Chapter 8
Path Integrals

One feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of calculus.

—R. Feynman

8.1 Postulates of Quantum Mechanics

Previously, we outlined how to quantize field theories with various spins using the canonical quantization approach. However, for increasingly complex systems, such as gauge theory, quantum gravity, and superstring theory, canonical quantization proves to be a very clumsy formalism since manifest Lorentz invariance is broken. Instead, we will explore a new method in this chapter.

Perhaps the most powerful quantization method is the *path integral* approach, which was developed by Feynman,[1,2] based on an idea of Dirac.[3] The path integral method is versatile enough to handle a variety of different types of gauge theories. The path integral approach has many advantages over the other techniques:

1. The path integral formalism yields a simple, covariant quantization of complicated systems with constraints, such as gauge theories. While calculations with the canonical approach are often prohibitively tedious, the path integral approach yields the results rather simply, vastly reducing the amount of work.
2. The path integral formalism allows one to go easily back and forth between the other formalisms, such as the canonical or the various covariant approaches. In the path integral approach, these various formalisms are nothing but different choices of gauge.
3. The path integral formalism is based intuitively on the fundamental principles of quantum mechanics. Quantization prescriptions, which may seem rather arbitrary in the operator formalism, have a simple physical interpretation in the path integral formalism.

4. The path integral formalism can be used to calculate nonperturbative as well as perturbative results.
5. The path integral formalism is based on c-number fields, rather than q-number operators. Hence, the formalism is much easier to manipulate.
6. At present, there are a few complex systems with constraints that can only be quantized in the path integral formalism.
7. Renormalization theory is much easier to express in terms of path integrals.

Our discussion of the path integral formalism begins with two deceptively simple principles:

8.1.1 *Postulate I*

The probability $P(b, a)$ of a particle moving from point a to point b is the square of the absolute value of a complex number, the transition function $K(b, a)$:

$$P(b, a) = |K(b, a)|^2 \tag{8.1}$$

8.1.2 *Postulate II*

The transition function $K(b, a)$ is given by the sum of a phase factor $e^{iS/\hbar}$, where S is the action, taken over all possible paths from a to b:

$$K(b, a) = \sum_{\text{paths}} k e^{iS/\hbar} \tag{8.2}$$

where the constant k can be determined by:

$$K(c, a) = \sum_{\text{paths}} K(c, b)K(b, a) \tag{8.3}$$

where we sum over all intermediate points b connecting a and c.

These postulates incorporate the essence of the celebrated double slit experiment, where a beam of electrons passes through a barrier with two small holes. A screen is placed behind the barrier to detect the presence of the electrons. As a point particle, an electron cannot, of course, go through both holes simultaneously. Classically, therefore, we expect that the electrons will go through one slit or the other, leaving two distinct marks on the screen just behind the two holes.

However, experiments show that the pattern created on the screen by repeated passages of the electrons through these holes is an interference pattern, associated with wave-like, not particle-like, behavior. Classically, we are therefore left with

a paradox. A point particle cannot go through both holes at once, yet the passage of a large number of electrons successively going past the barrier clearly leaves an interference pattern, with minima and maxima, as if the electron somehow went through both holes.

In the path integral approach, as in quantum mechanics, this puzzle can be resolved. The postulates of the path integral approach and quantum mechanics do not allow us to calculate the precise motion of a single point particle. They only allow us to calculate probability amplitudes. The probability that an electron will go from the source past the slits to the screen is given by summing over *all* possible paths. These probabilities, in turn, may have wave-like behavior, even if the electron itself is a point particle.

The sum over paths reproduces the interference pattern that is experimentally seen on the screen. Thus, the path integral approach incorporates the philosophy behind the double-slit experiment, which, in turn, embodies the essence of the quantum principle.

