Path Integrals in Quantum Theories: A Pedagogic First Step

As I have mentioned elsewhere (Quantum Field Theory: Introduction and Background)[1], I strongly believe it far easier, and more meaningful, for students to learn QFT first by the canonical quantization method, and once that has been digested, move on to the path integral (many paths) approach. Hopefully, the material below will help such students, as well as those who are forced to begin their study of QFT via path integrals.

1 Background Math: Examples and Definitions

1.1 Functionals

Functionals form the mathematical roots of Feynman's many paths approach to quantum theories. A functional is simply a function of another function.

Example: Kinetic energy $T = \frac{1}{2}mv^2$ where $v = v(t)$. T is a function of v, and v is a function of t.

Definition(s): A functional is 1) a function of a function, OR equivalently, 2) a function of a dependent variable, OR equivalently, 3) a mapping of a function to a number.

Symbolism:

$$F\left[x(t)\right] \text{ or } F\left[x(t), \dot{x}(t)\right] \tag{1}$$

The square bracket notation is common, but not always used. Mathematically, x and t represent any function and its independent variable, though in physical problems, they are typically spatial position and time, respectively. Functionals are often dependent on the derivative(s) of a function, as well as the function itself, as in the RHS of (1). Total energy, with potential energy dependent on $x(t)$ and kinetic energy dependent on $\dot{x}(t)$, is one example. Additionally, a functional could also be a direct (rather than indirect as in (1)) function of t, i.e.

$$F\left[x(t), \dot{x}(t), t\right] \ . \tag{2}$$

1.2 Functional Derivative

Definition: A functional derivative is simply the derivative of a functional (F above) with respect to a function upon which it depends (x above).

Symbolism:

$$\frac{\delta F}{\delta x(t)} \text{ or } \frac{\delta F}{\delta x} \tag{3}$$

The δ notation is common, though the partial derivative symbol ∂ is often used instead.

1.3 Functional Integral

Definition: A functional integral is the integral of a functional with respect to a function upon which it depends.

Symbolism:

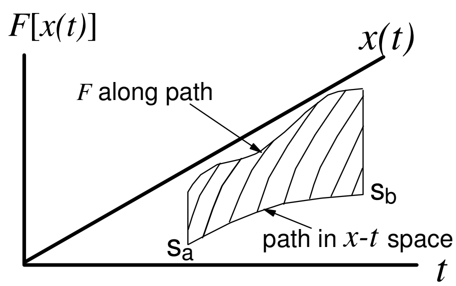
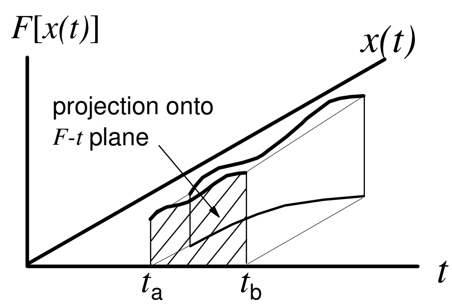
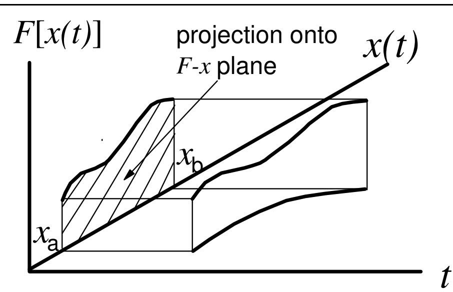
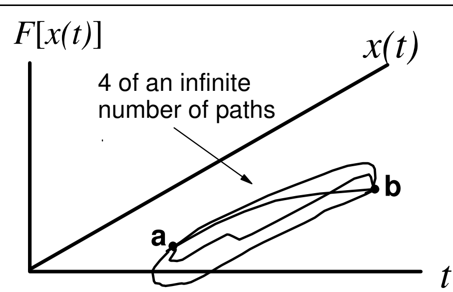
$$\int_{x_a}^{x_b} F\,\delta x \text{ or } \int_{x_a}^{x_b} F\,\delta x(t) \tag{4}$$

In the literature, one may find use of the usual differential symbol d instead of δ.

2 Different Kinds of Integration with Functionals

The value of a functional F of a physical system, such as a particle, is dependent on where it is in space and time, i.e., $x(t)$ and t in (1) are then considered spatial position and time. Further, one can integrate a functional F in different ways over its path in space and time, or over projections of that path. Several of these are depicted in Table 1 below. The first three kinds of integration shown below are fairly self explanatory. We comment on the fourth after the table.

Table 1. Some Ways to Integrate Functionals

<table>
<tr><th></th><th>Type of Integration</th><th>Graphically</th><th>Math</th><th>Comment & Use in Physics</th></tr>
<tr><td>1.</td><td>Area over the path in x(t) vs. t space</td><td>
</td><td>$\int_{s_a}^{s_b} F ds$
where s is spacetime distance along path</td><td>No real physical application.</td></tr>
<tr><td>2.</td><td>Projection of the area in 1 onto the F-t plane</td><td>
</td><td>$\int_{t_a}^{t_b} F dt$</td><td>If $F=L$, the Lagrangian, then this integral = S, the action.
Classically, S = minimum (or stationary) for physical paths</td></tr>
<tr><td>3.</td><td>Projection of the area in 1 onto the F-x(t) plane</td><td>
</td><td>$\int_{x_a}^{x_b} F \delta x(t)$</td><td>This is the usual definition of "functional integral"
This is starting point for 4, below</td></tr>
<tr><td>4.</td><td>Simultaneous integration over all possible paths in 3</td><td>
</td><td>$\int_{x_a}^{x_b} F \mathcal{D} x(t)$</td><td>QM & QFT Feynman path integral approach. $\mathcal{D}$ symbol implies a sum of the integrals of all paths in 3, not just the classical path</td></tr>
</table>

The fourth way to integrate above is not simple, nor is its purpose at all obvious at this point. We devote entire sections below to explaining its origin, its value, and means to evaluate it. So, for now, just let it float easily through your head and don't bother straining to understand it.

Alternative nomenclature: Because functional integrals are integrated over particular paths (in *x*-*t* space in above examples), they are often also referred to as path integrals.

3 The Transition Amplitude

3.1 General Wave Functions (States)

Recall from QM wave mechanics, that for a general normalized wave function ψ equal to a superposition of energy eigenfunction waves (which are each also normalized),

$$\psi = A_1\psi_1 + A_2\psi_2 + A_3\psi_3 \,, \tag{5}$$

A_1 is the amplitude of ψ_1, so the probability of finding ψ_1 upon measuring is

$$A_1^* A_1 = \left| A_1 \right|^2 . \tag{6}$$

If we were to start with ψ initially, and measure ψ_1 later, the wave function would have collapsed, i.e., underwent a transition to a new state. (6) would be the transition probability.

Definition: The transition amplitude is that complex number, the square of the absolute magnitude of which is the probability of measuring a transition from a given initial state to a specific final state.

Symbolism: The transition amplitude is often written as

$$U(\psi_i, \psi_f; T) \,, \tag{7}$$

implying an initial state ψ_i, a final state ψ_f, and an elapsed time between measurements of the two of T.

This terminology carries over to QFT when particles change types. For example, the probability that an electron and a positron would annihilate to create two photons would be the square of the absolute value of the transition amplitude between the initial (e^- , e^+) and final (2 γ) states. (Almost all of QFT is devoted ultimately to determining the transition amplitudes for the different possible interactions between particles.)

Schroedinger Approach Amplitudes

We can't get into explaining it here (for those who may not know it already), but the Schroedinger approach to QM leads to an expression of the transition amplitude of form

$$U(\psi_i, \psi_f; T) = \underbrace{\left\langle \psi_f \right|}_{\substack{\text{final state} \\ \text{measured} \\ \text{at } T+t_a}} \underbrace{e^{-iHT/\hbar} \underbrace{\left| \psi_i \right\rangle}_{\substack{\text{inital state} \\ \text{at } t_a}}}_{\substack{\text{evolved state} \\ \text{at } T+t_a}} \,, \tag{8}$$

where H is the Hamiltonian operator.

Alternative nomenclature: The transition amplitude U is sometimes called the propagator (though *not* the "Feynman propagator" of QED) because it is the contribution to the wave function at f at time T from that at i at time 0. It "propagates" the particle from i to f.

