/v_a$. $|\mathcal{H}|^2 \propto \exp(-G_n - G_b)$. $\exp(-G_b)$ is $\approx$ constant, since it depends on the almost constant energy of the outgoing particle, or, in the case of an uncharged "b", is 1 exactly. Also $\exp(-G_n) = 1$. Therefore
$\sigma \sim 1/v_n$ (the "1/v" law)



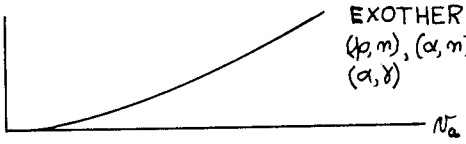
3) INELASTIC (n,n')
The nucleus is left in an excited state. The process is endothermic and -Q is the excitation energy of the nucleus. For incident neutron energies slightly above the threshold, $v_n \approx$ constant, since the fractional change in incident energy is small. But $v_{n'}$ changes relatively greatly in this region: $v_{n'}^2 \propto$ excess of energy above the threshold. Therefore $p_{n'}^2/v_n v_{n'} \propto v_{n'} \propto \sqrt{\text{energy excess}}$. Therefore near the threshold $\sigma \propto \sqrt{\text{energy excess}}$.



4) ENDOTHERMIC, CHARGED OUTGOING particle, as in (n,α), (n,p). Exactly as in case 3), except that the factor $\exp(-G_b)$ operates and is dominant. $\sigma \propto \sqrt{\text{energy excess}} \times \exp(-G_b)$

5) EXOTHERMIC, CHARGED INCOMING particle, as in (p,n), (α,n), (α,γ), (p, γ). For incident energies << Q, the factor $p_b^2/v_a v_b \propto 1/v_a$. The barrier factor $\exp(-G_a)$ operates on the incoming particle.

$\sigma \propto 1/v_a \exp(-G_a)$



In all of the above, no account has been taken of resonance phenomena.

C. Inverse Processes

Consider the transition A + a → B + b, where "A" and "B" are nuclei and "a" and "b" are, in general, lighter particles. From equation VIII.9 the cross section for this transition is (neglecting spins):

$$\sigma_{A\to B} = \frac{1}{\pi\hbar^4}\left|\Omega\mathcal{H}\right|^2 \frac{p_b^2}{v_a v_b} \qquad \text{VIII.9'}$$

The inverse reaction is B + b → A + a. Its cross section is

$$\sigma_{B\to A} = \frac{1}{\pi\hbar^4}\left|\Omega\mathcal{H}\right|^2 \frac{p_a^2}{v_a v_b} \qquad \text{VIII.9"}$$

$\left|\Omega\mathcal{H}\right|^2$ is the same in both cases, because the operator of the perturbation is Hermitian, i.e., $\left|\int \psi_B^* \mathcal{H} \psi_A \, d\tau\right| = \left|\int \psi_A^* \mathcal{H} \psi_B \, d\tau\right|$

therefore,

$$\frac{\sigma_{A\to B}}{\sigma_{B\to A}} = \frac{p_b^2}{p_a^2} \qquad \text{(neglecting spin)} \qquad \text{VIII.14}$$

The same result may be looked at from a different aspect. Suppose we have a box filled with arbitrary numbers of particles "A", "a", "B", "b". The transitions A + a ⇌ B + b occur. Statistical mechanics asserts that at equilibrium all possible states of the system consistent with the specification of the energy of the system are occupied with equal probability. If a state consisting of a pair of particles A + a is called an "A" state, and similarly for "B" state, then the occupied states in the energy range ΔE may be divided into the two types, A and B. Since all states in ΔE are equally probably occupied, this division is such that

$$\frac{\text{No. occupied A states}}{\text{No. occupied B states}} = \frac{\text{No. possible A states in } \Delta E}{\text{No. possible B states in } \Delta E} \qquad .15$$

The number of possible A states = maximum number of (A + a) pairs times the number of states in ΔE for one pair = $\eta \frac{4\pi p_a^2 \Omega}{(2\pi\hbar)^3 v_a} \Delta E$

where η = maximum number of (A + a) pairs formable with the particular numbers of particles put into the box initially. Similarly, the number of possible B states = $\eta \frac{4\pi p_b^2 \Omega}{(2\pi\hbar)^3 v_b} \Delta E$

where η is the same. Therefore

$$\frac{\text{No. of occupied A states}}{\text{No. of occupied B states}} = \frac{p_a^2 v_b}{p_b^2 v_a} \qquad \text{VIII.16}$$

Now at equilibrium the number of transitions A→B equals the number of transitions in reverse, per unit time.

$$\text{No. transitions A}\rightarrow\text{B/sec} = (\text{No. A states occupied})\,\sigma_{A\rightarrow B}\, v_a$$
$$\text{No. transitions B}\rightarrow\text{A/sec} = (\text{No. B states occupied})\,\sigma_{B\rightarrow A}\, v_b \qquad \text{VIII.17}$$

Combining VIII.16 with VIII.17,

$$\frac{\sigma_{B\rightarrow A} v_b}{\sigma_{A\rightarrow B} v_a} = \frac{p_a^2 v_b}{p_b^2 v_a} \qquad \text{VIII.14'}$$

as before.

If the particles have spins, the density of states is increased. If the spins are I_A, I_a, I_B, I_b, the density of A states is increased by the factor $(2I_A+1)(2I_a+1)$, and similarly for B states. Then the rate of transition A⟶B is proportional to

$$(2I_A+1)(2I_a+1)p_a^2\sigma_{A\rightarrow B}$$

and B⟶A to

$$(2I_B+1)(2I_b+1)p_b^2\sigma_{B\rightarrow A}$$

therefore

$$\boxed{(2I_A+1)(2I_a+1)p_a^2\sigma_{A\rightarrow B} = (2I_B+1)(2I_b+1)p_b^2\sigma_{B\rightarrow A}} \qquad \text{VIII.18}$$

Note that in this formula, σ is an average over the various kinds (spin orientations) of A states, and a sum of partial σ's for various possible final states. *

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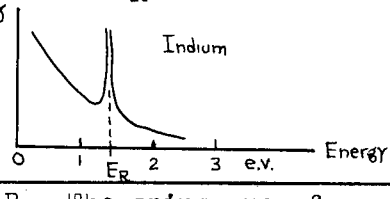
* This may be elucidated by writing $\sigma_{A\rightarrow B}$ more explicitly. Divide σ into contributions $\sigma(S)$ due to various relative orientations of I_A and I_a. The number of states represented by each relative orientation is $2S+1$, where S = resultant angular momentum of particles "A" and "a". In this discussion, orbital angular momentum is neglected. It is included in a discussion in the appendix of Bethe and Placzek's paper, Phys.Rev. 51 450. The total number of A states is $(2I_A+1)(2I_a+1)$. The total cross section for transition to any B state is $\sigma_{A\rightarrow B} = \frac{1}{(2I_A+1)(2I_a+1)}\sum_S (2S+1)\,\sigma(S)$ which is an average over spin states. (S takes on $2I_a+1$ values if $I_a < I_A$; $(2I_A+1)$ if $I_A < I_a$.) Now $\sigma(S)$, the partial cross section for various initial values of S, may be written as a sum of contributions to various possible final spin states, i.e., $\sigma(S) = \sum_i \sigma(S)_i$, where i denotes a particular final spin state of the B + b system. $\sigma(S)_i$ contains in addition to the density of states in energy, the squared matrix element for the particular transition represented by $\sigma(S)_i$. For transitions not conserving total vector angular momentum, $\sigma(S)_i = 0$. For example,

Problem: Design an experiment to detect the inverse reaction to $Be^9 + H^1 \longrightarrow Li^6 + He^4$.

