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1

INTRODUCTION

The main purpose of this book is to provide a mathematical treatment of the Feynman path integral and the related subject of Feynman's operational calculus for noncommuting operators. The former subject is more widely known than the latter and has the reputation of being a formidable and rather elusive mathematical topic.

We will keep this introductory chapter, especially Section 1.1, nontechnical and relatively brief as far as possible. A detailed table of contents is provided and additional introductory chapters are included in the book in appropriate places. The main two are Chapters 7 and 14, dealing, respectively, with the first and second subjects:

Chapter 7, entitled "The Feynman integral: Heuristic ideas and mathematical difficulties", provides an introduction to quantum theory mainly from the perspective of the physicist Richard Feynman. Further, it points out why the Feynman "integral" is a difficult subject and shows how Feynman's ideas have led to the mathematical approaches to the Feynman integral which are used in Chapters 11–13 and 15–18.

Chapter 14 provides an introduction to Feynman's operational calculus for noncommuting operators, the subject of Chapters 15–19, and indicates how the Feynman integral and Feynman's operational calculus are related both in the present theory and in their historical development.

1.1 General introductory comments

Feynman's path integral

I find Feynman's formula to be very beautiful. It connects the quantum mechanical propagator, which is a twentieth-century concept, with the classical mechanics of Newton and Lagrange in a uniquely compelling way.

Mark Kac, 1984 [Kac5, p. 116]

Bohr got up and said: "Already in 1925, 1926, we knew that the classical idea of a trajectory or a path is not legitimate in quantum mechanics; one could not talk about the trajectory of an electron in the atom, because it was something not observable." In other words, he was telling me about the uncertainty principle. It became clear to me that there was no communication between what I was trying to say and [what] they were thinking. Bohr thought that I didn't know the uncertainty principle, and was actually not doing quantum mechanics right either. He didn't understand at all what I was saying. I got a terrible feeling of resignation. I said to myself, "I'll just have to write it all down and publish it, so that they can read it and study it, because I know it's right! That's all there is to it.

Richard P. Feynman, reminiscing about the 1948 Pocono conference.

(Quoted in [Me, p. 248].)

We begin with Feynman's famous heuristic formula [Fey1,2] for the evolution of a nonrelativistic quantum system:

$$\frac{1}{K}\int_{C_{u,v}^{0,t}} \exp\left\{\frac{i}{\hbar}S(x)\right\}\mathcal{D}x, \qquad (1.1.1)$$

where $i = \sqrt{-1}$. We will make some comments about this formula here, but a much more thorough discussion will be given in Chapter 7.

In (1.1.1), $C_{u,v}^{0,t}$ is the space of all real-valued (more generally, $\mathbb{R}^d$-valued) continuous functions x on $[0, t]$ such that $x(0) = u$ and $x(t) = v$. Further, $\mathcal{D}x$ represents a measure on $C_{u,v}^{0,t}$ which weighs all paths x equally (in much the same way as Lebesgue measure weighs all points in $\mathbb{R}$ equally), $\hbar$ is Planck's constant divided by 2π, and $S(x)$ is the *action integral* associated with the path x; that is,

$$S(x) = \int_0^t \left\{\frac{m}{2}\left[\frac{dx}{ds}\right]^2 - V(x(s))\right\} ds. \qquad (1.1.2)$$

The integrand in (1.1.2) is the *Lagrangian*; it equals, for each s in the time interval $[0, t]$, the kinetic energy minus the potential energy at the point $x(s)$.

Note that the potential V in (1.1.2) is real-valued, so that the integrand in (1.1.1) has a constant absolute value of one. Hence, it is the net interference effect as x ranges over the space of paths that determines the value of the oscillatory integral.

Feynman's ideas on the path integral (or "sum over histories") were ingenious and have had far-reaching consequences in many parts of physics, and more recently, of mathematics as well. At first, however, they seemed "crazy" to many physicists, including some famous ones (see [Me, §2.4]). Paths—and concepts that depend on paths, such as the Lagrangian and the action integral—play a crucial role in Feynman's formulation, whereas they had been "banned" (in light of the Heisenberg uncertainty principle) from the standard Hamiltonian approach to quantum dynamics (see Chapter 6).

The formula (1.1.1) seems hopeless at first to most mathematicians who come in contact with it. The "integral" in (1.1.1) is over a space of functions x "most" of which are nowhere differentiable, and yet the formula for the action $S(x)$ in (1.1.2) involves calculating the derivative of x. Further, there is a mathematical theorem which implies that there is no countably additive measure on $C_{u,v}^{0,t}$ which weighs all paths equally. (See Section 3.1 for a closely related result.)

We should add that Feynman had some awareness of the mathematical difficulties just described above and concentrated throughout much of [Fey2] on a second approach (see Section 7.4) that begins with a discretization of the time interval $[0, t]$. (It enabled him, in particular, to replace the normalization constant K—which is ill-defined and for all practical purposes, infinite—by a suitable sequence of finite normalization constants.) This alternative approach involves fewer but still substantial mathematical difficulties.

The path integral of Feynman is not a Lebesgue integral; indeed, *there is no "Feynman measure"* (see Section 4.6, especially Theorem 4.6.1). At least for functions of physical interest, conditional convergence—instead of absolute convergence (as in the Lebesgue theory)—is at the heart of the matter. Additionally, since the domain of integration of this oscillatory "integral" is a set of paths, the subject is intrinsically infinite dimensional. (Physically, the *cancellation effects* caused by the oscillatory nature of the Feynman integral correspond to interference effects between quantum-mechanical matter waves.)

The Feynman integral has been approached from many different points of view by mathematicians and physicists with varied background and interests. The resulting diversity has led to many different definitions of "the" Feynman integral. In this book, we address several (certainly not all) of these approaches in a setting appropriate for nonrelativistic quantum mechanics. In each of the cases considered, the existence of the Feynman integral is established under very general assumptions. The different approaches have their own domain of validity as well as their own strengths and weaknesses, as will be discussed further on in the book, especially in Chapters 11 and 13. However, under more restrictive but still quite general hypotheses, we will show that there is far more agreement than seems to have been previously realized between three of these approaches to the Feynman integral and the standard Hamiltonian approach to quantum dynamics. (See Section 13.4.)

Results on the Feynman integral for highly singular potentials are given in Chapters 11–13. Chapter 7, which was mentioned earlier, is crucial to an understanding of the Feynman integral. There, the physical background for nonrelativistic quantum mechanics is discussed from Feynman's point of view along with the way in which his ideas on the subject have led to several of the definitions of the Feynman integral which are used in Chapters 11–13. (Chapter 6 provides an extremely brief discussion of a few of the ideas which are common both to the usual Hamiltonian approach to quantum dynamics and to Feynman's approach.)

We close this part of the general introductory comments by providing more specific information on some issues that are central to the subject matter of this book through Chapter 13.

The following are shortcomings of many of the mathematical theories of the Feynman integral which are often pointed out:

(1) The existence theory is not sufficiently general. In particular, many of the standard real-valued, time independent potentials ($V : \mathbb{R}^d \to \mathbb{R}$) which are used in modeling quantum systems are singular (for example, the attractive Coulomb potential) and do not fit within the theory.

(2) Not much information is given about how the various approaches to "the" Feynman integral are related to one another or to the unitary group which gives the evolution of the quantum systems in the standard approach to quantum dynamics.

(3) There is a shortage of satisfactory limiting theorems. Indeed, in some cases, no such theorems are available, while in others, the results do not seem natural from a physical point of view.

One of the strong points of the work here is that we give quite satisfactory responses to all three of these objections, especially for three of the four approaches to the Feynman integral which are developed in detail in this book. The Feynman integral defined via the Trotter product formula is shown to exist under very general conditions in

Corollary 11.2.22. Both the modified Feynman integral and the analytic-in-time operator-valued Feynman integral are shown to exist under even more general conditions in Corollary 11.4.5 and Theorem 13.3.1, respectively. Further, under the common conditions for their existence in the corollary and theorem just referred to, the modified Feynman integral and the analytic-in-time operator-valued Feynman integral not only exist but agree with each other and with the unitary group, as is shown in Corollary 13.4.1. Under the somewhat more restrictive conditions of Corollary 11.2.22, we will see in Corollary 13.4.2 that the Feynman integral via the Trotter Product Formula can be added to the list so that all three of these Feynman integrals exist and agree with one another and with the unitary group associated with the usual Hamiltonian approach to quantum dynamics.

Our limiting theorems for the three approaches to the Feynman integral referred to above are "dominated-type" convergence theorems. Since cancellation effects are intrinsic to the Feynman integral, there cannot be dominated convergence theorems in this subject that exactly parallel the Lebesgue dominated convergence theorem. However, in the most frequently used models in nonrelativistic quantum mechanics, it is only the potential energy function that may vary and our assumptions are that the sequence of functions $\{V_m\}$ is pointwise convergent (Lebesgue almost everywhere) and "dominated" in an appropriate sense (see (11.5.20) and (11.5.21) for example). The result for the modified Feynman integral, Theorem 11.5.19 [La12], is the key. The corresponding result for the analytic in time operator-valued Feynman integral, Corollary 13.4.3, is an easy corollary of Theorem 11.5.19 and Corollary 13.4.1. The convergence result for the Feynman integral via the Trotter product formula, Corollary 13.4.6, rests on Theorem 11.5.19 and Corollary 13.4.2 but also on some further considerations.

Although it is not especially difficult, Section 13.4 is quite pleasing because it brings together all of the positive results associated with items (1)–(3). (Note that we have omitted from the present discussion the analytic-in-mass operator-valued Feynman integral as studied in Sections 13.5 and 13.6. This material is interesting in its own right, but it is not readily compared with the three approaches above.)

