8

The Path Integral Formulation of Quantum Theory

We consider here an alternate formulation of quantum mechanics invented by Feynman in the forties.‡ In contrast to the Schrödinger formulation, which stems from Hamiltonian mechanics, the Feynman formulation is tied to the Lagrangian formulation of mechanics. Although we are committed to the former approach, we discuss in this chapter Feynman's alternative, not only because of its aesthetic value, but also because it can, in a class of problems, give the full propagator with tremendous ease and also give valuable insight into the relation between classical and quantum mechanics.

8.1. The Path Integral Recipe

We have already seen that the quantum problem is fully solved once the propagator is known. Thus far our practice has been to first find the eigenvalues and eigenfunctions of H, and then express the propagator $U(t)$ in terms of these. In the path integral approach one computes $U(t)$ directly. For a single particle in one dimension, the procedure is the following.

To find $U(x, t; x', t')$:

(1) Draw all paths in the x–t plane connecting (x', t') and (x, t) (see Fig. 8.1).
(2) Find the action $S[x(t)]$ for each path $x(t)$.
(3) $$U(x, t; x', t') = A \sum_{\text{all paths}} e^{iS[x(t)]/\hbar} \tag{8.1.1}$$

where A is an overall normalization factor.

‡ The nineteen forties that is, and in his twenties. An interesting account of how he was influenced by Dirac's work in the same direction may be found in his Nobel lectures. See, *Nobel Lectures—Physics*, Vol. III, Elsevier Publication, New York (1972).

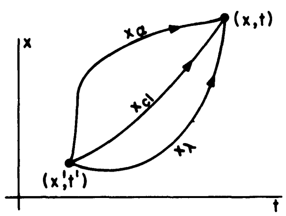

Figure 8.1. Some of the paths that contribute to the propagator. The contribution from the path $x(t)$ is $Z=\exp\{iS[x(t)]/\hbar\}$.

8.2. Analysis of the Recipe

Let us analyze the above recipe, postponing for a while the proof that it reproduces conventional quantum mechanics. The most surprising thing about it is the fact that every path, including the classical path, $x_{cl}(t)$, gets the same weight, that is to say, a number of unit modulus. How are we going to regain classical mechanics in the appropriate limit if the classical path does not seem favored in any way?

To understand this we must perform the sum in Eq. (8.1.1). Now, the correct way to sum over all the paths, that is to say, path integration, is quite complicated and we will discuss it later. For the present let us take the heuristic approach. Let us first pretend that the continuum of paths linking the end points is actually a discrete set. A few paths in the set are shown in Fig. 8.1.

We have to add the contributions $Z_\alpha = e^{iS[x_\alpha(t)]/\hbar}$ from each path $x_\alpha(t)$. This summation is done schematically in Fig. 8.2. Since each path has a different action, it contributes with a different phase, and the contributions from the paths essentially cancel each other, until we come near the classical path. Since S is stationary here, the Z's add constructively and produce a large sum. As we move away from $x_{cl}(t)$, destructive interference sets in once again. It is clear from the figure that $U(t)$ is dominated by the paths near $x_{cl}(t)$. Thus the classical path is important, not because it contributes a lot by itself, but because in its vicinity the paths contribute coherently.

How far must we deviate from x_{cl} before destructive interference sets in? One may say crudely that coherence is lost once the phase differs from the stationary value $S[x_{cl}(t)]/\hbar \equiv S_{cl}/\hbar$ by about π. This in turn means that the action for the coherence paths must be within $\hbar\pi$ of S_{cl}. For a macroscopic particle this means a very tight constraint on its path, since S_{cl} is typically $\simeq 1$ erg sec $\simeq 10^{27}\hbar$, while for an electron there is quite a bit of latitude. Consider the following example. A free particle leaves the origin at $t=0$ and arrives at $x=1$ cm at $t=1$ second. The classical path is

$$x = t \tag{8.2.1}$$

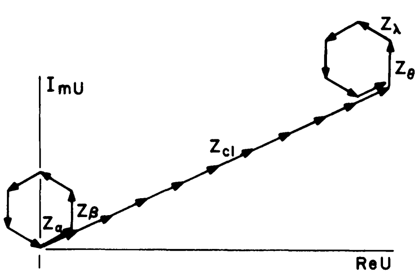

Figure 8.2. Schematic representation of the sum ΣZ_α. Paths near $x_{cl}(t)$ contribute coherently since S is stationary there, while others cancel each other out and may be ignored in the first approximation when we calculate $U(t)$.

