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Feynman's path to quantum mechanics

10.1 Introduction

Feynman began his 1948 paper in the *Reviews of Modern Physics*, entitled 'space–time approach to non-relativistic quantum mechanics', by stating: 'It is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schrödinger, and the matrix algebra of Heisenberg [as well as the q-number formulation of P. A. M. Dirac]. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

'This paper will describe what is essentially a *third formulation* of non-relativistic quantum theory. This formulation was suggested by some of Dirac's remarks concerning the relation of classical action to quantum mechanics.[1] The probability amplitude is associated with [the] entire motion of a particle at a particular time.'[2]

With these words Richard Feynman introduced one of his now most well known papers. In the spring of 1947 he decided to publish the most important parts of his Ph.D. thesis. Feynman had thought about publishing this work in a regular journal earlier, but World War II intervened, and he was not able to do so: 'The war interrupted the work.'[3]

'During the war, I didn't have time to work on these things very extensively, but wandered about on buses and so forth, with little pieces of paper (in my pockets), and struggled to work on it and discovered indeed that there was something wrong. . ..'[4] He discovered that he was not able to exclude the possibility of a complex value for the energy in his quantum action-at-a-distance theory. And if this was to be the case, one would obtain obviously wrong results for the probabilities of the events in this theory. In particular, the sum of the probabilities of all possible independent events in such a theory would not be equal to one. But this sum must be equal to one, because in every case some of these events should happen.

Feynman had started to think about these things in the fall of 1946 at Cornell, but at that time he found it quite difficult to write it all down in the form of a paper. At the January 1947 annual meeting of the American Physical Society Feynman encountered Herbert C. Corben, his friend from the early days in Princeton; Corben had discouraged Feynman at that time from getting too deeply involved in the ideas of A.S. Eddington. Corben had returned to Australia in 1942, with his California-born wife Mulaika (née Barclay), where he taught at the University of Melbourne. The Corbens had returned to the United States in the fall of 1946, after Herbert had accepted the offer of an appointment at the Carnegie Institute of Technology at Pittsburgh. Feynman told them about the problems he was working on, how his work was going, and the difficulties he was having in getting it all down on paper; 'but after three weeks the paper emerged.'[5]

This was not the whole story. In April 1970, Feynman recalled: 'I wrote up this paper and sent it to the *Physical Review*, and they suggested that I publish it in the *Reviews of Modern Physics*, which I did. At first, it was returned; they said it was too long, that this stuff was old hat, and that the first part in the paper was well known, which could be left out. Hans Bethe taught me a trick. He said, "You have to emphasize that this part is known, and others that are new. It will take only a few paragraphs. In fact, I will shorten it for you." Then Bethe took out one sentence, and said, "If you make a small effort in this direction you don't have to take the whole thing out." That worked. They published it.' In the paper, Feynman thanked H. C. Corben and his wife. 'She more than cooked. She was very enthusiastic, cooperating with my writing and trying to encourage me.'[3]

In his 1948 paper (herein later referred to as the RMP (1948) article), Feynman first described that part of his Ph.D. thesis that did not produce any difficulties. 'All the ideas which appear in the RMP (1948) article were written in such a form that if any generalization is possible, they can be translated. . .. The thesis contains a somewhat more detailed analysis of the general relation of the invariance properties of the (action) functional and constants of the motion. Also the problems of elimination of the intermediate harmonic oscillators is done more completely than is done [in the RMP (1948) article]. The reason I did not publish everything in the thesis is this. I met with a difficulty. An arbitrary action functional S produces results which do not conserve probability; for example, the energy values come out complex. I do not know what this means, nor was I able to find that class of action functionals which would be guaranteed to give real eigenvalues for the energies.'[6]

10.2 The path-integral

In the RMP (1948) article Feynman presented in detail his new approach to quantum mechanics. Feynman's *third way* of formulating quantum mechanics was based on the new physical interpretation of the mathematical method

which he had developed in his thesis. The 'key words' which led to the new conceptual advances were 'superposition of probability amplitudes'. The important equation (6.11) can be interpreted physically as a Huygens's principle† for matter waves, and it describes the evolution of the wave function during a small time interval. In the case of quantum mechanics, the classical trajectories play the role of rays in geometrical optics. Then, instead of Fermat's principle of least time, we can apply Hamilton's principle of least action (see equations (6.1) and (6.4) in Chapter 6) for classical or 'geometrical' mechanics. The analogy between geometrical optics and wave optics was used by Schrödinger when he derived the wave equation in quantum mechanics.[8] Feynman used this analogy in a more direct way to reach the right physical interpretation of his new method. The formula (6.11) says that 'if the amplitude ψ of the wave is known on a given "surface", in particular the "surface" consisting of all x at time t, its value at a particular nearby point $t+\varepsilon$, is a sum of contributions from all points of the "surface" at t. Each contribution is delayed in phase by an amount proportional to the *action* it would require to get from the "surface" to the point along the path of least *action* of classical mechanics.'[9]

It ought to be emphasized that Huygens's principle is actually not completely correct in optics, and has to be replaced by some modification, which was given by Kirchhoff. Since the wave equation in optics is of second order with respect to time derivatives, one should—in accordance with Kirchhoff's modification—give both the amplitude and its time derivative on the adjacent surface. It is curious that Huygens's principle actually may be

† Huygens's principle was established by Christiaan Huygens in 1678. The wave properties of light were discovered in quite a long chain of investigations, which were started by the work of Francesco Grimaldi (1618–63) leading to the discovery of the diffraction of light. His experiments were repeated by Robert Hooke and then by Isaac Newton. In 1675 Olaf Roemer established the finite speed of light, by measuring the periods of the eclipses due to Jupiter's shadow on its innermost moon. Starting from Roemer's discovery, Huygens was able to explain the propagation of light as a wave phenomenon. This explanation says that when the wave reaches a given point in space, this point becomes the source of spherical waves, which spread out with a finite speed. The spherical waves from all space points result in the interference of the spreading waves. Thus Huygens was able to prove the rectilinear nature of light rays as a consequence of his spherical waves.