3.2 Position Eigenstates

When the particle has a definite position, e.g., x_i, the wave function is an eigenstate of position, and the ket is written $|x_i>$. The transition amplitude for measuring a particle initially at x_i, and finally at x_f, would take the form

$$U(x_i,x_f;T)=\langle x_f\,|\underbrace{e^{-iHT/\hbar}|x_i\rangle}_{\text{evolved state }\psi}\ . \tag{9}$$

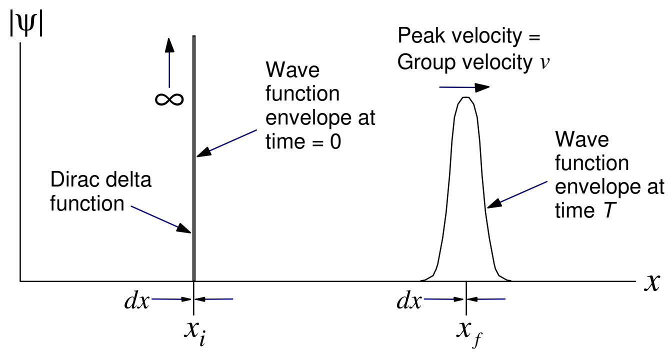


Figure 1. Propagation of a Position Eigenstate Quantum Wave

In wave mechanics notation, $|x_i>$ and $|x_f>$ are both delta functions of form $\delta(x\text{-}x_i)$ and $\delta(x\text{-}\ x_f)$, the first of which is represented schematically on the left in Figure 1. As the initial state evolves into ψ, however, it, like wave packets generally do, spreads, and its peak diminishes (see wave function envelope on right side of Figure 1.) The amplitude for measuring the particle at time T at x_f, i.e., for measuring a delta function $|x_f>$ that collapsed from ψ, is (9).

We can re-write (9), in wave mechanics notation as

$$U(x_i,x_f;T)=\int_{-\infty}^{+\infty}\delta(x-x_f)\psi(x,T)dx=\psi(x_f,T)\ . \tag{10}$$

Thus,

$$\left|U(x_i,x_f;T)\right|^2=\psi^*(x_f,T)\psi(x_f,T)=\text{probability density at }x_f\ . \tag{11}$$

Modification to definition: Hence, the square of the absolute value of the transition amplitude for eigenstates of position is *probability density*, *not probability*, as was the case for energy eigenstate wave functions of form (5).

As we will see, the value found using the RHS of (9), i.e., that of the Schroedinger approach, is the same as the value found using Feynman's many paths approach.

4 Expressing the Wave Function Peak in Terms of the Lagrangian

4.1 Background

One of Feynman's assumptions for his path integral approach to QM and QFT was to express the wave function value at the peak of a wave packet (see Figure 1) in terms of the Lagrangian (exact relation shown at the end of this section 4). I have never seen much justification for this in the literature, other than it is simply an assumption that works (so learn to live with it and move on!)

In the present section I have taken a different tack, by providing rationale for why we could expect Feynman's form of the wave function peak to work. The logic herein may well parallel what went on in Feynman's mind as he was developing his path integral approach.

4.2 Deducing Feynman's Phase Peak Relationship

4.2.1 The Simplified, Heuristic Argument

In QM, the plane wave function solution to the Schroedinger equation,

$$\psi = Ae^{-i(Et-\mathbf{p}\cdot\mathbf{x})/\hbar} \quad , \tag{12}$$

means the phase angle, at any given **x** and t, is

$$\phi = -(Et-\mathbf{p}\cdot\mathbf{x})/\hbar \quad . \tag{13}$$

If we have a particle wave packet, it is an aggregate of many such waves, so it is not in an energy or momentum eigenstate. However, it does have energy and momentum expectation values that correspond to the classical values for the particle. The wave packet peak travels at the wave packet group velocity, which corresponds to the classical particle velocity.

Now, imagine that we approximate the wave packet with a (spatially short) wave function such as ψ, where E and **p** take on the values of the wave packet expectation values for energy and momentum, respectively. If **x** represents the position of the wave packet peak (the middle of our approximated wave function ψ), the time rate of change of phase at the peak is then

$$\frac{d\phi}{dt} = \frac{-(E-\mathbf{p}\cdot\mathbf{v})}{\hbar} = \frac{-T-V+\mathbf{p}\cdot\mathbf{v}}{\hbar} \quad , \tag{14}$$

where **v** is the velocity of the wave peak, T is kinetic energy, and V is potential energy. Non-relativistically,

$$T = \tfrac{1}{2}mv^2 \qquad \mathbf{p} = m\mathbf{v} \quad \rightarrow \quad \mathbf{p}\bullet\mathbf{v} = 2T \, , \tag{15}$$

so, in terms of the classical Lagrangian L,

$$\frac{d\phi}{dt} = \frac{T-V}{\hbar} = \frac{L}{\hbar} \quad . \tag{16}$$

More formally, using the Legendre transformation

$$L = p_i\dot{q}_i - H \quad (L = \mathbf{p}\cdot\mathbf{v} - E \quad \text{here}) \, , \tag{17}$$

directly in (14), we get (16).

Thus, from (16), the phase difference between two events the particle traverses is

$$\phi = \int \frac{L}{\hbar} dt = \frac{S}{\hbar} \quad , \tag{18}$$

where S is the classical action of Hamilton. The classical path between two events is that for which the Hamiltonian action is least. Note that (18) is an integral of type 2 in Table 1.

Hence, the wave function at the peak could be written in terms of the Lagrangian as

$$\psi_{peak} = Ae^{i\int\frac{L}{\hbar}dt} = Ae^{i\frac{S}{\hbar}} \; . \qquad (19)$$

This is the typical starting point assumption when teaching the Feynman path integral approach (still to be developed beginning in Section 5.)

In relativistic quantum mechanics (RQM) and quantum field theory (QFT), we get a solution form similar to (12) (differing only in the normalization factor A), and thus (14) is also true relativistically. Further, since (17) is true relativistically, as well, then so are (16), (18), and (19).

4.2.2 More Precise Argument

The precise expression for a QM particle wave packet[2], where overbars designate expectation (classical) values; v_g, the group (peak, classical) velocity; and $g(p)$, the momentum space distribution is

$$\psi(x,t) = e^{-\frac{i}{\hbar}\left(\bar{E}t-\bar{p}x\right)} \underbrace{\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \underbrace{g(p)}_{\text{real}} \underbrace{e^{-\frac{i}{\hbar}\left(v_g t-x\right)(p-\bar{p})}}_{\substack{=1 \text{ for } x=\text{ peak,} \\ \text{i.e., for } x=v_g t}} \underbrace{e^{-\frac{i}{2\hbar}\frac{t}{m}(p-\bar{p})^2}}_{\substack{\text{time depend} \\ \&\text{complex}}} dp}_{A(t) \text{ for } x=x_{peak}} \, . \qquad (20)$$

We are interested in the value of (20) at the peak, $\psi(x_{peak},t)$, where $x_{peak} = v_g\, t$. To begin, note that with $x{=}x_{\text{peak}}$ inside the integral, the exponent of the second factor in the integrand equals zero, and so that factor equals one. The function $g(p)$ is typically a real, Gaussian distribution in $p-\bar{p}$, and independent of time. The third factor in the integrand is complex and time dependent.

Thus, with $x{=}x_{peak}$, the integral in (20) is a function (generally complex) only of time, which, along with the factor in front, we will designate as $A(t)$. Thus, for the entire history of the wave packet, the wave function value at the peak is

$$\psi(x_{peak},t) = A(t)e^{-\frac{i}{\hbar}\left(\bar{E}t-\bar{p}x_{peak}\right)} \, . \qquad (21)$$

Except for the time dependence in $A(t)$, this is equivalent to (12), as the expectation values for E and p equal the classical values for the particle. So, with regard to the exponent factor in (21), all of the logic from (13) through (19) applies here as well. The final result is so important, we repeat it below, with L being the classical particle Lagrangian, T representing the time when the peak is detected, and phase at $t = 0$ taken as zero. The RHS comes from (10).

$$\boxed{\psi\left(x_{peak},T\right) = A(T)e^{i\int_0^T\frac{L}{\hbar}dt} = A(T)e^{i\frac{S}{\hbar}} = U(x_i,x_f,T)} \qquad (22)$$

We evaluate $A(t)$ exactly in the Appendix.

Definition: Borrowing a term from electrical engineering, we will herein refer to $e^{i\phi}$ as a phasor.

5 Feynman's Path Integral Approach: The Central Idea

Feynman's remarkable idea takes a little getting used to. He reasoned that a particle/wave (such as an electron) traveling a path (world line in spacetime) between two events could actually be considered to be traveling along all possible paths (infinite in number) between those events.

Difficult as it may be, initially, to believe, we will see below that the result from superimposition of the phasors from all of these paths gives us the same result as if we used the standard QM theory of Schroedinger with a single wave. The two different approaches are equivalent.

Definition: Feynman's method is called the "path integral", "many paths", or "sum over histories" approach to QM (and QFT).

Note that the paths do not have to satisfy physical laws like conservation of energy, **F**=m**a**, least action, etc. Moreover, each possible path is considered equally probable.

We will lead into the formal mathematics of the many paths approach by first examining simple situations with a finite number of paths between two events.

6 Superimposing a Finite Number of Paths[3]

6.1 The Rotating Phasor

The phasor of (22) can be expressed in the complex plane as a unit length vector with angle ϕ relative to the real positive (horizontal) axis. As time evolves this vector rotates at the rate $L/\hbar$, i.e., the total phase $\phi = \int \frac{L}{\hbar} dt$. So we can picture the phasor as a unit length vector rotating like a hand on a clock in a 2D complex plane (though it is a counterclockwise rotation).