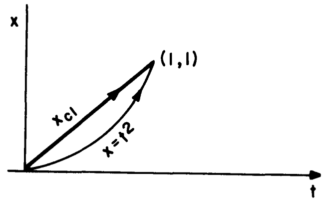

Figure 8.3. Two possible paths connecting (0, 0) and (1, 1). The action on the classical path $x=t$ is $m/2$, while on the other, it is $2m/3$.

Consider another path

$$x=t^2 \tag{8.2.2}$$

which also links the two space-time points (Fig. 8.3.)

For a classical particle, of mass, say 1 g, the action changes by roughly $1.6\times 10^{26}\hbar$, and the phase by roughly 1.6×10^{26} rad as we move from the classical path $x=t$ to the nonclassical path $x=t^2$. We may therefore completely ignore the nonclassical path. On the other hand, for an electron whose mass is $\simeq 10^{-27}$ g, $\delta S\simeq \hbar/6$ and the phase change is just around a sixth of a radian, which is well within the coherence range $\delta S/\hbar \lesssim \pi$. It is in such cases that assuming that the particle moves along a well-defined trajectory, $x_{cl}(t)$, leads to conflict with experiment.

8.3. An Approximation to $U(t)$ for a Free Particle

Our previous discussions have indicated that, to an excellent approximation, we may ignore all but the classical path and its neighbors in calculating $U(t)$. Assuming that each of these paths contributes the same amount $\exp(iS_{cl}/\hbar)$, since S is stationary, we get

$$U(t)=A'\, e^{iS_{cl}/\hbar} \tag{8.3.1}$$

where A' is some normalizing factor which "measures" the number of paths in the coherent range. Let us find $U(t)$ for a free particle in this approximation and compare the result with the exact result, Eq. (5.1.10).

The classical path for a free particle is just a straight line in the x–t plane:

$$x_{cl}(t'')=x'+\frac{x-x'}{t-t'}(t''-t') \tag{8.3.2}$$

corresponding to motion with uniform velocity $v=(x-x')/(t-t')$. Since $\mathscr{L}=mv^2/2$ is a constant,

$$S_{cl}=\int_{t'}^{t}\mathscr{L}\,dt''=\frac{1}{2}m\frac{(x-x')^2}{t-t'}$$

so that

$$U(x, t; x', t') = A' \exp\left[\frac{im(x-x')^2}{2\hbar(t-t')}\right] \tag{8.3.3}$$

To find A', we use the fact that as $t-t'$ tends to 0, U must tend to $\delta(x-x')$. Comparing Eq. (8.3.3) to the representation of the delta function encountered in Section 1.10 (see footnote on page 61),

$$\delta(x-x') \equiv \lim_{\Delta\to 0} \frac{1}{(\pi\Delta^2)^{1/2}} \exp\left[-\frac{(x-x')^2}{\Delta^2}\right]$$

(valid even if Δ is imaginary) we get

$$A' = \left[\frac{m}{2\pi\hbar i(t-t')}\right]^{1/2}$$

so that

$$U(x, t; x', 0) \equiv U(x, t; x') = \left(\frac{m}{2\pi\hbar i t}\right)^{1/2} \exp\left[\frac{im(x-x')^2}{2\hbar t}\right] \tag{8.3.4}$$

which is the exact answer! We have managed to get the exact answer by just computing the classical action! However, we will see in Section 8.6 that only for potentials of the form $V = a + bx + cx^2 + d\dot{x} + ex\dot{x}$ is it true that $U(t) = A(t)\, e^{iS_{cl}/\hbar}$. Furthermore, we can't generally find $A(t)$ using $U(x, 0; x') = \delta(x-x')$ since A can contain an arbitrary dimensionless function f such that $f \to 1$ as $t \to 0$. Here $f \equiv 1$ because we can't construct a nontrivial dimensionless f using just m, $\hbar$, and t (check this).

8.4. Path Integral Evaluation of the Free-Particle Propagator

Although our heuristic analysis yielded the exact free-particle propagator, we will now repeat the calculation without any approximation to illustrate path integration.