The modern idea of interference was given by Thomas Young in 1801, and then the theory of these phenomena was developed by Augustin Fresnel in 1818. Fresnel's theory was expressed in purely geometrical terms by using the famous zone construction.[7] Finally, in 1883 Gustav Kirchhoff wrote the solution of the equation for the light waves in the correct form as an integral like equation (6.11), but involving a wave derivative in time. For the partial differential equations of more general type, Kirchhoff's approach was developed by Jacques-Salomon Hadamard in 1923.

The change from Feynman's work to Schrödinger's looks like a change from Grimaldi to Kirchhoff, but only as an evolution in the opposite direction.

applied to the quantum wave equation without any modification, as Feynman showed. 'The wave equation of quantum mechanics is of first order in the time; therefore, Huygens's principle *is* correct for matter waves, the action replacing the time.'[9]

Thus the complete and clear physical interpretation of equation (6.11) was obtained. But the more important step was to arrive at the correct physical interpretation of equation (6.13), which gives the amplitude K for a finite time as the limit of the integration performed multiple times on the coordinates. What can this procedure mean physically? After some general considerations of the relation between probabilities and quantum magnitudes, Feynman arrived at an extremely nice and simple answer to this principal question. To explain how this can be done, he assumed that he had a particle moving in one dimension, which can take up various values of a coordinate x. Then he wrote the formula (6.11) in the form

$$K = \lim_{\varepsilon \to 0} \int_R \exp\left(\frac{\mathrm{i}}{\hbar} \sum_i S(x_{i+1}, x_i)\right) \cdots \frac{dx_{i+1}}{A} \frac{dx_i}{A} \cdots , \tag{10.1}$$

where A is a normalization factor. Here Feynman divided the time interval from the initial instant to the final instant into a large number of small intervals, given by successive times $t_1, t_2, t_3, \ldots$, where $t_{i+1} = t_i + \varepsilon$. Then the coordinates $x_1, x_2, x_3, \ldots$, which lie in some region R, could be considered as coordinates of the positions of the particle at corresponding times $t_1, t_2, t_3, \ldots$. 'From the classical point of view, the successive values $x_1, x_2, x_3, \ldots$ of the coordinates practically define the path $x(t)$. Eventually, we expect to go to the limit $\varepsilon \to 0$.'[10] By varying the values of a coordinate x_i, we will have various paths in the range R.

The quantity $S(x_{i+1}, x_i)$ in equation (10.1) is simply the classical action on the corresponding path from point x_{i+1} to point x_i. One can obtain this action function from the the formula (6.10). Hence, the sum in the exponent in equation (10.1) in the limit $\varepsilon \to 0$ goes to the classical action on the path $x(t)$: $S = \lim_{\varepsilon \to 0} \sum_i S(x_{i+1}, x_1)$. Finally, the many-time integration in equation (10.1) obviously means a summation over all possible paths in the range R, since by varying $x - s$ we will have all possible paths in this range. But this means just the *interference of the terms* $\exp(iS/\hbar)$, which corresponds to every possible path in R. Hence, Feynman's main postulate was: 'The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of $\hbar$), i.e. the time integral of the Lagrangian taken along the path.'[11]

Later on, Feynman explained this postulate as follows: 'The total amplitude can be written as the sum of the amplitudes of each path—for each way of arrival. For every $x(t)$ that we could have—for every possible imaginary

trajectory—we have to calculate an amplitude. Then we add them all together. What do we take for the amplitude for each path? Our action integral tells us what the amplitude for a single path ought to be. The amplitude is proportional to some constant times $\exp(iS/\hbar)$, where S is the action for the path. That is, if we represent the phase of the amplitude by a complex number, the phase angle is $S/\hbar$. The action S has dimensions of energy times time, and Planck's constant $\hbar$ has the same dimensions. It is the constant $\hbar$ that determines when quantum mechanics is important.'[12] 'I could see the paths . . . each path got an amplitude. (So the) clarity came from writing up the RMP (1948) article.'[13]

As a straightforward consequence of Feynman's extremely important and completely new viewpoint concerning the relation (10.1),† one can answer a three-century-old question about the meaning of the principle of least action (Section 6.2). 'Here is how it works: Suppose that for all paths, S is very large compared to $\hbar$. One path contributes a certain amplitude. For a nearby path, the phase is quite different, because with an enormous S even nearby paths will normally cancel their different phases—because $\hbar$ is so tiny. So, nearby paths will normally cancel their effects out in taking the sum—except for one region, and that is when a path and a nearby path all give the same phase in the first approximation (more precisely, the same action within $\hbar$). Only those paths will be the important ones. So in the limiting case in which Planck's constant $\hbar$ goes to zero, the correct quantum mechanical laws can be summarized by simply saying: "Forget about all these probability amplitudes. The particle does go on a special path, namely, that one for which S does not vary in the

† Considering the formula (6.1) from a purely mathematical point of view, one ought to emphasize that Feynman was not the first to discover such types of relations. In pure mathematics, an analogous idea was first developed by Vito Volterra.[14] He studied an ordinary linear differential equation which is similar to Schrödinger's equation, but in infinite-dimensional spaces. Volterra proved rigorously that one can represent the solution of these equations in a form similar to equation (10.1), but instead of integration there occurred summation on certain discrete indices, and instead of terms like $\exp[iS(x_{i+1}, x_i)/\hbar]$ there were matrices. Then the corresponding type of the limit in equation (10.1) will yield a so-called 'multiplicative of Volterra', which was studied by many mathematicians. Volterra also considered the case of the infinite-dimensional linear functional space like $\psi(q)$.[15]

Feynman did not know this mathematical result. He was not looking for rigorous formal proofs of his new method, but only for clear intuitive arguments. Before Feynman, nobody had ever made the attempt to visualize the summation in equation (10.1) in the intuitively clear way he did it.