The questions raised in (1)–(3) above are clearly central to the mathematical theory of the Feynman integral, but the answers provided in this book are not the only possible ones. Moreover, there are other important issues besides those implicit in (1)–(3). For example, the method of stationary phase is one of the heuristically appealing features of the Feynman path integral (see Chapter 7) but is not discussed rigorously anywhere in this book. However, it has been justified in the context of the Fresnel integral approach to the Feynman integral (see, for example, [AlHo2, Rez, AlBr1]).

Feynman's operational calculus

We turn now to the second topic in the title of this book, Feynman's operational calculus for noncommuting operators. A fuller introduction to this topic is given in Chapter 14.

It is easy to form functions of operators if the operators commute with one another. However, the subject becomes far more difficult when the operators fail to commute. Motivated by problems arising in quantum mechanics and quantum electrodynamics, Feynman ([Fey8], 1951) gave heuristic "rules" for forming functions of noncommuting operators. One of these "rules" says to treat the operators as though they commuted, once a suitable time-ordering convention has been adopted. For example, Feynman writes such "equalities" as

$$\exp(A + B) = \exp(A)\exp(B), \tag{1.1.3}$$

even when A and B fail to commute.

The process of appropriately restoring the conventional ordering of the operators after the use of "equalities" such as in (1.1.3) above is referred to as "disentangling". This "disentangling process" is central to Feynman's operational calculus.

Feynman's "rules", as strange as they may seem, have led to useful results, notably the time-ordered perturbation series (or Dyson series) of quantum theory.

Feynman's work on his operational calculus is far from mathematically rigorous, as he himself noted. One of the challenges to mathematicians is to suitably interpret Feynman's ideas and to put them on a firm mathematical basis. Our work in Chapters 15–18 and in Chapter 19, respectively, discusses two ways of carrying this out and also further develops the subject in several directions.

What led Feynman to his operational calculus? He wanted a path "integral" in order to calculate perturbation series in quantum electrodynamics, but he had no such integral in that setting. His operational calculus was motivated initially by a desire to find methods of calculation which would generalize those which could be carried out in nonrelativistic quantum mechanics via his path "integral".

The operational calculus for noncommuting operators which Feynman discovered generalizes some aspects of path integration. *This suggests that in settings where mathematically rigorous path integrals are available, it might be possible to use such integrals to interpret and make rigorous Feynman's operational calculus. Indeed, this is what we do in Chapters 15–18 using the Wiener and Feynman path integrals.*

Feynman's operational calculus via the Feynman and Wiener integrals

Feynman's operational calculus, the Feynman integral and the Wiener integral all come together in Chapters 15–18 as well as in Sections 14.3–14.5. Chapters 15, 16 and 18 are based on joint work of the authors; much of this material can be found in [JoLa1] and [JoLa4], respectively. Chapter 17 is adapted from the following papers of the second author [La15, La18, La16].

The Wiener process (or Brownian motion) does not appear in the title of this book, but it—along with the associated Wiener measure and integral—appears repeatedly in this work. It plays an especially important role in Chapters 7 and 12–18. Chapters 3 and 4 present the information that we will need about Wiener measure from an analyst's point of view. A short Chapter 2 discusses physical Brownian motion and relates it to its mathematical model, the Wiener process. In Chapter 5, another short chapter, we give a very brief discussion of a more probabilistic approach to the Wiener process.

The main emphasis in Chapters 15–18 is on using the Feynman and Wiener integrals to study Feynman's operational calculus in the quantum-mechanical and diffusion (alternatively, heat or probabilistic) settings, respectively. However, many of the results in Chapters 15–18 have an interest of their own as contributions to the Feynman and Wiener integrals, apart from their connection with Feynman's operational calculus.

We will now describe more precisely than above our approach to the operational calculus in this context. A more detailed overview of Chapters 15–18 is provided in Chapter 14, especially in Sections 14.3–14.5.

The functions on the space of continuous paths on $[0, t]$ that are Wiener and Feynman integrated in Chapters 15–18 belong, for each time $t > 0$, to the "disentangling algebra" $\mathcal{A}_t$. This commutative Banach algebra consists of certain infinite sums of finite products of functions of the form

$$F(x) = F_{\theta,\eta}(x) := \int_{[0,t]} \theta(s, x(s))\eta(ds), \tag{1.1.4}$$

where θ (often thought of as a time-dependent potential) is a complex-valued function on $[0, t) \times \mathbb{R}^d$ and η is a bounded Borel measure on $[0, t)$. The function $\exp(F)$ is an important example of a function in $\mathcal{A}_t$. (It is called the "Feynman–Kac functional with Lebesgue–Stieltjes measure" η; see Chapter 17. More generally, the elements of $\mathcal{A}_t$ will often be referred to as "Wiener functionals" in Chapters 14–18.)

The operator-valued path integral of $F \in \mathcal{A}_t$ is denoted $K_\lambda^t(F)$. For $\lambda > 0$ (the diffusion case), $K_\lambda^t(F)$ is defined as a Wiener integral and then extended first via analytic continuation in λ to $\mathbb{C}_+$, the open right half-plane, and then via continuity to $\mathbb{C}_+^\sim := \overline{\mathbb{C}_+}\backslash\{0\}$. When λ is purely imaginary (the quantum-mechanical case), $K_\lambda^t(F)$ is the "Feynman integral" of F. [This is the analytic (in mass) operator-valued Feynman integral of F; see Definition 15.2.1 for a more precise statement.]

The disentangling process is carried out in Chapters 15–18 by calculating the path integral $K_\lambda^t(F)$ for $\lambda > 0$ and then extending the result to $\lambda \in \mathbb{C}_+^\sim$. One need not invoke Feynman's "rules" explicitly in this setting; the necessary time-ordering is done naturally (but not always easily) while calculating the functional integrals.

The disentangled operators $K_\lambda^t(F)$ are expressed as time-ordered perturbation expansions or "generalized Dyson series". Generalized Feynman diagrams (see Section 15.6) provide a visual aid for keeping track of the terms of a generalized Dyson series. (These diagrams can be complicated in their own right but they generalize the simple diagrams of nonrelativistic quantum mechanics and not those of quantum electrodynamics.)

The work in Chapters 15–18 (and also in Chapter 19) not only interprets Feynman's ideas and makes them rigorous, but also extends them in several different ways. Noncommutative operations $*$ and $\dot{+}$ on the family of disentangling algebras $\{\mathcal{A}_t\}_{t>0}$ are introduced in Chapter 18. They can be thought of as a noncommutative multiplication and addition, respectively, on the space of Wiener functionals; see Section 18.3. Such operations—introduced by the authors in [JoLa3,4]—were not envisioned by Feynman but they fit nicely into the operational calculus in various ways. If $F \in \mathcal{A}_{t_1}$ and $G \in \mathcal{A}_{t_2}$, then we know that the operators $K_\lambda^{t_1}(F)$ and $K_\lambda^{t_2}(G)$ can be disentangled via generalized Dyson series. It is natural to ask if the product of $K_\lambda^{t_1}(F)$ and $K_\lambda^{t_2}(G)$ can also be disentangled. It can; in fact (Theorem 18.5.6 and Corollary 18.5.7), $F * G \in \mathcal{A}_{t_1+t_2}$ and for all $\lambda \in \mathbb{C}_+^\sim$,

$$K_\lambda^{t_1+t_2}(F * G) = K_\lambda^{t_1}(F)K_\lambda^{t_2}(G). \tag{1.1.5}$$

Since we can show that

$$\exp(F \dot{+} G) = \exp(F) * \exp(G) \tag{1.1.6}$$

on the level of the functionals, we immediately deduce from (1.1.5) that, on the level of the operators,

$$K_\lambda^{t_1+t_2}(\exp(F \dot{+} G)) = K_\lambda^{t_1}(\exp(F))K_\lambda^{t_2}(\exp(G)). \tag{1.1.7}$$

Note that (1.1.6) formally resembles Feynman's paradoxical formula (1.1.3) but involves the noncommutative operations $*$ and $\dot{+}$ on the disentangling algebras.

The family of commutative disentangling algebras $\{\mathcal{A}_t\}_{t>0}$—equipped with the noncommutative operations $*$ and $\dot{+}$ along with the (operator-valued, analytic-in-mass) Feynman integrals $K_\lambda^t(\cdot)$—forms a rich interlocking algebraic and analytic structure that enables us to explore more deeply the noncommutative aspects of Feynman's operational calculus.

Our systematic use of measures as in (1.1.4) contributes significantly to the richness of Feynman's operational calculus. Different measures can provide different directions for disentangling. For example, what is one exponential function of a sum of commuting operators becomes infinitely many different exponential functions of a sum of noncommuting operators. This leads in Chapter 17, entitled "The Feynman–Kac formula with a Lebesgue–Stieltjes measure and Feynman's operational calculus" and based on work of Lapidus in [La14–18], to the solution of a wide variety of evolution equations which can incorporate both discrete and continuous phenomena.

Feynman's operational calculus and evolution equations

Another approach to Feynman's operational calculus is considered in Chapter 19, based on joint work of the authors with Brian DeFacio ([dFJoLa1] and especially [dFJoLa2]). The setting is much more general than in Chapters 15–18, but, on the other hand, attention is focused almost exclusively on exponentials of sums of noncommuting operators. In [Fey8] and in the papers which led up to it, the emphasis was also on such exponential functions. This particular focus came from Feynman's desire to calculate formulas for the evolution of physical systems.

The operators that appeared as the arguments of the exponential function in Feynman's work were associated with the different forces involved in the physical problem. Feynman seemed to have complete confidence that applying his "rules" to such exponential expressions would yield a formula for the evolution of the physical system at hand. The main results of Chapter 19, Theorems 19.5.1 and 19.6.1, justify (in a mathematical sense) Feynman's confidence (under a certain rather general set of hypotheses) by showing that the disentangled exponential expression gives the unique solution to the associated evolution equation. Our method is to use Feynman's heuristic ideas to "disentangle" the exponential expression; we then prove that the disentangled expression makes sense and satisfies the evolution equation.

