1 Introduction

> Thirty-one years ago, Dick Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes', he said. 'It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wavefunction.' I said to him, 'You're crazy'. But he wasn't.
>
> F.J. Dyson[1]

When we write down Feynman diagrams in quantum field theory, we proceed with the mind-set that our system will take on every configuration imaginable in traveling from the initial to final state. Photons will split in to electrons that recombine into different photons, leptons and anti-leptons will annihilate one another and the resulting energy will be used to create leptons of a different flavour; anything that can happen, will happen. Each distinct history can be thought of as a path through the configuration space that describes the state of the system at any given time. For quantum field theory, the configuration space is a Fock space where each vector represents the number of each type of particle with momentum $\mathbf{k}$. The key to the whole thing, though, is that each path that the system takes comes with a probabilistic *amplitude*. The probability that a system in some initial state will end up in some final state is given as a sum over the amplitudes associated with each path connecting the initial and final positions in the Fock space. Hence the perturbative expansion of scattering amplitudes in terms of Feynman diagrams, which represent all the possible ways the system can behave.

But quantum field theory is rooted in ordinary quantum mechanics; the essential difference is just the number of degrees of freedom. So what is the analogue of this "sum over histories" in ordinary quantum mechanics? The answer comes from the path integral formulation of quantum mechanics, where the amplitude that a particle at a given point in ordinary space will be found at some other point in the future is a sum over the amplitudes associated with all possible trajectories joining the initial and final positions. The amplitude associated with any given path is just e^{iS}, where S is the classical action $S = \int L(q, \dot{q})\, dt$. We will derive this result from the canonical formulation of quantum mechanics, using, for example, the time-dependent Schrödinger equation. However, if one *defines* the amplitude associated with a given trajectory as e^{iS}, then it is possible to derive the Schrödinger equation[2]. We can even "derive" the classical principle of least action from the quantum amplitude e^{iS}. In other words, one can view the amplitude of traveling from one point to another, usually called the propagator, as the fundamental object in quantum theory, from which the wavefunction follows. However, this formalism is of little

[1] Shamelessly lifted from page 154 of Ryder [1].

[2] Although, the procedure is only valid for velocity-independent potentials, see below.

use in quantum mechanics because state-vector methods are so straightforward; the path integral formulation is a little like using a sledge-hammer to kill a fly.

However, the situation is a lot different when we consider field theory. The generalization of path integrals leads to a powerful formalism for calculating various observables of quantum fields. In particular, the idea that the propagator Z is the central object in the theory is fleshed out when we discover that all of the n-point functions of an interacting field theory can be derived by taking derivatives of Z. This gives us an easy way of calculating scattering amplitudes that has a natural interpretation in terms of Feynman diagrams. All of this comes without assuming commutation relations, field decompositions or anything else associated with the canonical formulation of field theory. Our goal in this paper will to give an account of how path integrals arise in ordinary quantum mechanics and then generalize these results to quantum field theory and show how one can derive the Feynman diagram formalism in a manner independent of the canonical formalism.

2 Path integrals in quantum mechanics

To motivate our use of the path integral formalism in quantum field theory, we demonstrate how path integrals arise in ordinary quantum mechanics. Our work is based on section 5.1 of Ryder [1] and chapter 3 of Baym [2]. We consider a quantum system represented by the Heisenberg state vector $|\psi\rangle$ with one coordinate degree of freedom q and its conjugate momentum p. We adopt the notation that the Schrödinger representation of any given state vector $|\phi\rangle$ is given by

$$|\phi, t\rangle = e^{-iHt}|\phi\rangle, \tag{1}$$

where $H = H(q, p)$ is the system Hamiltonian. According to the probability interpretation of quantum mechanics, the wavefunction $\psi(q, t)$ is the projection of $|\psi, t\rangle$ onto an eigenstate of position $|q\rangle$. Hence

$$\psi(q, t) = \langle q|\psi, t\rangle = \langle q, t|\psi\rangle, \tag{2}$$

where we have defined

$$|q, t\rangle = e^{iHt}|q\rangle. \tag{3}$$

$|q\rangle$ satisfies the completeness relation

$$\langle q|q'\rangle = \delta(q - q'), \tag{4}$$

which implies

$$\langle q|\psi\rangle = \int dq' \langle q|q'\rangle\langle q'|\psi\rangle, \tag{5}$$

or

$$1 = \int dq' |q'\rangle\langle q'|. \tag{6}$$

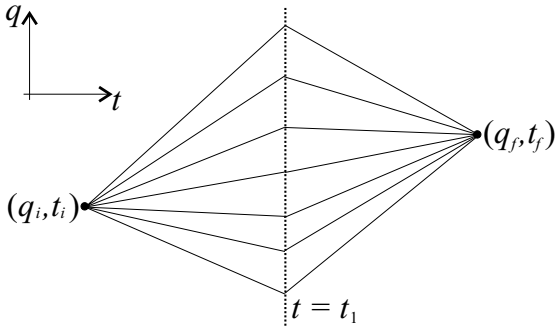

Figure 1: The various two-legged paths that are considered in the calculation of $\langle q_f, t_f | q_i, t_i \rangle$