The purely mathematical considerations show why the path-integral method should be extremely useful in various scientific domains, although it was invented by Feynman for quantum mechanics. The reason for this is that from the mathematical point of view the path-integral method gives the solution of a linear differential equation in a linear space with any dimension whatsoever. The basic equations in many scientific domains are of this type, hence the generality of Feynman's method.

first approximation." That's the relation between the principle of least action and quantum mechanics.'†[12]

Thus Feynman's postulate leads to the principle of least action (6.1) and gives us the right explanation as to where this principle is coming from. As far as the principle of least action, *the* most fundamental principle of classical physics, is concerned, one which leads to the classical dynamical equations in all the fundamental classical theories, one can truly say that it was Feynman who discovered the deepest import of this principle.

10.3 The new operator algebra

In several sections of his RMP (1948) paper, Feynman developed the new formalism of quantum mechanics, and proved its equivalence to the older formulations of Heisenberg and Schrödinger. He showed how one can introduce the wave function in his path-integral approach, and derived the Schrödinger equation for this wave function. We have explained Feynman's derivation of the Schrödinger equation in Section 6.4. Then he introduced his new so useful notion of the 'transition amplitude', which can now be found in textbooks on quantum mechanics. Given two quantum states with wave functions $\psi(x, t)$ and $\chi(x, t)$, Feynman called the expression $\int \chi^*(x, t'')\psi(x, t')\, dx$ the 'transition amplitude'. Here $\chi^*(x, t'')$ is the function conjugate to $\chi(x, t)$ at the instant of time t'', and $\psi(x, t)$ is taken at another

† A derivation of the principle of least action from quantum mechanical reasoning was first given by Dirac.[1] This served as the point of departure for Feynman's investigations on the path-integral method. Considering equation (10.1) as an approximation to the exact quantum transition function from the initial to the final instants of time, Dirac discovered that the quantum analog of Hamilton's action principle (equation (6.1)) is absorbed in the composition law (10.1), and the classical requirement that the values of the intermediate coordinates shall make the action stationary corresponds to the condition in quantum mechanics that all values of the intermediate coordinates are important in proportion to the integral (10.1). Then Dirac considered the limiting case when $\hbar$ tends to zero and stated that the integrand in equation (10.1) is a rapidly oscillating function when $\hbar$ is small. Thus the multiple integral (10.1) 'contains the quantum analog of the action principle (as far as) the importance of our considering any set of values for the intermediate [coordinates] is determined by the importance of this set of values in the integration. If we now make make $\hbar$ tend to zero, this statement goes over into the classical statement that . . . the importance of our considering any set of values for the intermediate [coordinates] is zero unless these values make the action stationary.'[1]

From the above remarks it is clear that Dirac was very close to the interpretation of equation (10.1) as a summation over all virtual paths, and he had found the new formulation as an extremely nice and important way to explain the principle of least action as a result of quantum laws. However, Dirac was not able to complete this line of his investigation on quantum mechanics because his point of view was that the exponent of the classical action in the form of Hamilton's principal function is only an approximate semiclassical relation. Dirac was interested only in a general question: What is the quantum analog of the classical principle of least action?

instant of time t'. Thus the transition amplitude gives us the quantum amplitude for the transition from the quantum state ψ at the time t' to the quantum state χ at the time t''. Feynman showed that, in his path-integral method, the average of the transition amplitude may be regarded as unity, that is

$$\langle \chi_{t''} | 1 | \psi_{t'} \rangle_S = \lim_{\varepsilon \to 0} \int \ldots \int \chi^*(x'', t'') \times exp(iS/\hbar) \psi(x', t') \frac{dx_0}{A} \ldots \frac{dx_{j-1}}{A} dx_j. \tag{10.2}$$

In the language of ordinary quantum mechanics, if the quantum Hamiltonian operator **H** does not depend on time, this transition amplitude is the matrix element of the quantum evolution operator $\exp[-(t''-t')\mathbf{H}/\hbar]$, between the quantum states $\chi_{t''}$ and ψ_t. This operator describes the evolution of the wave function ψ from the instant t' to the instant t'': $\psi(x, t'') = \exp[-(t''-t')\mathbf{H}/\hbar]\psi(x, t')$.

As a generalization of formula (10.2), Feynman introduced the formula for the averages of any functional F of the coordinates x_i for $t' < t_i < t''$. He defined the 'transition element' of the functional F between the states ψ at t' and χ'' at t'' for the action S as

$$\langle \chi_{t''} | F | \psi_{t'} \rangle_S = \lim_{\varepsilon \to 0} \int \ldots \int \chi^*(x'', t'') F(x_0, x_1, \ldots, x_j) \times \exp\left(\frac{i}{\hbar} \sum_i S(x_{i+1}, x_i)\right) \psi(x', t') \frac{dx_0}{A} \ldots \frac{dx_{j-1}}{A} dx_j. \tag{10.3}$$

Then he used these basic formulas to obtain several fundamental results from his new formulation of quantum mechanics.

The first application was the new formulation of the so-called perturbation theory in quantum mechanics.† Suppose we consider a second problem which

† It turns out that, historically, this was the fundamental result of Feynman's path-integral method, which was first used in important physical problems. Feynman himself used this part of his method very soon in his papers on quantum electrodynamics.

The perturbation theory plays a very important role in many physical problems. Very often one cannot solve the exact problem because of its complexity. But it may turn out that we can solve some other problem, which differs slightly from the initial one. In this case, one can say that some small 'perturbation' leads from the unperturbed solvable problem to the perturbed one that we actually wish to solve. In such a situation, the method which permits one to reach the solution of the complicated problem, using the solution of the simpler one, is needed. This method is called 'perturbation theory'. Feynman's path-integral method leads to the now very successful formulation of the perturbation theory in quantum mechanics, based on formulas like equation (10.4). Very soon after the RMP (1948) article, Feynman proposed a proper generalization of his new perturbation theory in quantum electrodynamics. Since then Feynman's perturbation theory has been one of the most useful methods in the quantum theory of various physical fields.

differs from the first because, for example, the potential is augmented by a small amount $\lambda U(x, t)$. Then in the new problem the quantity replacing S is $S' = S + \sum_i \lambda U(x_i, t)$. Substituting it into equation (10.2) leads directly to an expression, which after performing some algebra leads to an important perturbation formula. If the effect of U is small, we find

$$\langle \chi_{t''}|1|\psi_{t'}\rangle_{S'} = \langle \chi_{t''}|1|\psi_{t'}\rangle_S + \frac{i}{\hbar}\left\langle \chi_{t''}\left|\sum_i \lambda U(x_i, t_i)\right|\psi_{t'}\right\rangle_S . \tag{10.4}$$

Formula (10.4) permits one to calculate the effects of the perturbation $\lambda U(x, t)$ to first order with respect to the small parameter λ, using the solution of the more simple problem with the action function S.