We hope that the combination of some simple examples of disentangling found in Chapter 14, the more complicated calculations from Chapter 19 that were just referred to above, along with some additional examples that are provided in Section 19.7, will help to clarify Feynman's heuristic "rules" for the reader. Chapters 15–18 will also be helpful in this regard. Although the disentangling is carried out in these chapters in the process of calculating the Wiener and Feynman integrals, one can see clearly the connections with Feynman's time-ordering ideas both in the details of the calculations and in the resulting answers.

Further work on or related to the Feynman integral: Chapter 20

Chapter 20, our last chapter, has a very different character from the rest of this book. Our main focus in regard to the Feynman integral will be on operator-valued approaches. However, in Section 20.1, we will give a brief expository account (without proofs) of *scalar-valued* approaches to the Feynman integral which involve "transform assumptions". A great deal of work on the Feynman integral has been along these lines since the 1976 monograph of Albeverio and Hoegh-Krohn [AlHo1] on the "Fresnel integral".

In Section 20.2, our main concern is with the connections between the "heuristic Feynman integral" and a variety of further topics in contemporary mathematics and physics. The greatest emphasis will be on Section 20.2.A where we discuss Witten's heuristic Feynman integral [Wit14] and its influence on the subjects of knot theory and low-dimensional topology. In Section 20.2.B, we briefly discuss the relationship between heuristic path integrals and four additional topics: The Atiyah–Singer index theorem, deformation quantization, gauge field theory, and string theory. We should stress that the mathematical existence of the "Feynman integrals" used in Section 20.2 has usually not been established. We should also caution the reader that the authors are far from being experts on the subjects involved in Section 20.2.

Given its special nature, Chapter 20 will be excluded from our discussion in the remainder of this introduction.

Section 1.1, with the exception of its last subsection, has been a brief introduction to the main topics of this book. Next we turn to a discussion of some of the themes that are repeated in several places in this work. An ordered (rather than thematic) and quite detailed list of the topics treated in this book can be found in the list of contents; the latter has been written partly with this goal in mind. Section 1.2 below is somewhat more technical than Section 1.1. Depending on their background, some readers may wish initially to go over parts of this material quickly and then return to it at a later time.

1.2 Recurring themes and their connections with the Feynman integral and Feynman's operational calculus

There are a number of subjects related to those in the title of this book which will play an important role and will reappear frequently; the Wiener process has already been mentioned in this connection. Product formulas, such as the Trotter Product Formula and the product formula for imaginary resolvents discussed in detail in Chapter 11, certainly fall into this category as well.

Product formulas and applications to the Feynman integral

Perhaps the approach to the Feynman integral which is most straightforwardly motivated by Feynman's original paper ([Fey2], 1948), is the approach using the Trotter product formula. It is Trotter's formula for the case of unitary groups that is used. Ignoring some technicalities, this result says that if A and B are (unbounded, noncommuting) self-adjoint operators on a Hilbert space $\mathcal{H}$ and if $A + B$ is essentially self-adjoint (i.e., if it has a unique self-adjoint extension), then

$$e^{-it(\overline{A+B})} = \lim_{n\to\infty} \left(e^{-i\frac{t}{n}A}e^{-i\frac{t}{n}B}\right)^n, \tag{1.2.1}$$

where here, by the operator $\overline{A+B}$ on the left-hand side of (1.2.1), we mean the unique self-adjoint extension of the algebraic sum $A + B$.

When (1.2.1) is applied to the Feynman integral, the Hilbert space $\mathcal{H}$ will be $L^2(\mathbb{R}^d)$, and we will take, after normalizing the physical constants, $A = -\frac{1}{2}\Delta = H_0$ (the free Hamiltonian), where Δ denotes the Laplacian on $\mathbb{R}^d$. Further, we will let $B = V$, the operator of multiplication by the potential energy function. (The "potential" $V : \mathbb{R}^d \to \mathbb{R}$ is a suitable real-valued function on $\mathbb{R}^d$.) Finally, we let $H = \overline{A+B} = \overline{H_0 + V}$ denote the Hamiltonian or energy operator associated with V. Then, when applied to an appropriate wave function φ, the left-hand side of (1.2.1), namely, $\psi(t, \cdot) := e^{-itH}\varphi$, yields the unique solution of the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta\psi + V(\xi)\psi = H\psi \ (\xi \in \mathbb{R}^d, t \in \mathbb{R}), \tag{1.2.2}$$

with initial state $\psi(0, \cdot) = \varphi$ in the domain of H.

The approach to the Feynman integral via the Trotter product formula is the first of two approaches which appeared in Nelson's paper ([Ne1], 1964). An informal explanation of the connection between [Fey2] and the Trotter product formula is given in Sections 7.2 and 7.5 and a precise discussion with proofs of a result which is more general than the one in [Ne1] appears in Sections 11.1 and 11.2.

Inspired by the product formula of Trotter and the work of Nelson, Lapidus found [La11] a "product formula for imaginary resolvents" and used it to define and establish the existence of the "modified Feynman integral". The result of Lapidus goes well beyond the case where $A + B$ is essentially self-adjoint; in fact, his formula involves the "form sum" $A \dot{+} B$ of the operators A and B. Also, the unitary operators $e^{-i\frac{t}{n}A}$ and $e^{-i\frac{t}{n}B}$ on the right-hand side of (1.2.1) are replaced by the imaginary resolvents $[I + i(t/n)A]^{-1}$ and $[I + i(t/n)B]^{-1}$, respectively. Thus, we have the "product formula for imaginary resolvents" (see Section 11.3, especially Theorem 11.3.1 and Corollary 11.3.7):

$$e^{-it(A\dot{+}B)} = \lim_{n\to\infty}([I + i(t/n)A]^{-1}[I + i(t/n)B]^{-1})^n. \tag{1.2.3}$$

(If $A+B$ is essentially self-adjoint, as in the hypothesis of the product formula for unitary groups (1.2.1), then the form sum $A \dot{+} B$ coincides with $\overline{A+B}$, the unique self-adjoint extension of the algebraic sum $A + B$—and so the left-hand side of (1.2.3) coincides with that of (1.2.1); see Proposition 11.2.10(ii).)

When the product formula (1.2.3) is applied to define and establish the convergence of the modified Feynman integral, we obtain much as before a solution to the Schrödinger equation, but now with the Hamiltonian given by the form sum of H_0 and V; i.e., $H = H_0 \dotplus V$. (See Section 11.4, including Definition 11.4.4.)

In the setting we have been considering, the potential is a real-valued and time-independent function V and the Hamiltonian is obtained by "adding" V to H_0, with the sum allowed to be a form sum. The maximum domain of validity for V in this framework is (as we will see in Section 11.4) exactly the same for the modified Feynman integral as it is in the Hamiltonian approach to quantum dynamics. Further, this maximum domain of validity is physically natural; the "form domain" of the Hamiltonian $H = H_0 \dotplus V$ consists precisely of those functions $\varphi \in L^2(\mathbb{R}^d)$ which have finite total (i.e., kinetic + potential) energy. Looking ahead and considering the same setting, the maximal domain of validity for V in the case of the analytic-in-time operator-valued Feynman integral agrees with the other two. It should be added, however, that the modified Feynman integral extends nicely to the case of complex potentials V (see Section 11.6) whereas a corresponding theorem has not been proved (and may not be true) for the analytic-in-time operator-valued Feynman integral considered in Sections 13.3 and 13.4.

An advantage of the generality of the modified Feynman integral is that it allowed Lapidus to obtain in [La12] a very general stability theorem (relative to the potential) and to deduce a "dominated-type convergence theorem" appropriate for this context. (See Section 11.5.)

The results leading to the definition of the "Feynman integral via TPF" [Ne1] are discussed in Section 11.2, while those concerning the "modified Feynman integral" [La1–2, La6–13] and various extensions of its definition (notably, to $\mathbb{C}$-valued potentials [BivLa]) are presented in Sections 11.3–11.6. In addition, we mention that Sections 13.5 and 13.6, respectively, describe analytic (in mass) versions of these two approaches to the Feynman integral. Product formulas of various types—not themselves consequences of (1.2.1) or of (1.2.3)—also play a prominent role in these sections.

Feynman–Kac formula: Analytic continuation in time and mass

Mark Kac heard Feynman speak about his path integral in 1947. Kac realized that if time t in Feynman's formula is replaced by $-it$ ("imaginary time" from the perspective of quantum physics), then the expression involved before the limit is taken is equal to a Wiener integral, a true integral in the Lebesgue sense with respect to the countably additive Wiener measure $\mathfrak{m}$. The powerful results of the Lebesgue theory of integration can then be used to rigorously justify the calculation of the limit. One outcome of all this is the famous Feynman–Kac formula. (A detailed proof of a very general version of the result is given in Chapter 12, based on work of B. Simon in [Si9].)

Kac's discovery expresses the solution to the heat equation as a certain Wiener integral. More precisely, if the "Feynman–Kac functional" F is given by

$$F(x) := \exp\left\{-\int_0^t V(x(s))ds\right\}, \tag{1.2.4}$$

then for $t \geq 0$ and $\xi \in \mathbb{R}^d$, we have

$$u(t,\xi) := (e^{-tH}\varphi)(\xi) = \int_{C_0^t} F(x+\xi)\varphi(x(t)+\xi)d\mathfrak{m}(x), \tag{1.2.5}$$

where $\mathfrak{m}$ denotes Wiener measure on the space C_0^t of continuous paths x such that $x(0) = 0$. The left-hand side of (1.2.5) yields the unique solution, $u(t,\cdot) = e^{-tH}\varphi$, at time $t \geq 0$ of the heat (or diffusion) equation

$$-\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + V(\xi)u = Hu \ (\xi \in \mathbb{R}^d, t \geq 0), \tag{1.2.6}$$

with initial condition $u(0,\cdot) = \varphi$. Here, as before, $H = H_0 \dot{+} V$, with $H_0 = -\frac{1}{2}\Delta$. Note, however, that we now use the heat semigroup e^{-tH} to represent the solution to the heat equation (1.2.6) whereas we have used earlier the unitary group e^{-itH} to represent the solution to the Schrödinger equation (1.2.2).