Multiplying by $e^{iHt'}$ on the left and $e^{-iHt'}$ on the right yields that

$$1 = \int dq' |q', t'\rangle\langle q', t'|. \tag{7}$$

Now, using the completeness of the $|q, t\rangle$ basis, we may write

$$\begin{aligned} \psi(q_f, t_f) &= \int dq_i \langle q_f, t_f | q_i, t_i \rangle \langle q_i, t_i | \psi \rangle \\ &= \int dq_i \langle q_f, t_f | q_i, t_i \rangle \psi(q_i, t_i). \end{aligned} \tag{8}$$

The quantity $\langle q_f, t_f | q_i, t_i \rangle$ is called the *propagator* and it represents the probability amplitudes (expansion coefficients) associated with the decomposition of $\psi(q_f, t_f)$ in terms of $\psi(q_i, t_i)$. If $\psi(q_i, t_i)$ has the form of a spatial delta function $\delta(q_0)$, then $\psi(q_f, t_f) = \langle q_f, t_f | q_0, t_i \rangle$. That is, if we know that the particle is at q_0 at some time t_i, then the probability that it will be later found at a position q_f at a time t_f is

$$P(q_f, t_f; q_0, t_i) = |\langle q_f, t_f | q_i, t_0 \rangle|^2. \tag{9}$$

It is for this reason that we sometimes call the propagator a *correlation function*.

Now, using completeness, it is easily seen that the propagator obeys a *composition equation*:

$$\langle q_f, t_f | q_i, t_i \rangle = \int dq_1 \langle q_f, t_f | q_1, t_1 \rangle \langle q_1, t_1 | q_i, t_i \rangle. \tag{10}$$

This can be understood by saying that the probability amplitude that the position of the particle is q_i at time t_i and q_f at time t_f is equal to the sum over q_1 of the probability that the particle traveled from q_i to q_1 (at time t_1) and then on to q_f. In other words, the probability amplitude that a particle initially at q_i will later be seen at q_f is the sum of the probability amplitudes associated with all possible

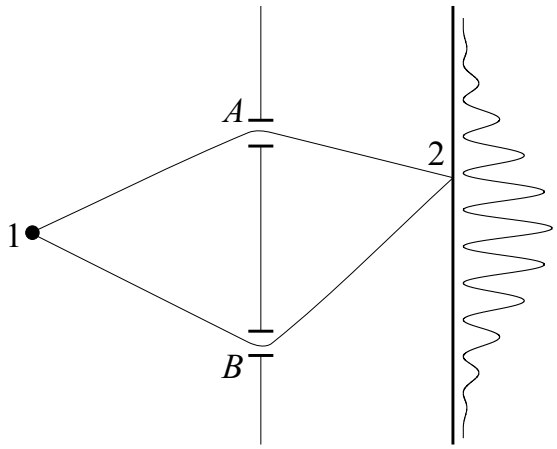

Figure 2: The famous double-slit experiment

two-legged paths between q_i and q_f, as seen in figure 1. This is the meaning of the oft-quoted phrase: "motion in quantum mechanics is considered to be a sum over paths". A particularly neat application comes from the double slit experiment that introductory texts use to demonstrate the wave nature of elementary particles. The situation is sketched in figure 2. We label the initial point (q_i, t_i) as 1 and the final point (q_f, t_f) as 2. The amplitude that the particle (say, an electron) will be found at 2 is the sum of the amplitude of the particle traveling from 1 to A and then to 2 and the amplitude of the particle traveling from 1 to B and then to 2. Mathematically, we say that

$$\langle 2|1\rangle = \langle 2|A\rangle\langle A|1\rangle + \langle 2|B\rangle\langle B|1\rangle. \tag{11}$$

The presence of the double-slit ensures that the integral in (10) reduces to the two-part sum in (11). When the probability $|\langle 2|1\rangle|^2$ is calculated, interference between the $\langle 2|A\rangle\langle A|1\rangle$ and $\langle 2|B\rangle\langle B|1\rangle$ terms will create the classic intensity pattern on the screen.

There is no reason to stop at two-legged paths. We can just as easily separate the time between t_i and t_f into n equal segments of duration $\tau = (t_f - t_i)/n$. It then makes sense to relabel $t_0 = t_i$ and $t_n = t_f$. The propagator can be written as

$$\langle q_n, t_n|q_0, t_0\rangle = \int dq_1 \cdots dq_{n-1}\, \langle q_n, t_n|q_{n-1}, t_{n-1}\rangle \cdots \langle q_1, t_1|q_0, t_0\rangle. \tag{12}$$

We take the limit $n \to \infty$ to obtain an expression for the propagator as a sum over infinite-legged paths, as seen in figure 3. We can calculate the propagator for small time intervals $\tau = t_{j+1} - t_j$ for some j between 1 and $n-1$. We have

