

Path Integral Methods and Applications*

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Abstract

These lectures are intended as an introduction to the technique of path integrals and their applications in physics. The audience is mainly first-year graduate students, and it is assumed that the reader has a good foundation in quantum mechanics. No prior exposure to path integrals is assumed, however.

The path integral is a formulation of quantum mechanics equivalent to the standard formulations, offering a new way of looking at the subject which is, arguably, more intuitive than the usual approaches. Applications of path integrals are as vast as those of quantum mechanics itself, including the quantum mechanics of a single particle, statistical mechanics, condensed matter physics and quantum field theory.

After an introduction including a very brief historical overview of the subject, we derive a path integral expression for the propagator in quantum mechanics, including the free particle and harmonic oscillator as examples. We then discuss a variety of applications, including path integrals in multiply-connected spaces, Euclidean path integrals and statistical mechanics, perturbation theory in quantum mechanics and in quantum field theory, and instantons via path integrals.

For the most part, the emphasis is on explicit calculations in the familiar setting of quantum mechanics, with some discussion (often brief and schematic) of how these ideas can be applied to more complicated situations such as field theory.

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1 Introduction

1.1 Historical remarks

We are all familiar with the standard formulations of quantum mechanics, developed more or less concurrently by Schroedinger, Heisenberg and others in the 1920s, and shown to be equivalent to one another soon thereafter.

In 1933, Dirac made the observation that the action plays a central role in classical mechanics (he considered the Lagrangian formulation of classical mechanics to be more fundamental than the Hamiltonian one), but that it seemed to have no important role in quantum mechanics as it was known at the time. He speculated on how this situation might be rectified, and he arrived at the conclusion that (in more modern language) the propagator in quantum mechanics “corresponds to” $\exp iS/\hbar$, where S is the classical action evaluated along the classical path.

In 1948, Feynman developed Dirac’s suggestion, and succeeded in deriving a third formulation of quantum mechanics, based on the fact that the propagator can be written as a sum over all possible paths (not just the classical one) between the initial and final points. Each path contributes $\exp iS/\hbar$ to the propagator. So while Dirac considered only the classical path, Feynman showed that all paths contribute: in a sense, the quantum particle takes all paths, and the amplitudes for each path add according to the usual quantum mechanical rule for combining amplitudes. Feynman’s original paper,¹ which essentially laid the foundation of the subject (and which was rejected by Physical Review!), is an all-time classic, and is highly recommended. (Dirac’s original article is not bad, either.)

1.2 Motivation

What do we learn from path integrals? As far as I am aware, path integrals give us no dramatic new results in the quantum mechanics of a single particle. Indeed, most if not all calculations in quantum mechanics which can be done by path integrals can be done with considerably greater ease using the standard formulations of quantum mechanics. (It is probably for this reason that path integrals are often left out of undergraduate-level quantum mechanics courses.) So why the fuss?

As I will mention shortly, path integrals turn out to be considerably more useful in more complicated situations, such as field theory. But even if this were not the case, I believe that path integrals would be a very worthwhile contribution to our understanding of quantum mechanics. Firstly, they provide a physically extremely appealing and intuitive way of viewing quantum mechanics: anyone who can understand Young’s double slit experiment in optics should be able to understand the underlying ideas behind path integrals. Secondly, the classical limit of quantum mechanics can be understood in a particularly clean way via path integrals.

It is in quantum field theory, both relativistic and nonrelativistic, that path integrals (functional integrals is a more accurate term) play a much more important role, for several

¹References are not cited in the text, but a short list of books and articles which I have found interesting and useful is given at the end of this article.

reasons. They provide a relatively easy road to quantization and to expressions for Green's functions, which are closely related to amplitudes for physical processes such as scattering and decays of particles. The path integral treatment of gauge field theories (non-abelian ones, in particular) is very elegant: gauge fixing and ghosts appear quite effortlessly. Also, there are a whole host of nonperturbative phenomena such as solitons and instantons that are most easily viewed via path integrals. Furthermore, the close relation between statistical mechanics and quantum mechanics, or statistical field theory and quantum field theory, is plainly visible via path integrals.

In these lectures, I will not have time to go into great detail into the many useful applications of path integrals in quantum field theory. Rather than attempting to discuss a wide variety of applications in field theory and condensed matter physics, and in so doing having to skimp on the ABCs of the subject, I have chosen to spend perhaps more time and effort than absolutely necessary showing path integrals in action (pardon the pun) in quantum mechanics. The main emphasis will be on quantum mechanical problems which are not necessarily interesting and useful in and of themselves, but whose principal value is that they resemble the calculation of similar objects in the more complex setting of quantum field theory, where explicit calculations would be much harder. Thus I hope to illustrate the main points, and some technical complications and hangups which arise, in relatively familiar situations that should be regarded as toy models analogous to some interesting contexts in field theory.