(Design of the alpha particle source will depend on the threshold energy for the inverse reaction. From Allison, Skaggs and Smith, Phys.Rev. 57 550, or from Hornyak and Lauritsen, Rev. Mod.Phys. 20, 202, we find that Q for the forward reaction is 2.115 Mev. In the reverse reaction, in order to get 2.115 Mev into the center of mass coordinate system we must give the alpha an energy of about 3.5 Mev, and this is the threshold for the inverse reaction (See section A). Design of the Li^6 target and of the detector, and determining the required alpha beam strength require knowing the cross section. This is got by detail balancing arguments from $\sigma_{Be^9(p,\alpha)Li^6}$, taking into account a spin factor of 8/3. This cross section is found in Livingston and Bethe, C, Rev.Mod.Phys. 9 245, p. 310, or in the original source, Allen, Phys.Rev. 51 182 (1937), and is 5×10^{-29} cm^2 at 0.1 Mev. The cross section for the inverse reaction increases rapidly as the volume of phase space available to the proton is increased, therefore it is advantageous to use alpha energies an Mev or more above the threshold of 3.5 Mev. Higher energy protons also penetrate the Coulomb barrier readily, and are easier to detect. A qualitative curve of cross-section for the forward reaction as a function of energy is given in Hornyak and Lauritsen, Rev. Mod.Phys. 20 191, p. 201.

D. The Compound Nucleus

In the diagrams of section B it was assumed the $|\mathcal{H}|^2$ was approximately constant, except for the Coulomb barrier factor. Often, perhaps in most cases, the matrix element has irregular variations. This phenomenon is called resonance. For example, in the (n, γ) process in indium, there is an extremely pronounced peak in σ at a neutron energy of 1.44 e.v. σ reaches 27,000 barns at this energy. (one barn is 10^{-24} cm^2.) The half-width of this resonance peak is 0.042 e.v. $\equiv \Gamma_{1/2}$ Near the resonance, the curve of σ vs. energy has the form $1/(E-E_R)^2$. Another example is the resonance at $E_R = 5.2$ e.v. σ for the (n, γ) reaction in silver. In this case σ reaches 24,000 barns, and the peak has a half-width $\Gamma_{1/2} = 0.063$ e.v.



consider the reaction $n + A \longrightarrow \alpha + B$. The spins are, for n, 1/2; for α, 0; assume for A, 1; and for B, 3/2. The total number of initial spin states = (2(1)+1)(2(1/2)+1) = 6. The number of initial spin states for total angular momentum S = 3/2 is (2(3/2)+1) = 4; for S = 1/2, (2(1/2)+1) = 2.

$$\sigma_{A(n,\alpha)B} = \frac{4}{6} \times \sigma_{(S=3/2)} + \frac{2}{6} \times \sigma_{(S=1/2)}$$

Now the first term represents transitions to any of the final spin states having S = 3/2. For a given initial orientation, there is only one. Similarly, the second term represents transitions to any final state having total angular momentum 1/2. But, since the spin of the α = 0, there are none, so $\sigma(1/2) = 0$. When orbital angular momentum is involved, there may be more than one way in which the given initial state can form a final state, so that $\sigma(3/2)$, for example, is a sum over the various possibilities. See Bethe and Placzek, Phys. Rev. 51 450, appendix.

The explanation of this phenomenon is based on the assumption that the transition $A + a \rightarrow B + b$ occurs through an intermediate state C:

$$A + a \longrightarrow C \longrightarrow B + b$$

State C is the "Compound nucleus". The idea of the compound nucleus is due to Bohr.*

The idea of how resonances in cross section result from this assumption can be obtained from the quantum mechanics of second order transitions. The probability of transition, per unit time, is given by "Golden Rule No. 1":**

$$\text{trans. prob./sec} = \frac{2\pi}{\hbar}\left|\frac{\mathcal{H}_{CA}\mathcal{H}_{BC}}{E_A - E_C}\right|^2 \times \begin{pmatrix}\text{energy}\\ \text{density of}\\ \text{states}\end{pmatrix} \qquad \text{VIII.19}$$

provided there are no direct transitions from A to B. The cross section is, from VIII.9,

$$\sigma_{A\rightarrow B} = \frac{1}{\pi\hbar^4}\left|\Omega\mathcal{H}\right|^2 \frac{p_b^2}{v_a v_b} \qquad \text{VIII.9'}$$

which becomes, analogously,

$$\sigma_{A\rightarrow B} = \frac{1}{\pi\hbar^4}\left|\frac{\mathcal{H}_{CA}\mathcal{H}_{BC}}{E_A - E_C}\right|^2 \frac{p_b^2}{v_a v_b} \qquad \text{VIII.20}$$

Near $E_A = E_C$, (resonance), σ is large. This formula gives infinite σ at the resonance energy, but the formula does not take into account the short lifetime of the compound state. A correct formula is derived in section F.

The life-time of the compound state is long enough for the nucleus C to "forget" how it was formed,*** and this results in a basic simplification in the interpretation.

From the Heisenberg relation $\Delta t \Delta E \gtrsim \hbar$, the lifetime of the compound nucleus and the uncertainty Γ in its energy are related by

$$\Gamma \gtrsim \frac{\hbar}{\text{lifetime}} \qquad \text{VIII.21}$$

The reasons why the compound nucleus has a lifetime greater than zero are the following:

1) For charged particle decay, the barrier factor (VIII.12) reduces the rate of decay.

2) Decay by γ radiation is very slow compared to the times in which the nucleus changes its organization: the lifetime against γ emission is $\sim 10^{-13} - 10^{-14}$ sec. The characteristic time of the nucleus, i.e., the time for a nucleon to cross the nucleus, is $\sim$(size)/(velocity) $\approx 10^{-13}/10^{9}$, or about 10^{-22} sec.

3) A particularly important reason is the tendency toward equipartition of energy in the nucleus. The excess energy due to the absorption of the bombarding particle is distributed among all the nucleons. It is rare that there is a fluctuation in which a large fraction of the excess energy is on one nucleon.

4) Selection rules forbid some modes of decay.

---

* Bohr, Nature 137 344 (1936)

** Schiff, p. 196, eq. (29.20)

*** Discussed in Peierl's review article in Reports on the Progress in Physics VIII (1940), Phys. Soc. of London, 1941.

E. Example of an Unstable Nucleus

An example of a nucleus which plays the role of an intermediate-state compound-nucleus for several well known nuclear reactions is Be^8.