Consider $U(x_N, t_N; x_0, t_0)$. The peculiar labeling of the end points will be justified later. Our problem is to perform the path integral

$$\int_{x_0}^{x_N} e^{iS[x(t)]/\hbar} \mathscr{D}[x(t)] \tag{8.4.1}$$

where

$$\int_{x_0}^{x_N} \mathscr{D}[x(t)]$$

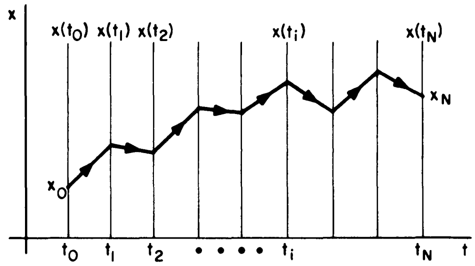

Figure 8.4. The discrete approximation to a path $x(t)$. Each path is specified by $N-1$ numbers $x(t_1), \ldots, x(t_{N-1})$. To sum over paths we must integrate each x_i from $-\infty$ to $+\infty$. Once all integrations are done, we can take the limit $N\to\infty$.

is a symbolic way of saying "integrate over all paths connecting x_0 and x_N (in the interval t_0 and t_N)." Now, a path $x(t)$ is fully specified by an infinity of numbers $x(t_0), \ldots, x(t), \ldots, x(t_N)$, namely, the values of the function $x(t)$ at every point t in the interval t_0 to t_N. To sum over all paths we must integrate over all possible values of these infinite variables, except of course $x(t_0)$ and $x(t_N)$, which will be kept fixed at x_0 and x_N, respectively. To tackle this problem, we follow the idea that was used in Section 1.10: we trade the function $x(t)$ for a discrete approximation which agrees with $x(t)$ at the $N+1$ points $t_n = t_0 + n\varepsilon$, $n=0, \ldots, N$, where $\varepsilon = (t_N - t_0)/N$. In this approximation each path is specified by $N+1$ numbers $x(t_0), x(t_1), \ldots, x(t_N)$. The gaps in the discrete function are interpolated by straight lines. One such path is shown in Fig. 8.4. We hope that if we take the limit $N\to\infty$ at the end we will get a result that is insensitive to these approximations.‡ Now that the paths have been discretized, we must also do the same to the action integral. We replace the continuous path definition

$$S=\int_{t_0}^{t_N} \mathscr{L}(t)\, dt=\int_{t_0}^{t_N} \frac{1}{2} m\dot{x}^2\, dt$$

by

$$S=\sum_{i=0}^{N-1} \frac{m}{2}\left(\frac{x_{i+1}-x_i}{\varepsilon}\right)^2 \varepsilon \tag{8.4.2}$$

where $x_i = x(t_i)$. We wish to calculate

$$\begin{aligned} U(x_N, t_N; x_0, t_0) &= \int_{x_0}^{x_N} \exp\{iS[x(t)]/\hbar\}\mathscr{D}[x(t)] \\ &= \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2}\sum_{i=0}^{N-1}\frac{(x_{i+1}-x_i)^2}{\varepsilon}\right] \\ &\quad \times dx_1\cdots dx_{N-1} \end{aligned} \tag{8.4.3}$$

‡ We expect that the abrupt changes in velocity at the points $t_0 + n\varepsilon$ that arise due to our approximation will not matter because $\mathscr{L}$ does not depend on the acceleration or higher derivatives.

It is implicit in the above that x_0 and x_N have the values we have chosen at the outset. The factor A in the front is to be chosen at the end such that we get the correct scale for U when the limit $N\to\infty$ is taken.

Let us first switch to the variables

$$y_i=\left(\frac{m}{2\hbar\varepsilon}\right)^{1/2} x_i$$

We then want

$$\lim_{N\to\infty} A'\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\sum_{i=0}^{N-1}\frac{(y_{i+1}-y_i)^2}{i}\right]dy_1\cdots dy_{N-1} \qquad (8.4.4)$$

where

$$A'=A\left(\frac{2\hbar\varepsilon}{m}\right)^{(N-1)/2}$$

Although the multiple integral looks formidable, it is not. Let us begin by doing the y_1 integration. Considering just the part of the integrand that involves y_1, we get

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{i}[(y_2-y_1)^2+(y_1-y_0)^2]\right\}dy_1=\left(\frac{i\pi}{2}\right)^{1/2} e^{-(y_2-y_0)^2/2i} \qquad (8.4.5)$$