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle = \langle q_{j+1}|e^{-iHt_{j+1}}e^{+iHt_j}|q_j\rangle$$

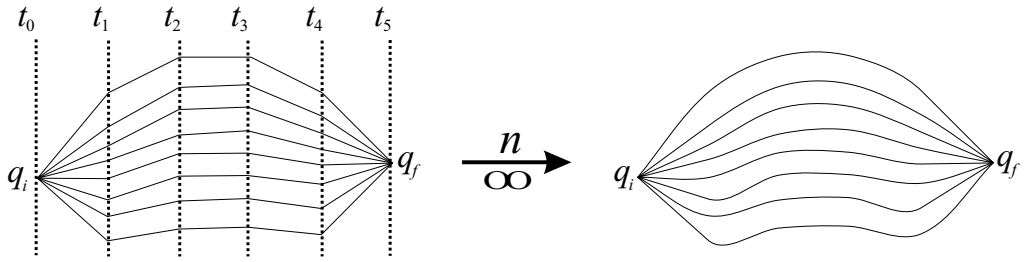

Figure 3: The continuous limit of a collection of paths with a finite number of legs

$$\begin{aligned}
&= \langle q_{j+1}|(1 - iH\tau + O(\tau^2)|q_j\rangle \\
&= \delta(q_{j+1} - q_j) - i\tau\langle q_{j+1}|H|q_j\rangle \\
&= \frac{1}{2\pi}\int dp\, e^{ip(q_{j+1}-q_j)} - \frac{i\tau}{2m}\langle q_{j+1}|p^2|q_j\rangle \\
&\quad -i\tau\langle q_{j+1}|V(q)|q_j\rangle, \qquad (13)
\end{aligned}$$

where we have assumed a Hamiltonian of the form

$$H(p,q) = \frac{p^2}{2m} + V(q). \qquad (14)$$

Now,

$$\langle q_{j+1}|p^2|q_j\rangle = \int dp\, dp'\, \langle q_{j+1}|p'\rangle\langle p'|p^2|p\rangle\langle p|q_j\rangle, \qquad (15)$$

where $|p\rangle$ is an eigenstate of momentum such that

$$p|p\rangle = |p\rangle p, \quad \langle q|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipq}, \quad \langle p|p'\rangle = \delta(p - p'). \qquad (16)$$

Putting these expressions into (15) we get

$$\langle q_{j+1}|p^2|q_j\rangle = \frac{1}{2\pi}\int dp\, p^2 e^{ip(q_{j+1}-q_j)}, \qquad (17)$$

where we should point out that p^2 is a number, not an operator. Working on the other matrix element in (13), we get

$$\begin{aligned}
\langle q_{j+1}|V(q)|q_j\rangle &= \langle q_{j+1}|q_j\rangle V(q_j) \\
&= \delta(q_{j+1} - q_j)V(q_j) \\
&= \frac{1}{2\pi}\int dp\, e^{ip(q_{j+1}-q_j)}V(q_j).
\end{aligned}$$

Putting it all together

$$\begin{aligned}\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle &= \frac{1}{2\pi}\int dp\, e^{ip(q_{j+1}-q_j)}\left[1 - i\tau H(p, q_j) + O(\tau^2)\right] \\ &= \frac{1}{2\pi}\int dp\, \exp\left[i\tau\left(p\,\frac{\Delta q_j}{\tau} - H(p, q_j)\right)\right],\end{aligned}$$

where $\Delta q_j \equiv q_{j+1} - q_j$. Substituting this expression into (12) we get

$$\langle q_n, t_n|q_0, t_0\rangle = \int dp_0 \prod_{i=1}^{n-1}\frac{dq_i\, dp_i}{2\pi}\exp\left[i\sum_{j=0}^{n-1}\tau\left(p_j\,\frac{\Delta q_j}{\tau} - H(p_j, q_j)\right)\right]. \tag{18}$$

In the limit $n \to \infty$, $\tau \to 0$, we have

$$\sum_{j=0}^{n-1}\tau \to \int_{t_0}^{t_n} dt, \quad \frac{\Delta q_j}{\tau} \to \frac{dq}{dt} = \dot{q}, \quad dp_0 \prod_{i=1}^{n}\frac{dq_i\, dp_i}{2\pi} \to [dq]\,[dp], \tag{19}$$

and

$$\langle q_n, t_n|q_0, t_0\rangle = \int [dq]\,[dp]\,\exp\left\{i\int_{t_0}^{t_n} dt\,[p\,\dot{q} - H(p, q)]\right\}. \tag{20}$$

The notation $[dq]\,[dp]$ is used to remind us that we are integrating over *all possible paths* $q(t)$ and $p(t)$ that connect the points (q_0, t_0) and (q_n, t_n). Hence, we have succeed in writing the propagator $\langle q_n, t_n|q_0, t_0\rangle$ as a *functional integral* over the all the phase space trajectories that the particle can take to get from the initial to the final points. It is at this point that we fully expect the reader to scratch their heads and ask: what exactly is a functional integral? The simple answer is a quantity that arises as a result of the limiting process we have already described. The more complicated answer is that functional integrals are beasts of a rather vague mathematical nature, and the arguments as to their standing as well-behaved entities are rather nebulous. The philosophy adopted here is in the spirit of many mathematically controversial manipulations found in theoretical physics: we assume that everything works out alright.