The ground state Be^8 decays as follows:

$$Be^8 \longrightarrow 2\ He^4 + 110\ Kev.*$$

The reaction is barely exothermic. The Gamow exponent for decay into α's is low due to low nuclear charge, see equation VIII.57, p.163. The theoretical estimate of the lifetime is 10^{-16} sec.**, corresponding to a width of between 1 and 100 e.v. This time is long compared with the nuclear characteristic time of 10^{-22} sec.; hence the width of the level is small.

Information on the excited levels of Be^8 can be obtained from study of those nuclear reactions for which Be^8 is the intermediate compound nucleus state, such as $Li^7(p,\gamma)Be^8 \longrightarrow 2\alpha$, $Li^7(p,n)Be^7$. These reactions are discussed here. The energy levels are plotted in FIG. VIII.1.

1) α-α scattering. For two Coulomb centers, the total scattering cross section is ∞. We may study the scattering at some angle not near 0 (90° in center of mass system is best). We expect peaks in the value of σ when the incident relative energy equals the energy of excitation of an excited state. For α's scattered on α's, the first such resonance should come at 0.110 Mev (in center of mass system), corresponding to the Be^8 ground state. This resonance is presumably very sharp, a few e.v. wide, as mentioned above. It has never been observed experimentally.

Problem. Discuss the possibility of experimentally observing the resonance expected in alpha-helium scattering at an energy corresponding to the Be^8 ground state, i.e., 0.110 Mev in the center of mass frame.
(The Coulomb barrier keeps alphas of this energy at least 5 x 10^{-12} cm apart classically, so the effect of nuclear forces is probably undetectable. Also the experiment is difficult because the range of 200 Kev alphas is so short that it is hard to shoot them through an appreciable number of scattering centers and detect them. Any attempt to detect a resonance might be guided by the experimental procedure of Devons (Proc.Roy.Soc. A 172 127 and 559 (1939)), who investigated alpha-helium scattering at higher energies. The theory of α-α scattering and its relation to the Be^8 nucleus is given in Wheeler, Phys.Rev. 59 16 and 27, (1941).)

A second resonance, this one experimentally observed, is at ~3 Mev. The barrier factor is lower at 3 Mev, hence the state has shorter lifetime and greater width. The half-width is estimated to be 0.8 Mev.

Further resonances in α-α scattering are so broad as to be scarcely recognizable as resonances. All the resonances mentioned so far correspond to states of even parity. This is because α's obey Bose-Einstein statistics and have symmetric wave

* Hemmendinger; quoted in Seaborg and Perlman table of isotopes, Rev. Mod. Phys. 20 585.
** Wheeler, Phys.Rev. 59 27.

functions.* The incident α's will have angular momentum 0,2,4,.. with respect to a target α particle. Therefore states of Be^8 detectable by α scattering in helium are even states.

Not all states of Be^8 are even. Odd states of Be^8 cannot decay directly into two α's or into the even Be^8 states mentioned above. Emission of electromagnetic radiation must occur first, because an $\begin{matrix}\text{odd}\\\text{even}\end{matrix}$ state cannot change to an $\begin{matrix}\text{even}\\\text{odd}\end{matrix}$ state by "mechanical", i.e., non-radiative, interactions. Change of parity occurs in emission of photons.

2) $Li^7(p,\gamma)Be^8$. There is a prominent and narrow resonance at a proton energy of 440 Kev. This indicates that the lifetime of the excited Be^8 is long and thus that it is an odd state. It decays through the relatively slow process of γ emission to the much lower even Be^8 states. The energy of a Li^7 and a proton separated and at rest is 17.2 Mev higher than that of the ground state of Be^8. The γ's given off in decay from the excited Be^8 state produced in the $Li^7(p,\gamma)$ reaction have energies of 17.5 Mev and ≈ 14.5 Mev, indicating γ decay to the two even states mentioned in the paragraph on α-α scattering.

That the excited Be^8 state produced in $Li^7(p,\gamma)$ is odd accords with the following considerations. The most probable case is for the bombarding proton to be in a high S state, which is even. The Li^7 is odd, as is suggested by the arguments in the following paragraph. Then Li^7(odd) + proton (S state) is an odd Be^8 state.

The picture of a nucleus as built up of "shells" of protons and neutrons, somewhat like atomic electron shells, suggests that Li^7 in the ground state is odd.

Suppose the average potential for the nucleons is a square well. The single particle approximate quantum mechanical solution to the problem leads to orbits which may be designated: 1s, 1p, 1d, etc. The 1s orbit accomodates 2 neutrons (spins opposed) and two protons (spins opposed); 1p accomodates 6 neutrons and 6 protons, etc. ${}_3Li_4^7$ would have the configuration:

protons: $1s^2\ 1p^1$
neutrons: $1s^2\ 1p^2$

or a total configuration: $(1s^2 1p; 1s^2 1p^2)$, which has $\Sigma|\ell| = 3$, and is hence an odd state. This model of the nucleus is discussed in

---

* For two identical particles, parity of the state and symmetry of the wave functions are simply related: If the wave function $\Psi\begin{pmatrix}\text{changes}\\\text{does not change}\end{pmatrix}$ sign when space is inverted by the operation $x\to -x, y\to -y, z\to -z$, then the Ψ has $\begin{pmatrix}\text{odd}\\\text{even}\end{pmatrix}$ parity. If $\Psi^{\text{changes}}_{\text{does not change}}$ sign when the two particles are interchanged in position, then Ψ is $\begin{pmatrix}\text{antisymmetrical}\\\text{symmetrical}\end{pmatrix}$.

Operation of inversion of space: $\Psi(\underline{r}_1,\underline{r}_2) \longrightarrow \pm\, \Psi(-\underline{r}_1,-\underline{r}_2)$
Operation of particle interchange: $\Psi(\underline{r}_1,\underline{r}_2) \longrightarrow \pm\, \Psi(\underline{r}_2,\underline{r}_1)$
But for identical particles, $\underline{r}_1 = -\underline{r}_2$, $\underline{r}_2 = -\underline{r}_1$, so that $\Psi(-\underline{r}_1,-\underline{r}_2) = \Psi(\underline{r}_2,\underline{r}_1)$, and inversion is equivalent to particle interchange.