Consider next the integration over y_2. Bringing in the part of the integrand involving y_2 and combining it with the result above we compute next

$$\left(\frac{i\pi}{2}\right)^{1/2}\int_{-\infty}^{\infty} e^{-(y_3-y_2)^2/i}\cdot e^{-(y_2-y_0)^2/2i}\,dy_2$$

$$=\left(\frac{i\pi}{2}\right)^{1/2} e^{-(2y_3^2+y_0^2)/2i}\left(\frac{2\pi i}{3}\right)^{1/2} e^{(y_0+2y_3)^2/6i}$$

$$=\left[\frac{(i\pi)^2}{3}\right]^{1/2} e^{-(y_3-y_0)^2/3i} \qquad (8.4.6)$$

By comparing this result to the one from the y_1 integration, we deduce the pattern: if we carry out this process $N-1$ times so as to evaluate the integral in Eq. (8.4.4), it will become

$$\frac{(i\pi)^{(N-1)/2}}{N^{1/2}}e^{-(y_N-y_0)^2/Ni}$$

or

$$\frac{(i\pi)^{(N-1)/2}}{N^{1/2}} e^{-m(x_N-x_0)^2/2\hbar\varepsilon Ni}$$

Bringing in the factor $A(2\hbar\varepsilon/m)^{(N-1)/2}$ from up front, we get

$$U = A\left(\frac{2\pi\hbar\varepsilon i}{m}\right)^{N/2}\left(\frac{m}{2\pi\hbar iN\varepsilon}\right)^{1/2}\exp\left[\frac{im(x_N-x_0)^2}{2\hbar N\varepsilon}\right]$$

If we now let $N\to\infty$, $\varepsilon\to 0$, $N\varepsilon\to t_N-t_0$, we get the right answer provided

$$A = \left[\frac{2\pi\hbar\varepsilon i}{m}\right]^{-N/2} \equiv B^{-N} \tag{8.4.7}$$

It is conventional to associate a factor $1/B$ with each of the $N-1$ integrations and the remaining factor $1/B$ with the overall process. In other words, we have just learnt that the precise meaning of the statement "integrate over all paths" is

$$\int \mathscr{D}[x(t)] = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \frac{1}{B}\int_{-\infty}^{\infty}\int\int\cdots\int_{-\infty}^{\infty}\frac{dx_1}{B}\cdot\frac{dx_2}{B}\cdots\frac{dx_{N-1}}{B}$$

where

$$B = \left(\frac{2\pi\hbar\varepsilon i}{m}\right)^{1/2} \tag{8.4.8}$$

8.5. Equivalence to the Schrödinger Equation

The relation between the Schrödinger and Feynman formalisms is quite similar to that between the Newtonian and the least action formalisms of mechanics, in that the former approach is local in time and deals with time evolution over infinitesimal periods while the latter is global and deals directly with propagation over finite times.

In the Schrödinger formalism, the change in the state vector $|\psi\rangle$ over an infinitesimal time ε is

$$|\psi(\varepsilon)\rangle - |\psi(0)\rangle = \frac{-i\varepsilon}{\hbar} H|\psi(0)\rangle \tag{8.5.1}$$

which becomes in the X basis

$$\psi(x,\varepsilon) - \psi(x,0) = \frac{-i\varepsilon}{\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,0)\right]\psi(x,0) \tag{8.5.2}$$

to first order in ε. To compare this result with the path integral prediction to the same order in ε, we begin with

$$\psi(x, \varepsilon) = \int_{-\infty}^{\infty} U(x, \varepsilon; x')\psi(x', 0)\, dx' \tag{8.5.3}$$

The calculation of $U(\varepsilon)$ is simplified by the fact that there is no need to do any integrations over intermediate x's since there is just one slice of time ε between the start and finish. So

$$U(x, \varepsilon; x') = \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} \exp\left\{i\left[\frac{m(x-x')^2}{2\varepsilon} - \varepsilon V\left(\frac{x+x'}{2}, 0\right)\right]\Big/\hbar\right\} \tag{8.5.4}$$

where the $(m/2\pi\hbar i\varepsilon)^{1/2}$ factor up front is just the $1/B$ factor from Eq. (8.4.8). We take the time argument of V to be zero since there is already a factor of ε before it and any variation of V with time in the interval 0 to ε will produce an effect of second order in ε. So

$$\psi(x, \varepsilon) = \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[\frac{im(x-x')^2}{2\varepsilon\hbar}\right] \exp\left[-\frac{i\varepsilon}{\hbar} V\left(\frac{x+x'}{2}, 0\right)\right] \times \psi(x', 0)\, dx' \tag{8.5.5}$$