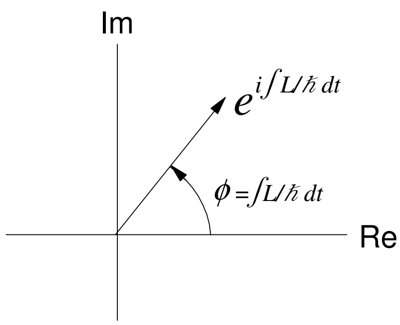


Figure 2. Rotating Phasor

For the purposes of Feynman's approach, we can consider the particle as a wave packet with phase at the peak determined by (22), and our final measurement a position eigenstate measured at the packet peak. We then imagine a different wave packet following each one of the infinitely many paths between two specific events. We visualize the phasor at the peak for each of these paths as a vector rotating in the complex plane as time passes (i.e., as the wave packet peak moves along the path), eventually having a particular value at the final event, the arrival place and time. Each path will have a different final phase.

6.2 Several Paths Graphically

Fig. 24 in Feynman's book *QED: The Strange Theory of Light and Matter*[4], is an insightful, somewhat heuristic, illustration of the many paths concept for light. Since we wish to focus, for the time, on non-relativistic quanta, we employ a similar, and at least equally heuristic, illustration in Fig. 3 for an electron rather than a photon. In Fig. 3 an electron is emitted at event a, reflected, like light from a mirror, off of a scattering surface, and detected at point b. The scattering surface might be difficult to construct in practice, but one can imagine a surface densely packed with tightly bound negative charge.

We look at a representative 15 different paths for the electron, out of the infinite number in the many paths approach, and label them with letters A to O. Each path takes the same time *T*. Note that path H is the classical path, having equal angles of incidence and reflection. Since it is the shortest, particle speed for that path is lowest.

The Lagrangian here is simply the kinetic energy, and this is constant, though different, for each path. Since speed is least for the classical path H, it has the smallest Lagrangian, and thus

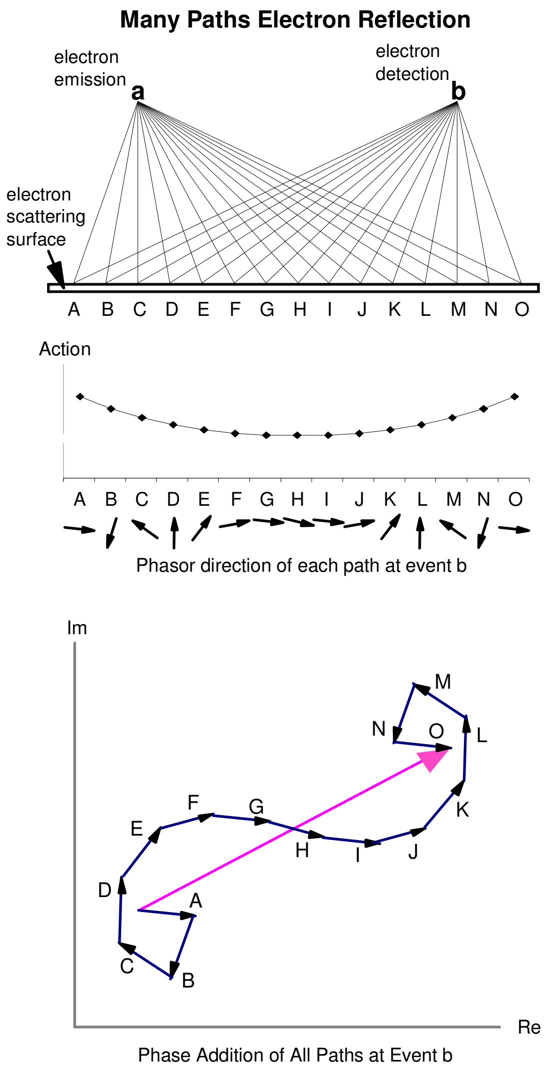


Figure 3. Graphical Justification for Many Paths Approach

the least action. The other paths do not obey the usual classical laws, such as least action, equal angles of incidence and reflection, etc. But according to Feynman's approach, we have to include all of them.

From (22) and Fig. 2, we can determine the phasor $e^{iS/\hbar}$ of (22) for the particle/wave arriving at event b, for each path, where $S = LT = ½\ mv^2T$. The phasor direction in complex space for each path at the detection event b is depicted in the middle of Fig. 3.

The bottom part of Fig. 3 shows the addition of the final event phasors for all 15 paths. Note that the paths further from the classical path H tend to cancel each other out, because they are out of phase. Conversely, H and the paths close to H are close to being in phase, and thus, reinforce each other via constructive interference. So, the primary contributions to the phasor sum are from those paths close to the classical path.

If we were to increase the number of paths, the jaggedness of the curve formed by the 15 phasors would smooth out, but its basic overall shape would remain essentially the same. If we were to increase the Lagrangian, while keeping speed the same for each path (i.e., increasing mass of the particle), phasors now near the middle of the curve would shift towards the ends, and thus, be cancelled out via interference. In other words, increasing mass brings us closer to the classical case, and the paths closer to classical then make greater contributions to the final sum. A similar effect would occur if the value for Planck's constant were smaller. As $\hbar \rightarrow 0$, all paths but H would tend to cancel out.

<u>Clarification</u>

I used to think that increasing mass, and thus getting closer to the classical situation, would bring the phase angle of the sum-of-all-paths phasor in directional alignment with H, the classical path phasor (or at least with U of (22).) However, this is not the case. The important thing in Feynman's approach is not the phase of the sum-of-all-paths phasor, but its *length*, which is proportional to $|U|$. And this length gets greater contribution from paths nearer H than from those further away.

Note that in order to get a graphically significant Fig. 3, I had to use a value for $\hbar$ almost eight orders of magnitude greater than the actual value. Otherwise the phase angles between adjacent paths, for the relatively large spacing between paths of the figure, would have resulted in a seemingly random hodgepodge of phasors, and obscured, rather than illumined, the real physics involved.

If you would like to experiment with changing values for mass, $\hbar$, and number of paths yourself, download the Excel spreadsheet Many Paths Graphic Electron Reflection[5] .

End of Clarification

Feynman intuited that the amplitude of the final phasor sum was extremely meaningful. That is, the square of its absolute value (i.e., the square of its length in complex space) was proportional (approximately, for a finite number of paths; exactly, for an infinite number) to the probability density for measuring the photon/particle at event b. What we mean by "proportional" should become clearer after the following three sections.

6.3 Many Paths Mathematically

Consider particle paths similar to those of Fig. 3, where the wave function peak for path number 1, with $A_1(T)$ as in (22), as

$$\psi_1^{peak} = A_1(T)e^{iS_1/\hbar} \,. \tag{23}$$

In the spirit of the prior section, one considers the phasor of (23) *without* $A_1(t)$ as representing the particle, AND that particle is considered to simultaneously travel many paths between events a and b. Then, the summation of the final phasors for each path is expressed mathematically as

$$e^{iS_1/\hbar} + e^{iS_2/\hbar} + e^{iS_3/\hbar} + ... = A_b e^{i\phi_{\text{sum}}} \tag{24}$$

where A_b is the amplitude of the sum. As the number of paths approaches infinity, $|A_b|^2$ becomes proportional to the probability density of measuring the particular final state at event b. That is,

$$\lim_{N\to\infty}\sum_{j=1}^{N} e^{iS_j/\hbar} = A_b e^{i\phi_{\text{sum}}} \propto U(x_i, x_f, T) \qquad \left|A_b e^{i\phi_{\text{sum}}}\right|^2 = \left|A_b\right|^2 \propto \left|U\right|^2 \text{ (probability density)}. \tag{25}$$

We will learn how to evaluate the limit in (25).

6.4 Another Example

Consider a double slit experiment with a classical Huygen's wave analysis showing alternating fringes of light and dark, which via the classical interpretation is caused by constructive and destructive interference of light/electron waves.

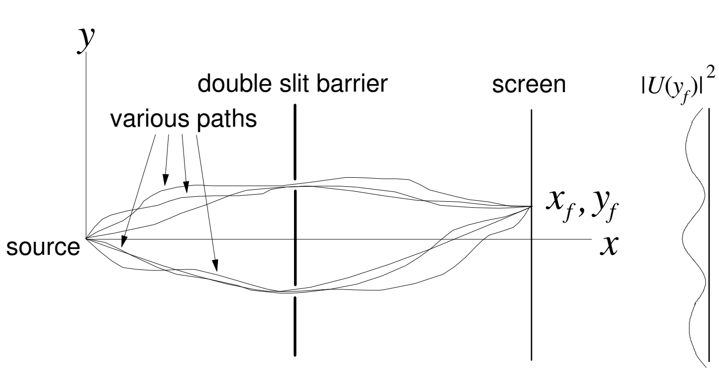


Figure 4. Double Slit Experiment in Many Paths Approach

By the Schroedinger wave approach, a single quantum wave travels through both slits, interferes with itself, either constructively or destructively, to result in a wave amplitude that varies with location along the receiving screen. The probability density (square of the amplitude absolute value) of finding a photon/electron also varies with that screen location. So as the quantum waves collapse, one at a time, on the screen, they tend to collapse more often in the high probability (high magnitude amplitude) regions. These correspond to the bright fringe regions, which, with enough individual quanta collapsing on the screen, are seen by the human eye.