The argument of the exponential in (20) ought to look familiar. We can bring this out by noting that

$$\begin{aligned}\frac{1}{2\pi}\int dp_i\, e^{i\tau\left[p_i\frac{\Delta q_i}{\tau} - H(p_i, q_i)\right]} &= \frac{1}{2\pi}\exp\left\{i\tau\left[\frac{m}{2}\left(\frac{\Delta q_i}{\tau}\right)^2 - V(q_i)\right]\right\} \\ &\quad \times \int dp_i \exp\left[-\frac{i\tau}{2m}\left(p - \frac{m\Delta q_i}{\tau}\right)^2\right] \\ &= \left(\frac{m}{2\pi i\tau}\right)^{1/2}\exp\left\{i\tau\left[\frac{m}{2}\left(\frac{\Delta q_i}{\tau}\right)^2 - V(q_i)\right]\right\}.\end{aligned}$$

Using this result in (18) we obtain

$$\langle q_n, t_n | q_0, t_0 \rangle = \left(\frac{m}{2\pi i\tau}\right)^{n/2} \int \prod_{i=1}^{n-1} dq_i \exp\left\{ i \sum_{j=0}^{n-1} \tau \left[\frac{m}{2} \left(\frac{\Delta q_j}{\tau} \right)^2 - V(q_j) \right] \right\}$$
$$\to N \int [dq] \exp\left[i \int_{t_0}^{t_n} dt \left(\frac{1}{2} m\dot{q}^2 - V(q) \right) \right], \tag{21}$$

where the limit is taken, as usual, for $n \to \infty$ and $\tau \to 0$. Here, N is an infinite constant given by

$$N = \lim_{n\to\infty} \left(\frac{m}{2\pi i\tau}\right)^{n/2}. \tag{22}$$

We won't worry too much about the fact that N diverges because we will later normalize our transition amplitudes to be finite. Recognizing the Lagrangian $L = T - V$ in equation (21), we have

$$\langle q_n, t_n | q_0, t_0 \rangle = N \int [dq] \exp\left[i \int_{t_0}^{t_n} L(q, \dot{q})\, dt \right] = N \int [dq]\, e^{iS[q]}, \tag{23}$$

where S is the classical action, given as a *functional* of the trajectory $q = q(t)$. Hence, we see that the propagator is the sum over paths of the amplitude $e^{iS[q]}$, which is the amplitude that the particle follows a given trajectory $q(t)$. Historically, Feynman demonstrated that the Schrödinger equation could be derived from equation (23) and tended to regard the relation as the fundamental quantity in quantum mechanics. However, we have assumed in our derivation that the potential is a function of q and not p. If we do indeed have velocity-dependent potentials, (23) fails to recover the Schrödinger equation. We will not go into the details of how to fix the expression here, we will rather heuristically adopt the generalization of (23) for our later work in with quantum fields[3].

An interesting consequence of (23) is seen when we restore $\hbar$. Then

$$\langle q_n, t_n | q_0, t_0 \rangle = N \int [dq]\, e^{iS[q]/\hbar}. \tag{24}$$

The classical limit is obtained by taking $\hbar \to 0$. Now, consider some trajectory $q_0(t)$ and neighbouring trajectory $q_0(t) + \delta q(t)$, as shown in figure 4. The action evaluated along q_0 is S_0 while the action along $q_0 + \delta q$ is $S_0 + \delta S$. The two paths will then make contributions $\exp(iS_0/\hbar)$ and $\exp[i(S_0 + \delta S)/\hbar]$ to the propagator. For $\hbar \to 0$, the phases of the exponentials will become completely disjoint and the contributions will in general destructively interfere. That is, unless $\delta S = 0$ in which case all neighbouring paths will constructively interfere. Therefore, in the classical limit the propagator will be non-zero for points that may be connected by a trajectory

[3]The generalization of velocity-dependent potentials to field theory involves the quantization of non-Abelian gauge fields

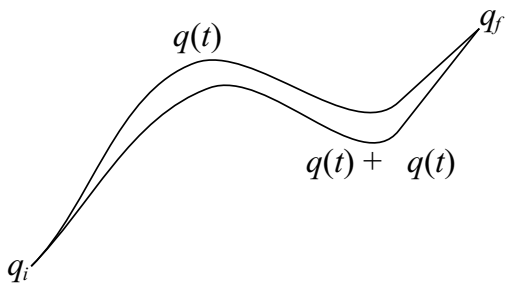

Figure 4: Neighbouring particle trajectories. If the action evaluated along $q(t)$ is stationary (i.e. $\delta S = 0$), then the contribution of $q(t)$ and it's neighbouring paths $q(t) + \delta q(t)$ to the propagator will constructively interfere and reconstruct the classical trajectory in the limit $\hbar \to 0$

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satisfying $\delta S[q]|_{q=q_0}$; i.e. for paths connected by *classical* trajectories determined by Newton's 2nd law. We have hence seen how the classical principle of least action can be understood in terms of the path integral formulation of quantum mechanics and a corresponding principle of *stationary phase*.