Consider the factor $\exp[im(x-x')^2/2\varepsilon\hbar]$. It oscillates very rapidly as $(x-x')$ varies since ε is infinitesimal and $\hbar$ is so small. When such a rapidly oscillating function multiplies a smooth function like $\psi(x', 0)$, the integral vanishes for the most part due to the random phase of the exponential. Just as in the case of the path integration, the only substantial contribution comes from the region where the phase is stationary. In this case the only stationary point is $x = x'$, where the phase has the minimum value of zero. In terms of $\eta = x' - x$, the region of coherence is, as before,

$$\frac{m\eta^2}{2\varepsilon\hbar} \lesssim \pi$$

or

$$|\eta| \lesssim \left(\frac{2\varepsilon\hbar\pi}{m}\right)^{1/2} \tag{8.5.6}$$

Consider now

$$\psi(x, \varepsilon) = \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp(im\eta^2/2\hbar\varepsilon) \cdot \exp\left[-\left(\frac{i}{\hbar}\right)\varepsilon V\left(x+\frac{\eta}{2}, 0\right)\right] \times \psi(x+\eta, 0)\, d\eta \tag{8.5.7}$$

We will work to first order in ε and therefore to second order in η [see Eq. (8.5.6) above]. We expand

$$\psi(x+\eta, 0)=\psi(x, 0)+\eta \frac{\partial \psi}{\partial x}+\frac{\eta^2}{2} \frac{\partial^2 \psi}{\partial x^2}+\cdots$$

$$\begin{aligned}\exp\left[-\left(\frac{i}{\hbar}\right)\varepsilon V\left(x+\frac{\eta}{2}, 0\right)\right]&=1-\frac{i\varepsilon}{\hbar} V\left(x+\frac{\eta}{2}, 0\right)+\cdots\\ &=1-\frac{i\varepsilon}{\hbar} V(x, 0)+\cdots\end{aligned}$$

since terms of order $\eta\varepsilon$ are to be neglected. Equation (8.5.7) now becomes

$$\begin{aligned}\psi(x, \varepsilon)=\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\bigg[\psi(x, 0)-\frac{i\varepsilon}{\hbar} V(x, 0)\psi(x, 0)\\ +\eta \frac{\partial \psi}{\partial x}+\frac{\eta^2}{2} \frac{\partial^2 \psi}{\partial x^2}\bigg] d\eta\end{aligned}$$

Consulting the list of Gaussian integrals in Appendix A.2, we get

$$\begin{aligned}\psi(x, \varepsilon)=\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2}\bigg[\psi(x, 0)\left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{1/2}-\frac{\hbar\varepsilon}{2im}\left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{1/2}\frac{\partial^2 \psi}{\partial x^2}\\ -\frac{i\varepsilon}{\hbar}\left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{1/2} V(x, 0)\psi(x, 0)\bigg]\end{aligned}$$

or

$$\psi(x, \varepsilon)-\psi(x, 0)=\frac{-i\varepsilon}{\hbar}\left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2}+V(x, 0)\right]\psi(x, 0) \tag{8.5.8}$$

which agrees with the Schrödinger prediction, Eq. (8.5.1).

8.6. Potentials of the Form $V=a+bx+cx^2+d\dot{x}+ex\dot{x}$‡

We wish to compute

$$U(x, t; x')=\int_{x'}^{x} e^{iS[x(t'')]/\hbar}\mathscr{D}[x(t'')] \tag{8.6.1}$$

‡ This section may be omitted without loss of continuity.