The "Feynman–Kac formula" (1.2.5) has been extremely useful for a variety of purposes, both in mathematics and in physics (see Section 7.6 for a brief discussion of this along with some references), but it does not by itself resolve the problem of making sense of the Feynman integral since the change from t to $-it$ takes us from quantum theory and the Schrödinger equation to the heat equation. The Feynman–Kac formula does, however, suggest an approach to the Feynman integral. Start with imaginary time and the theoretically powerful Wiener integral and define the Feynman integral by analytically continuing to real time. Indeed, operator-valued analytic continuation in time is another of the approaches to the Feynman integral which will be discussed in detail in this book. These results on the analytic-in-time Feynman integral (at the level of generality found here) are due to Johnson [Jo6] and are the subject of Sections 13.2 and 13.3. We should mention that what is imaginary time from the point of view of quantum theory is real time from the perspective of the heat equation. We shall adopt the latter point of view in Chapter 13 (Sections 13.2, 13.3 and 13.7) and analytically continue from real time to purely imaginary time—going in the process from the Wiener integral to the Feynman integral.

We remark that Section 13.7 gives a brief discussion of an extension (see [AlJoMa]) of the analytic-in-time operator-valued Feynman integral which is based on the theory of "additive functionals of Brownian motion" (see [Fuk, FukOT]) and Feynman–Kac formulas in which, for example, the potential V can be replaced by a suitable measure on $\mathbb{R}^d$.

The last of the approaches to the Feynman integral which will be treated in detail in this book is operator-valued analytic continuation in mass. Again, one starts with the Wiener integral but this time, the analytic continuation is in a mass parameter (or alternately, in a variance parameter). The connection between Feynman's ideas and the approach to the Feynman integral via operator-valued analytic continuation in mass is discussed in an informal way in Section 7.6, with the approach via the Trotter product formula serving to link the two.

The precise discussion of the analytic-in-mass operator-valued Feynman integral is given in Section 13.5. The crucial starting point for this work is Nelson's second approach developed in [Ne1]. An earlier paper by Cameron ([Ca1], 1960) used *scalar-valued* analytic continuation in mass; the key contribution of [Ca1] was the proof that there is no countably additive "Wiener measure" with a complex variance parameter (see Theorem 4.6.1). This result corrected an error in [GelYag], an interesting and even earlier paper which used analytic continuation.

Various extensions of Nelson's results are given in Sections 13.5 and 13.6. Among them, the reader will find hybrids which combine a suitable product formula with analytic continuation in mass. A comparison of the resulting analytic in mass Feynman integrals within their common domain of validity is provided towards the end of Section 13.6.

We remind the reader that the analytic-in-mass operator-valued Feynman integral will also be used in Chapters 15–18. Unlike the approaches in Chapter 13 via analytic continuation in mass, this Feynman integral exists for *every* (rather than Lebesgue almost every) value of the mass parameter. The class of functionals treated in Chapters 15–18 is, *in some respects*, much larger than in Chapter 13. However, in Chapters 15–18, no attempt is made to deal with potential functions with strong spatial singularities.

There are four different versions of the analytic-in-mass Feynman integral discussed in this book, as was alluded to above; in addition, three other approaches to the Feynman integral have already been discussed in this introduction. In the next two paragraphs, we indicate briefly what these are and where they are to be found.

The approaches to the Feynman integral that are discussed at any length in this book are all operator-valued. (Recall that we are not taking Chapter 20 into account in our present discussion.) Two of the analytic-in-mass approaches start from the Wiener integral when the mass parameter is real. One of these is discussed in the first part of Section 13.5; the other, which has quite different features, is defined in Section 15.2 and used throughout Chapters 15–18. The last two begin with product formulas for semigroups (in Section 13.5) and resolvents (in Section 13.6) to yield analytic-in-mass versions of the Feynman integral via TPF ([Kat7, BivPi]) and of the modified Feynman integral [BivLa], respectively.

The Feynman integral defined via the Trotter product formula for unitary groups is discussed in Section 11.2 and the modified Feynman integral (defined via a product formula for imaginary resolvents established in Section 11.3) is treated in Sections 11.4–11.6. Finally, the analytic-in-time Feynman integral appears in Sections 13.2 and 13.3, with an extension given in Section 13.7.

The role of operator theory

As mentioned above, the approaches to the Feynman integral that will be discussed in detail in this book are all operator-valued. Further, there is always at least one unbounded operator involved; much of the time, it is $H_0 = -\frac{1}{2}\Delta$, the free Hamiltonian, although various physically meaningful substitutes for H_0 are allowed in Sections 11.4, 11.6, and Sections 13.5–13.6, and more abstract generators are considered in Chapters 11 and 19. In Sections 11.2–11.5, Chapter 12, Sections 13.2–13.4 and 13.7, the theory of (not necessarily bounded) self-adjoint operators and functions of them is sufficient for our needs. These needs include various forms of the spectral theorem for unbounded self-adjoint operators as well as basic results about unbounded quadratic forms and form sums of operators. This background material is provided in Chapter 10 which is titled "Unbounded self-adjoint operators and quadratic forms". (See also Section 9.6 for introductory material on unbounded self-adjoint operators and the associated semigroups.) The spectral theorem enables us to define the functions e^{-itH} (the unitary group) and, if the spectrum of the self-adjoint operator H is bounded from below, the (self-adjoint) semigroup e^{-tH}. For us, in most applications, H is the Hamiltonian (or energy operator), a suitable self-adjoint extension of $H_0 + V$, where V is the potential. (More specifically, in Section 11.2, H is the unique self-adjoint extension of $H_0 + V$, and, more generally, it is the form sum of H_0 and V in Sections 11.3–11.5, Chapter 12, Sections 13.2–13.3 and 13.7.)

Self-adjoint operators—and the associated unitary groups or self-adjoint semigroups—are not adequate for everything that we will do. Strongly continuous (or (C_0)) semigroups of operators will be discussed in Chapter 9 (and in the brief and informal chapter that precedes it). Such semigroups (not necessarily associated with self-adjoint operators) will be used in Sections 11.1, 11.6, 13.5, 13.6, parts of Chapter 14 and throughout Chapter 19. They will also frequently be present in Chapters 15–18 but will be used in a more straightforward way there.

Connections between the Feynman–Kac and Trotter product formulas

The Feynman–Kac and Trotter product formulas have already been discussed above, but there are additional places in the book where these related formulas or variations of them appear. The Trotter product formula is the main tool in the crucial first step of the proof of the Feynman–Kac formula in Chapter 12. A variation of the Feynman–Kac formula, the "Feynman–Kac formula with a Lebesgue–Stieltjes measure", is—along with its connection with Feynman's operational calculus—the topic of Chapter 17, which describes part of the work in [La14–18]. A related product integral, a relative of the Trotter product formula, is discussed in Section 17.6 [La18,16]. Example 16.2.7 (in conjunction with Example 15.5.5) looks at the relationship between the Trotter product formula and the Feynman–Kac formula from the point of view of weak (or vague) convergence of measures. This broad perspective is informative even though the results are far less general than those proved in Chapters 11 and 12. A version of the Feynman–Kac formula which substantially extends the one in Chapter 12 is discussed briefly in Section 13.7. There, for example, the potential energy function can be replaced by certain measures (in the space rather than in the time variable, as in Chapter 17) which are singular with respect to Lebesgue measure. Finally, a Feynman–Kac formula for certain complex potentials is contained in the work presented in Sections 13.5 and 13.6.

Evolution equations

A fundamental concept of quantum mechanics is a quantity called the propagator, and the standard way of finding it (in the non-relativistic case) is by solving the Schrödinger equation. Feynman found another way based on what became known as the Feynman path integral or "the sum over histories" . . .

Mark Kac, 1984 [Kac5, p. 116]

The evolution of physical systems concerns us throughout this book, so it is not surprising that the subject of evolution equations is another recurring theme. Our point of view (following Feynman) is not, however, the usual one. Typically, the evolution equation comes first and is regarded as the model for the physical system. One then looks for a method to solve the evolution equation and the solution gives the evolving state. Our deviation from this point of view is perhaps seen most clearly in Chapter 19. The idea there is: Given the forces involved in the problem, write down and then "disentangle" the exponential of a sum of integrals (from, say, 0 to t) of associated time-ordered operators (see (19.4.8)). The resulting time-ordered perturbation series (see (19.3.14)) should give the evolution of the physical system. Of course, it is of mathematical and physical interest to know if this series solves some related evolution equation. Theorem 19.5.1 shows that this is so under a quite general set of assumptions.

As remarked earlier in this introduction, the approach to quantum dynamics provided by "the" Feynman path integral differs in several ways from the standard Hamiltonian approach. The point we wish to make here is that the path integral itself should give the evolving state. No evolution equation is needed ahead of time. Of course, it is of interest to know conditions under which the evolving state given by the Feynman integral satisfies the Schrödinger equation or some variation of it.