In the many paths approach, for any particular spot on the screen, we would add the phases of every "possible" path from the emission point, through one slit, to that spot (x_f,y_f), plus all paths through the other slit to the spot. See Figure 4. The result would be proportional to the amplitude at the spot found in the Schroedinger approach. That is, the sum of all phasors at x_f,y_f (see (25)) yields

$$C \lim_{N \to \infty} \sum_{j=1}^{N} e^{iS_j/\hbar} = U(x_i, y_i; x_f, y_f; T), \tag{26}$$

where C is some constant.

We would then repeat that procedure for every other point on the screen. Since, for a fixed source at x_i,y_i , and a fixed x_f for the screen, the amplitude would be spatially only a function of y_f, and we could express it simply as $U(y_f)$.

6.5 Finding the Proportionality Constant: By Example

The square of the absolute value of the amplitude U is the probability density. So we can normalize U over the length of the screen, i.e.,

$$\int_{y_f=-\infty}^{y_f=+\infty} \left| C \lim_{N \to \infty} \sum_{j=1}^{N} e^{iS_j/\hbar} \right|^2 dy_f = \int_{y_f=-\infty}^{y_f=+\infty} \left| U(y_f) \right|^2 dy_f = 1, \tag{27}$$

and thus, once the value of the limit is determined, readily find the proportionality constant C.

7 Summary of Approaches

7.1 Feynman's Postulates

Richard Feynman was probably well aware of much of the foregoing when he speculated on the viability of the following three postulates for his many paths approach. Subsequent extensive analysis by Feynman and many others has validated his initial speculation.

The postulates of the many paths approach to quantum theories are:

1. The phasor value at any final event is equal to $e^{iS/\hbar}$ where the action S is calculated along a particular path beginning with a particular initial event.
2. The probability density for the final event is given by the square of the magnitude of a typically complex amplitude.
3. That amplitude is found by adding together the phasor values at that final event from all paths between the initial and final events, including classically impossible paths. The amplitude of the resultant summation must then be normalized relative to all other possible final events, and it is this normalized form of the amplitude that is referred to in 2.

Note two things.

First, there is no weighting of the various path phasors. The nearly classical paths are not weighted more heavily than the paths that are far from classical. That is, the different individual paths in the summation do not have different amplitudes (see (24) and Fig. 3). The correlation with the classical result comes from destructive interference among the paths far from classical, and constructive interference among the paths close to classical.

Second, time on all paths (all histories) must move forward. This is implicit in the exponent phase value of (19), where the integral of L is over time, with time moving forward. Our paths do not include particles zig-zagging backward and forward through time.

Footnote: Caveat: A famous quote by Freeman Dyson states that Feynman, while speculating on this approach, told him that one particle travels all paths, including those going backward in time. But the usual development of the theory (see Section 8) only includes paths forward in time. Perhaps all paths backward in time sum to zero and so are simply ignored. In such case, Dyson's quote would be accurate. But I don't know for sure. End footnote.

7.2 Comparison of Approaches to QM

Unifying Chart 1 summarizes the major similarities and differences between alternative approaches to QM.

Unifying Chart 1. Equivalent Approaches to Non-relativistic Quantum Mechanics

	Schroedinger Wave Mechanics	Heisenberg Matrix Mechanics	Feynman Many Paths
Probability Density of Position Eigenstates	$\lvert\text{amplitude}\rvert^2$	Same results as other two approaches.	$\lvert\text{amplitude}\rvert^2$
Transition Amplitude	$U(x_i, x_f; T) = \langle x_f \rvert e^{-iHT/\hbar} \lvert x_i \rangle$		$U(x_i, x_f; T) \propto \lim_{N\to\infty} \sum_{j=1}^{N} e^{iS_j/\hbar}$ $= \int_{x_i}^{x_f} e^{i\int_0^T \frac{L}{\hbar}dt} \mathcal{D}x(t)$
Comments	Above assumes normalized states.		RHS above must be normalized for $\propto \rightarrow =$. We haven't done the integral part yet.

8 Finite Sums to Functional Integrals

8.1 Time Slicing: The Concept

After all of the foregoing groundwork, it is time to extend the phasor sum of a finite number of paths, such as we saw in Fig. 3 and (24), over into an infinite sum, or in other words, an integral. To do this, we first consider finite "slices" of time, for a finite number of paths in one spatial dimension, as shown in Fig. 5 where, for convenience, we plot time vertically and space horizontally. As opposed to our spatially 2D example in Fig. 3, different paths in Fig. 5 actually refer to the particle traveling along the x axis only between i and f, though at varying (both positive and negative) velocities. The paths between each slice are straight lines, but there is no loss in generality, as one can take the time between slices $\Delta t \rightarrow dt$, and thus, any possible shape path can be included.

As noted earlier, for any <u>single path</u>, the

$$\text{phasor at } \mathbf{f} = \underbrace{e^{i\int_{t_i}^{t_f}\frac{L}{\hbar}dt} = e^{iS/\hbar}}_{\text{one path}} \;, \tag{28}$$

The amplitude U for the transition from **i** to **f** is proportional to the sum of (28) for <u>all paths</u>,

$$\text{sum of } \infty \text{ phasors at } \mathbf{f} = \lim_{N\rightarrow\infty}\sum_{j=1}^{N} e^{iS_j/\hbar} \;. \tag{29}$$

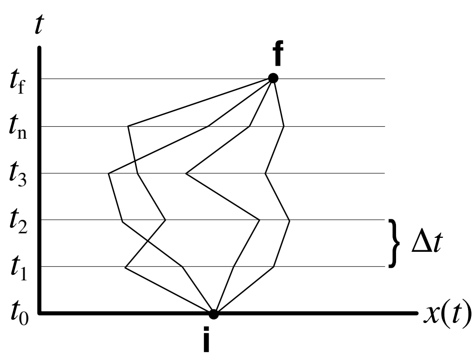


Figure 5. Time Slicing for Finite Number of Paths

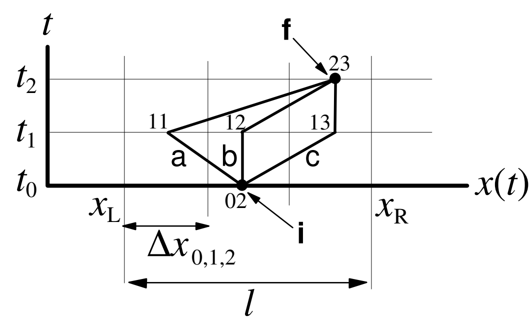


Figure 6. Space Slicing for Three Dicrete Paths

8.2 Space Slicing: Simple Paths with Discrete Approximation

To evaluate (29), we next also discretize ("slice") space, and consider a small number (three) of paths over a small number of discrete events in spacetime, as in Figure 6. We label the paths a, b, and c, and the events with two numbers, such that the first number represents the time slice, and the second the space slice. The continuous range of x values at time t_1 will be designated x_1; at t_2, x_2; etc. We limit the spatial range for paths considered to $x_R - x_L = l$, where the number of paths $N = 3 = l/\Delta x_1$. Each path passes through the center of one Δx_1 segment.

We then assume the phase ϕ_{02} at **i** is zero, and find the phasors at **f** for each of the three paths by subsequently adding the phase difference between discrete events along a given path, as in the second column of Table 2 below.

Note that in the last line of column two in Table 2, the Lagrangian L without subscript is assumed to be the L for the particular subpath being integrated, and this is common notation.

In column three, we approximate the integrals of L over t, such that, for example, for path a over an interval Δt,

$$S_a \approx L_a^{apprx}\Delta t \tag{30}$$

where, for the first subpath,

$$L_a = \tfrac{1}{2}m\dot{x}^2 - V(x) \approx \tfrac{1}{2}m\left(\frac{x_{11}-x_{02}}{\Delta t}\right)^2 - V\left(\frac{x_{11}+x_{02}}{2}\right) = L_a^{apprx}(x_{11},x_{02}) \qquad (31)$$

Similar relations hold for the other subpaths, and are shown in Table 2.

Note that (31) is solely a function of x_{11} and x_{02}. The summation of all three paths in the last row of column three in Table 2 is solely a function of x_{02}, x_{23}, and the three intermediate event x values x_{11}, x_{12}, and x_{13}. Since x_{02} and x_{23} are the initial and final events, which are fixed and the same for all paths, the final summation approximation in Table 2 are really only functions of the three x_{1j}. It will, however, serve a future purpose if we keep x_{02} and x_{23} in the relationship for the time being.