3 Perturbation theory, the scattering matrix and Feynman rules

In practical calculations, it is often impossible to solve the Schrödinger equation exactly. In a similar manner, it is often impossible to write down analytic expressions for the propagator $\langle q_f, t_f | q_i, t_i \rangle$ for general potentials $V(q)$. However, if one assumes that the potential is small and that the particle is nearly free, one makes good headway by using perturbation theory. We follow section 5.2 in Ryder [1].

In this section, we will go over from the general configuration coordinate q to the more familiar x, which is just the position of the particle in a one-dimensional space. The extension to higher dimensions, while not exactly trivial, is not difficult to do. We assume that the potential that appears in (23) is "small", so we may perform an expansion

$$\exp\left[-i\int_{t_0}^{t_n} V(x,t)\,dt\right] = 1 - i\int_{t_0}^{t_n} V(x,t)\,dt - \frac{1}{2!}\left[\int_{t_0}^{t_n} V(x,t)\,dt\right]^2 + \cdots. \quad (25)$$

We adopt the notation that $K = K(x_n, t_n; x_0, t_0) = \langle x_n, t_n | x_0, t_0 \rangle$. Inserting the expansion (25) into the propagator, we see that K possesses and expansion of the form:

$$K = K_0 + K_1 + K_2 + \cdots \quad (26)$$

The K_0 term is

$$K_0 = N \int [dx] \, \exp\left[i \int \frac{1}{2} m \dot{x}^2 \, dt \right]. \tag{27}$$

If we turned off the potential, the full propagator would reduce to K_0. It is for this reason that we call K_0 the *free particle propagator*, it represents the amplitude that a free particle known to be at x_0 at time t_0 will later be found at x_n at time t_n. Going back to the discrete expression:

$$K_0 = \lim_{n\to\infty} \left(\frac{m}{2\pi i \tau} \right)^{n/2} \int \prod_{i=1}^{n-1} dx_i \, \exp\left[\frac{im\tau}{2} \sum_{j=0}^{n-1} (x_{j+1} - x_j)^2 \right]. \tag{28}$$

This is a doable integral because the argument of the exponential is a simple quadratic form. We can hence diagonalize it by choosing an appropriate rotation of the x_j Cartesian variables of integration. Conversely, we can start calculating for $n = 2$ and solve the general n case using induction. The result is

$$K_0 = \lim_{n\to\infty} \left(\frac{m}{2\pi i \tau} \right)^{n/2} \frac{1}{n^{1/2}} \left(\frac{2\pi i \tau}{m} \right)^{(n-1)/2} \exp\left[\frac{im(x_n - x_0)^2}{2n\tau} \right]. \tag{29}$$

Now, $(t_n - t_0)/n = \tau$, so we finally have

$$K_0(x_n - x_0, t_n - t_0) = \left[\frac{m}{2\pi i (t_n - t_0)} \right]^{1/2} \exp\left[\frac{im(x_n - x_0)^2}{2(t_n - t_0)} \right], \quad t_n > t_0. \tag{30}$$

Here, we've noted that the substitution $n\tau = (t_n - t_0)$ is only valid for $t_n > t_0$. In fact, if K_0 is non-zero for $t_n > t_0$ it must be zero for $t_0 > t_n$. To see this, we note that the calculation of K_0 involved integrations of the form:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\alpha x^2} \, dx &= \frac{1}{2} \int_0^{\infty} e^{i\alpha x^2} \, dx \\ &= \frac{i^{-1/2}}{4} \int_0^{i\infty} \frac{e^{\alpha s}}{s^{1/2}} \, ds \\ &= \frac{i^{-1/2}}{4} \int_{-i\infty}^{i\infty} \Theta(-is) \frac{e^{\alpha s}}{s^{1/2}} \, ds, \end{aligned}$$

where $\alpha \propto \text{sign}(\tau) = \text{sign}(t_n - t_0)$. Now, we can either choose the branch of $s^{-1/2}$ to be in either the left- or righthand part of the complex s-plane. But, we need to complete the contour in the lefthand plane if $\alpha > 0$ and the righthand plane if $\alpha < 0$. Hence, the integral can only be non-zero for one case of the sign of α. The choice we have implicitly made is the the integral is non-zero for $\alpha \propto (t_n - t_0) > 0$, hence it must vanish for $t_n < t_0$. When we look at equation (8) we see that K_0 is little more than a type of kernel for the integral solution of the free-particle Schrödinger equation, which is really a statement about Huygen's principle. Our

choice of K_0 obeys causality in that the configuration of the field at prior times determines the form of the field in the present. We have hence found a retarded propagator. The other choice for the boundary conditions obeyed by K_0 yields the advanced propagator and a version of Huygen's principle where future field configurations determine the present state. The moral of the story is that, if we choose a propagator that obeys casuality, we are justified in writing

$$K_0(x,t) = \Theta(t)\left[\frac{m}{2\pi i t}\right]^{1/2} \exp\left[\frac{imx^2}{2t}\right]. \tag{31}$$