Let us write every path as

$$x(t'')=x_{\rm cl}(t'')+y(t'') \tag{8.6.2}$$

It follows that

$$\dot{x}(t'')=\dot{x}_{\rm cl}(t'')+\dot{y}(t'') \tag{8.6.3}$$

Since all the paths agree at the end points, $y(0)=y(t)=0$. When we slice up the time into N parts, we have for intermediate integration variables

$$x_i\equiv x(t_i'')=x_{\rm cl}(t_i'')+y(t_i'')\equiv x_{\rm cl}(t_i'')+y_i$$

Since $x_{\rm cl}(t_i'')$ is just some constant at t_i'',

$$dx_i=dy_i$$

and

$$\int_{x'}^{x}\mathscr{D}[x(t'')]=\int_0^0\mathscr{D}[y(t'')] \tag{8.6.4}$$

so that Eq. (8.6.1) becomes

$$U(x,t;x')=\int_0^0\exp\left\{\frac{i}{\hbar}S[x_{\rm cl}(t'')+y(t'')]\right\}\mathscr{D}[y(t'')] \tag{8.6.5}$$

The next step is to expand the functional S in a Taylor series about $x_{\rm cl}$:

$$\begin{aligned}S[x_{\rm cl}+y]&=\int_0^t\mathscr{L}(x_{\rm cl}+y,\dot{x}_{\rm cl}+\dot{y})\,dt''\\&\equiv\int_0^t\left[\mathscr{L}(x_{\rm cl},\dot{x}_{\rm cl})+\left(\frac{\partial\mathscr{L}}{\partial x}\bigg|_{x_{\rm cl}}y+\frac{\partial\mathscr{L}}{\partial\dot{x}}\bigg|_{x_{\rm cl}}\dot{y}\right)\right.\\&\quad\left.+\frac{1}{2}\left(\frac{\partial^2\mathscr{L}}{\partial x^2}\bigg|_{x_{\rm cl}}y^2+2\frac{\partial^2\mathscr{L}}{\partial x\,\partial\dot{x}}\bigg|_{x_{\rm cl}}y\dot{y}+\frac{\partial^2\mathscr{L}}{\partial\dot{x}^2}\bigg|_{x_{\rm cl}}\dot{y}^2\right)\right]dt''\end{aligned} \tag{8.6.6}$$

The series terminates here since $\mathscr{L}$ is a quadratic polynominal.

The first piece $\mathscr{L}(x_{\rm cl},\dot{x}_{\rm cl})$ integrates to give $S[x_{\rm cl}]\equiv S_{\rm cl}$. The second piece, linear in y and $\dot{y}$, vanishes due to the classical equation of motion. In the last piece, if we recall

$$\mathscr{L}=\tfrac{1}{2}m\dot{x}^2-a-bx-cx^2-d\dot{x}-ex\dot{x} \tag{8.6.7}$$

we get

$$\frac{1}{2}\frac{\partial^2 \mathscr{L}}{\partial x^2} = -c \tag{8.6.8}$$

$$\frac{\partial^2 \mathscr{L}}{\partial x\,\partial \dot{x}} = -e \tag{8.6.9}$$

$$\frac{1}{2}\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^2} = m \tag{8.6.10}$$

Consequently Eq. (8.6.5) becomes

$$U(x, t; x') = \exp\left(\frac{iS_{\text{cl}}}{\hbar}\right)\int_0^0 \exp\left[\frac{i}{\hbar}\int_0^t \left(\frac{1}{2}m\dot{y}^2 - cy^2 - ey\dot{y}\right)dt''\right] \times \mathscr{D}[y(t'')] \tag{8.6.11}$$

Since the path integral has no memory of x_{cl}, it can only depend on t. So

$$U(x, t; x') = e^{iS_{\text{cl}}/\hbar}A(t) \tag{8.6.12}$$

where $A(t)$ is some unknown function of t. Now if we were doing the free-particle problem, we would get Eq. (8.6.11) with $c = e = 0$. In this case we know that [see Eq. (8.3.4)]

$$A(t) = \left(\frac{m}{2\pi\hbar it}\right)^{1/2} \tag{8.6.13}$$

Since the coefficient b does not figure in Eq. (8.6.11), it follows that the same value of $A(t)$ corresponds to the linear potential $V = a + bx$ as well. For the harmonic oscillator, $c = \frac{1}{2}m\omega^2$, and we have to do the integral

$$A(t) = \int_0^0 \exp\left[i/\hbar \int_0^t \frac{1}{2}m(\dot{y}^2 - \omega^2 y^2)\right]dt''\mathscr{D}[y(t'')] \tag{8.6.14}$$

The evaluation of this integral is discussed in the book by Feynman and Hibbs referred to at the end of this section. Note that even if the factor $A(t)$ in $\psi(x, t)$ is not known, we can extract all the probabilistic information at time t.