Table 2. Adding Phasors at the Final Event for Three Discrete Paths

Path	Phasor at **f**	Phasor at **f** in Terms of Approx L
a	$e^{i\phi_{a23}} = e^{i(\phi_{02\to11}+\phi_{11\to23})} = e^{i\int_{02}^{11}\frac{L_a}{\hbar}dt}e^{i\int_{11}^{23}\frac{L_a}{\hbar}dt}$	$\approx e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_{11}-x_{02}}{\Delta t}\right)^2-V\left(\frac{x_{11}+x_{02}}{2}\right)\right\}\Delta t}e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_{23}-x_{11}}{\Delta t}\right)^2-V\left(\frac{x_{23}+x_{11}}{2}\right)\right\}\Delta t}$ $= e^{\frac{i}{\hbar}f(x_{02},x_{11})}e^{\frac{i}{\hbar}f(x_{11},x_{23})}$
b	$e^{i\phi_{b23}} = e^{i(\phi_{02\to12}+\phi_{12\to23})} = e^{i\int_{02}^{12}\frac{L_b}{\hbar}dt}e^{i\int_{12}^{23}\frac{L_b}{\hbar}dt}$	$\approx e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_{12}-x_{02}}{\Delta t}\right)^2-V\left(\frac{x_{12}+x_{02}}{2}\right)\right\}\Delta t}e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_{23}-x_{12}}{\Delta t}\right)^2-V\left(\frac{x_{23}+x_{12}}{2}\right)\right\}\Delta t}$ $= e^{\frac{i}{\hbar}f(x_{02},x_{12})}e^{\frac{i}{\hbar}f(x_{12},x_{23})}$
c	$e^{i\phi_{c23}} = e^{i(\phi_{02\to13}+\phi_{13\to23})} = e^{i\int_{02}^{13}\frac{L_c}{\hbar}dt}e^{i\int_{13}^{23}\frac{L_c}{\hbar}dt}$	$\approx e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_{13}-x_{02}}{\Delta t}\right)^2-V\left(\frac{x_{13}+x_{02}}{2}\right)\right\}\Delta t}e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_{23}-x_{13}}{\Delta t}\right)^2-V\left(\frac{x_{23}+x_{13}}{2}\right)\right\}\Delta t}$ $= e^{\frac{i}{\hbar}f(x_{02},x_{13})}e^{\frac{i}{\hbar}f(x_{13},x_{23})}$
Sum of a, b, c	$= e^{i\int_{02}^{11}\frac{L_a}{\hbar}dt}e^{i\int_{11}^{23}\frac{L_a}{\hbar}dt} + e^{i\int_{02}^{12}\frac{L_b}{\hbar}dt}e^{i\int_{12}^{23}\frac{L_b}{\hbar}dt}$ $+e^{i\int_{02}^{13}\frac{L_c}{\hbar}dt}e^{i\int_{13}^{23}\frac{L_c}{\hbar}dt}$ $= \sum_{j=1}^{N=3} e^{i\int_{02}^{1j}\frac{L}{\hbar}dt}e^{i\int_{1j}^{23}\frac{L}{\hbar}dt}$	$= e^{\frac{i}{\hbar}f(x_{02},x_{11})}e^{\frac{i}{\hbar}f(x_{11},x_{23})} + e^{\frac{i}{\hbar}f(x_{02},x_{12})}e^{\frac{i}{\hbar}f(x_{12},x_{23})}$ $+e^{\frac{i}{\hbar}f(x_{02},x_{13})}e^{\frac{i}{\hbar}f(x_{13},x_{23})}$ $= \sum_{j=1}^{N=3} e^{\frac{i}{\hbar}f(x_{02},x_{1j})}e^{\frac{i}{\hbar}f(x_{1j},x_{23})}$

The final relationship in Table 2 is approximately proportional to the transition amplitude, i.e.,

$$U\left(i,f;T=t_f-t_i\right) \approx C\sum_{j=1}^{N=3} e^{\frac{i}{\hbar}f\left(x_{02},x_{1j}\right)}e^{\frac{i}{\hbar}f\left(x_{1j},x_{23}\right)} = C\sum_{j=1}^{N=3} e^{\frac{i}{\hbar}S^{apprx}\left(x_{02},x_{1j}\right)}e^{\frac{i}{\hbar}S^{apprx}\left(x_{1j},x_{23}\right)}, \qquad (32)$$

where C is some constant, and what we designated as a function f in Table 2, in order to emphasize its independent variables, is actually an approximation to the action S.

Since U is *proportional* to the sum of the phasors, we can multiply the RHS of (32) by any constant we like and the proportionality still holds. To aid us in taking limits to get an integral, we multiply (32) by Δx_1, and get

$$U\left(i,f;T\right)\approx C'\sum_{j=1}^{N=3} e^{\frac{i}{\hbar}S^{apprx}\left(x_{02},x_{1j}\right)} e^{\frac{i}{\hbar}S^{apprx}\left(x_{1j},x_{23}\right)}\Delta x_1\,, \tag{33}$$

where C' is a new constant. Taking the limit where $\Delta x_1 \rightarrow dx_1$ means taking the number of paths $N\rightarrow\infty$. And thus,

$$\begin{aligned} U\left(i,f;T\right) &\approx C'\lim_{N\to\infty}\sum_{j=1}^{N} e^{\frac{i}{\hbar}S^{apprx}\left(x_{02},x_{1j}\right)} e^{\frac{i}{\hbar}S^{apprx}\left(x_{1j},x_{23}\right)}\Delta x_1 \\ &= C'\int_{x_1=x_L}^{x_1=x_R} e^{\frac{i}{\hbar}S^{apprx}\left(x_{02},x_1\right)} e^{\frac{i}{\hbar}S^{apprx}\left(x_1,x_{23}\right)}dx_1 \approx C'\int_{x_1=x_L}^{x_1=x_R} e^{i\int_{t_{02}}^{t_{23}}\frac{L}{\hbar}dt}dx_1\,. \end{aligned} \tag{34}$$

where our discrete values x_{1j} have become a continuum x_1, and it is implicit that the L of the last part of (34) is that over the appropriate path corresponding to each increment of dx_1. (34) is still only approximately proportional to the amplitude because time is still discretized in Δt intervals and we limit the integration range to $x_L > x_1 > x_R$. Before extending those limits, however, we must consider a slightly more complicated set of paths.

8.3 From Simple Discrete Paths to the General Case

In Figure 7 we introduce one more time interval between the initial and final events, resulting in nine discrete paths.

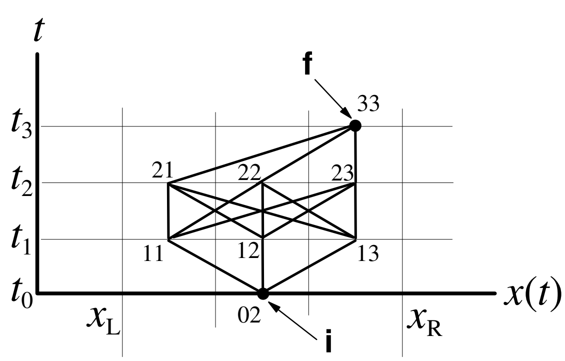


Figure 7. Nine Discrete Paths between Two Events

Repeating the logic from the previous section (use Table 2 as an aide), the phasor of the first path (02→11→21→33) is simply

$$\begin{aligned} \underbrace{e^{i\phi_{33}}}_{\substack{\text{1st}\\\text{path}\\\text{only}}} &= e^{i\left(\phi_{02\to11}+\phi_{11\to21}++\phi_{21\to33}\right)} \\ &= e^{i\int_{02}^{11}\frac{L}{\hbar}dt}\,e^{i\int_{11}^{21}\frac{L}{\hbar}dt}\,e^{i\int_{21}^{33}\frac{L}{\hbar}dt}\,. \end{aligned} \tag{35}$$

We repeat this for the other eight paths, approximate L along subpaths as before, and take k below to indicate the kth Δx_2 segment. This results in a phasor summation from all paths at event **f** (= 33) [compare with last row, last column of Table 2 and (32)] proportional to the amplitude, i.e.,

$$U\left(i,f;T\right)\approx C\sum_{j=1}^{N=3}\sum_{k=1}^{N=3}e^{\frac{i}{\hbar}S^{apprx}\left(x_{02},x_{1j}\right)}e^{\frac{i}{\hbar}S^{apprx}\left(x_{1j},x_{2k}\right)}e^{\frac{i}{\hbar}S^{apprx}\left(x_{2k},x_{33}\right)}. \tag{36}$$

Note that (36) depends on the discrete values of both x_1 and x_2. So, as we did with (33), we can multiply (36) by one or more constants without changing the proportionality. We choose to multiply by Δx_1 and Δx_2. We follow by taking limits where $\Delta x_1 \rightarrow dx_1$ and $\Delta x_2 \rightarrow dx_2$ (i.e., $N \rightarrow \infty$), [compare with (34)] which results in

$$\begin{aligned} U\left(i,f;T\right) &\approx C' \lim_{N\rightarrow\infty}\sum_{j=1}^{N=3}\sum_{k=1}^{N=3}e^{\frac{i}{\hbar}S^{apprx}\left(x_{02},x_{1j}\right)}e^{\frac{i}{\hbar}S^{apprx}\left(x_{1j},x_{2k}\right)}e^{\frac{i}{\hbar}S^{apprx}\left(x_{2k},x_{33}\right)}\Delta x_1\Delta x_2 \\ &= C'\int_{x_2=x_L}^{x_2=x_R}\int_{x_1=x_L}^{x_1=x_R}e^{\frac{i}{\hbar}S^{apprx}\left(x_{02},x_{1j}\right)}e^{\frac{i}{\hbar}S^{apprx}\left(x_{1j},x_{2k}\right)}e^{\frac{i}{\hbar}S^{apprx}\left(x_{2k},x_{33}\right)}dx_1dx_2 \\ &\approx C'\int_{x_2=x_L}^{x_2=x_R}\int_{x_1=x_L}^{x_1=x_R}e^{i\int_{t_{02}}^{t_{33}}\frac{L}{\hbar}dt}dx_1dx_2 . \end{aligned} \tag{37}$$

We can readily generalize (37) to any number of time slices as

$$\boxed{U\left(i,f;T=t_f-t_i\right)\approx C\int_{x_n=x_L}^{x_n=x_R}....\int_{x_2=x_L}^{x_2=x_R}\int_{x_1=x_L}^{x_1=x_R}e^{i\int_{t_i}^{t_f}\frac{L}{\hbar}dt}dx_1dx_2...dx_n}, \tag{38}$$

Approximation for Transistion Amplitude

where, as before, it is implicit that L in the integral is for the particular path that crosses the respective t slices at x_1, x_2,… x_n.