As in quantum mechanics, we make the transition to classical mechanics by taking the limit $\hbar \to 0$. For large values of S, the exponential of $iS/\hbar$ undergoes large fluctuations, and hence cancels out to zero. Hence, the contribution of the paths that maximize the action S do not contribute much to the sum over paths:

$$\delta S \gg \hbar : \quad \sum_{\text{paths}} e^{iS/\hbar} \sim 0 \tag{8.4}$$

In the classical limit, the paths that dominate the sum are the ones where $\delta S/\hbar$ is as small as possible. However, the path for which δS is minimized is just the classical path:

$$\delta S = 0 \quad \to \quad \text{classical mechanics} \tag{8.5}$$

Thus, we recover classical mechanics in the limit as $\hbar \to 0$. The picture that emerges from the path integral approach is therefore intuitively identical to the principles of quantum mechanics. To calculate the probability that a particle at point a goes to a point b, one must sum over all possible paths connecting these two points, including the classical one. The path preferred by classical mechanics is the one that minimizes the action for $\delta S \ll \hbar$ (Fig. 8.1).

Although the path integral method gives us an elegant formalism in which to reformulate all of quantum field theory, one should also point out the potential drawbacks of the formalism. One problem is that the path integral is not well defined in Minkowski space. In this chapter, we will assume that all path integrals are computed with the Euclidean metric. Then, the functional integral is taken over e^{-S}, which has much better convergence properties than integrals over e^{iS}. At the end of the calculation, we assume that we can analytically continue back

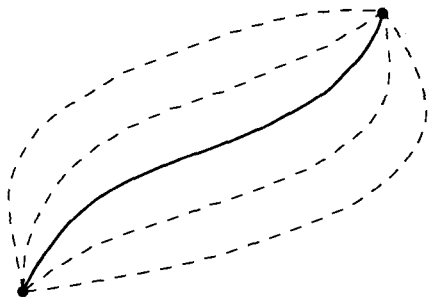

Figure 8.1. The path integral sums over all possible paths connecting two points, including the one favored by classical mechanics. In this way, the path integral sums over quantum corrections to classical mechanics.

to Minkowski space. (The question of whether this analytic continuation from Minkowski space to Euclidean space and back again is rigorously defined is a highly nontrivial question. This is a delicate matter, the subject of a field called axiomatic field theory, which is beyond the scope of this book.)

Another problem is that the transition between c numbers and operators becomes illdefined when the Hamiltonian has ordering problems. The path integral over a system with the Hamiltonian of the form $p^2 f(q)$, for example, becomes ambiguous when making the transition to the operator language, since p and q do not commute. For systems more complex than the harmonic oscillator, the integrals may not be Gaussian, and ordering problems may creep into the path integral. For complicated systems, one must often use "point splitting" methods, that is, separating two fields by a small infinitesimal amount in space-time in order to regularize the integrals. Unfortunately, a detailed elaboration of these delicate points is also beyond the scope of this book.

With these problems in mind, let us now compute with the path integral approach. We first divide up a path by discretizing space–time. Let us divide up each path in three-space into N points (Fig. 8.2). Then the "sum over all paths" can be transformed into a functional integral:

$$\sum_{\text{paths}} = \lim_{N\to\infty} \prod_{i=1}^{3} \prod_{n=1}^{N} \delta x_n^i \to \int Dx \tag{8.6}$$

The integral $\int Dx$ is not an ordinary integral. It is actually an infinite product of integrals, taken over all possible $dx(t)$. Whenever we use the differential symbol D, we should remember that it is actually an infinite product of differentials taken

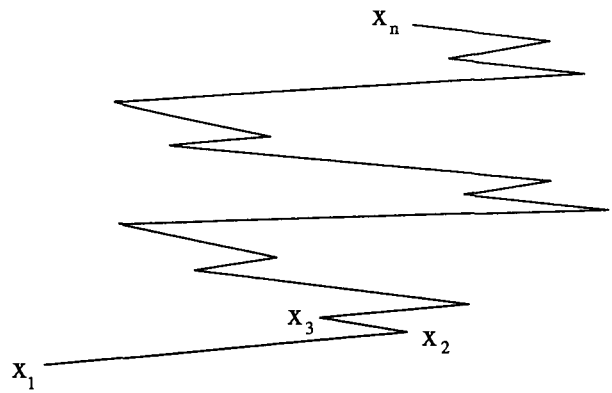

Figure 8.2. To calculate with path integrals, we break the path into a discrete number of intermediate points, and then integrate over the position of these intermediate points.