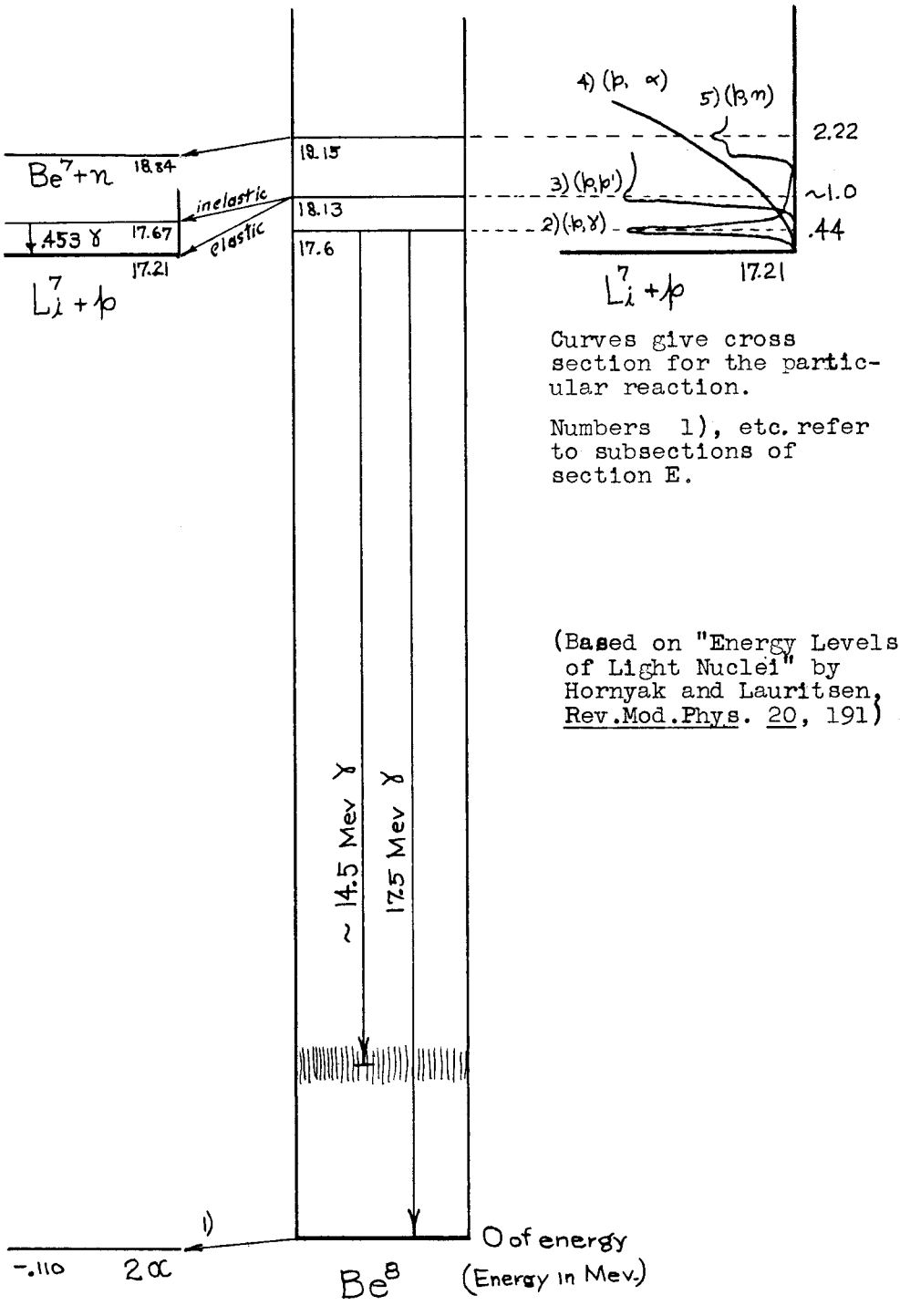
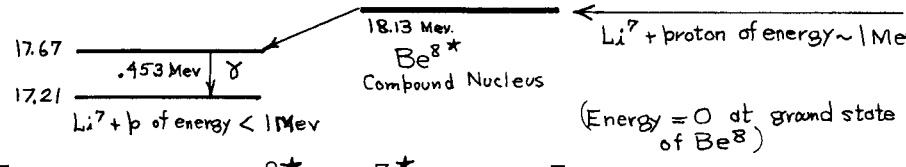

4) (p, α)
5) (p, n)
2.22
3) (p, p')
~1.0
2) (p, γ)
.44
18.84
$Be^7 + n$
18.15
inelastic
18.13
.453 γ
17.67
elastic
17.6
17.21
$Li^7 + p$
17.21
$Li^7 + p$
Curves give cross section for the particular reaction.
Numbers 1), etc. refer to subsections of section E.
(Based on "Energy Levels of Light Nuclei" by Hornyak and Lauritsen, Rev. Mod. Phys. 20, 191)
~ 14.5 Mev γ
17.5 Mev γ
1)
0 of energy
-.110
2α
Be^8
(Energy in Mev.)

FIG. VIII.1

greater detail in section K.

3) $Li^7(p,p')Li^{7*}$. This is similar to 2) except that a proton is emitted having less energy than the incident proton, leaving Li^7 in an excited state. The resonance in σ is observed at a proton energy of ~1.05 Mev. Li^{7*} decays by emitting a γ of about 0.45 Mev. The 0.45 Mev splitting between this excited state and the ground state Li^7 may be due to energy difference between $p_{1/2}$ and $p_{3/2}$ states of Li^7, on the nuclear shell model:



$$Li^7 + p(1\ Mev) \longrightarrow Be^{8*} \longrightarrow Li^{7*} + p' \longrightarrow Li^7 + \gamma + p'$$

4) $Li^7(p,\alpha)He^4$. Since two α's are in an even state, and since Li^7 is odd, the incident proton must be in an odd state, probably a p state with respect to the Li^7 nucleus. No resonances are observed. None is to be expected, since all even resonances are extremely broad. Note that all the observed resonances are odd levels.

5) $Li^7(p,n)Be^7$. A resonance is observed at a proton energy of ~ 2.22 Mev, corresponding to an odd Be^8 state 19.15 Mev above the ground state.

6) $Li^6(d,\alpha)He^4$ No resonances. Evidently $Li^6 + H^2$ form an even state, and quickly decay to two α's.

Problem: Design an experiment to observe the famous 440 Kev resonance in the reaction $Li^7 + p \longrightarrow Be^{8*} \longrightarrow \gamma + Be^8 \rightarrow 2\alpha$ (This experiment has been performed by Walker and McDaniel, using a gamma ray spectrometer which measures the energy of pairs produced by the gamma ray (Phys.Rev. 74 315) and by Delsasso, Fowler and Lauritsen (Phys.Rev. 51 (1937)) using a cloud chamber to detect the gammas by means of electrons produced in the chamber by pair production and Compton collisions. Recent electrostatic accelerators have been equipped with electrostatic velocity selectors which provide an energy spread in the proton beam of less than 300 e.v. at 1 Mev. In order to take advantage of this narrow energy range, very thin targets must be used. These problems are discussed in "Gamma-Radiation from Excited States of Light Nuclei" by Fowler, Lauritsen and Lauritsen, Rev.Mod.Phys. 20 236 (1948)).

F. Quantitative Development of Resonance Theory; Breit-Wigner Formula.

In this discussion we use as an example the (n,γ) reaction, i.e., radiative capture of neutrons, which is an important reaction.

As in the preceding sections, the energy levels of the initial and final states form a continuous distribution. The experment picks out the particular initial state.