8.4 From Approximate to Exact

To get a precise, not approximate, relation for the RHS of (38) we would have to do two things.

1. Take the x range from l to infinity, i.e., $x_L \rightarrow -\infty$ and $x_R \rightarrow \infty$, and
2. Take $\Delta t \rightarrow dt$ for the same T (time between events.)

Doing this, (38) would become

$$\boxed{U\left(i,f;T=t_f-t_i\right)=C\underbrace{\int_{x=x_i}^{x=x_f}e^{i\int_{t_i}^{t_f}\frac{L}{\hbar}dt}\mathcal{D}x}_{\substack{\text{integ limits along with}\\ \mathcal{D}\text{ symbol imply } all\\ \text{paths between i and f}}}} \tag{39}$$

Exact Expression for Transition Amplitude

The symbol $\mathcal{D}$, as noted earlier, represents integration over all paths. With this, the integration limits designate the initial and final x values and do not imply a constraint on the x dimension during the integration (as was the case with (38).) In (39) we have, at long last, obtained the relation of integration type #4 in Table 1, where

$$F = e^{i\int_{t_i}^{t_f}\frac{L}{\hbar}dt} . \tag{40}$$

8.5 Practicality and Calculations

Practically, for the first approximation addressed in Section 8.4, we really don't have to take l to infinity, as we know that paths outside of a reasonably large range from the initial and final spatial locations will sum to very close to zero. So we can live with significant, but not infinite, l.

For the second approximation, we only need small enough Δt such that taking a smaller value does not change our answer much.

If we use (38), with judicious choices for Δt and l, we can, in many cases, obtain valid closed form solutions for the amplitude. We can also obtain numerical solutions with a digital computer by using approximations for L between time slices, as we did previously. That is, we can approximate the RHS of (38) in the manner we did for the first line of (37), but extending the approximation of (37) from 2 to n time slices.

9 An Example: Free Particle

We will first determine the amplitude (and thus the detection probability density) of a free particle via the Schroedinger approach and then compare it to that for Feynman's many paths approach.

9.1 Schroedinger Transition Amplitude

Recall, from Section 3.2, that, in the Schroedinger approach, a position eigenstate is a delta function, and as it evolves, the wave function envelope spreads and the peak diminishes. $|U|^2$ for such functions is the probability density at the final point x_f, after time T, where the peak is located. We should then expect $|U|^2$ to decrease as T increases, and to equal infinity when $T = 0$.

We start with the Schroedinger transition amplitude relation (9),

$$U(x_i, x_f; T) = \left\langle x_f \left| e^{-iHT/\hbar} \right| x_i \right\rangle . \tag{41}$$

Since the bra and ket here are Dirac delta functions, with the well known relation

$$\delta(x - x_i) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-x_i)}dk = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{i\frac{p}{\hbar}(x-x_i)}dp , \tag{42}$$

we can re-write (41) as

$$U(x_i, x_f; T) = \int_{-\infty}^{\infty}\left(\delta(x - x_f)e^{-iHT/\hbar}\delta(x - x_i)\right)dx . \tag{43}$$

(For readers unfamiliar with operators in exponents, one can express the exponential quantity as a Taylor series expanded about T, i.e., $f(T) = e^{-iTH/\hbar} = 1 - iTH/\hbar + ½\, T^2H^2/\hbar^2 + \ldots$ Then, operate on the ket/state term by term [getting terms in $iET/\hbar$ to various powers], and finally re-express the resulting Taylor series as an exponential in $iET/\hbar$. We have taken the ket with time $t_i = 0$ to make things simpler, but even if you like to think of the Hamilton operator as a time derivative, when it acts on that ket, it functions as an energy operator and still yields the energy.)

For the exponential with the H operator acting on the initial state, and $E = p^2/2m$, (43) is

$$\begin{aligned} U(x_i,x_f;T) &= \int_{-\infty}^{\infty}\left(\left(\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{i\frac{p'}{\hbar}(x-x_f)}dp'\right)\left(\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}TH} e^{i\frac{p}{\hbar}(x-x_i)}dp\right)\right)dx \\ &= \int_{-\infty}^{\infty}\left(\left(\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{i\frac{p'}{\hbar}(x_f-x)}dp'\right)\left(\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-iTp^2/2m\hbar} e^{i\frac{p}{\hbar}(x-x_i)}dp\right)\right)dx. \end{aligned} \tag{44}$$

We then re-arrange (44) to get

$$\begin{aligned} U(x_i,x_f;T) &= \frac{1}{2\pi\hbar}\iint e^{-iTp^2/2m\hbar}\underbrace{\left(\frac{1}{2\pi\hbar}\int e^{i\frac{x}{\hbar}(p-p')}dx\right)}_{\delta(p-p')} e^{\frac{i}{\hbar}p'x_f} e^{-\frac{i}{\hbar}px_i}dp'dp \\ &= \frac{1}{2\pi\hbar}\int e^{-iTp^2/2m\hbar} e^{\frac{i}{\hbar}p(x_f-x_i)}dp. \end{aligned} \tag{45}$$

Using the integral formula

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{b^2/4a} \qquad \operatorname{Re}(a) \geq 0,\ |a| \neq 0\ , \tag{46}$$

we find

$$U(x_i,x_f;T) = \sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}\frac{m}{2T}(x_f-x_i)^2} \ . \tag{47}$$

The astute reader may question whether (46), with complex a and b, converges. It does because the integrand oscillation rate increases with larger $|p|$ in such a way as to make successive cycles shorter. As $|p|$ gets very large, the cycles become so short that the contribution from each cycle (think area under a sine curve) tends to zero, and it does so in a manner that allows the integral to converge. Said another way, the smaller and smaller contributions as $|p|$ gets large alternate between positive and negative values (for both real and complex portions), and thus convergence is assured.

From (47), the probability density at event **f** is

$$\left|U(x_i,x_f;T)\right|^2 = \frac{m}{2\pi\hbar T}, \tag{48}$$

which, as we said it must, decreases with increasing T, and equals infinity for $T = 0$. Note also, that increasing m increases the envelope height, and thus decreases the width (for constant area under the envelope = constant probability.) In other words, the wave packet approaches more classical behavior, i.e., a narrower, more well defined location. Further, if $\hbar$ were to go to zero, the peak would be infinite, i.e., we would have a delta function and an exact particle location, as in classical mechanics.

9.2 Many Paths Transition Amplitude

We now seek to derive (47) using the many paths approach.

A free, non-relativistic particle has Lagrangian

$$L = \tfrac{1}{2} m v^2 \approx \tfrac{1}{2} m \left(\frac{x(t+\Delta t) - x(t)}{\Delta t} \right)^2 , \tag{49}$$

where the RHS is an approximation between adjacent time slices. Taking $t_i = 0$, and $l \to \infty$ (see Fig. 6, pg 12), (38) becomes

$$\begin{aligned} U(i,f;T) &\approx C \int_{x_n=-\infty}^{x_n=\infty} \cdots \int_{x_2=-\infty}^{x_2=\infty} \int_{x_1=-\infty}^{x_1=\infty} e^{i\int_{t_n}^{t_f} \frac{L}{\hbar} dt} e^{i\int_{t_{n-1}}^{t_n} \frac{L}{\hbar} dt} \ldots e^{i\int_{t_1}^{t_2} \frac{L}{\hbar} dt} e^{i\int_{0}^{t_1} \frac{L}{\hbar} dt} dx_1 dx_2 \ldots dx_n \\ &\approx C \int_{x_n=-\infty}^{x_n=\infty} \cdots \int_{x_2=-\infty}^{x_2=\infty} \int_{x_1=-\infty}^{x_1=\infty} e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_f - x_n}{\Delta t}\right)^2\right\}\Delta t} \ldots e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_2 - x_1}{\Delta t}\right)^2\right\}\Delta t} e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_1 - x_i}{\Delta t}\right)^2\right\}\Delta t} dx_1 dx_2 \ldots dx_n . \\ &= C \int_{x_n=-\infty}^{x_n=\infty} \underbrace{e^{\frac{im}{2\hbar(\Delta t)}(x_f - x_n)^2}}_{f_\zeta} \ldots \int_{x_2=-\infty}^{x_2=\infty} \underbrace{e^{\frac{im}{2\hbar(\Delta t)}(x_3 - x_2)^2}}_{f_\gamma} \underbrace{\int_{x_1=-\infty}^{x_1=\infty} \underbrace{e^{\frac{im}{2\hbar(\Delta t)}(x_2 - x_1)^2}}_{f_\beta} \underbrace{e^{\frac{im}{2\hbar(\Delta t)}(x_1 - x_i)^2}}_{f_\alpha} dx_1}_{f(x_2)} dx_2 \ldots dx_n , \end{aligned} \tag{50}$$

where the underbracket notation will help us in subsequent sections.