over all points. In this functional language, the transition function becomes:

$$K(b,a) = k \int_a^b Dx\, e^{iS/\hbar} \tag{8.7}$$

where k can be determined as follows:

$$K(c,a) = \int K(c,b)K(b,a)\, Dx_b \tag{8.8}$$

where we integrate over all possible intermediate points x_b which link points a and c.

To give this approach some substance, let us begin with the simplest of all possible classical systems, the free nonrelativistic point particle in the first quantized formalism. Our discussion begins with the classical action:

$$S = \int dt\, \frac{1}{2} m \dot{x}_i^2 \tag{8.9}$$

Let us now discretize the paths. We take dt to be a small interval ϵ and then discretize the Lagrangian:

$$\begin{aligned} dt &\rightarrow \epsilon \\ \frac{1}{2} m \dot{x}_i^2 dt &\rightarrow \frac{1}{2} m (x_n - x_{n+1})_i^2 \epsilon^{-1} \end{aligned} \tag{8.10}$$

The transition function $K(b,a)$ can be written as the path integral over e^{iS}:

$$\begin{aligned} K(b,a) &= \lim_{\epsilon\to 0}\int\int\cdots\int dx_2\,dx_3\cdots dx_{N-1} \\ &\times\ k\exp\left(\frac{im}{2\epsilon}\sum_{n=1}^{N-1}(x_n - x_{n+1})^2\right) \end{aligned} \tag{8.11}$$

Unfortunately, one of the drawbacks of the path integral formalism is that embarrassingly few functional integrals can actually be performed. However, we will find that the simplest Gaussian path integral is also the one most frequently found for free systems. Specifically, we will repeatedly use the Gaussian integral:

$$\int_{-\infty}^{\infty} dx\, x^{2n} e^{-r^2x^2} = \frac{\Gamma(n+\frac{1}{2})}{r^{2n+1}} \tag{8.12}$$

To evaluate the expression for $K(b,a)$, we now perform one of these Gaussian integrations:

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_2\,\exp\left[-a(x_1-x_2)^2 - a(x_2-x_3)^2\right] \\ &\quad= \sqrt{\frac{\pi}{2a}}\exp\left[-\frac{1}{2}a(x_1-x_3)^2\right] \end{aligned} \tag{8.13}$$

The key point is that the Gaussian integral over x_2 has left us with another Gaussian integral over the remaining variables. This process can be repeated an arbitrarily large number of times: Each time we perform a Gaussian integral on an intermediate point, we find a Gaussian integral among the remaining variables.

After repeated integrations, we find:

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_2 \quad \cdots dx_{N-1}\,\exp\left[-a(x_1-x_2)^2 - \cdots - a(x_{N-1}-x_N)^2\right] \\ &\quad= \sqrt{\frac{\pi^{N-2}}{(N-1)a^{N-2}}}\exp\left[-\frac{a}{N-1}(x_1-x_N)^2\right] \end{aligned} \tag{8.14}$$

This, in turn, allows us to calculate the constant k:

$$k = \left(\frac{2\pi i\epsilon}{m}\right)^{-(1/2)N} \tag{8.15}$$

If we take the limit as the number of intermediate points goes to infinity, then we are left with the final result for the transition function:

$$K(b,a) = \left|\frac{m}{2\pi(t_b - t_a)}\right|^{1/2} \exp\frac{(1/2)im(x_b - x_a)^2}{t_b - t_a} \tag{8.16}$$