The different specific approaches to the Feynman integral discussed in this book have differing relationships with the standard Hamiltonian approach to quantum dynamics. Our first comments along these lines pertain to Chapter 17. Recall that in Chapters 15–18, the potentials can be time-dependent and complex-valued but are not allowed to have strong singularities in the space variables. If we take the appropriate Wiener integral involving the usual Feynman–Kac functional $\exp\{F_{\theta,l}(x)\}$, where $F_{\theta,l}$ is given by (1.1.4) and l is Lebesgue measure on the time interval $(0, t)$, we obtain a function of time and space which describes the evolution of a distribution of heat. By analytically continuing in mass (and making an adjustment in the potential), we arrive at a function giving a quantum evolution. These time evolutions are solutions to the heat and Schrödinger equations, respectively. In Chapter 17, we replace the Feynman–Kac functional $\exp\{F_{\theta,l}\}$ by the Feynman–Kac functional $\exp\{F_{\theta,\eta}\}$ (where $F_{\theta,\eta}$ is given by (1.1.4) and η is a Lebesgue-Stieltjes measure) and follow the procedure above. We show first that the resulting evolutions involve an interesting variety of discrete and continuous phenomena and then also that they are solutions to correspondingly adjusted versions of heat and Schrödinger equations which are quite different from the usual ones (see especially Sections 17.2 and 17.6).

Even though Feynman's approach to quantum dynamics does not depend *a priori* on the usual one, the method of proof for three of the specific approaches discussed in this book, the Feynman integral via the Trotter product formula (Section 11.2), the modified Feynman integral (Sections 11.3 and 11.4), and the operator-valued analytic-in-time Feynman integral (Sections 13.3 and 13.7), not only depend heavily on operator-theoretic results but also on the existence of the unitary group as established in the standard Hamiltonian approach. [In the case of the modified Feynman integral with complex (rather than real) potential studied in Section 11.6 ([BivLa]), the Schrödinger operator must be defined appropriately and the associated time evolution is dissipative but in general, not unitary.]

The situation is quite different for the analytic-in-mass operator-valued Feynman integral, whether you begin on the real line with a Wiener integral (Section 13.5) or with product formulas (Section 13.6 and the last part of Section 13.5). Although operator techniques are still heavily involved, they are not the ones based on self-adjointness that are used commonly in quantum mechanics. Moreover, knowledge of the existence of the unitary group from the usual approach to quantum dynamics is not needed in the proof. In fact, for *extremely singular potentials* (see Examples 13.6.13 and 13.6.18), the analytic-in-mass operator-valued Feynman integral exists but the Hamiltonian approach does not, at least not in an unambiguous way.

Functions of noncommuting operators

The formation of functions of noncommuting operators is a theme which is implicit in the title of this book and which is of direct concern to us throughout Chapters 14–19. Although it is less obvious, the same subject is also involved in Chapters 6–13. For example, if A and B are commuting self-adjoint operators, there is no need for the Trotter product formula (1.2.1); we simply have $e^{-it(A+B)} = e^{-itA}e^{-itB}$. The Trotter product formula has sometimes been referred to as the noncommutative exponential law. (In light of our later work, especially in Chapters 17 and 19, it would be more accurate to describe it as an especially important example but just one of many noncommutative exponential laws.) The spirit of the theory of semigroups of operators is that it is the theory of forming the "exponential function" of operators. In practice for us (and in general), the operator to be "exponentiated" is often of the form $A + B$, where A and B do not commute. The Feynman–Kac formula expresses the heat semigroup $e^{-tH} = e^{-t(H_0+V)}$ (where $H_0 = -\frac{1}{2}\Delta$ is the free Hamiltonian and V is the operator of multiplication by the potential V) as a certain Wiener integral. (See equations (1.2.5) and (1.2.4) above.) In some sense, this formula can be thought of as providing a way to handle the fact that the operators H_0 and V do not commute.

Time-ordered perturbation series

In [Fey8] and in the work in this book on Feynman's operational calculus, the disentangled functions of operators are more often than not expressed as time-ordered perturbation series. In Chapters 14–19, such series appear repeatedly. They are most often referred to as generalized Dyson series in Chapters 15–18. Indeed, special cases of the perturbation series in all of Chapters 14–19 coincide with the classical Dyson series of nonrelativistic quantum mechanics.

In Chapter 15 (and then throughout Chapters 16–18), our generalized Dyson series play a crucial role in *defining* the operators $K_\lambda^t(F)$ involved, especially in the quantum-mechanical (or Feynman) case where a bona fide path integral (such as the Wiener integral) is no longer available. In turn, these perturbation expansions—which can be thought of, in some extended sense, as providing a "sum over all possible histories" of a quantum particle—are very helpful mathematical tools and enable us to derive various properties with relative ease. (See, for example, Sections 15.3, 15.5, and Chapter 16.)

When F is an exponential functional (see Chapter 17), they also play a key role in deriving the evolution equation (either in differential or integral form) satisfied by $t \mapsto K_\lambda^t(F)$. (This is especially true in the quantum-mechanical case.)

[For simplicity, we will limit ourselves here to the setting of Chapters 15–18. We point out, however, that despite certain differences due to the generality of the assumptions made in Chapter 19 and the absence of any kind of path integral in that framework, our above comments regarding the *definition* of the operators involved and the derivation of a corresponding *evolution equation* remain valid in the setting of Chapter 19 as well.]

At this point, it may be helpful to recall from our discussion in Section 1.1 that in Chapter 15, given a Wiener functional F in the disentangling algebra $\mathcal{A}_t$, we associate with it an operator $K_\lambda^t(F)$, called the analytic (in mass) Feynman integral of F, which can be disentangled via a generalized Dyson series. [Briefly, the bounded linear operator $K_\lambda^t(F)$ is defined as a genuine Wiener (path) integral in the diffusion case when λ is real, and then, for complex λ, by analytic continuation followed by passage to the limit along the imaginary axis of the resulting perturbation expansion.] Consequently, the time-ordered perturbation series for $K_\lambda^t(F)$ has the same general expression as a function of the parameter λ both in the diffusion (or probabilistic) case (λ *real* and positive) and in the quantum-mechanical (or Feynman) case (λ *purely imaginary* and nonzero). This fact enables us to deal with these two situations in parallel in much of Chapters 15–18. (Notable exceptions occur in Sections 16.2 and 17.6.)

There is a last general comment that we wish to make about the "disentangling" provided by our generalized Dyson series: It is not necessarily unique; indeed, a given operator $K_\lambda^t(F)$ can be represented in many different ways via a time-ordered perturbation series, some of which may be more suitable than others in a given situation. (See especially Section 15.5 for various examples; see also, for instance, Section 17.6.) We stress that in spite of this fact, the *operator* $K_\lambda^t(F)$ associated with a *function* F in the "disentangling algebra" $\mathcal{A}_t$ *is always defined uniquely* (and hence unambiguously). In some suitable sense, the mapping K_λ^t (defined in Section 15.7) can be thought of as a *quantization map* from the commutative disentangling algebra $\mathcal{A}_t$ to a noncommutative algebra of (bounded linear) operators. [See especially Chapter 18 (including Appendix 18.6), where the action of the noncommutative operations $*$ and $\dot{+}$ on the family of disentangling algebras $\{\mathcal{A}_t : t > 0\}$ is taken into account.]

The use of measures

Measures and their associated integrals enter into this book in various ways. We mention some of these here and emphasize those which are less widely familiar but will be especially important to us.

Two of the definitions of "the" Feynman integral that are stressed in this book start with the Wiener integral. Our purpose in Chapter 3 is to give the reader who is not acquainted with Wiener measure some idea of how it can be constructed and some familiarity with the properties of the Wiener process that will be needed in subsequent work. The construction follows the pattern of Lebesgue measure on the line, a topic familiar to most mathematicians. It begins with the definition of the measure of an "interval" and ends with an application of the Carathéodory extension theorem.

We expect that many potential readers will be familiar with the results of Chapter 3 and with Lévy's quadratic variation law which is the subject of Section 4.1. However, we anticipate a much lower degree of familiarity with most of the rest of Chapter 4 which deals with such topics as the family of scaled Wiener measures $\{\mathfrak{m}_\sigma : \sigma > 0\}$, scale-invariant measurability [JoSk7] and the refined equivalence classes of functions that are needed for a careful discussion of the Feynman integral obtained via analytic continuation in mass. This definition of the Feynman integral will concern us in Section 13.5 (the second approach in [Ne1]) as well as throughout Chapters 15–18.

Measures on subintervals of $\mathbb{R}$ (Lebesgue–Stieltjes measures) are used systematically throughout Chapters 14–19 in connection with Feynman's operational calculus. They serve not only to assign weights but also to time-order the integrands which are usually (perhaps after some preliminary steps) products of noncommuting operators. The measures give directions for "disentangling", and a different set of measures can yield very different results. The first few pages of Section 14.2 (through Example 14.2.1) can be read independently of all of the earlier material in this book and will provide the reader with a discussion of Feynman's heuristic "rules" and an extremely simple example of the points made above.

1.3 Relationship with the motivating physical theories: background and quantum-mechanical models

What does this book have to say about the physical theories which motivate it? The reader will not find here applications to concrete and detailed physical problems of the mathematical results contained within. However, in certain respects, we do discuss in a number of places related physical theories and especially quantum mechanics.

Physical background

A discussion of the relevant physical background is provided in key places. Most importantly, Feynman's way of looking at quantum mechanics and his path integral and how this has led to the approaches to the Feynman integral found in this book is the subject of Chapter 7. Chapter 6 contains an extremely brief discussion of some parts of the usual Hamiltonian approach to quantum dynamics; this chapter is included partly for the sake of contrast but also because the two approaches have, of course, some common features. It seems to us that it is difficult to get an appreciation for the mathematics of the Feyman integral without at least some understanding of the physical background.