The next fundamental result of Feynman's new formulation was a completely new derivation of Newton's equations and the commutation relations. By using formula (10.3), Feynman had, in his dissertation, already derived the quantum Lagrange equation.[16] In the RMP (1948) paper, he wrote Newton's equations in the form

$$0 \underset{S}{\leftrightarrow} -\frac{m}{\varepsilon}\left(\frac{x_{k+1}-x_k}{\varepsilon} - \frac{x_k - x_{k-1}}{\varepsilon}\right) - V(x). \tag{10.5}$$

Here ε is the small difference between the successive times $t_1, t_2, t_3, \ldots$ that is, $t_{i+1} - t_i = \varepsilon$. Hence $(x_{k+1} - x_k)/\varepsilon$ is the velocity and the term $[(x_{k+1} - x_k)/\varepsilon - (x_k - x_{k-1})/\varepsilon]/\varepsilon$ presents the acceleration $\boldsymbol{a}$ of the particle. The derivative of the potential $V(x)$, taken with the minus sign, gives the force term $\boldsymbol{F}$. Thus, the right-hand side in the expression (10.5) represents just the classical term $-m\boldsymbol{a} + \boldsymbol{F}$, which according to Newton, must be equal to zero. Actually, in quantum mechanics, this expression corresponds to some quantum operator, which cannot be equal to the zero operator. But the average values of this operator (with respect to some given action S, according to equation (10.3)), are zero, and that is completely sufficient for the right physical interpretation of the meaning of this operator. Feynman employed the symbol $\overset{\leftrightarrow}{S}$ to emphasize the fact that two different functionals may give the same result for the transition amplitude between any two states or, in other words, they are equivalent under one action S but may not be equivalent under another.

So, equation (10.5) represents Newton's equations in Feynman's formulation of quantum mechanics.

It turns out that in Feynman's formulation of quantum mechanics, one can also derive the quantum commutation relations in exactly the same manner. The commutation relations between the quantum momentum operator $\boldsymbol{p}$ and the quantum position operator $\boldsymbol{x}$, in the usual formulation of quantum mechanics, reads: $[\boldsymbol{p}, \boldsymbol{x}] = \boldsymbol{p}\boldsymbol{x} - \boldsymbol{x}\boldsymbol{p} = \hbar/i$. Feynman showed that in his new formulation of quantum mechanics a new relation corresponds to the old one. This new relations reads

$$m\left(\frac{x_{k+1}-x_k}{\varepsilon}\right)x_k - m\left(\frac{x_k-x_{k-1}}{\varepsilon}\right)x_k \overset{\hbar}{\underset{s}{\leftrightarrow}} \frac{\hbar}{i}. \tag{10.6}$$

Since $(x_{k+1}-x_k)/\varepsilon$ corresponds to the velocity v of the particle, here $m(x_{k+1}-x_k)/\varepsilon$ corresponds to the classical momentum $p=mv$.

Taking into account the new form of the commutator relation, equation (10.6), as far as the new form of Newton's equations (10.5) are concerned, we see that 'the operators corresponding to the functions of x_{k+1} will appear to the left of the operators corresponding to the functions of x_k, i.e. *the order of terms in a matrix operator product* (in the old formalism of quantum mechanics) *corresponds* (in Feynman's new formulation) *to an order in time of the corresponding factors in a functional*. Thus, if the functional can be, and is, written in such a way that in each term factors correspond to earlier terms, the corresponding operator can immediately be written down if the order of the operators is kept the same as in the functional. Obviously, the order of factors in a function is of no consequence. The ordering just facilitates transition to conventional operator notation.'[17] *This is the essence of the new Feynman operator algebra, where the usual operator ordering is replaced by the ordering of the classical-like terms in time.*

The Hamiltonian operator is of central importance in the usual formulation of quantum mechanics. Therefore, Feynman studied in detail the functional corresponding to this operator. He preferred to define the Hamiltonian functional in a physical way by the changes made in a state when it is displaced in time. Feynman had to perform the calculations especially carefully, because of the squared dependence of the Hamiltonian on the velocity of the particle. He derived the following expression for the functional, which corresponds to the Hamiltonian operator:

$$H_k = \frac{m}{2}\left(\frac{x_{k+1}-x_k}{t_{k+1}-t_k}\right)^2 + \frac{\hbar}{2i(t_{k+1}-t_k)} + V(x). \tag{10.7}$$

The second term in this expression is proportional to Planck's constant $\hbar$. It is a very important term, because it is due to this term that the kinetic energy turns out to be finite. The first term in equation (10.7) leads only to the infinite kinetic energy because of the relation $(x_{k+1}-x_k)^2 \overset{\leftrightarrow}{s} (i\hbar/m)(t_{k+1}-t_k)$, which says that the root mean square of the 'velocity' $(x_{k+1}-x_k)/(t_{k+1}-t_k)$ is of order $(t_{k+1}-t_k)^{-1/2}$, hence it goes to infinity when $t_{k+1}-t_k$ goes to zero. From the physical point of view this leads to the important conclusion that, nevertheless, the paths which give an essential contribution to the Feynman path-integral for the transition amplitude (see equation (10.1)) are continuous paths, the velocity of the particle at each point of the paths is not well defined, and goes to infinity. One can imagine such a path as a continuous line, which breaks its direction at every point. From the mathematical point of view such a 'zigzag' path is not a differentiable path, hence the velocity of the particle is not

well defined. This fact does not make difficulties in the Hamiltonian, because of the second term in equation (10.7), which cancels the infinity from the first term exactly in the limit when $t_{k+1}-t_k$ goes to zero. Feynman derived the second term in equation (10.7) from the factor A in equation (10.1). One has to remember that this factor was just needed to establish the exact relation between the quantum transition amplitude and the exponent of the classical action S times $i/\hbar$, which was the starting point of Feynman's invention of the new formulation of quantum mechanics (see Section 6.5)