This is a pleasant result. This is exactly the Green's function we derived in Eq. (3.58) for nonrelativistic quantum mechanics. Beginning with only the postulates of the path integral approach and the simplest possible classical action, we have derived the Green's function found in quantum mechanics that propagates Schrödinger waves. It obeys the equation:

$$-\frac{1}{2m}\frac{\partial^2}{\partial x_b^2}K(b,a) = i\frac{\partial}{\partial t_b}K(b,a) \tag{8.17}$$

Our first exercise in the path integral formalism gave us encouraging results. Now let us tackle more general and more difficult problems, such as (1) the transition between the Lagrangian and the Hamiltonian approaches and (2) the transition from c-number expressions to operator q-number expressions. In the usual canonical approach, these two transitions appear rather ad hoc and counterintuitive.

In the path integral approach, the transition between the Lagrangian and Hamiltonian systems is easily performed by adding an infinite sequence of Gaussian integrations for the momentum p_i. For each infinitesimal integration, we use the fact that:

$$\int_{-\infty}^{\infty} dp\, e^{iap^2+ibp} = \sqrt{\frac{i\pi}{a}}e^{-ib^2/4a} \tag{8.18}$$

which can be proved by completing the square. If we let $a = -1/2m$ and $b = \dot{x}$ and integrate over an infinite number of these momenta, then we have:

$$\begin{aligned} K(b,a) &= \int_{x_a}^{x_b} Dx \exp i\int_{t_a}^{t_b} dt\left[\frac{1}{2}m(\dot{x}_i)^2 - V(x)\right] \\ &= \int_{x_a}^{x_b} Dx\, Dp \exp i\int_{t_a}^{t_b} dt\left(p\dot{x}_i - \frac{p_i^2}{2m} - V(x)\right) \end{aligned} \tag{8.19}$$

The Lagrangian appears on the first line, but the Hamiltonian, defined by $H(p,x) = p^2/2m + V(x)$, appears on the second line. By performing the functional integral over Dp, we can go back and forth between the Lagrangian and Hamiltonian formalisms. In the path integral formalism, the relationship between the Lagrangian and the Hamiltonian formalism is no mystery, but simply the byproduct of performing an additional functional integration over momentum.

(For clarity, because normalization factors, such as $1/2\pi$, appear repeatedly throughout our discussion, we have absorbed them into the definition of Dx and Dy. We will henceforth drop these trivial normalization factors, since they can can always be explicitly written out later.) Thus, in the path integral formalism, the difference between the two formalisms only lies in a Gaussian integration over momentum. The path integral formalism allows us to go between these formalisms with ease:

$$L = \frac{1}{2}m(\dot{x}_i)^2 - V(x) \leftrightarrow H = \frac{p_i^2}{2m} + V(x) \tag{8.20}$$

So far, everything has been defined in terms of c-number expressions. Operators, which are the basis of the canonical approach, do not enter into the picture at all. Now, let us make the second transition, this time from the path integral formalism to the operator formalism, to show that the operator formalism that we have patiently developed in Chapters 3 and 4 is nothing but a specific representation of the path integral.

We recall that in the canonical formalism, the starting point was the canonical equal-time commutation relation between fields $\phi(x)$ and their conjugates $\pi(x)$. Only later could we calculate the propagators and finally the S matrix. In the path integral formalism, the sequence is roughly the reverse. We begin with the S matrix as the starting point, and we later derive the operator formalism as a consequence.

To see how operators naturally emerge in a formalism defined entirely without operators, let us write the transition function between point x_1 at time t_1 to point x_N at time t_N in the Heisenberg representation. We will carefully divide the path into N intermediate points. In this formalism, the transition probability of a particle at point x_1 and time t_1 going to x_N and time t_N is given by the matrix element between eigenstates $|x, t\rangle$