9.2.1 Background Math

Look, for the moment, at the last two factors (functions f_α and f_β) in the integral. They must be integrated over x_1, and that result is a function of x_2. When one of the two functions in such a procedure is a function of $x_2 - x_1$, as it is here, the integral is called a *convolution integral*. (See http://www-structmed.cimr.cam.ac.uk/Course/Convolution/convolution.html.)

In mathematical circles (search "Borel's Theorem"), it is well known that the Fourier (and also, the Laplace) transform of such an integral equals the product of the Fourier (or Laplace) transforms of the two functions. That is, for $\mathcal{F}$ representing Fourier transform,

$$\mathcal{F}\left\{ \int f_\beta (x_2 - x_1) f_\alpha (x_1) dx_1 \right\} = \mathcal{F}\left\{ f_\beta \right\} \mathcal{F}\left\{ f_\alpha \right\} . \tag{51}$$

Note that although f_α is a function of x_1-x_i , we can write $f_\alpha(x_1)$ because x_i is fixed.

Each factor in the last row of (50), as one moves leftward, plays the part of f_β in the theorem above for the next convolution integral, where the prior convolution integral plays the role of f_α. We get, in essence, a series of nested convolution integrals. Using (51), you should be able to prove to yourself that the transform of (50) equals the product of the transforms of the exponential factors in (50). If you can't, or don't want to bother, proving it, then just accept that a corollary to (51) is

$$\mathcal{F}\left\{\int\dots\int\int f_\zeta(x_f-x_n)\dots f_\gamma(x_3-x_2)f_\beta(x_2-x_1)f_a(x_1)dx_1dx_2\dots dx_n\right\}$$
$$=\mathcal{F}\left\{f_\zeta\right\}\dots\mathcal{F}\left\{f_\gamma\right\}\mathcal{F}\left\{f_\beta\right\}\mathcal{F}\left\{f_a\right\}. \tag{52}$$

9.2.2 Evaluating the Amplitude

So, to evaluate (50), using (52), we i) transform each exponential factor f_μ , ii) multiply those transforms together, and iii) take the inverse transform of the result to get U (actually U/C of (50)). This is made simpler, because each f_μ has the same form, so each transform is the same, i.e.,

$$\mathcal{F}\left\{f_\alpha\right\}=\mathcal{F}\left\{f_\beta\right\}=\dots\dots=\mathcal{F}\left\{f_\zeta\right\} \ . \tag{53}$$

The Fourier transform of a function f_α is

$$\mathcal{F}\left\{f_\alpha(x_1)\right\}=\tilde{f}_\alpha(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}f_\alpha(x_1)e^{-\frac{i}{\hbar}px_1}dx_1 \,. \tag{54}$$

For the f_α of (50), and for convenience, taking the coordinate $x_i = 0$, this is

$$\tilde{f}_\alpha(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{\frac{i}{\hbar}\frac{m}{2(\Delta t)}x_1^2}e^{-\frac{i}{\hbar}px_1}dx_1 \,, \tag{55}$$

where here and throughout this section, p is merely a dummy variable allowing us to carry out the math. Using (46), we find (55) becomes

$$\tilde{f}_\alpha(p)=\sqrt{\frac{i(\Delta t)}{m}}e^{-\frac{i}{\hbar}\frac{\Delta t}{2m}p^2} \,, \tag{56}$$

and thus, from (50), (52), and (53),

$$\tilde{U}(p)\approx C\,\tilde{f}_\zeta(p)\dots\dots\tilde{f}_\beta(p)\tilde{f}_\alpha(p)=C\left(\frac{i(\Delta t)}{m}\right)^{N/2}e^{-\frac{i}{\hbar}\frac{T}{2m}p^2} \,. \tag{57}$$

The inverse Fourier transform of (57), is

$$U(x_i,x_f;T)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\tilde{U}(p)e^{\frac{i}{\hbar}p(x_f-x_i)}dp$$
$$\approx\frac{1}{\sqrt{2\pi\hbar}}\left(\frac{i(\Delta t)}{m}\right)^{N/2}C\int_{-\infty}^{\infty}e^{-\frac{i}{\hbar}\frac{T}{2m}p^2}e^{\frac{i}{\hbar}p(x_f-x_i)}dp. \tag{58}$$

In (58), we could have simply used x_f in the exponent, as we have been taking $x_i = 0$, and our result would have been in terms of x_f. In that case, x_f would have been the distance between x_i and x_f , i.e., $x_f - x_i$. In order to frame our final result in the most general terms, we re-introduced x_i as having any coordinate value in (58).

With (46) again, (58) becomes

$$U(x_i,x_f;T)\approx C\left(\frac{i(\Delta t)}{m}\right)^{N/2}\sqrt{\frac{m}{iT}}\,e^{\frac{i}{\hbar}\frac{m}{2T}(x_f-x_i)^2} \,. \tag{59}$$

By comparison with (47), we see the phase and dependence on T is the same as in the wave mechanics approach. Using that comparison, we can see that the constant of proportionality is

$$C = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m}{i(\Delta t)}\right)^{N/2} . \tag{60}$$

And thus, the probability density at the final event **f** is the same as (48), i.e.,

$$\left|U(x_i, x_f; T)\right|^2 = \frac{m}{2\pi\hbar T} , \tag{61}$$

where the equal sign is appropriate for $N \to \infty$.

Note that for $v = (x_f - x_i)/T$, the amplitude can be expressed in terms of the classical action as

$$U(x_i, x_f; T) = \sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}\frac{mv^2}{2}T} = \sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}LT} = \sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}S} , \tag{62}$$

which agrees with (22) if $A(t)$ there equals the root quantity. In the Appendix, we show it does.

9.3 The Message

It has probably not escaped the reader that the evaluation of a free particle using Feynman's many paths approach is considerably more complicated and lengthy than the Schroedinger approach. This is true for most, if not all, problems in QM.

The disadvantages of the many paths approach in QM are these.

1. It is generally more mathematically cumbersome and time consuming than the wave mechanics approach.
2. The quantity calculated is only proportional to the amplitude, and further analysis is required to determine the precise amplitude.
3. The approach is suitable primarily for position eigenstates and is not readily amenable to more general states, so it is generally not as all encompassing in nature.

The advantages of the many paths approach are these.

1. The approach also applies to quantum field theory (QFT). In a number of instances therein, development of the theory is more direct, and calculation of amplitudes is easier, than with the alternative approach (canonical quantization).
2. Philosophically, we see that there is more than one way to skin a cat. We learn anew that the physical world can be modeled in different, equivalent ways. We learn caution with regard to interpreting a given model as an actual picture of reality.

10 Quantum Field Theory via Path Integrals

So far, we have dealt primarily with non-relativistic quantum mechanics (NRQM), but the many paths approach is also applicable to relativistic quantum mechanics (RQM), and as noted above, to quantum field theory (QFT). (RQM is often confused with QFT. For a comparison of the similarities and differences between the two, see www.quantumfieldtheory.info/Chap01.pdf. Further similarities and differences are illustrated in Unifying Chart 2, below.)

We will not go deeply into QFT, and only outline, in a broad overview, how the theory presented herein is applicable therein. This should help those students who continue on to the standard texts for the many paths approach keep the forest in view while examining the trees.

10.1 Particle Theory (QM) vs Field Theory (QFT)

For the many paths approach, we want to make the jump from QM, which is a quantized version of particle theory, to QFT, which is a quantized version of field theory. Unifying Chart 2 below can help us do that. In it, the 2^{nd} and 3^{rd} columns compare particle theory entities/concepts to corresponding field theory entities/concepts. The upper half of the chart, as indicated, summarizes classical theory (non quantum, and implicitly including special relativity). The lower half summarizes quantum theory *via approaches other than many paths*. The chart should be relatively self explanatory, so we will not comment much on it.

We compare the quantum approaches of Unifying Chart 2 to the many paths approach in the next section.