In this way, Feynman demonstrated the self-consistency of his path-integral method and its equivalence to the old Schrödinger and Heisenberg formulations of quantum mechanics. At the end of the general discussion of the path-integral method, Feynman mentioned the shortcomings in his formulation of quantum mechanics. He noted: 'The formulation given here suffers from a serious drawback. The mathematical concepts needed are new. At present, it requires an unnatural and cumbersome subdivision of the time interval to make the meaning of the equations clear. Considerable improvement can be made through the use of the notation and concepts of the mathematics of functionals. However, it was thought best to avoid this in the first presentation. One needs, in addition, an appropriate measure for the space of the argument functions $x(t)$ of the functionals.

'It is also incomplete from the physical standpoint. One of the most important characteristics of quantum mechanics is its invariance under unitary transformations. These correspond to the canonical transformations of classical mechanics. Of course, the present formulation, being equivalent to ordinary formulations, can mathematically be demonstrated to be invariant under these transformations. However, it has not been formulated in such a way that it is *physically* obvious that it is invariant. This incompleteness shows itself in a definite way. No direct procedure has been outlined to describe measurements of quantities other than position. Measurements of momentum, for example, of one particle, can be defined in terms of measurements of positions of other particles. The result of the analysis of such a situation does show the connection of momentum measurements to the Fourier transform of the wave function. But this is a rather roundabout method of obtaining such an important physical result. It is to be expected that the postulates can be generalized by the replacement of the ideas of "paths in a region of space–time R", or to "paths of class R", or "paths having the property R". But which properties correspond to which physical measurements has not been formulated in a general way.' [18]

Feynman then gave an outline of the generalization of his path-integral method and how it could be used to solve certain problems. We shall discuss these matters in Section 10.5, but before doing so we will give a brief historical account of functional integration before Feynman, something which he discovered independently, without any knowledge of what had been done before.

10.4 Functional integration before Feynman

As we have noted, Feynman's principal mathematical result was the new method for the calculation of averages of quantum mechanical quantities with the help of formulas like (10.2) and, in more general cases, like (10.3). From the mathematical point of view, these formulas give us the averages of certain functionals $F\ [(x, t)]$ on the paths $x(t)$ in the configuration space of the classical mechanical system, where the time t runs from some initial instant t_{in} to some final instant t_{fin}. As a weight in the averaging procedure, one may use the other functional, namely, the exponent of the classical action times $i/\hbar$. We will now present an outline of some landmarks in the theory of functionals, which has developed and been applied very extensively after Feynman's work.

(*a*) *Vito Volterra*

The general theory of functionals was developed in the works of Vito Volterra, long before Feynman's investigations on the new quantum mechanical formalism. We have already mentioned the contribution of Volterra in the solution of linear equations in multi- or infinite-dimensional linear spaces by means of his multiplicative integral. Developing the general theory of the functional calculus,[19] Volterra invented the way to reduce the calculations with functionals to calculations with usual functions of many variables. This procedure was of just the type which Feynman used later: namely, one has to divide the interval from the initial time t_{in} to the final time t_{fin} into a large, but finite number N of time instants t_i, and then to approximate the functional $F[(x, t)]$ with the function $F(\ldots, x_{i+1}, x_i \ldots)$, where x_i gives the value of $x(t_i)$. Then, one has to work with this function, instead of the functional $F[(x, t)]$. This procedure is called a *finite-dimensional approximation*, or discretization, of the functional $F\ [(x,t)]$, which itself may be considered as a function of infinitely many variables $x(t)$, with a continuous label t. After performing operations on the function $F(\ldots, x_{i+1}, x_i, \ldots)$ in the final result one has to take the limit $N \to \infty$, keeping t_{in} and t_{fin} fixed. And this is just the procedure which Feynman employed.

(*b*) *Norbert Wiener and others*

The first considerations of the average value of a functional in pure mathematics, without any connection with quantum mechanics, were given by P.J. Daniel,[20] R. Gâteaux,[21] P. Lévy,[22] and in several papers by Norbert Wiener.[23]

The first physical application of the functional machinery was Wiener's approach to the Brownian motion of very small particles in a colloidal suspension.

Brownian motion was discovered by Robert Brown, a Scottish botanist, in

1827, when he saw through the microscope the strange chaotic movement of very small particles in colloidal suspensions. The analysis of this phenomenon, given by many physicists, shows that this movement is caused by small particles being struck by the molecules of the liquid in which they were suspended. Since these hits are random, the movement is too chaotic and leads to the diffusion of the particles in the liquid. The path of a single particle looks like a zigzag line, consisting of straight lines with random orientation and length. These zigzag straight lines describe the motion of the free particle between sequential impacts.