Now, we turn to the calculation of K_1:

$$K_1 = -iN\int [dx]\, \exp\left[i\int \frac{1}{2}m\dot{x}^2\, dt\right]\int dt\, V(x(t),t). \tag{32}$$

Moving again to the discrete case:

$$K_1 = -i\beta^{n/2}\int dx_1\cdots dx_{n-1}\, \exp\left[\frac{im\tau}{2}\sum_{j=0}^{n-1}(x_{j+1}-x_j)^2\right]\sum_{i=1}^{n-1}\tau V(x_i,t_i), \tag{33}$$

where $\beta = m/2\pi i\tau$ and the limit $n\to\infty$ is understood. Let's take the sum over i (which has replaced the integral over t) in front of the spatial integrals. Also, let's split up the sum over j in the exponential to a sum running from 0 to $i-1$ and a sum running from i to $n-1$. Then

$$\begin{aligned} K_1 = -i\sum_{i=1}^{n-1}\tau\int dx_i\,\beta^{i/2}\int dx_1\cdots dx_{i-1}\exp\left[\frac{im\tau}{2}\sum_{j=0}^{i-1}(x_{j+1}-x_j)^2\right]V(x_i,t_i) \\ \times\beta^{(n-i)/2}\int dx_{i+1}\cdots dx_{n-1}\exp\left[\frac{im\tau}{2}\sum_{j=i}^{n-1}(x_{j+1}-x_j)^2\right]. \end{aligned} \tag{34}$$

We recognize two factors of the free-particle propagator in this expression, which allows us to write

$$K_1 = -i\sum_{i=1}^{n-1}\tau\int dx\, K_0(x-x_0,t_i-t_0)V(x,t_i)K_0(x_n-x,t_n-t_i). \tag{35}$$

Now, we can replace $\sum_{i=1}^{n-1}\tau$ by $\int_{t_0}^{t_n}dt$ and $t_i\to t$ in the limit $n\to\infty$. Since $K_0(x-x_0,t-t_0)=0$ for $t<t_0$ and $K_0(x_n-x,t_n-t)$ for $t>t_n$, we can extend the limits on the time integration to $\pm\infty$. Hence,

$$K_1 = -i\int dx\, dt\, K_0(x_n-x,t_n-t)V(x,t)K_0(x-x_0,t-t_0). \tag{36}$$

In a similar fashion, we can derive the expression for K_2:

$$K_2 = \frac{(-i)^2}{2!}\beta^{n/2}\int dx_1\cdots dx_{n-1}\,\exp\left[\frac{im\tau}{2}\sum_{j=0}^{n-1}(x_{j+1}-x_j)^2\right] \tag{37}$$

$$\times\sum_{i=1}^{n-1}\tau V(x_i,t_i)\sum_{k=1}^{n-1}\tau V(x_k,t_k). \tag{38}$$

We would like to play the same trick that we did before by splitting the sum over j into three parts with the potential terms sandwiched in between. We need to construct the middle j sum to go from an early time to a late time in order to replace it with a free-particle propagator. But the problem is, we don't know whether t_i comes before or after t_k. To remedy this, we split the sum over k into a sum from 1 to $i-1$ and then a sum from i to $n-1$. In each of those sums, we can easily determine which comes first: t_i or t_k. Going back to the continuum limit:

$$\begin{aligned}K_2 = \frac{(-i)^2}{2!}\int dx_1\,dx_2\int_{t_0}^{t_n}dt_1\Bigg[&\int_{t_0}^{t_1}dt_2 K_0(x_n-x_1,t_n-t_1)\\ &\times V(x_1,t_1)K_0(x_1-x_2,t_1-t_2)V(x_2,t_2)K_0(x_2-x_0,t_2-t_0)\\ &+\int_{t_1}^{t_n}dt_2 K_0(x_n-x_2,t_n-t_2)V(x_2,t_2)K_0(x_2-x_1,t_2-t_1)\\ &\times V(x_1,t_1)K_0(x_1-x_0,t_1-t_0)\Bigg]\end{aligned} \tag{39}$$

But, we can extend the limits on the t_2 integration to $t_0 \to t_n$ by noting the middle propagator is zero for $t_2 > t_1$. Similarly, the t_2 limits on the second integral can be extended by observing the middle propagator vanishes for $t_1 > t_2$. Hence, both integrals are the same, which cancels the 1/2! factor. Using similar arguments, the limits of both of the remaining time integrals can be extended to $\pm\infty$ yielding our final result:

$$\begin{aligned}K_2 = (-i)^2\int dx_1\,dx_2\,dt_1\,dt_2\,&K_0(x_n-x_2,t_n-t_2)V(x_2,t_2)\\ &\times K_0(x_2-x_1,t_2-t_1)V(x_1,t_1)K_0(x_1-x_0,t_1-t_0).\end{aligned} \tag{40}$$