The Heisenberg representation, we recall, is based on a complete set of position eigenstates $|x\rangle$ of the position operator $\hat{x}$, which is now treated as an operator with eigenvalue x:

$$\hat{x}|x\rangle = x|x\rangle \tag{8.21}$$

We also introduce eigenstates of the momentum operator p:

$$\begin{aligned} 1 &= \int |x\rangle\, dx\, \langle x| \\ 1 &= \int |p\rangle\, dp\, \langle p| \end{aligned} \tag{8.22}$$

such that they are normalized as follows:

$$\begin{aligned} \langle x|y\rangle &= \delta(x-y) \\ \langle p|x\rangle &= \frac{e^{ipx}}{\sqrt{2\pi}}; \quad \langle x|p\rangle = \frac{e^{-ipx}}{\sqrt{2\pi}} \end{aligned} \tag{8.23}$$

To check the consistency of this normalization, we perform the following manipulations:

$$\begin{aligned} \langle x|y\rangle &= \langle x|p\rangle \int dp \langle p|y\rangle \\ &= \int dp \frac{e^{-ipx}}{\sqrt{2\pi}} \frac{e^{ipy}}{\sqrt{2\pi}} \\ &= \int \frac{dp}{2\pi} e^{-ip(x-y)} \\ &= \delta(x-y) \end{aligned} \tag{8.24}$$

Our normalizations are thus consistent.

Our task is now to rewrite the functional integration over p at an intermediate point along the path in terms of an operator expression defined in the Heisenberg picture. We will use the fact that the transition element between two neighboring points can be written as:

$$\langle x_2|e^{-iH(t_2-t_1)}|x_1\rangle = \langle x_2, t_2|x_1, t_1\rangle \tag{8.25}$$

Let us take a specific value of $\dot{x} \sim (x_1 - x_2)\delta t$ and dp that appears within the functional integral and carefully rewrite the integral over dp and its integrand as follows:

$$\begin{aligned} \int \frac{dp}{2\pi} e^{i(p\dot{x}-H(x,p))\delta t} &= \int \frac{dp}{2\pi} e^{-iH(x,p)\delta t} e^{ip(x_1-x_2)} \\ &= e^{-iH(x,\partial_x)\delta t} e^{-ix_2 p} \int \frac{dp}{\sqrt{2\pi}} e^{ipx_1} \\ &= e^{-iH(x,\partial_x)\delta t} \langle x_2|p\rangle \int dp \, \langle p|x_1\rangle \\ &= e^{-iH(x,\partial_x)\delta t} \langle x_2|x_1\rangle \\ &= \langle x_2|e^{-iH(x,\partial_x)\delta t}|x_1\rangle \\ &= \langle x_2, t_2|x_1, t_1\rangle \end{aligned} \tag{8.26}$$

We have now made the transition between a Lagrangian defined in terms of x and $\dot{x}$ and a Hamiltonian defined in terms of x and its derivative ∂_x. The transition was made possible because the derivative of the exponential brings down a p:

$$\begin{aligned} \partial_x e^{ipx} &= ip\, e^{ipx} \\ e^{-iH(x,\partial_x)\delta t}\, e^{ipx} &= e^{-iH(x,p)\delta t}\, e^{ipx} \end{aligned} \tag{8.27}$$

In the path integral formalism, this is the origin of the transition between c numbers and q-number operators; that is, the insertion of intermediate states defined in p space allows us to replace the p variable with a ∂_x operator. Thus, we have made the transition between:

$$\begin{aligned} H(x, p) &\leftrightarrow H(x, \partial_x) \\ p &\leftrightarrow -i\frac{\delta}{\delta x} \end{aligned} \tag{8.28}$$

In summary, we have now shown that the path integral formalism can express the propagator $K(b, a)$ in three different ways, in the Lagrangian or Hamiltonian formalism, or in the operator formalism in the Heisenberg picture. This can be summarized by the following identity:

$$\begin{aligned} K(N, 1) &= \langle x_N, t_N | x_1 t_1 \rangle \\ &= \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \int dx_{N-1} \langle x_{N-1}, t_{N-1} | \\ &\quad \ldots | x_2, t_2 \rangle \int dx_2 \langle x_2, t_2 | x_1, t_1 \rangle \\ &= \int Dx\, \exp\left(i \int_{t_1}^{t_N} dt\, L(x, \dot{x}) \right) \\ &= \int Dx\, Dp\, \exp\left(i \int_{t_1}^{t_N} dt\, (p\dot{x} - H(x, p)) \right) \end{aligned} \tag{8.29}$$