As noted earlier, this book contains a good deal of information about the Wiener integral (see Chapters 2–5, 7 and 12–18). Much of this material, apart from Chapter 3, Section 4.1 and Chapter 5, is not the standard fare but consists of special topics related to the two items in the title of this book. Chapter 2 discusses the character of physical Brownian motion and the way in which that led Norbert Wiener, through the work of Brown, Einstein and Perrin, to what is now known as the Wiener process, the mathematical model of Brownian motion.

Chapter 14 is an introduction to Feynman's operational calculus. Some discussion of the physical problems that led Feynman to this calculus can be found there, but much less than one might guess. Why is that?

The primary purpose of the paper [Fey8], "An operator calculus having applications in quantum electrodynamics", was to present the ideas and rules which Feynman had developed in connection with [Fey5–7] for forming functions of noncommuting operators. While most of the examples in [Fey8] are from quantum theory, Feynman was well aware that he had found a computational technique with implications beyond that particular setting. [In fact, this point was stressed repeatedly by Richard Feynman himself in a number of conversations with the second-named author (M. L. L.), during the first of which (in about 1981) Feynman mentioned his paper [Fey8] on the subject and urged M. L. L. to develop his operational calculus and to put it on a firm mathematical basis.] Chapter 14 is an exposition of these *mathematical* (but not mathematically rigorous) ideas of Feynman and how they will be interpreted, extended and developed with mathematical rigor in Chapters 15–19.

[The reader may be aware of Feynman's sometimes negative comments about some of the mathematicians' musings (see, for example, [Fey16,17]). However, he/she may wish to contrast this impression with Feynman's comments in [Fey8, p. 108] regarding the need for mathematical rigor and for further mathematical exploration of his "operator calculus". (See the second quote from [Fey8] at the very beginning of Chapter 14, which is in complete agreement with the second author's conversations with Feynman.) Perhaps it is appropriate at this point to add two more personal recollections. When asked by a physics Ph.D. student how much mathematics he needed to learn, Feynman answered without hesitation: "As much as possible." (This was witnessed by the second author in Los Angeles in 1981.) Finally, and to give a more balanced view, when during his 1983 UCLA public lectures for a general scientifically curious audience (of which his book *QED*, [Fey15], is an edited version), he was asked what were the relationships between mathematicians and physicists, he began his answer (approximately) as follows: "They are very good friends, but they do not consider the same problems, and they do not have the same point of view. The mathematician looks at a very broad area and is interested in everything related to it. The physicist, on the other hand, who is interested in certain specific questions, can go much further in some particular directions..."]

The discussion of physical background and physical interpretation of results goes beyond the introductory chapters mentioned above. It can be found in various places throughout the book. We mention Chapters 11, 13, 15, 16 and especially, Chapters 17 and 19.

Highly singular potentials

A variety of quantum-mechanical models are discussed in this book. These include in Chapters 11 and 13 highly singular potentials V and the standard Hamiltonian

$$H = -\tfrac{1}{2}\Delta + V. \tag{1.3.1}$$

In (1.3.1), V denotes the operator of multiplication by a time independent, real-valued potential energy function V. [The precise form of H when the mass m and $\hbar =$ (Planck's constant)/2π are not normalized is given in (6.4.1). For the case of an N-particle system where the jth particle has mass m_j, $j = 1, \ldots, N$, see (6.4.2).] The inclusion of highly singular potentials in the approaches to the Feynman integral discussed in Chapters 11 and 13 is a major advantage of those approaches. Some of the most basic potentials of quantum mechanics such as the Coulomb potential are singular in the space variables. (See [FrLdSp] for a detailed account from a physicist's point of view of the role of singular potentials in quantum theory.)

A discussion of highly singular central potentials is provided in Example 11.4.7 and pursued in Example 13.6.13. The interesting special case of the inverse-square potential is treated in Example 13.6.18.

We give in Example 11.4.12 and in parts of Sections 13.5 and 13.6 a brief discussion of a refined and highly singular Hamiltonian which is obtained by supplementing H in (1.3.1) by a magnetic vector potential. This corresponds to the Schrödinger equation associated with a magnetic as well as an electric field. Further, in Example 11.4.10, we consider the case where a d-dimensional Riemannian manifold replaces Euclidean space $\mathbb{R}^d$.

Time-dependent potentials

The operator-valued Feynman integral used in Chapters 15–18 is defined via analytic continuation in mass. In those chapters, the emphasis is on Feynman's operational calculus and, in particular, on disentangling via time-ordered perturbation series by using the Wiener and Feynman integrals. The "potentials" allowed there are very general in most respects; they can be time-dependent and complex-valued and no smoothness assumptions are made. However, they are required to be essentially bounded in the space variables; that is, no spatial singularities are permitted. (Hence, for instance, the Coulomb potential is not allowed in this setting since it has a blow-up singularity at the origin.)

Potentials which are bounded and may be time-dependent appear in various places in the physics literature. Forces that are under the control of an experimenter provide a natural source of examples of potentials that are both time-dependent and bounded.

It is not just the potentials θ that influence the possible physical models in Chapters 15–18, but also the Lebesgue–Stieltjes measure η as in (1.1.4). These measures determine the disentangling (as noted earlier) and, when combined appropriately with an exponential function, determine the evolution of an associated physical system (see Chapter 17). [We refer, in particular, to Section 17.5 for possible physical interpretations of the corresponding results both in the quantum-mechanical (or Feynman) case and in the diffusion (or probabilistic) case.] The fact that such measures may have continuous and/or discrete parts allows us to study both continuous and discrete phenomena and their relationships with one another. This considerably broadens our approach to Feynman's operational calculus via Wiener and Feynman path integrals in Chapters 15–18. Mathematically, it also gives a rich combinatorial structure to the time-ordered perturbation expansions (or generalized Dyson series) and the associated generalized Feynman graphs introduced in Chapter 15 and used throughout the above chapters.

A brief discussion is given in Section 13.5 of Haugsby's extension of Nelson's second approach to the Feynman integral. This is the only place in the book where potentials are treated which can be both singular in the space variables and time-dependent.

Phenomenological models: complex and nonlocal potentials

We are also able to treat certain phenomenological models. By a phenomenological model, we mean one that does *not* arise from the basic principles of quantum mechanics but has, nevertheless, been found useful in modeling certain quantum systems. We have already mentioned complex potentials above. Such potentials are used in modeling dissipative (or open) quantum systems. An extensive discussion of this topic—including its strengths and weaknesses and its relationship with "the" Feynman integral—can be found in Exner's book [Ex], *Open Quantum Systems and the Feynman Integral.* Complex potentials are permitted in some of the results in Chapters 11 and 13 (see especially Sections 11.6 and 13.6, as well as the end of Section 13.5) and in nearly all of the results in Chapters 15–18. The setting of Chapter 19 is more general, but operators of multiplication by a potential can be considered, and, when they are, the potentials involved can be both time-dependent and complex-valued.

Chapter 19 deals with time-dependent families $\{\beta(s) : 0 \leq s < \infty\}$ of bounded operators on a Hilbert space. (A strongly continuous semigroup of operators on the Hilbert space and the generator of that semigroup are also involved but are not particularly relevant to the present comments.) *Nonlocal* potentials are used phenomenologically in many body problems in several areas of quantum physics (see [Tab, ChSa, Mc] and the relevant references therein). The operator is an integral operator whose kernel $V(x, y)$ (or $V(s; x, y)$ if we have time-dependence) is referred to as a "nonlocal potential". It is nonlocal in that this "potential" does *not* depend on one sharp choice for the space coordinates (see formula (19.7.15)). Such nonlocal potentials are used, for example, in nuclear physics where the kernels used to model various situations are surprisingly simple; they are, in practice, separable kernels of finite (and low) rank (see Example 19.7.5).

Finally, we mention that some of the highly singular potentials discussed just above and treated in Section 11.4 and Sections 13.5–13.6 can also be viewed as providing suitable phenomenological models for certain problems occurring in quantum physics or in molecular chemistry. (See, for example, [LL, Ne1, FrLdSp].) For instance, the attractive inverse-square potential (Example 13.6.18) and more generally, highly singular attractive or repulsive central potentials (as in Examples 11.4.7 and 13.6.13), can be used to model problems occurring in quantum field theory or in polymer physics. They are often considered as "nonphysical" or only of academic interest because, in particular, they may lead (as in Example 13.6.18) to nonunitary evolutions and thus to Schrödinger operators which are no longer self-adjoint—in contradiction with one of the basic tenets of standard Hamiltonian quantum mechanics. (This unusual aspect is apprehended naturally within the context of the various approaches to analytic-in-mass Feynman integrals discussed in Sections 13.5 and 13.6; see [Ne1, Kat7, BivPi, BivLa].) Actually, the situation is somewhat more complicated than that and a suitable dose of pragmatism is needed to decide which model (whether of Feynman type or of Hamiltonian type, say) is most appropriate for a given physical situation; see, for instance, [Case, R, FrLdSp] and Example 13.6.18. In spite of these drawbacks, it can be argued convincingly that such highly singular potentials provide better approximate (or "phenomenological") models of suitable physical systems than their more regular counterparts. (See especially the review article [FrLdSp] as well as, for example, [LL, Ne1, PariZi, MarPari] and the relevant references therein.)

In closing the main part of this introduction, we briefly return to Chapter 20 which, as was mentioned earlier, is of a very different nature than the rest of this book. We recall, in particular, that in Section 20.2, we discuss some of the relationships between heuristic Feynman-type integrals (as well as aspects of Feynman's operational calculus) and a variety of subjects from contemporary physics (or mathematics). In addition, in Section 20.2.A, several mathematical or physical models inspired by quantum field theory (specifically, "Chern–Simons gauge theory" defined in terms of a heuristic Feynman–Witten functional integral [Wit14]), are used to gain insight into (and extend) the celebrated Jones polynomial, along with other topological invariants that are central to modern knot theory and low-dimensional topology.