The resonance phenomenon that we wish to describe is very energy-sensitive. We shall attribute it to the existence of a compound nucleus state C, connecting with the initial and final states by matrix elements $\mathcal{H}_{ac}$ and $\mathcal{H}_{bc}$, where the notation is

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CHAPTER VIII

APPROXIMATION METHODS FOR TIME-DEPENDENT PROBLEMS

It is generally impossible to obtain exact solutions of the Schrödinger equation when the Hamiltonian depends on the time. The three approximation methods that we consider in this chapter all start from the assumption that there is a time-independent Hamiltonian that approximates the actual Hamiltonian in some sense, for which the Schrödinger equation can be solved. The time-dependent part of the actual Hamiltonian may be small compared to the stationary part, in which case a perturbation method can be used. Or there may be time-dependent parameters in the actual Hamiltonian that change very slowly (*adiabatic approximation*) or very rapidly (*sudden approximation*) in comparison with the periods of the approximate stationary solutions.

29. TIME-DEPENDENT PERTURBATION THEORY

The perturbation theory of a system for which the Hamiltonian depends on the time[1] is sometimes called the *method of variation of constants*. It starts from the assumption of Sec. 25 that

$$H = H_0 + H', \qquad H_0 u_n = E_n u_n \tag{29.1}$$

where the unperturbed Hamiltonian H_0 can be solved for its normalized eigenfunctions u_n and its energy eigenvalues E_n, and the perturbation H' is small. Since H' now depends on the time, stationary solutions of the actual Schrödinger equation do not exist, and we must work with the time-dependent equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{29.2}$$

Expansion in Unperturbed Eigenfunctions. Our procedure is to express ψ as an expansion in the eigenfunctions $u_n e^{-\frac{iE_n t}{\hbar}}$ of the unperturbed time-dependent wave equation, where the expansion coefficients evidently depend on the time.

$$\psi = \sum a_n(t) u_n e^{-\frac{iE_n t}{\hbar}} \tag{29.3}$$

[1] P. A. M. Dirac, *Proc. Roy. Soc.*, **A112**, 661 (1926); **A114**, 243 (1927).

$\sum$ denotes a summation over the discrete set together with an integration over the continuous set of eigenfunctions. Substitution of (29.3) into (29.2) gives

$$\sum i\hbar \dot{a}_n u_n e^{-\frac{iE_n t}{\hbar}} + \sum a_n E_n u_n e^{-\frac{iE_n t}{\hbar}} = \sum a_n (H_0 + H') u_n e^{-\frac{iE_n t}{\hbar}}$$

where the dot denotes differentiation with respect to the time.

We replace $H_0 u_n$ by $E_n u_n$ on the right side, multiply through on the left by $\bar{u}_k$, and integrate over all space, making use of the orthonormality of the u's

$$i\hbar \dot{a}_k e^{-\frac{iE_k t}{\hbar}} = \sum a_n e^{-\frac{iE_n t}{\hbar}} \int \bar{u}_k H' u_n d\tau$$

The integral on the right is the matrix element H'_{kn} of the perturbation. We define the Bohr (angular) frequency

$$\omega_{kn} \equiv \frac{E_k - E_n}{\hbar} \tag{29.4}$$

and obtain

$$\dot{a}_k = (i\hbar)^{-1} \sum H'_{kn} a_n e^{i\omega_{kn} t} \tag{29.5}$$

The group of Eqs. (29.5) for all k's is exactly equivalent to the Schrödinger equation (29.2); the amplitude a_n of a particular unperturbed eigenfunction u_n in the expansion of ψ has replaced the amplitude ψ at a particular point in space. Because of the choice of the representation, which is determined by the eigenfunctions of the unperturbed Hamiltonian, H_0 does not appear explicitly in (29.5).

The perturbation approximation consists in replacing H' by $\lambda H'$ in (29.1) and (29.5), and expressing the a's as power series in λ:

$$a_n = a_n^{(0)} + \lambda a_n^{(1)} + \lambda^2 a_n^{(2)} + \cdots \tag{29.6}$$

As in Sec. 25, we assume that these series are analytic for λ between 0 and 1. We can therefore substitute (29.6) into (29.5), equate coefficients of equal powers of λ, and set $\lambda = 1$ in the final results. The substitution yields the set of equations

$$\dot{a}_k^{(0)} = 0; \qquad \dot{a}_k^{(s+1)} = (i\hbar)^{-1} \sum H'_{kn} a_n^{(s)} e^{i\omega_{kn} t}, \quad s = 0, 1, 2, \ldots \tag{29.7}$$

These can in principle be integrated successively to obtain approximate solutions to any desired order in the perturbation.

First-order Perturbation. The first of Eqs. (29.7) shows that the zero-order coefficients $a_k^{(0)}$ are constant in time. Their values are the initial conditions of the problem, which specify the state of the system before the perturbation is applied. We assume throughout this section

that all except one of the $a_k^{(0)}$ are zero, so that the system is in a definite unperturbed energy state when the perturbation is applied.[1] The results that we shall obtain can easily be generalized to situations in which more than one of the zero-order coefficients is different from zero.

We thus put $a_k^{(0)} = \delta_{km}$ or $\delta(k - m)$, according as the state m is one of a discrete or a continuous set. Integration of the first-order equation gives

$$a_k^{(1)}(t) = (i\hbar)^{-1} \int_{-\infty}^{t} H'_{km}(t')e^{i\omega_{km}t'}dt' \tag{29.8}$$

where the constant of integration is taken to be zero in order that $a_k^{(1)}$ be zero at $t = -\infty$ (before the perturbation is applied). If H' is of finite duration, the amplitude of a state u_k ($k \neq m$) after the perturbation has disappeared is proportional to the time Fourier component of the matrix element of the perturbation between this state and the initial state, that corresponds to the angular frequency ω_{km} given in (29.4). This result is analogous to that obtained for the scattered amplitude in the Born approximation [see the discussion of Eq. (26.18)].

Equation (29.8) takes a particularly simple form if the perturbation H' is independent of the time except for being turned on at one time and off at a later time. We call these two times 0 and t, respectively, and obtain for the first-order amplitudes at the time t (these are also the amplitudes at any subsequent time)

$$a_k^{(1)}(t) = -\frac{H'_{km}}{\hbar}\frac{e^{i\omega_{km}t} - 1}{\omega_{km}} \tag{29.9}$$

Thus the probability of finding the system in the state k at t is

$$|a_k^{(1)}(t)|^2 = \frac{4|H'_{km}|^2 \sin^2 \frac{1}{2}\omega_{km}t}{\hbar^2\omega_{km}^2}$$

The factor $\sin^2 \frac{1}{2}\omega_{km}t/\omega_{km}^2$ is plotted in Fig. 27 as a function of ω_{km}.