Higher order contributions to the propagator follow in a similar fashion. The general j^{th} order correction to the free propagator is

$$\begin{aligned}K_j = (-i)^j\int dx_1\ldots dx_j\,dt_1\ldots dt_j\,&K_0(x_n-x_j,t_n-t_j)\\ &\times V(x_j)\cdots V(x_1)K_0(x_1-x_0,t_1-t_0).\end{aligned} \tag{41}$$

We would like to apply this formalism to scattering problems where we assume that the particle is initially in a plane wave state incident on some localized potential.

As $t \to \pm\infty$, we assume the potential goes to zero, which models the fact that the particle is far away from the scattering region in the distant past and the distant future. We go over from one to three dimensions and write

$$\begin{aligned}\psi(\mathbf{x}_f, t_f) = & \int d\mathbf{x}_i K_0(\mathbf{x}_f - \mathbf{x}_i, t_f - t_i)\psi(\mathbf{x}_i, t_i) \\ & -i \int d\mathbf{x}_i \, d\mathbf{x} \, dt \, K_0(\mathbf{x}_f - \mathbf{x}, t_f - t) \\ & \times V(\mathbf{x}, t) K_0(\mathbf{x} - \mathbf{x}_i, t - t_i)\psi(\mathbf{x}_i, t_i) + \cdots \end{aligned} \tag{42}$$

We push t_i into the distant past, where the effects of the potential may be ignored, and take the particle to be in a plane wave state:

$$\psi_{\text{in}}(\mathbf{x}_i, t_i) = \frac{1}{\sqrt{V}} e^{-ip_i \cdot x_i}, \tag{43}$$

where we have used a box normalization with V being the volume of the box and $p_i \cdot x = E_i t_i - \mathbf{p}_i \cdot \mathbf{x}_i$. The "in" label on the wavefunction is meant to emphasize that it is the form of ψ before the particle moves into the scattering region. We want to calculate the first integral in (42) using the 3D generalization of (31):

$$K_0(\mathbf{x}, t) = -i\Theta(t) \left(\frac{\lambda}{\pi}\right)^{3/2} e^{\lambda \mathbf{x}^2}, \tag{44}$$

where $\lambda = im/2t$. Hence,

$$\begin{aligned}\int d\mathbf{x}_i K_0(\mathbf{x}_f - \mathbf{x}_i, t_f - t_i)\psi_{\text{in}}(\mathbf{x}_i, t_i) = & -\frac{i}{\sqrt{V}} \left(\frac{\lambda}{\pi}\right)^{3/2} \\ & \times e^{-iE_i t_i} \int d\mathbf{x}_i e^{\lambda(\mathbf{x}_f - \mathbf{x}_i)^2 + i\mathbf{p}_i \cdot \mathbf{x}_i}. \end{aligned} \tag{45}$$

This integral reduces to $\psi_{\text{in}}(\mathbf{x}_f, t_f)$ as should have been expected, because K_0 is the free particle propagator and must therefore propagate plane waves into the future without altering their form. We also push t_f into the infinite future where the effects of the potential can be ignored. Then,

$$\begin{aligned}\psi^+(\mathbf{x}_f, t_f) = & \psi_{\text{in}}(\mathbf{x}_f, t_f) - i \int d\mathbf{x}_i \, d\mathbf{x} \, dt \, K_0(\mathbf{x}_f - \mathbf{x}, t_f - t) \\ & \times V(\mathbf{x}, t) K_0(\mathbf{x} - \mathbf{x}_i, t - t_i)\psi_{\text{in}}(\mathbf{x}_i, t_i) + \cdots \end{aligned} \tag{46}$$

The "+" notation on ψ is there to remind us that ψ^+ is the form of the wave function after it interacts with the potential. What we really want to do is Fourier analyze $\psi^+(\mathbf{x}_f, t_f)$ into momentum eigenstates to determine the probability amplitude for a particle of momentum $\mathbf{p}_i$ becoming a particle of momentum $\mathbf{p}_f$ after interacting

with the potential. Defining $\psi_{\text{out}}(\mathbf{x}_f, t_f)$ as a state of momentum p_f in the distant future:

$$\psi_{\text{out}}(\mathbf{x}_f, t_f) = \frac{1}{\sqrt{V}} e^{-ip_f \cdot x_f}, \tag{47}$$

we can write the amplitude for a transition from $\mathbf{p}_i$ to $\mathbf{p}_f$ as

$$S_{fi} = \langle \psi_{\text{out}} | \psi^+ \rangle. \tag{48}$$

Inserting the unit operator $1 = \int d\mathbf{x}_f |\mathbf{x}_f, t_f\rangle\langle \mathbf{x}_f, t_f|$ into (48) and using the propagator expansion (46), we obtain

$$\begin{aligned} S_{fi} = \delta(\mathbf{p}_f - \mathbf{p}_i) - i \int d\mathbf{x}_i \, d\mathbf{x}_f \, d\mathbf{x} \, dt \, \psi^*_{\text{out}}(\mathbf{x}_f, t_f) K_0(\mathbf{x}_f - \mathbf{x}, t_f - t) \\ \times V(\mathbf{x}, t) K_0(\mathbf{x} - \mathbf{x}_i, t - t_i) \psi_{\text{in}}(\mathbf{x}_i, t_i) + \cdots \end{aligned} \tag{49}$$