Unifying Chart 2. Comparing Particle Theory to Field Theory: Classical and Quantum

	Particle Theory		**Field Theory**
	Classical Theory		
Indep variables	1D t	3D t	3D x,y,z,t
Depend variables	$x(t)$ position	$x(t)$, $y(t)$, $z(t)$	$\phi(x,y,z,t)$ field
Dynamic variables (functionals)	Particle total value: $\mathbf{p}$, E, L functions of $x, \dot{x}, t$ (or $\mathbf{r}, \dot{\mathbf{r}}, t$)		Density values (per unit vol): $\mathcal{p}, \mathcal{E}, \mathcal{L}$ functions of $\phi, \dot{\phi}, x, y, z, t$ $E = \oint \mathcal{E} d^3x$, etc.
Equations of motion	$\mathbf{F} = m\mathbf{a}$ or equivalently, Euler-Lagrange formulation, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$		$\mathcal{f} = \rho\mathbf{a}$ (force/vol) for media; Maxwell's eqs for e/m, or equivalently, for $\mathcal{L}$ of either, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$
Variable correspondences particle ↔ field	$t \leftrightarrow x,y,z,t$ $x \leftrightarrow \phi$ total values ↔ density values		

	Particle Theory	**Field Theory**
	Quantum Theories	
	QM and RQM via Wave Mechanics	QFT via Wave Mechanics = Canonical Quantization
Quantum character change	x and all dynamic variables → operators	ϕ and all dynamic variables → operators
New quantum entity	state $\lvert\psi\rangle$ = wave function ψ	state $\lvert\phi\rangle$ different from (operator) field ϕ
Note		Fields create & destroy states. States can be multiparticle ($\lvert\phi_1,\phi_{2,}...\rangle$)
Operators	functions of $x,\dot{x},t$	functions of $\phi,\dot{\phi},t$
Expectation values of operators	$\bar{E}=\langle\psi\lvert H\rvert\psi\rangle$ etc. for other opers	$\bar{E}=\langle\phi\lvert H\rvert\phi\rangle$ or for multiparticle state $\bar{E}=\langle\phi_1,\phi_2...\lvert H\rvert\phi_1,\phi_2...\rangle$
Equations of motion	For wave function ψ QM: Schroedinger eq RQM: Klein-Gordon, Dirac, Proça eqs or equivalently, Euler-Lagrange formulations	For quantum field ϕ QFT: Klein-Gordon, Dirac, Proça eqs or equivalently, Euler-Lagrange formulations
Macro equations of motion	Deduced from above and expectation values of force, acceleration	Deduced from above and expectation values of relevant quantities
Transition amplitude U	$U(x_i,x_f;T)=\langle x_f\lvert e^{-iHT}\rvert x_i\rangle$ i & f are eigen states of position	$U(\phi_i,\phi_f;T)=\langle\phi_f\lvert e^{-iHT}\rvert\phi_i\rangle$ i & f states can be multiparticle
$\lvert U\rvert^2$ =	probability density	probability

10.2 "Derivation" of Many Paths Approach for QFT

From the next to last row of Unifying Chart 2, we see that the transition amplitude for the QFT canonical approach, which is essentially a wave mechanics approach for relativistic fields, is similar in form to that of the QM wave mechanics approach, given that we note the correspondence $x\rightarrow\phi$ between QM and QFT. An additional fundamental difference between the two is the form of the Hamiltonian H. In QM, H is a non-relativistic function of x, dx/dt, and (rarely) t. In QFT, it is a relativistic function of ϕ, $d\phi/dt$, and (rarely) t.

Since the canonical (wave mechanics) QFT approach mirrors the wave mechanics QM approach, one could postulate (and Feynman probably did) that the many paths approach in QFT would mirror the many paths approach in QM. (See Unifying Chart 1 in Section 7.2 for the corresponding QM transition amplitudes using each approach.) Simply using the same correspondences $x \rightarrow \phi$ and $H_{\text{nonrel}} \rightarrow H_{\text{rel}}$ (and thus, $L_{\text{nonrel}} \rightarrow L_{\text{rel}}$) for the many paths approach yields Unifying Chart 3.

Unifying Chart 3. Comparing QM to QFT for the Many Paths Approach

	Quantum Theories	
	QM and RQM via Many Paths	QFT via Many Paths
Transition amplitude	$U(x_i, x_f; T) \propto \lim_{N\to\infty} \sum_{j=1}^{N} e^{iS_j/\hbar}$ $= \int_{x_i}^{x_f} e^{i\int_0^T \frac{L}{\hbar} dt} \mathcal{D}x(t)$	$U(\phi_i, \phi_f; T) \propto \lim_{N\to\infty} \sum_{j=1}^{N} e^{iS_j/\hbar}$ $= \int_{\phi_i}^{\phi_f} e^{i\int_0^T \frac{\mathcal{L}}{\hbar} d^4x} \mathcal{D}\phi(x^\mu)$
Note	Above is from Unifying Chart 1 in Section 7.2	Above is a simplified example for a single scalar field.

In the RH column above, all paths, comprising all configurations of the entire field ϕ over all space between its initial and final configurations, are added (integrated). S here is the action for the entire field. $\mathcal{L}$ is the (relativistic) Lagrangian *density* for the field, which, integrated as it is above over all space d^3x, yields L.

Of course, the many paths transition amplitude of Unifying Chart 3 is, at this point, only a guess. However, decades of research, first by Feynman and then by many others, have proven that it is completely valid.

To summarize, briefly

Unifying Chart 4. Super Simple Summary

Correspondences	$x \rightarrow \phi$ $H_{\text{nonrel}} \rightarrow H_{\text{rel}}$	
Wave mechanics amplitude	QM → QFT	<u>canonical quantization</u> QFT
Many paths amplitude	QM → QFT	<u>functional quantization</u> QFT

10.3 Time Slicing in QFT

Using the same correspondences as in Unifying Chart 4, and the time slicing approximation for QM of (38), we find, for QFT,

$$U\left(i,f;T=t_f-t_i\right)\approx C\int \ ....\int \int e^{i\int_{t_i}^{t_f}\frac{\mathcal{L}}{\hbar}d^4x}d\phi_1 d\phi_2 ... d\phi_n \quad \text{QFT Approximation for Transition Amplitude} \quad , \tag{63}$$

where the subscripts refer to different time slices, not to different fields. This example is for only a single field.

The exact form of the transition amplitude, obtained from (39), is given in Unifying Chart 3, and is repeated here,

$$U(i,f;T)=C\int_{\phi_i}^{\phi_f} e^{i\int_0^T \frac{\mathcal{L}}{\hbar}d^4x}\mathcal{D}\phi(x^\mu) \quad \text{QFT Exact Expression for Transition Amplitude} \quad . \tag{64}$$

10.4 More Ahead in Path Integral QFT

Of course, we have only scratched the surface of the many paths approach to QFT. There is a great deal more, including some fairly fundamental concepts. However, hopefully, all of the above will provide a solid foundation for that, by explaining more simply, more completely, and in smaller steps of development what traditional introductions to the subject often treat rapidly and in somewhat less than transparent fashion.

If you find errors (typographical or otherwise), or have suggestions on how to make anything herein easier to understand, please help those who come after you by letting me know, so I can make appropriate corrections/modifications. I can be reached via the email address in the home page, Pedagogic Aids to Quantum Field Theory[6] (www.quantumfieldtheory.info), for which this material is a sub-section. Thank you.

Appendix

From (20), with $x=x_{\text{peak}}$,

$$\psi(x_{peak},t)=e^{-\frac{i}{\hbar}\left(\bar{E}t-\bar{p}x\right)}\underbrace{\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}g(p)e^{-\frac{i}{2\hbar}\frac{t}{m}(p-\bar{p})^2}dp}_{A(t)}\ . \tag{65}$$

We note that (20), and thus (65), are derived from the general wave packet relation (see ref 2)

$$\psi(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} g(p)e^{-\frac{i}{\hbar}(Et-px)}dp\,. \tag{66}$$

At our initial event, take $t=t_i = 0$, so the above becomes

$$\psi_i(x,t=0) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} g(p)e^{\frac{i}{\hbar}px}dp\,. \tag{67}$$

If (67) is a delta function centered at $x_i = 0$, then, from the definition of the delta function,

$$\psi_i(x,t=0) = \delta(x) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}px}dp\,. \tag{68}$$

Comparing (68) to (67), we see that for an initial delta function measured at x_i

$$g(p) = 1\,. \tag{69}$$

Using (69) in (65), we obtain

$$A(t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-\frac{i}{2\hbar}\frac{t}{m}(p-\bar{p})^2}dp\,\,. \tag{70}$$

With the integral formula

$$\int_{-\infty}^{+\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}\,, \tag{71}$$

we find

$$A(t) = \sqrt{\frac{m}{i2\pi\hbar t}}\,\,. \tag{72}$$

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[1] http://www.quantumfieldtheory.info/Chap01/pdf

[2] Merzbacher, E., *Quantum Mechanics*. 2nd ed. John Wiley & Sons (1970). See Chap 2, Sec 3.

[3] Much of the material in this section parallels "Action on Stage: Historical Introduction", Ogborn, J., Hanc, J. and Taylor, E.F., and "A First Introduction to Quantum Behavior", Ogborn, J., both from The Girep Conference 2006, Modeling Physics and Physics Eduction, Universiteit van Amsterdam.

[4] Feynman, R., *QED: The strange theory of light and matter*. Penguin Books, London (1985).

[5] http://www.quantumfieldtheory.info/manypathsgraphicelectronreflection.xls

[6] http://www.quantumfieldtheory.info