Finally, let us reanalyze, from the point of view of path integrals, how the time ordering operator T enters into the propagator. Let us analyze the matrix element of an initial state $|x_i, t_i\rangle$ with a final state $\langle x_n, t_n|$, with the operators $x_j(t_j)$ and $x_k(t_k)$ sandwiched between them. We will assume that $t_j > t_k$. As before, we will take time slices and insert a series of complete intermediate states between the states at each slice:

$$\langle x_n, t_n | x(t_j) x(t_k) | x_1, t_1 \rangle = \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \int dx(t_{n-1}) \langle x_{n-1}, t_{n-1} |$$

$$\cdots |x_{j+1}, t_{j+1}\rangle \int dx(t_{j+1}) \langle x_{j+1}, t_{j+1}|x_j, t_j\rangle x(t_j)$$

$$\cdots |x_{k+1}, t_{k+1}\rangle \int dx(t_{k+1}) \langle x_{k+1}, t_{k+1}|x_k, t_k\rangle x(t_k)$$

$$\cdots |x_2, t_2\rangle \int dx(t_2) \langle x_2, t_2|x_1, t_1\rangle \tag{8.30}$$

Taking the limit as the number of time slices goes to infinity, we have:

$$\begin{aligned} \langle x_n, t_n|x(t_j)x(t_k)|x_1, t_1\rangle &= \int Dx\,Dp\, x(t_j)x(t_k) \\ &\quad \times \exp\left(i \int_{t_1}^{t_n} (p\dot{x} - H(p,x))\, dt\right) \\ &= \int Dx\, x(t_j)x(t_k) \exp i \left(\int_{t_1}^{t_n} L(x,\dot{x}) dt\right) \end{aligned} \tag{8.31}$$

Now, let us reverse the order of the times, such that $t_j < t_k$. In this case, the previous formula must be modified because we can no longer take time slices. Thus, whenever $t_j < t_k$, we cannot make the transition from operators to path integrals unless we reverse the ordering of the operators. In order for this formalism to make sense, we will always reverse the order of the operators, such that the later times always appear to the left, so that we can proceed with taking time slices. To enforce this condition, we must use the time ordered product in this case:

$$\begin{aligned} &\langle x_n, t_n|T\left[x(t_j)x(t_k)\right]|x_1, t_1\rangle \\ &\quad = \int Dx\,Dp\, x(t_j)x(t_k) \exp\left(i \int_{t_1}^{t_n} dt\, (p\dot{x} - H(p,x))\right) \end{aligned} \tag{8.32}$$

For a large number of insertions, we have obviously:

$$\begin{aligned} &\langle x_n, t_n|T\left[x(t_j)x(t_k)\cdots x(t_m)\right]|x_1, t_1\rangle \\ &\quad = \int Dx\,Dp\, x(t_j)x(t_k)\cdots x(t_m) \exp\left(i \int_{t_1}^{t_n} [p\cdot x - H(p,x)]\right) \end{aligned} \tag{8.33}$$

We emphasize that the left-hand side consists of operators, so the ordering of the times is important. However, on the right-hand side we have a c-number expression, where the ordering of the $x(t_i)$ makes no difference. The correspondence between operators and these c-number expressions in the path integral only holds

when we can make time slices, that is, when the operators are time ordered. From the path integral point of view, this is the origin of the time ordering in the matrix elements.