Let us consider, for simplicity, a particle which wanders along the X axis only. Let $\phi(x, t)$ denote the probability distribution of the particle to be at the point x at at time t. In his paper on Brownian motion, Albert Einstein[24] showed that if one supposes that the particle wanders a given distance in a given time that is independent of (1) the starting position of the particle, (2) the initial instant of time when it begins to wander, (3) the direction in which it starts to wander, then the probability distribution that after a time t the particle has wandered from the origin to a position lying between x and $x+dx$ is given by

$$\phi(x, t) = (4\pi Dt)^{-1/2} \exp(-x^2/4Dt),$$

and this probability distribution satisfies the so-called diffusion equation:

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}.$$

Here D is called the 'diffusion coefficient', and it is related to the size and the mass of the particle and to the temperature and the viscosity of the liquid. It may be connected, too, with Boltzmann's constant k and hence with the fundamental Avogadro number $N = 6.02213 \times 10^{23}\ \text{mol}^{-1}$, being the number of molecules in one gram-molecule of matter. In fact, the possibility of obtaining the value of the fundamental constant from experiments stimulated Einstein and Smoluchowski to develop the theory of this phenomenon at the beginning of the century. However, from the mathematical point of view, the value of the diffusion coefficient is not essential and by a proper choice of the units one may put it equal to $\frac{1}{4}$ for simplicity. Then

$$\phi(x, t) = (\pi t)^{-1/2} \exp(-x^2/t).$$

Wiener investigated the history of the wandering particle. He assumed that this history is represented by the equation $x = x(t)$, $x(t)$ being a continuous function; then he considered 'particle histories' or 'time paths', these being the key notions, which Wiener used in the proper sense for the first time, and said: 'There are certain assemblies of time paths to which we can immediately assign a measure, a probability. These assemblies are obtained by restricting the position of the particle at certain specified times, finite in number, to certain

specified finite intervals.'[23] Using Feynman's notation (see equation (10.1)), we can write this probability as

$$P_{\mathrm{W}} = \lim_{\varepsilon \to 0} \int_R \left[-\sum_i (x_{i+1} - x_i)^2/(t_{i+1} - t_i) \right] \cdots \frac{dx_{i+1}}{A} \frac{dx^i}{A} \cdots .$$

In the limit $N \to \infty$, when the time intervals $t_{i-1} - t_i$ go to zero, this expression defines a measure on the set of all paths $x = x(t)$. Wiener showed that this measure is concentrated on the set of continuous paths, which are not differentiable. In such a limit, the term in the exponent gives just twice the integral on the time of the kinetic energy of the particle with unit mass: $\int_0^1 \dot{x}(\tau)^2 \, d\tau$. Then we can denote the so-called 'Wiener measure' as

$$d_{\mathrm{W}}[x(t)] = 1/C \exp\left(-\int_0^1 \dot{x}(\tau)^2 \, d\tau \right) \prod_0^t dx(t),$$

and calculate the average value of any functional $F[x(t)]$ on the path $x(t)$ according to the formula $\langle F \rangle_{\mathrm{W}} = \int_R F[x(t)] \, d_{\mathrm{W}}[x, t]$. In the Wiener measure, C is a normalization constant, which normalizes the infinite-dimensional integral under the condition $\int_R d_{\mathrm{W}}[x(t)] = 1$.

Thus we can see that Wiener represented the corresponding probability and the average values of the functionals for Brownian particles as functional integrals, i.e. as weighted sums on all the possible paths or, in other words, on the histories of the particle.

But there exists an essential difference between Wiener's formula given above and equation (10.3) for Feyman's averages. Wiener's averages $\langle F \rangle_{\mathrm{W}}$ look almost like Feynman's averages $\langle F \rangle_S$ (see equation (10.2)) for a free particle, when the action functional reduces to the integral over the time of the kinetic energy of the particle. But in the exponent of Feynman's averages there remains an imaginary factor $i/\hbar$, which makes it impossible to interpret Feynman's averages as averages with respect to some real measure. In contrast to Wiener's path integral, in Feynman's averages there exist rapidly oscillating complex exponential functions, owing to which, as we know, there exist the least action principle and classical mechanics. These rapidly oscillating integrals are not convergent and one must understand them in a proper sense. In contrast, Wiener's integrals are very well convergent because of the properties of the Wiener measure.

Taking into account the finite-dimensional case one may ask how one should include the exponent of the kinetic energy with the minus sign in Wiener's measure. Is it impossible to define the measure in the infinite case as $\prod_0^t dx(t)$? Such an expression would be completely analogous to the expression $\prod_{i=1}^N dx_i$, which gives the measure, i.e. the volume of the ranges in N-dimensional space. If this were possible, one could include the exponential factor in the functional, and one would have to take the average both in

Feynman's and Wiener's cases. Moreover, if this were possible we shall have obtained a translational invariant additive measure, like Lebesgue's, in an infinite-dimensional case, and it would be possible to treat both cases in a similar manner. Unfortunately, one can easily prove that this is impossible and such a measure does not exist in infinite-dimensional spaces.

The reason is quite simple. Suppose we have a translational invariant additive measure in the infinite-dimensional linear space, and the volume of the bounded bodies with respect to this measure is infinite. This means that the volume of each body does not depend on its position in the infinite-dimensional space and this volume is the sum of the volumes of all parts of the body. Let us put on every axis in this space one ball with radius 1, centered at a distance 1 from the origin. The volume of every such ball is a finite number V_0, but the volume of the whole set of such balls is infinite because the space is infinite-dimensional, and one will have an infinite set of balls, one in every direction, which does not intersect with another (see Fig. 10.1). Now we may consider a new ball of radius 10 with its center at the origin. It is evident that all the infinitely many initial balls will lie within this bigger ball, which must also have a finite volume V, since we have a Lebesgue-like measure. But this is impossible, since the volume of the big ball is larger than the common volume of the entire initial set of balls, which now lie within the larger one. The common volume of the small balls is infinite.

This consideration shows that it is impossible to treat Wiener's and Feynman's averages in a similar way. The fast decreasing exponential, included in the Wiener measure, allows one to use measure theory. But in the case of Feynman's averages this is impossible and one needs some completely

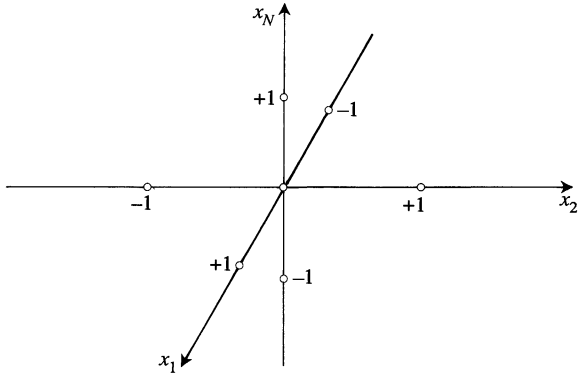

Fig. 10.1. Three-dimensional illustration of the geometrical picture we have to consider to prove the absence of the translational invariant additive measure in infinite-dimensional space. In the infinite-dimensional case there are infinitely many axes, hence, infinitely many small balls in the big ball.

different ideas to give his procedure a rigorous meaning. In spite of this problem, we should note that now there exist different approaches which can deal with Feynman's averages involving certain pseudomeasures. There exists a very suggestive short conference address given by Kirkwood,[25] in which he mentioned that one could apply to quantum physics the integration of functionals in the Wiener sense for the calculation of the statistical sum. This is the first known attempt to connect the functional integral method with quantum problems.