Physical Interpretation. The height of the main peak in Fig. 27 increases in proportion to t^2, while its breadth decreases inversely as t, so that the area under the curve is proportional to t. Thus if there is a group of states k that have energies nearly equal to that of the initial state m, and for which H'_{km} is roughly independent of k, the probability of finding the system in one or another of these states is proportional to t. This is the physically interesting situation, since what we wish to calculate eventually is a *transition probability per unit time* w, and this implies that

[1] This need not conflict with the uncertainty relation (3.3), since the infinite lapse of time prior to the application of the perturbation makes it possible to determine the original energy of the system with arbitrarily great precision.

the probability that a transition has taken place when the perturbation has been on for a time t is proportional to t.[1]

It follows that a definite value of w exists only when the final state k is one of a continuous or nearly continuous set of states. The spread in energy of the final states to which transitions occur, shown in Fig. 27

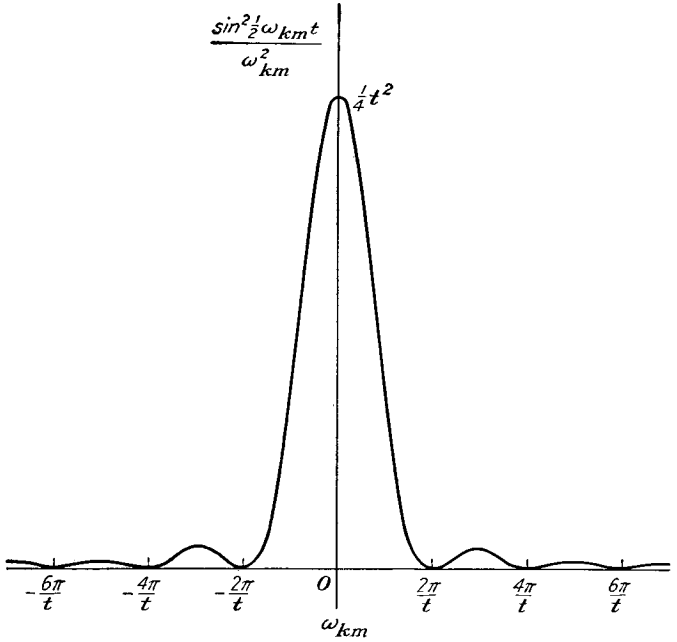

FIG. 27. The ordinate is proportional to the first-order perturbation probability of finding a system in a state that has an energy different from that of the initial state by $\hbar\omega_{km}$; the scales for ordinate and abscissa depend on the duration t of the perturbation in the manner indicated.

($E_k = E_m + \hbar\omega_{km}$), is connected with the uncertainty relation (3.3) for energy and time in the following way. We can regard the perturbation H' as a device that measures the energy of the system (which is not necessarily its initial energy since the system is disturbed) by transferring it to one of the states k. The time available for the measurement is t, so that the uncertainty in energy predicted by (3.3) is of order $\hbar/t$, in agreement with the breadth of the main peak in Fig. 27. It is interesting

[1] We assume that the total transition probability to all states k is small enough in comparison with unity so that the initial state m is not significantly depleted. This is equivalent to the original assumption that the perturbation is small, which means that for times t of physical interest, there is little change in the initial state. There can still be an effect of observable magnitude if a large number of independent systems receive identical treatment.

to note that conservation of energy, suitably modified by the uncertainty principle, is an automatic consequence of the calculation and does not have to be inserted as a separate assumption.

Transition Probability. In order to obtain an explicit expression for w, it is convenient to assume that the system is contained in a large cubical box of dimensions L that has periodic boundary conditions at its walls (Sec. 10). Then the eigenfunctions u_n form a discrete set and can be normalized to unity in the volume L^3. We now consider a particular group of final states k that have nearly the same energy as the initial state m and for which the matrix element H'_{km} of the perturbation is a slowly varying function of k. We define a density of final states $\rho(k)$ such that $\rho(k)dE_k$ is the number of such states in the energy range dE_k, and assume that $\rho(k)$ is also a slowly varying function of k.

The transition probability per unit time to one or another of this group of states can then be written

$$w = t^{-1} \sum_k |a_k^{(1)}(t)|^2 = t^{-1} \int |a_k^{(1)}(t)|^2 \rho(k) dE_k \tag{29.10}$$

when the box L is large enough so that the summation over k can be replaced by the integration over E_k. Since H'_{km} and $\rho(k)$ are slowly varying and most of the contribution to the integral comes from a narrow range of energy about $E_k = E_m$, they can be taken outside of the integral, and (29.10) can be rewritten as

$$w = \frac{1}{t} \frac{4|H'_{km}|^2}{\hbar} \rho(k) \int_{-\infty}^{\infty} \frac{\sin^2 \frac{1}{2}\omega_{km} t}{\omega_{km}^2} d\omega_{km} \tag{29.11}$$

where the index k now refers to a typical one of the group of states having about the energy E_m. The integral in (29.11) is $\frac{1}{2}t \int_{-\infty}^{\infty} x^{-2} \sin^2 x dx = \frac{1}{2}\pi t$, so that we finally obtain

$$w = \frac{2\pi}{\hbar} \rho(k) |H'_{km}|^2 \tag{29.12}$$

which is independent of t, as expected.

There may be several different groups of final states k, all of which have about the energy E_m but for which the perturbation matrix elements H'_{km} and the densities of states $\rho(k)$, while nearly constant within a group, differ from one group to another. Then (29.12) gives the transitions per unit time to a particular group; similar expressions of the same form give the rates of transition to other groups.

Scattering Cross Section. As a first application of Eq. (29.12), we calculate w when the initial and final states are free-particle momentum

eigenfunctions (plane waves) and the perturbation is a potential energy $V(\mathbf{r})$. The result can be interpreted in terms of an elastic scattering cross section, and then agrees with the Born approximation result (Sec. 26), as expected. We take for the initial and final states

$$u_m(\mathbf{r}) = L^{-\frac{3}{2}} \exp i\mathbf{k}_0 \cdot \mathbf{r}, \qquad u_k(\mathbf{r}) = L^{-\frac{3}{2}} \exp i\mathbf{k} \cdot \mathbf{r}$$

where $\mathbf{k}_0$ and $\mathbf{k}$ are the initial and final propagation vectors, respectively. Thus the perturbation matrix element is

$$H'_{km} = L^{-3}\int \exp(-i\mathbf{k} \cdot \mathbf{r})V(\mathbf{r}) \exp(i\mathbf{k}_0 \cdot \mathbf{r})d\tau$$
$$= L^{-3}\int V(\mathbf{r}) \exp(i\mathbf{K} \cdot \mathbf{r})d\tau \quad (29.13)$$

where $\mathbf{K} = \mathbf{k}_0 - \mathbf{k}$.