Notice the ease with which the Feynman formalism yields the full propagator in these cases. Consider in particular the horrendous alternative of finding the eigenfunctions of the Hamiltonian and constructing from them the harmonic oscillator propagator.

The path integral method may be extended to three dimensions without any major qualitative differences. In particular, the form of U in Eq. (8.6.12) is valid for potentials that are at most quadratic in the coordinates and the velocities. An interesting problem in this class is that of a particle in a uniform magnetic field. For further details on the subject of path integral quantum mechanics, see R. P. Feynman and A. R. Hibbs, *Path Integrals and Quantum Mechanics*, McGraw-Hill (1965), and Chapter 21.

*Exercise 8.6.1.** Verify that

$$U(x, t; x', 0) = A(t) \exp(iS_{cl}/\hbar), \; A(t) = \left(\frac{m}{2\pi\hbar i t}\right)^{1/2}$$

agrees with the exact result, Eq. (5.4.31), for $V(x) = -fx$. Hint: Start with $x_{cl}(t'') = x_0 + v_0 t'' + \frac{1}{2}(f/m)t''^2$ and find the constants x_0 and v_0 from the requirement that $x_{cl}(0) = x'$ and $x_{cl}(t) = x$.

Exercise 8.6.2. Show that for the harmonic oscillator with

$$\mathscr{L} = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} m\omega^2 x^2$$

$$U(x, t; x') = A(t) \exp\left\{\frac{im\omega}{2\hbar \sin \omega t}[(x^2 + x'^2)\cos \omega t - 2xx']\right\}$$

where $A(t)$ is an unknown function. (Recall Exercise 2.8.7.)

Exercise 8.6.3. We know that given the eigenfunctions and the eigenvalues we can construct the propagator:

$$U(x, t; x', t') = \sum_n \psi_n(x)^* \psi_n(x')\, e^{-iE_n(t-t')/\hbar} \tag{8.6.15}$$

Consider the reverse process (since the path integral approach gives U directly), for the case of the oscillator.

(1) Set $x = x' = t' = 0$. Assume that $A(t) = (m\omega/2\pi i\hbar \sin \omega t)^{1/2}$ for the oscillator. By expanding both sides of Eq. (8.6.15), you should find that $E = \hbar\omega/2, 5\hbar\omega/2, 9\hbar\omega/2, \ldots$, etc. What happened to the levels in between?

(2) (Optional). Now consider the extraction of the eigenfunctions. Let $x = x'$ and $t' = 0$. Find E_0, E_1, $|\psi_0(x)|^2$, and $|\psi_1(x)|^2$ by expanding in powers of $\alpha = \exp(i\omega t)$.

*Exercise 8.6.4.** Recall the derivation of the Schrödinger equation (8.5.8) starting from Eq. (8.5.4). Note that although we chose the argument of V to be the midpoint $x + x'/2$, it did not matter very much: any choice $x + \alpha\eta$, (where $\eta = x' - x$) for $0 \le \alpha \le 1$ would have given the same result since the difference between the choices is of order $\eta\varepsilon \simeq \varepsilon^{3/2}$. All this was thanks to the factor ε multiplying V in Eq. (8.5.4) and the fact that $|\eta| \simeq \varepsilon^{1/2}$, as per Eq. (8.6.5).

Consider now the case of a vector potential which will bring in a factor

$$\exp\left[\frac{iq\varepsilon}{\hbar c}\frac{x-x'}{\varepsilon}A(x+\alpha\eta)\right]\equiv\exp\left[-\frac{iq\varepsilon}{\hbar c}\frac{\eta}{\varepsilon}A(x+\alpha\eta)\right]$$

to the propagator for one time slice. (We should really be using vectors for position and the vector potential, but the one-dimensional version will suffice for making the point here.) Note that ε now gets canceled, in contrast to the scalar potential case. Thus, going to order ε to derive the Schrödinger equation means going to order η^2 in expanding the exponential. This will not only bring in an A^2 term, but will also make the answer sensitive to the argument of A in the linear term. Choose $\alpha=1/2$ and verify that you get the one-dimensional version of Eq. (4.3.7). Along the way you will see that changing α makes an order ε difference to $\psi(x, \varepsilon)$ so that we have no choice but to use $\alpha=1/2$, i.e., use the *midpoint prescription*. This point will come up in Chapter 21.