(c) *Subrahmanyan Chandrasekhar*

Wiener's treatment of the Brownian motion of the particles in liquids was approximate, because he neglected the inertia of the particle and external forces, such as the gravitational force and others, which can act on the particles. The next important step in this direction was taken by Subrahmanyan Chandrasekhar,[26] who treated Brownian motion entirely by the functional integral method, although nowhere did he mention this name.

In 1908 Paul Langevin[27] had proposed to treat Brownian motion as the motion of a classical particle of mass m under the action of a random force $f(t)$, and a frictional force $-(m/\tau)\dot{x}$, which is proportional to the velocity $\dot{x}$ of the particle. The frictional force describes the interaction of the particle with the medium, and τ is the relaxation time of the particle. In addition, there exists an external force F. Thus the so-called Langevin equation for the particles reads;

$$m\frac{d^2x}{dt^2} = f(t) - (m/\tau)\frac{dx}{dt} + F(x, t).$$

One can solve this equation with respect to the velocity $v(t) = \dot{x}$ of the particle. The result shows that the velocity of the particle is a functional of the random force $f(t)$: $v = v[f(t)]$. Then, in order to calculate the physical quantities for the Brownian particle, one has to obtain the averages of this or other functionals. To do so Chandrasekhar used Wiener's method. He assumed the probability distribution of the Wiener force to be Wiener-like; that is, the probability of finding the value of the force between $f(t)$ and $f(t) + df(t)$ in a short interval of time from t to $t + \Delta t$ to be $(4\pi a/\Delta t)^{1/2} \exp(-f^2\,\Delta t)\,df$, where $a = kT\ (m/\tau)$ is a constant, k being Boltzmann's constant and T the temperature of the medium. By a proper choice of the units one may put a equal to $\frac{1}{4}$ for simplicity; then the above probability becomes equal to:

$$(\pi/\Delta t)^{-1/2} \exp(-f^2\,\Delta t)\,df.$$

Chandrasekhar introduced a Wiener-like measure:

$$d_W[f(t)] = (1/C) \exp\left(\int_0^t d\tau [f(t)]^2\right) \prod_0^t df(t).$$

By calculating the functional average $\langle V\rangle_{\mathrm{W}}$, Chandrasekhar showed that the average velocity of the Brownian particle does not depend on the random force $f(t)$, and the mean square displacement $\langle \Delta x^2\rangle_{\mathrm{W}}$ along any direction is proportional to the time t.

For the relaxation processes for times larger than τ, the relaxation time of the particle, we have to go back to Einstein's approximation, neglecting the inertia of the particles. This means that we have to write down the Langevin equation in a more simple form: $(m/\tau)\, dx/dt = f(t) + F(x, t)$. The solution of this equation is a functional of the random force $f(t)$: $x(t) = x[t; x_0, t_0; f(t)]$. Here x_0 is the initial position of the particle at the initial instant t_0. Chandrasekhar invented the new approach to evaluate averages of the functions $\Phi(x, t)$ of the position x of the Brownian particle at time t; this approach now has a large number of applications both in quantum mechanics and in statistical physics. The average value of the function $\Phi[x(t), t]$,

$$\langle \Phi[x(t), t]\rangle_{\mathrm{W}} = \int \Phi\{x[t; x_0, t_0; f(t)]\}\, d_{\mathrm{W}}[\mathrm{f(t)}],$$

according to Chandrasekhar, may be represented in the form

$$\langle \Phi[x(t), t]\rangle_{\mathrm{W}} = \int \Phi(x, t)\phi(x, t; x_0, t_0)\, dx,$$

where

$$\phi(x, t; x_0, t_0) = \langle \delta[x - x(t)]\rangle_{\mathrm{W}} = \int \delta\{x - x[t; x_0, t_0; f(t)]\}\, d_{\mathrm{W}}[f(t)],$$

is the average value of Dirac's delta-function of the corresponding argument. The point is that one can calculate this average quite simply by using certain techniques. Thus, when the external force equals zero, one obtains the result that the above function, $\phi(x, t; x_0, t_0)$, is exactly the probability distribution, first established by Einstein, and the diffusion coefficient is $D = kT(\tau/m)$.

Chandrasekhar showed that the function $\phi(x, t)$ obeys the diffusion equation for the free particle, which we have written above. This was the first derivation of this equation by the method of functional integration. Chandrasekhar used a procedure which is exactly the same as Feynman's method for the derivation of the Schrödinger equation in his dissertation (see Section 6.5). Chandrasekhar also used the same method for the derivation of the Fokker–Planck equation in the general case:

$$\partial W/\partial t = \beta\, \mathrm{div}_v(Wv) + q\, \nabla_v^2 W.$$

Here $W(v, t)$ is the probability distribution in velocity space, $\beta = m/\tau$, and $q = kT/\tau$ is a constant.

Thus Chandrasekhar showed how one could obtain the phenomenological

results of Einstein and Smoluchowski from a microscopical point of view by using functional integration. We should note that nowadays a similar approach is used in so-called statistical quantization to arrive at quantum mechanics from classical mechanics.

Chandrasekhar also solved the problem of a particle subject to both a random force and an external force, for example a harmonic oscillator under a random force. Then he used his method and results in many physical and astrophysical problems: the theory of density fluctuations, colloid statistics, thermodynamical irreversible processes, effects of gravity on Brownian motion, the phenomenon of sedimentation, the theory of coagulation in colloids, the escape of a particle over potential barriers, and various problems of stellar dynamics.