We hope that despite its relative brevity, Section 20.2 will prove helpful to a reader interested in getting a sense of the fascinating interplay between heuristic Feynman path integrals and a number of topics lying at the border of mathematics and theoretical physics.

Prerequisites, new material, and organization of the book

We end this introduction by making some specific comments about the content and the structure of this book, along the lines suggested in the title of the present italicized subsection.

As was mentioned in the preface and further explained earlier on in this chapter, much of the background material needed for the main part of this book is provided here; see especially Chapters 3,4, 6, 8–10, along with Chapter 7.

Detailed proofs—based mainly on the background material just referred to—are given for nearly all the theorems which deal with the main topics of this book. Most of the exceptions come in Sections 13.6, 13.7 and in the last part of Section 13.5, as well as in Section 17.6.

The reader will see that proofs are provided even for a good portion of the background material itself; see, in particular, Sections 3.1–3.4, 4.1–4.2, 4.5–4.6, and 10.2–10.3. We remark that if the reader is willing to forego the proofs in Sections 11.6, 13.5 and 13.6, then the operator-theoretic background needed for the book (especially through Chapter 13) is reduced to the information about self-adjoint operators and quadratic forms found in Section 9.6 and in Chapter 10 plus relatively few basic facts about semigroups of operators provided in Chapters 8 and 9.

We should mention that the comments in the preceding paragraphs do not apply to Chapter 20; no attempt there is made to supply proofs. (In the case of Section 20.2, in which much of the material connected with Feynman path integrals is of a heuristic nature, rigorous mathematical proofs are usually not known.)

The Lebesgue theory of measure and integration is employed in many places in this book. Precise references are typically given for the results used, but no systematic presentation of measure-theoretic prerequisites is provided. Brief discussions of this subject can be found in the books [BkExH, Appendix A, pp. 531–544], [Ru2, pp. 5–75] and [ReSi1, pp. 12–26]. Fuller treatments are given in many places, for example, in [Roy, Fol2, Coh, Du].

The list of references provided at the end of this book is extensive but certainly not complete. (We note that a significant fraction of the references is connected in some way with Chapter 20, which deals with a broad selection of topics.) When the references are given in the main body of the text, they are typically presented in enough detail so that the relevant material can be easily located. The topics discussed in this book are interrelated in a variety of ways; we try to keep track of these relationships by systematic cross-referencing.

A substantial part of the material in this book other than the background material has appeared previously only in the research literature and, in a number of cases, only in the recent research literature. A few results that will play a prominent role come from sources that are not widely available. The primary example of that is the Sherbrooke monograph of the first author [Jo6] which plays a key role in the operator-valued analytic continuation in time results in Chapter 13.

There is a significant amount of novelty to the exposition in several places. For example, in Section 4.6, we discuss in detail what is meant by the "nonexistence of Feynman's measure" as well as related issues. Chapter 14 is an introduction to Feynman's operational calculus for noncommuting operators. This subject extends certain aspects of the Feynman integral, a fact that does not seem to be widely understood in the mathematical literature. We explain this in some detail in Chapter 14, and the idea is developed further in Chapters 15–19.

It will be clear to the reader of this book that the research interests of the authors have influenced much of the content. However, the influence went the other way as well; the desire to fill in missing pieces of the book directed some of our research in recent years. (For instance, reference [dFJoLa2]—on which Chapter 19 is based—was very much written with our book in mind.) A portion of that work is new here. We wish to call the reader's attention to a few such items.

In Section 13.4, we show that under rather general conditions, three of the four approaches that are discussed in this book are closely related. Most of this material is new.

The last part of Section 13.5 and nearly all of Section 13.6 deal with product formulas and operator-valued analytic continuation in mass from such formulas (rather than from the Wiener integral). A good portion of this material is new as well, particularly with regard to the explicit connections with the Feynman integral. Further, a comparison of the various analytic-in-mass Feynman integrals is provided, along with related results; see Theorems 13.6.10 and 13.6.11, along with Corollary 13.5.18. In addition, a detailed treatment of highly singular central potentials is given in this context; see Examples 13.6.13 and 13.6.18.

In Section 15.7, a "time-reversal map" is introduced and studied for our disentangling algebras in Chapters 15 and 18; see Definition 15.7.5, Theorem 15.7.6 and Corollary 15.7.8. This enables us, in particular, to clarify the connections with the usual "physical ordering" in the context of Feynman's operational calculus. In turn, these changes have repercussions in Chapter 17. We expect that more work will be done along related lines in the future.

New examples are provided in several places. We mention one, Example 15.5.3, that may be of particular interest. This example involves the purely continuous but singular measure associated with the Cantor function.

At the end of Chapter 16, we indicate how results from our earlier work (contained in [JoLa1]) on "stability" in the measures can be extended. We carry this out in detail in one case (see Proposition 16.2.14) and indicate how to go further in another case.

In Remark 11.5.15(d), we point out that the requirements on the negative part of the dominating "potential" in Theorem 11.5.13 (from [La12]) can be reduced to membership in the "Kato class" of functions on $\mathbb{R}^d$. (Theorem 11.5.13 is a dominated-type convergence theorem for the modified Feynman integral; it is the subject of Section 11.5 and is also applied in Chapter 13 to other approaches to the Feynman integral.)

Additional work on Feynman's operational calculus by Brian Jefferies and the first author ([JeJo]) is discussed briefly in III of Section 14.4. That work provides a nice supplement to the treatment given in Chapters 15–19 of Feynman's operational calculus (and based on [JoLa1–4, La14–18, dFJoLa1,2]). However, some aspects of the new material still need further development and so a fuller discussion of this topic could not be included in this book.

Several exercises or problems are proposed throughout the book. They are of varying degrees of difficulty. Typically, the exercises are mainly intended to illustrate a new concept, apply a new technique, or supplement some material in the text. Most of them should be accessible to graduate students. However, in a few instances, some of the proposed problems are extremely difficult and not yet solved in the literature (e.g. Problem 11.3.9). In other cases, they correspond to results already published but the proof of which is not discussed fully in the book (e.g. Problem 17.3.6 or 17.6.28). In addition, a few open-ended problems—the precise interpretation or formulation of which is left to the reader—are provided either formally (e.g. Problem 17.6.31) or in various comments or remarks scattered in the text. When appropriate, we have usually indicated the nature or the difficulty of the problem at hand.

The numbering system used in this book is straightforward. For example, Theorem 11.5.13 is the thirteenth numbered item in Section 5 of Chapter 11; a similar comment applies to the numbering of equations. Further, Section 15.4 is the fourth section of Chapter 15. Frequently, unnumbered subsections with italicized headings are used within a given section in order to delineate or highlight certain topics.

Indexes for symbols or notation, authors, and subjects are provided just after the bibliography. Along with the detailed list of contents, we hope that they will prove to be a useful guide to the reader throughout this book.

2

THE PHYSICAL PHENOMENON OF BROWNIAN MOTION

While examining the form of these particles immersed in water, I observed many of them very evidently in motion; their motion consisting not only of a change of place in the fluid, manifested by alterations of their relative positions, but also not infrequently of a change in form of the particle itself; a contraction or curvature taking place repeatedly about the middle of one side, accompanied by a corresponding swelling or convexity on the opposite side of the particle. In a few instances, the particle was seen to turn on its longer axis. These motions were such as to satisfy me, after frequently repeated observations, that they arose neither from currents in the fluid, nor from its gradual evaporation, but belonged to the particle itself.

Robert Brown, 1827

In this paper it will be shown that according to the molecular-kinetic theory of heat, bodies of microscopically-visible size suspended in a liquid will perform movements of such magnitude that they can be easily observed in a microscope, on account of the molecular motions of heat. It is possible that the movements to be discussed here are identical with the so-called "Brownian molecular motion"; however, the information available to me regarding the latter is so lacking in precision that I can form no judgment in the matter.

Albert Einstein, 1905 [Ei, p. 1]

We will still stay within the realm of experimental reality if, putting our eye on a microscope, we observe the Brownian motion which agitates every small particle suspended in a fluid. In order to obtain a tangent to its trajectory, we should find a limit, at least approximatively, to the direction of the line which joins the positions of this particle at two successive instants very close to each other. But, as long as one can perform the experiment, this direction varies wildly when we let the duration that separates these two instants decrease more and more. So that what is suggested by this study to the observer without prejudice, is again the function without derivative, and not at all the curve with a tangent.

Jean Perrin, 1913 [Per, p. 27]

2.1 A brief historical sketch

The distinguished English botanist Robert Brown made the first careful study of "Brownian motion". In 1827, he was investigating the fertilization process in a certain species of flower. While looking at the pollen in water through a microscope, he observed small particles in "rapid oscillatory motion". He examined the pollen of other species with similar results. He first hypothesized that the motion was particular to the male sexual cell of plants and next that the motion involved living matter. His experiments with inorganic material showed that both of these are wrong. What does cause the motion? Brown did an experiment which refutes some explanations that were put forth well after his study; he immersed a drop of water containing *one* particle in oil and looked at it under the microscope. The motion is observed as before. It is clear then for one thing that the source of the motion is not a force of attraction between the particles. Brown also noted that when several particles are present, they appear to move independently of one another even when they are very close together. Brown's conjecture, carefully labeled as such, was that matter is composed of small particles, which he calls active molecules, which exhibit a rapid, irregular motion having its origin in the particles themselves and not in the surrounding fluid. Considerably later, it became clear that the surrounding fluid *is* the source of the motion.