8.2 Derivation of the Schrödinger Equation

In a first quantized formalism, where the action is a function of x^i and not fields, the path integral formalism gives us an added bonus: It gives us a derivation of the Schrödinger equation. Usually, introductory courses in quantum mechanics begin by postulating the Schrödinger wave equation. Certain conventions, such as the quantization of x and p, seem rather arbitrary. Only later emerges the probabilistic interpretation. Here, we reverse this order: we begin with the probabilistic postulates of quantum mechanics and derive the Schrödinger wave equation as a consequence, thus giving a new physical interpretation to that equation.

In the path integral approach, the evolution of a state is given by the transition function $K(b,a)$. From a classical point of view, this can be viewed as the analogue of Huygen's principle, where the evolution of a wave can be determined by assuming that each point along a wave front emits a new wave front. The integration over all these infinitesimal wave fronts then gives us the overall evolution of the wave front. Mathematically, this is given by:

$$\psi(x_j,t_j)=\int_{-\infty}^{\infty}K(x_j,t_j;x_i,t_i)\psi(x_i,t_i)\,dx_i \tag{8.34}$$

Earlier, we derived, assuming only the Lagrangian $(1/2)mv_i^2$, an expression for the nonrelativistic transition function. Now let us calculate how wave fronts move with this transition function. The time evolution, from t to $t+\delta t$, is given by:

$$\psi(x,t+\epsilon)=\int_{-\infty}^{\infty}A^{-1}\exp\left(\frac{im(x-y)^2}{2\epsilon}\right)\psi(y,t)\,dy \tag{8.35}$$

where:

$$A=\left(\frac{2\pi i\epsilon}{m}\right)^{1/2} \tag{8.36}$$

To perform this integration, let dy be replaced by $d\eta$, where $\eta=y-x$:

$$\psi(x,t+\epsilon)=\int_{-\infty}^{\infty}A^{-1}e^{im\eta^2/2\epsilon}\psi(x+\eta,t)\,d\eta \tag{8.37}$$

Now Taylor expand the left-hand side in terms of t, and the right-hand side in terms of η:

$$\psi(x,t)+\epsilon\frac{\partial\psi}{\partial t} = \int_{-\infty}^{\infty} A^{-1}e^{im\eta^2/2\epsilon} \times\left(\psi(x,t)+\eta\frac{\partial\psi}{\partial x}+\frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial x^2}+\cdots\right)d\eta \quad (8.38)$$

The integration over $d\eta$ is easily performed. The integration over the linear term in η vanishes because it is linear, and the integration over the higher terms vanish in the limit $\epsilon \to 0$. This gives us:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} \quad (8.39)$$

This is the Schrödinger wave equation, as desired. It is straightforward to insert a potential into the path integral, in which case we derive the Schrödinger wave equation in a potential, which is the traditional starting point for quantum mechanics.

8.3 From First to Second Quantization

So far, we have only investigated the path integral formalism in the first-quantized formalism, reproducing known results. The reader may complain that the path integral formalism is an elaborate, powerful machinery that has only rederived simple results. However, when we make the transition to the second quantized formalism and eventually to gauge theory, we will find that the path integral approach is the preferred formalism for quantum field theory. We saw earlier that the integration over all intermediate points along a path was enforced by inserting the number "1" at each intermediate point:

$$1 = |x_i, t_i\rangle \int dx_i \langle x_i, t_i| \quad (8.40)$$

The transition to field theory is made by introducing yet another expression for the number "1," this time based on an integration over an infinite number of degrees of freedom. We will use the familiar Gaussian integration:

$$\delta_{ij} = \frac{2}{(\sqrt{\pi})^n}\int\prod_{l=1}^{n}dx_l\ (x_i\,x_j)\exp\left(-\sum_{k=1}^{n}(x_k)^2\right) \quad (8.41)$$

Now let us replace the variable x_i with a function $\psi(x)$, which is temporarily viewed as a discretized number ψ_x, where x is now seen as an infinite discrete index.