Feynman did not know anything about the achievements of functional analysis in the calculation of functional averages. He was unfamiliar with the mathematical articles in this domain, but he invented all the things he needed. 'In order to do [the path-integral method] I had to invent this new kind of mathematical structure I had thought of, which was similar to a thing called the Wiener integral . . . . I didn't know anything about that. Actually, if you look up the Wiener integral you will find, if I'm not mistaken, that what Wiener did was to suppose that the e (raised) to minus the kinetic energy piece was a standard weight, and you could do integrals with that weight, whereas the way I was looking at at it, that weight as well as the rest of the function was what you integrate. There's a lot of difference, maybe not as much as I thought. Anyway, never mind! I didn't know anything about Wiener or the source of the Wiener integral, except the complex plane.'[28]

'There was much in my thesis [and in the RMP 1948 article] which was of that kind, things other people had done, and I never even checked the references, but presumably other people had done them. There was, of course, a discussion of principles of least action in classical mechanics, and the problems of the definition of energy and momentum under these circumstances. [In my thesis] they were defined in a general way. I think this was quite early for this definition, but I don't know if it wasn't published earlier—certainly, it has been published since then.'[28]

10.5 Possible generalization and some applications

At the end of the RMP (1948) paper, Feynman proposed some generalization of the path-integral method, which was connected with his action-at-distance theory. For the case of the theory with a time delay he briefly sketched the generalization of the method for calculating averages of functionals of a general type.

Next, Feynman presented the main application of the path-integral method, namely, the elimination of the field oscillators. This problem had arisen from

electrodynamics, which Feynman wished to modify in order to avoid the infinities at the quantum level (see Section 6.3). Feynman briefly illustrated how one could do this in the simple case of a particle with Lagrangian $L(\dot{x}, x)$, which interacts with an oscillator with Lagrangian $\frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$, with the help of a term $\gamma(x, t)q(t)$. Here x is the coordinate of the particle, q is the coordinate of the oscillator, and $\gamma(x, t)$ is an arbitrary function of the coordinate $x(t)$ of the particle at time t. Then the entire classical action of this system may be written as $S = S_p + S_o + S_i$, where S_p is the action of the particle, S_o is the action of the oscillator, and S_i is the interaction term in the action of the system. Feynman showed that the solution of this complicated problem may be divided into two parts. First, one can take the path integral only on the paths connected with the oscillator degree of freedom. The result is just the elimination of the oscillator at the quantum level. Thus one obtains an intermediate problem, without the oscillator degree of freedom, which has to be solved separately. It turns out that this intermediate problem is just the action-at-a-distance theory, which Feynman first considered at the classical level in his dissertation (see Section 6.3).

The power of the new technique for the calculations of the quantum amplitudes lies in the possibility one has of separating the complex system into pieces and 'integrate out parts of it'. It is very hard to do this in the ordinary Schrödinger differential or the Heisenberg matrix form of quantum mechanics. '. . . Drawing on the classical analogue we shall expect that the system with the oscillator is not equivalent to the system without the oscillator for all possible motions of the oscillator, but only for those for which some property [i.e. the initial and final position] of the oscillator is fixed. These properties, in the cases discussed, are not properties of the system at just one time, so we shall not expect to find the equivalence simply by specifying the state of the oscillator at a certain time, by means of a particular wave function. *It is just for this reason that the ordinary methods of quantum mechanics do not suffice to solve this problem.*'[29]

However, there is one more extremely important result, which must be stressed. In the classical elimination of the oscillator degree of freedom there was one ambiguous choice. There we can choose different solutions of the classical oscillator equation. To make the right choice in the classical problem, Feynman had to add an additional requirement: one must derive the equations of motion of particles without the oscillator from the new principle of least action. Then it follows that one must choose just this solution of the classical oscillator equation, which contains one-half advanced and one-half retarded interaction.

Now, in the quantum problem, there is no need to require the principle of least action as an additional postulate, because it follows directly from the path-integral formulation of quantum mechanics for every system (see Section 10.2). In particular, if we have quantum mechanics for the particles without the oscillator, we shall automatically have a principle of least action for this

system. Hence, Feynman's treatment leads unambiguously to the one-half advanced and one-half retarded interaction in the action-at-a-distance theory. This result is of great importance for action-at-a-distance theories.

Feynman's final result of this investigation of the particles interacting through an intermediate oscillator was that at the quantum level, as at the classical level, one can eliminate the oscillator and describe the particles as interacting at a distance, using—in both quantum and classical theories, respectively—an effective action with delay to describe such an interaction.

Feynman's method of solving complicated quantum problems, by integrating by parts the corresponding path integral, is by now a well-known powerful technique which has got a large number of applications in different kinds of theoretical investigations. In the last section of his RMP (1948) article, Feynman gave some suggestions for the applications of the path-integral method in quantum statistical mechanics and in certain relativistic problems. He considered the general problem as to how one may include the spin of the particles in the path-integral method, and suggested a formal way to do it. 'These results for spin and relativity are purely formal and add nothing to the understanding of these (problems). There are other ways of obtaining the Dirac equation which offer some promise of giving a clear physical interpretation to that important and beautiful equation.'[30]

The complete realization of this belief of Feynman's has been reached only recently in theories making use of the so-called Grassmann variables.[31]

At first, Feynman's fundamental article (RMP, 1948) did not arouse much interest among theoretical physicists, who were not familiar with Feynman's new approach to doing quantum mechanics. As Feynman recalled: 'At the Shelter Island Conference [which we shall discuss later on], a lot of exciting things were discussed and talked about. But in spite of all this, the physicists ran out of ideas. They asked me if I would explain my path-integral method for doing quantum mechanics, so I did. I must have been preparing the manuscript of my paper [RMP, 1948], so that everything was organized and I explained it. It's hard to pay attention to some new idea, and they didn't pay much attention to it.'[3] However, nowadays Feynman's RMP (1948) paper is one of the most well known and widely cited papers; it is one of the cornerstones of modern theoretical physics.

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