A brief discussion of the work between Brown and Einstein is given in [Ne2, pp. 11–13] and by Fürth in Note 1 of [Ei, pp. 86–88]. These make interesting reading and illustrate well that progress in science does not always proceed linearly. We restrict our attention to a few positive contributions. Cantoni and Oehl found in 1865 that the movement continued unchanged for a year. S. Exner observed in 1867 that the motion is most rapid with the smallest particles and is increased by light and heat rays. In 1877, Delsaux expressed the now commonly accepted idea that Brownian motion is caused by the impacts of the molecules of the liquid on the particles. Guoy found in 1888 that the motion becomes more lively as the viscosity of the fluid is decreased and that, on the other hand, a strong electromagnetic field has no effect. Like Delsaux, Guoy attributed the motion of the particle to the molecular motions of the fluid.

The main points made by observing Brownian motion are as follows:

(1) The path followed by the particle is continuous but extremely jagged.

(2) The particles move independently of one another.

(3) The motion is more active the smaller the particles.

(4) The composition and density of the particles have no effect.

(5) The motion is more active the less viscous the fluid.

(6) The motion is more active the higher the temperature.

(7) The motion never ceases.

It has been generally accepted for a long while now that the source of the motion is the bombardment of the particle by the molecules of the surrounding fluid. The following remark addresses the extent to which (1)–(7) above are consistent with this explanation.

Remark 2.1.1 *(a) It is certainly reasonable that a particle under molecular bombardment would follow a continuous path. The extreme jaggedness of the path seems less easy to account for but is plausible given the random and almost continuous bombardment of the particle.*

(b) Items (2) and (7) above are quite consistent with the molecular-kinetic theory.

(c) It is not immediately evident that (3) is consistent. How does one attempt to explain it? Individual molecular hits do not produce observable motion. Such motion occurs when there is a preponderance of hits in one or the other direction. The larger the particle, the less often will the hits be sufficiently unbalanced to produce observable motion.

(d) If the motion comes from the bombardment of the particles by the molecules of the fluid, is it reasonable that the density of the particles has no effect as (4) asserts? Not

entirely it seems. What probably happens is that within the range of densities, viscosities, etc. where Brownian motion is observed, (4) is approximately true. Relatively high viscosity is present when Brownian motion is observed and so the velocity caused by a bump (or bumps) is quickly damped out. This will also be relevant to our discussion of "independent increments" in the next section.

(e) The viscosity of a fluid is a measure of internal friction and so it affects the ease with which a particle moves through the fluid. As such, (5) is not surprising.

(f) Since increasing the temperature speeds up molecular action, (6) is certainly consistent with the molecular-kinetic theory.

The quantitative theory of Brownian motion began with Albert Einstein. His five papers on the subject between 1905 and 1908 made careful quantitative predictions based on the molecular-kinetic theory of heat.

Einstein's first paper in 1905 was a prediction motivated by the molecular-kinetic theory that a phenomenon with properties similar to Brownian motion ought to be observable in nature. He was apparently just becoming aware of the earlier studies on Brownian motion. He says, "It is possible that the movements to be discussed here are identical with the so-called 'Brownian molecular motion'; however, the information available to me regarding the latter is so lacking in precision that I can form no judgment in the matter." On the other hand, his second paper in 1906 is titled "On the theory of the Brownian movement". See [Ei], where his five papers on Brownian motion are reprinted. [In 1905, Einstein formulated the special theory of relativity, studied the photoelectric effect and wrote his first paper on Brownian motion—one of the most productive years in the history of science; see [Srt], where his five 1905 papers are reprinted.]

Let $\rho = \rho(u, t)$ be the probability density that a Brownian particle starting at the origin at time 0 is at u at time t. Through physical reasoning, Einstein derived the diffusion equation

$$\frac{\partial \rho}{\partial t} = D\Delta\rho,$$

where D is a positive constant called the coefficient of diffusion. A solution to this equation is given by

$$\rho(u, t) = \frac{1}{(4\pi Dt)^{3/2}} e^{-\frac{\|u\|^2}{4Dt}}.$$

Hence, *the probability that the particle is in say a cube E at time t*

$$= \int_E \rho(u, t)du. \tag{2.1.1}$$

Remark 2.1.2 *(a) That Einstein expressed position at time t in probabilistic terms was not surprising since this is the framework of the molecular-kinetic theory. There are too many particles involved to use Newtonian mechanics effectively.*

(b) In his 1923 paper Differential space ([Wi1], reprinted in [Wi3, pp. 455–498]), Wiener obtained the formula (2.1.1) quite differently. He combined relatively elementary

physical considerations with the central limit theorem to obtain (2.1.1). He does not specifically invoke the central limit theorem, but it is clear that he had it in mind. We will give a closely related discussion in the next section.

Contained in the formula (2.1.1) is the information that the mean square displacement of a particle in time t is $2Dt$. We will return to this fact shortly.

Assuming that the particles are spheres of radius a, Einstein also derived the formula

$$D = \frac{kT}{6\pi \eta a}, \tag{2.1.2}$$

where η is the coefficient of viscosity, T is the temperature and k is Boltzmann's constant. Note how points (3)–(6) earlier are reflected in the formula (2.1.1). [Einstein's papers contain much more, but the key for us will be (2.1.1).]

In 1909, the French physicist Jean Perrin observed that the Brownian trajectories appear to have no tangents and mentions the nowhere differentiable curves of Weierstrass. This comment led to a beautiful theorem of Norbert Wiener.

Wiener [Wi1] quoting Perrin [Per]: "One realizes from such examples how near the mathematicians are to the truth in refusing, by a logical instinct, to admit the pretended geometrical demonstrations, which are regarded as experimental evidence for the existence of a tangent at each point of a curve."

Are the paths traced out by the *physical* Brownian particles actually nondifferentiable? We suppose not. Consider what is likely to happen to the difference quotient $\frac{x(t+h)-x(t)}{h}$ when h is orders of magnitude smaller than the average time between molecular hits. On the other hand, the paths of the Wiener process, the mathematical model of Brownian motion, *will* turn out to be nondifferentiable. We remark that there is another model for Brownian motion, due to Ornstein and Uhlenbeck, such that the associated paths are differentiable [Ne2].

The Brownian motion studies played an important role in supporting the atomistic theory at a time when this theory was much in doubt. In Einstein's formula (2.1.2), T and η can be calculated and it can be arranged that the particles are all spheres of the same radius a. Further, the mean square displacement $2Dt$ (see above) at time $t = 1$ can be estimated statistically. The idea is simple: Observe the paths $x_j(\cdot)$, $j = 1, \ldots, n$, in $\mathbb{R}^3$ of n independent Brownian particles starting at the origin at time 0 and record their positions when $t = 1$. We then have

$$2D \approx \frac{1}{n} \sum_{j=1}^{n} \|x_j(1)\|^2.$$

Hence one arrives at a statistical estimate for Boltzman's constant k,

$$k = \frac{6\pi\eta a D}{T} \approx \frac{3\pi\eta a}{T} \left[\frac{1}{n} \sum_{j=1}^{n} \|x_j(1)\|^2 \right],$$

and so also for Avogadro's number $N_A = R/k$, where R is the universal gas constant. This was done by Perrin (and others) and the value was sufficiently close to the value obtained by another quite different approach to be regarded as a significant confirmation of the atomistic theory. (Recall that Avogadro's number, $N_A \approx 6 \times 10^{23}$, is defined as the number of atoms or molecules in a mole of substance.) Perrin received the Nobel prize in 1926 in recognition of his work on Brownian motion.

Remark 2.1.3 *Many of the conclusions drawn by Einstein concerning Brownian motion were reached at about the same time by the Polish physicist, M. Smoluchowski [Smo]. Even earlier, L. Bachelier in his 1900 dissertation [Bac] arrived heuristically at some of the same mathematical formulas. Bachelier's study was motivated by a very different problem; he was attempting to analyze the French stock market. It was primarily Einstein's work that influenced Norbert Wiener's later brilliant study of "Brownian motion" ([Wi1,2], [Wi3, §IC]).*

We note that much of the material in this section is adapted from d'Abro [dA, pp. 411–415], Einstein [Ei] (including the notes by Fürth on pp. 86–88), and especially from Sections 2 and 3 of Nelson's book *Dynamical Theories of Brownian Motion* [Ne2].

2.2 Einstein's probabilistic formula

The starting point in the introduction of Wiener measure is Einstein's probabilistic formula (2.1.1) which we now recall:

The probability that a particle which starts at the origin at time 0 is in a (Lebesgue measurable) set E at time t

$$= \int_E \frac{1}{(4\pi Dt)^{3/2}} e^{-\frac{\|u\|^2}{4Dt}}\, du. \tag{2.2.1}$$

In order to simplify matters, we concern ourselves for now with the motion of the particle in a single direction. Further, we normalize the diffusion coefficient, taking $2D = 1$, and we take the set E ($E \subseteq \mathbb{R}$ now) to be the interval $(\alpha, \beta]$. Formula (2.1.1) then becomes:

The probability that a particle which starts at the origin at time 0 is in $(\alpha, \beta]$ at time t

$$= \int_\alpha^\beta \frac{1}{(2\pi t)^{1/2}} e^{-\frac{u^2}{2t}}\, du. \tag{2.2.2}$$

Letting $x = x(t)$ be the location of the particle at time t, we can rewrite (2.2.2) as

$$\text{Prob.}(\{\alpha < x(t) \le \beta\}) = \int_\alpha^\beta \frac{1}{(2\pi t)^{1/2}} e^{-\frac{u^2}{2t}}\, du. \tag{2.2.3}$$

In words, the right-hand side of (2.2.3) gives the probability that the path traced out by the particle goes through the gap $(\alpha, \beta]$ at time t. (See Figure 2.2.1.)

We will have to go far beyond the simple sets of paths involved in (2.2.3) to develop a satisfactory mathematical theory. Before doing this, however, *we wish to give a largely heuristic discussion of the appropriateness of (2.2.3) as a starting point for a mathematical model of Brownian motion*. Some knowledge of introductory probability theory will be needed for what immediately follows but will not be necessary as we continue.