The density of final states can be found from the permitted values of $\mathbf{k}$ in a box: $k_x = 2\pi n_x/L$, etc., where the n's are positive or negative integers or zero. Thus there are $(L/2\pi)^3 dk_x dk_y dk_z$ states in the range $dk_x dk_y dk_z$ of propagation vector. Now there are many different final states $\mathbf{k}$ with the same energy, corresponding to different directions of $\mathbf{k}$ with a given magnitude. The matrix element (29.13) usually depends on the direction of $\mathbf{k}$, so that we have to consider only a small range of directions at a time. We therefore ask for the rate of transition into an infinitesimal element of solid angle $\sin\theta d\theta d\phi$ about some direction that is specified by the polar angles θ, ϕ. Then $\rho(k)dE_k$ is equal to the number of states in the range $d\tau_k$ given by the above solid angle element and the magnitude element dk that corresponds to the energy element dE_k.

$$\rho(k)dE_k = \left(\frac{L}{2\pi}\right)^3 k^2 dk \sin\theta d\theta d\phi$$

Since $E_k = \hbar^2k^2/2\mu$, $dE_k/dk = \hbar^2k/\mu$, and we obtain for $\rho(k)$

$$\rho(k) = \frac{\mu L^3}{8\pi^3\hbar^2} k \sin\theta d\theta d\phi \quad (29.14)$$

The value of w obtained in this way is the number of particles scattered into the element of solid angle per unit time when there is one incident particle in the volume L^3. This is an incident flux of v/L^3 particles per unit area and time, where $v = \hbar k/\mu$ is the speed of the incident or scattered particle (since energy is conserved). Since the differential scattering cross section is defined as the scattering per unit incident flux, we have that

$$\sigma(\theta,\phi) \sin\theta d\theta d\phi = \frac{\mu L^3}{\hbar k} w \quad (29.15)$$

Substitution of (29.12), (29.13), and (29.14) into (29.15) gives

$$\sigma(\theta,\phi) = \left(\frac{\mu}{2\pi\hbar^2}\right)^2 \left| \int V(\mathbf{r}) \exp(i\mathbf{K}\cdot\mathbf{r}) d\tau \right|^2 \tag{29.16}$$

This agrees with the Born approximation result (26.18) and (26.19), and has the same range of validity.

Harmonic Perturbation. Another situation for which Eq. (29.8) assumes a simple form occurs when the perturbation depends harmonically on the time, except for being turned on at zero time and off at time t. If we put $H'_{km}(t') = H'^0_{km} \sin \omega t'$, the first-order amplitudes at time t are

$$a_k^{(1)}(t) = -\frac{H'^0_{km}}{2i\hbar}\left[\frac{e^{i(\omega_{km}+\omega)t}-1}{\omega_{km}+\omega} - \frac{e^{i(\omega_{km}-\omega)t}-1}{\omega_{km}-\omega}\right] \tag{29.17}$$

The probability of finding the system in the state k is appreciable only when the denominator of one or the other of the two terms in (29.17) is practically zero. Thus there is no interference between the two terms, and the perturbation can produce transitions for which $\omega_{km} \cong \pm\omega$ if the corresponding matrix element does not vanish. The energy-conservation condition $E_k \cong E_m$ obtained earlier is now replaced by the condition

$$E_k \cong E_m \pm \hbar\omega \tag{29.18}$$

Equation (29.18) suggests that the first-order effect of a perturbation that varies harmonically in the time with angular frequency ω is to transfer to or receive from the system on which it acts an amount of energy $\hbar\omega$. This concept will be used for a qualitative treatment of radiation processes in Chap. X.

Second-order Perturbation. The series of equations (29.7) can readily be solved to second order for a perturbation that is constant in time. We take the equation with $s = 1$, and substitute from (29.9) on the right side.

$$\dot{a}_k^{(2)} = \frac{i}{\hbar^2} \sum \frac{H'_{kn}H'_{nm}}{\omega_{nm}} (e^{i\omega_{km}t} - e^{i\omega_{kn}t})$$

Integration of this equation subject to the initial condition $a_k^{(2)}(0) = 0$ gives for the second-order amplitudes at time t

$$a_k^{(2)}(t) = \hbar^{-2} \sum \frac{H'_{kn}H'_{nm}}{\omega_{nm}}\left[\frac{e^{i\omega_{km}t}-1}{\omega_{km}} - \frac{e^{i\omega_{kn}t}-1}{\omega_{kn}}\right] \tag{29.19}$$

Equation (29.19) indicates that transitions for which the probability increases linearly with the time can occur either for $\omega_{km} \cong 0$ or $\omega_{kn} \cong 0$. While the first type of transition conserves energy between the initial

state m and the final state k, the second need not. It is not difficult to see that the second bracket term arises from the 1 in the numerator of (29.9), which in turn comes from the initial condition at zero time. This initial condition means that the perturbation is turned on suddenly; thus the mathematical formulation suggests that the second-order transitions that do not conserve energy are caused by the sudden turning on of the perturbation. This is in agreement with Eqs. (29.8) and (29.17), which show that a perturbation that has nonzero frequency Fourier components can give up energy to or absorb energy from the system that it perturbs. In the case we are now considering, these Fourier components are not marked enough to produce in first order a transition probability that is proportional to the time, but they do in second order.

In most practical problems, the sudden turning on of the perturbation is introduced as a mathematical artifice that simplifies the calculation. Actually, in such cases, the perturbation either is always present, or is turned on very slowly, and we are concerned with transitions that conserve energy between initial and final states. Problems that can be treated by the sudden approximation (see end of Sec. 31) are an exception; there energy need not be conserved. Throughout this section and the next, we assume that only transitions that conserve the energy actually occur ($\omega_{km} \cong 0$).

Suppose now that the perturbation produces no transitions in first order; this means that there are no states n that conserve energy ($\omega_{nm} \cong 0$) for which the matrix element $H'_{nm} \neq 0$. Since $\omega_{km} \cong 0$, this means also that $H'_{nm} = 0$ whenever $\omega_{kn} \cong 0$. In this case, the second term in the bracket of (29.19) is never appreciable. The calculation of w is carried through as before, except that $a_k^{(2)}$ replaces $a_k^{(1)}$; thus (29.12) can be used if the matrix element H'_{km} is replaced by the second-order matrix element

$$\mathsf{S} \frac{H'_{kn}H'_{nm}}{E_m - E_n} \tag{29.20}$$

Effect of First-order Transitions. In the event that transitions can occur in first order, but they are not to the state in which we are interested, we can proceed as follows. It is still true that the second term in the bracket of (29.19) is negligible for states n that have energies appreciably different from E_k (or E_m), since then ω_{kn} is large. However, there may now be states n for which E_n, E_m, and E_k are all close together and neither H'_{kn} nor H'_{nm} is zero. The second bracket term cannot be ignored, for without it the summation or integration over n would have a singularity when ω_{nm} is zero. It is not difficult to see that for any value of ω_{km} (zero or otherwise), the entire bracket is proportional to ω_{nm} (which is equal to $\omega_{km} - \omega_{kn}$) when ω_{nm} is small; this cancels out the ω_{nm}

in the denominator, and makes the summand or integrand finite where $\omega_{nm} = 0$.[1]

We now show how an explicit evaluation of (29.19) is obtained in this case if S can be represented by an integral over E_n or ω_{nm}. We divide the integral into parts according as $|\omega_{nm}|$ is large or is not large in comparison with $1/t$. In the first region, the second bracket term in (29.19) can be neglected, since $|\omega_{kn}| = |\omega_{km} - \omega_{nm}|$ is also large in comparison with $1/t$ ($\omega_{km} \cong 0$ means that $\omega_{km}t$ is not large in comparison with unity). We thus obtain for this part of the integral

$$\frac{e^{i\omega_{km}t} - 1}{\omega_{km}} \int' \frac{H'_{kn}H'_{nm}}{\omega_{nm}} \rho(n)\hbar d\omega_{nm} \tag{29.21}$$

Here $\rho(n)dE_n$ is the number of states of the particular group n under consideration in the energy range dE_n; the prime on the integral implies that the region $-c/t \leqq \omega_{nm} \leqq c/t$ is excluded from the integration, where c is a constant number that is large in comparison with unity. If there are two or more distinct groups of states n for which the matrix elements or densities of states differ, a further summation over these different groups must eventually be carried out.