The amplitude S_{fi} is the fi component of what is known as the S or scattering matrix. This object plays a central rôle in scattering theory because it answers all the questions that one can experimentally ask about a physical scattering process. What we have done is expand these matrix elements in terms of powers of the scattering potential. Our expansion can be given in terms of *Feynman diagrams* according to the rules:

1. The vertex of this theory is attached to two legs and a spacetime point $(\mathbf{x}, t)$.
2. Each vertex comes with a factor of $-iV(\mathbf{x}, t)$.
3. The arrows on the lines between vertices point from the past to the future.
4. Each line going from $(\mathbf{x}, t)$ to $(\mathbf{x}', t')$ comes with a propagator $K_0(\mathbf{x}' - \mathbf{x}, t' - t)$.
5. The past external point comes with the wavefunction $\psi_{\text{in}}(\mathbf{x}_i, t_i)$, the future one comes with $\psi^*_{\text{out}}(\mathbf{x}_f, t_f)$.
6. All spatial coordinates and internal times are integrated over.

Using these rules, the S matrix element may be represented pictorially as in figure 5. We note that these rules are for configuration space only, but we could take Fourier transforms of all the relevant quantities to get momentum space rules. Obviously, the Feynman rules for the Schrödinger equation do not result in a significant simplification over the raw expression (49), but it is important to notice *how* they were derived: using simple and elegant path integral methods.

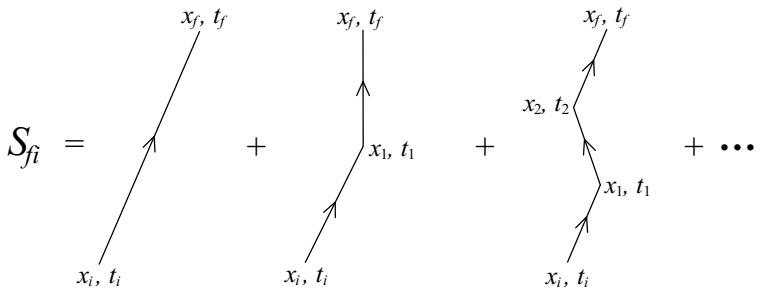

Figure 5: The expansion of S_{fi} in terms of Feynman diagrams

4 Sources, vacuum-to-vacuum transitions and time-ordered products

We now consider a alteration of the system Lagrangian that models the presence of a time-dependent "source". Our discussion follows section 5.5 of Ryder [1] and chapters 1 and 2 of Brown [3]. In this context, we call any external agent that may cause a non-relativistic system to make a transition from one energy eigenstate to another a "source". For example, a time-dependent electric field may induce a charged particle in a one dimensional harmonic oscillator potential to go from one eigenenergy to another. In the context of field theory, a time-dependent source may result in spontaneous particle creation[4]. In either case, the source can be modeled by altering the Lagrangian such that

$$L(q, \dot{q}) \to L(q, \dot{q}) + J(t)q(t). \tag{50}$$

The source $J(t)$ will be assumed to be non-zero in a finite interval $t \in [t_1, t_2]$. We take $T_2 > t_2$ and $T_1 < t_1$. Given that the particle was in it's ground state at $T_1 \to -\infty$, what is the amplitude that the particle will still be in the ground state at time $T_2 \to \infty$?

To answer that question, consider

$$\begin{aligned}
\langle Q_2, T_2 | Q_1, T_1 \rangle_J &= \int dq_1\, dq_2 \langle Q_2, T_2 | q_2, t_2 \rangle \langle q_2, t_2 | q_1, t_1 \rangle_J \langle q_1, t_1 | Q_1, T_1 \rangle \\
&= \int dq_1\, dq_2 \langle Q_2 | e^{-iHT_2} e^{iHt_2} | q_2 \rangle \langle q_2, t_2 | q_1, t_1 \rangle_J \\
&\quad \times \langle q_1 | e^{-iHt_1} e^{iHT_1} | Q_1 \rangle \\
&= \sum_{mn} \int dq_1\, dq_2 \langle Q_2 | e^{-iHT_2} | m \rangle \langle m | e^{iHt_2} | q_2 \rangle \langle q_2, t_2 | q_1, t_1 \rangle_J \\
&\quad \times \langle q_1 | e^{-iHT_1} | n \rangle \langle n | e^{iHt_1} | Q_1 \rangle
\end{aligned}$$

[4] *cf.* PHYS 703 March 14, 2000 lecture