The transition from finite degrees of freedom to infinite degrees of freedom is then made by inserting the following expression for "1" into the path integral:

$$\delta_{x,y} \sim \int D\psi \; D\psi^* \; \psi_x^* \psi_y \exp - \left(\sum_z \psi_z^* \psi_z \right) \tag{8.42}$$

where:

$$D\psi = \prod_z d\psi_z \tag{8.43}$$

Written in terms of functions $\psi(x)$ rather than discretized variables ψ_z, we now have:

$$\delta(x - y) = \int D\psi \; D\psi^* \; \psi^*(x)\psi(y) \exp - \left(\int Dx |\psi(x)|^2 \right) \tag{8.44}$$

These expressions can also be rewritten in terms of bra and ket vectors as follows:

$$\begin{aligned} \psi(x) &= \langle x|\psi\rangle \\ \psi^*(x) &= \langle \psi|x\rangle \end{aligned} \tag{8.45}$$

This allows us to write:

$$\begin{aligned} \delta(x - y) &= \int D^2\psi \; \psi^*(x)\psi(y) \exp - \left(\int Dx \; \psi^*(z)\psi(z) \right) \\ &= \langle x|\psi\rangle \int D^2\psi \; \exp - \left(\int \langle \psi|z\rangle \; Dz \; \langle z|\psi\rangle \right) \langle \psi|y\rangle \\ &= \langle x|1|y\rangle \end{aligned} \tag{8.46}$$

Written in this language, the number "1" now becomes:

$$1 = |\psi\rangle \int d^2\psi \, e^{-\langle \psi|\psi\rangle} \langle \psi| \tag{8.47}$$

We can now repeat all the steps used in making the transition from the Lagrangian approach to the operator approach in the Heisenberg picture by inserting this new expression for the number "1" into the path integral. When we do this, we then

have an expression for the transition function written entirely in terms of $\psi(x)$. A straightforward insertion of this new set of intermediate states yields:

$$\begin{aligned} K(b,a) &= \int_{x_a}^{x_b} Dx\, e^{i\int dt\, \frac{1}{2}m\dot{x}_i^2} \\ &= \int D\psi\, D\psi^*\, \psi(x_a)^*\psi(x_b) e^{i\int dt\, \psi^*(i\partial_t - H)\psi} \end{aligned} \tag{8.48}$$

where the Lagrangian is equal to:

$$L = \psi^* \left(i\frac{\partial}{\partial t} - H \right) \psi \tag{8.49}$$

At this point, we have now derived a second quantized version of the nonrelativistic Schrödinger equation. This may seem odd, since usually quantum field theory is associated with the merger of relativity and quantum mechanics. But quantum field theory can be viewed independently from relativity; that is, the essence of quantum field theory is that it has an infinite number of quantum degrees of freedom. In this sense, the path integral formalism can accomodate a nonrelativistic Schrödinger field theory.

Next, we would like to compute the familiar expressions found in Chapter 3 and 4 in terms of the path integral approach. First, we define the average $\langle O \rangle$ of the expression O by inserting it into the integral:

$$\langle O \rangle \equiv N \int DX \exp\left(-\sum_{i,j=1}^{n} \frac{1}{2} x_i D_{ij} x_j \right) O \tag{8.50}$$

Our goal is to find an expression for $\langle O \rangle$, and later make the transition to an infinite number of degrees of freedom ($n \to \infty$). To find an expression for this average, we will find it convenient to introduce an intermediate stage in the calculation. We define the generating functional as follows:

$$I(D, J) \equiv \int \prod_{i=1}^{n} dx_i \exp\left(-\sum_{i,j=1}^{n} \frac{1}{2} x_i D_{ij} x_j + \sum_{i=1}^{n} J_i x_i \right) \tag{8.51}$$

where we fix N by setting $I(D, 0) = N^{-1}$.

To find an expression for $I(D, J)$, we first make a similarity transformation $x_i' = S_{ij} x_j$, such that S diagonalizes the matrix D. We are then left with the eigenvalues of the matrix D in the integral. The integration separates into a product of independent integrations over x_i'. We then perform each integration separately, giving us the square root of the eigenvalues of D matrix. The product