In the second region, where $|\omega_{nm}| \leqq c/t$, we assume that t is large enough so that $H'_{kn}H'_{nm}\rho(n)$ can be regarded as a constant, taken outside of the integral, and evaluated at $\omega_{nm} = 0$. We must now use both terms in the bracket of (29.19) in order that the integrand be finite. This part of the integral is then

$$[\hbar H'_{kn}H'_{nm}\rho(n)]_{\omega_{nm}=0} \cdot \int_{-\frac{c}{t}}^{\frac{c}{t}} \left[\frac{e^{i\omega_{km}t} - 1}{\omega_{km}} - \frac{e^{i(\omega_{km}-\omega_{nm})t} - 1}{\omega_{km} - \omega_{nm}}\right] \frac{d\omega_{nm}}{\omega_{nm}} \tag{29.22}$$

The integral that appears in (29.22) can be evaluated by considering the contour in the complex ω_{nm} plane shown in Fig. 28, which contains no poles of the integrand. Thus the integral over the closed contour is zero, and the integral in (29.22) is equal to the integral around the semicircle of radius c/t taken in the counterclockwise direction. The magnitude of ω_{nm} is great enough over this semicircle so that the contribution

[1] This result follows quite generally from the structure of the whole perturbation calculation, since there is no way in which a singularity can appear. Thus if the perturbation is turned on slowly rather than suddenly, so that the energy-conservation difficulties mentioned above do not occur, the second bracket term of (29.19) has a more complicated form but still cancels out the singularity at $\omega_{nm} = 0$. This can be verified by direct calculation.

of the second term in the integrand can be neglected in comparison with the first. The integral in (29.22) is then easily evaluated and becomes

$$\pi i \frac{e^{i\omega_{km}t} - 1}{\omega_{km}} \tag{29.23}$$

For large t, the prime on the integral in (29.21) is equivalent to taking its principal value.[1] Thus if we substitute (29.23) into (29.22) and add the result to (29.21), we obtain an expression like (29.21) except that the primed integral is replaced by the principal value of the integral plus πi times the residue of the integrand at the pole $\omega_{nm} = 0$. This is equivalent to evaluating the integral along a contour in the complex ω_{nm} plane

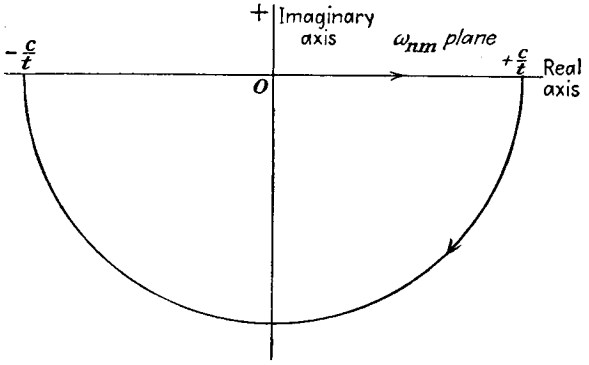

FIG. 28. Contour for the evaluation of the integral in Eq. (29.22).

that passes along the real axis from $-\infty$ to ∞ except for passing beneath the origin. We thus obtain finally

$$a_k^{(2)}(t) = \frac{e^{i\omega_{km}t} - 1}{\hbar\omega_{km}} \int_C \frac{H'_{kn}H'_{nm}}{E_n - E_m} \rho(n)dE_n \tag{29.24}$$

where the contour C is over the real axis of E_n except for passing under the pole of the integrand at $E_n = E_m$. Equation (29.24) is to be used in place of (29.19) whenever S can be represented by $\int \rho(n)dE_n$. Comparison of Eqs. (29.24) and (29.9) shows that we can use the expression (29.12) for w if we replace the matrix element H'_{km} by the integral in (29.24), which we sometimes call the second-order matrix element. An example of this is given in the next section.

Intermediate States. We see that the time-dependent perturbation theory gives a result in first order if there is a nonvanishing matrix

[1] E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," 4th ed., pp. 75, 117 (Cambridge, London, 1935).

element of H' that connects the initial state m and the final state k. If $H'_{km} = 0$, but there are one or more states n for which neither H'_{nm} nor H'_{kn} is zero, the transition occurs in second order.

It is then convenient to think of one of the states n as an *intermediate state:* the perturbation transfers the system from m to k in two steps, through a state n. Energy need not be conserved for an intermediate state, since it has only a transient existence and according to the uncertainty relation (3.3) it is impossible to determine the energy of such a short-lived state with any precision. If some of the intermediate states do conserve energy, the summation (29.20) over these states must be interpreted in accordance with the integral in (29.24).

In some cases, a perturbation can produce a particular transition only through two or more different intermediate states; this corresponds to a third or higher order of the perturbation calculation. If the perturbation is small, it usually happens that the result of a calculation to the lowest order in which the transition occurs gives a useful result, while higher order calculations do not improve on this and may even be quite misleading.

30. INELASTIC COLLISIONS

The expression for the scattering cross section given in the preceding section is easily generalized to a description of inelastic collisions, in which internal as well as kinetic energy can be transferred between the colliding systems. In this section we apply the result to two problems that are typical of first-order and of second-order processes.[1] The latter calculation is of unusual theoretical interest, for it shows explicitly how a particle that is described entirely in terms of a plane wave (momentum eigenfunction) can produce a sharp track in a Wilson cloud chamber.

Expression for the Cross Section. The expression (29.12) for the rate of transition w is applicable to inelastic collisions if the matrix element is defined accordingly. We consider here the collision of a fast electron with a hydrogen atom in its ground state, and wish to calculate the cross section for scattering of the electron through a definite angle accompanied by excitation of the hydrogen atom to a definite state. We leave out of consideration the possibility that the incident electron changes places with the atomic electron; such exchange collisions will be taken up in Chap. IX.

[1] The examples considered in this section can also be treated by an extension of the Born approximation; such an extension to first-order rearrangement collisions is given in Sec. 34. For the treatment of second-order processes, it is more convenient to work with the method of variation of constants.