1.3 Outline

The outline of the lectures is as follows. In the next section I will begin with an introduction to path integrals in quantum mechanics, including some explicit examples such as the free particle and the harmonic oscillator. In Section 3, I will give a “derivation” of classical mechanics from quantum mechanics. In Section 4, I will discuss some applications of path integrals that are perhaps not so well-known, but nonetheless very amusing, namely, the case where the configuration space is not simply connected. (In spite of the fancy terminology, no prior knowledge of high-powered mathematics such as topology is assumed.) Specifically, I will apply the method to the Aharonov-Bohm effect, quantum statistics and anyons, and monopoles and charge quantization, where path integrals provide a beautifully intuitive approach. In Section 5, I will explain how one can approach statistical mechanics via path integrals. Next, I will discuss perturbation theory in quantum mechanics, where the technique used is (to put it mildly) rather cumbersome, but nonetheless illustrative for applications in the remaining sections. In Section 7, I will discuss Green's functions (vacuum expectation values of time-ordered products) in quantum mechanics (where, to my knowledge, they are not particularly useful), and will construct the generating functional for these objects. This groundwork will be put to good use in the following section, where the generating functional for Green's functions in field theory (which *are* useful!) will be elucidated. In Section 9, I will discuss instantons in quantum mechanics, and will at least pay lip service to important applications in field theory. I will finish with a summary and a list of embarrassing omissions.

4 Topology and Path Integrals in Quantum Mechanics: Three Applications

In path integrals, if the configuration space has holes in it such that two paths between the same initial and final point are not necessarily deformable into one another, interesting effects can arise. This property of the configuration space goes by the following catchy name: non-simply-connectedness. We will study three such situations: the Aharonov-Bohm effect, particle statistics, and magnetic monopoles and the quantization of electric charge.

4.1 Aharonov-Bohm effect

The Aharonov-Bohm effect is one of the most dramatic illustrations of a purely quantum effect: the influence of the electromagnetic potential on particle motion even if the particle is *perfectly* shielded from any electric or magnetic fields. While classically the effect of electric and magnetic fields can be understood purely in terms of the forces these fields create on particles, Aharonov and Bohm devised an ingenious thought-experiment (which has since been realized in the laboratory) showing that this is no longer true in quantum mechanics. Their effect is best illustrated by a refinement of Young's double-slit experiment, where particles passing through a barrier with two slits in it produce an interference pattern on a screen further downstream. Aharonov and Bohm proposed such an experiment performed with charged particles, with an added twist provided by a magnetic flux from which the particles are perfectly shielded passing between the two slits. If we perform the experiment

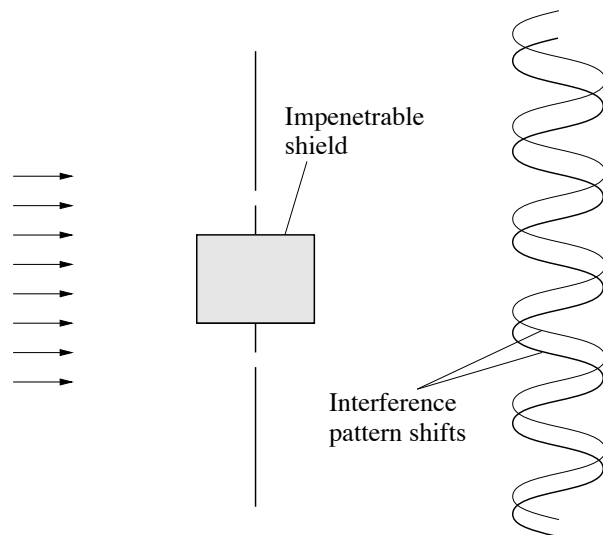


Figure 4: Aharonov-Bohm effect. Magnetic flux is confined within the shaded area; particles are excluded from this area by a perfect shield.

first with no magnetic flux and then with a nonzero and arbitrary flux passing through the shielded region, the interference pattern will change, *in spite of the fact that the particles are perfectly shielded from the magnetic field and feel no electric or magnetic force whatsoever.*

Classically we can say: no force, no effect. Not so in quantum mechanics. PIs provide a very attractive way of understanding this effect.

Consider first two representative paths $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$ (in two dimensions) passing through slits 1 and 2, respectively, and which arrive at the same spot on the screen (Figure 5). Before turning on the magnetic field, let us suppose that the actions for these paths are $S[\mathbf{q}_1]$ and $S[\mathbf{q}_2]$. Then the interference of the amplitudes is determined by

$$e^{iS[\mathbf{q}_1]/\hbar} + e^{iS[\mathbf{q}_2]/\hbar} = e^{iS[\mathbf{q}_1]/\hbar} \left(1 + e^{i(S[\mathbf{q}_2]-S[\mathbf{q}_1])/ \hbar} \right).$$

The relative phase is $\phi_{12} \equiv (S[\mathbf{q}_2] - S[\mathbf{q}_1])/\hbar$. Thus these two paths interfere constructively if $\phi_{12} = 2n\pi$, destructively if $\phi_{12} = (2n + 1)\pi$, and in general there is partial cancellation between the two contributions.

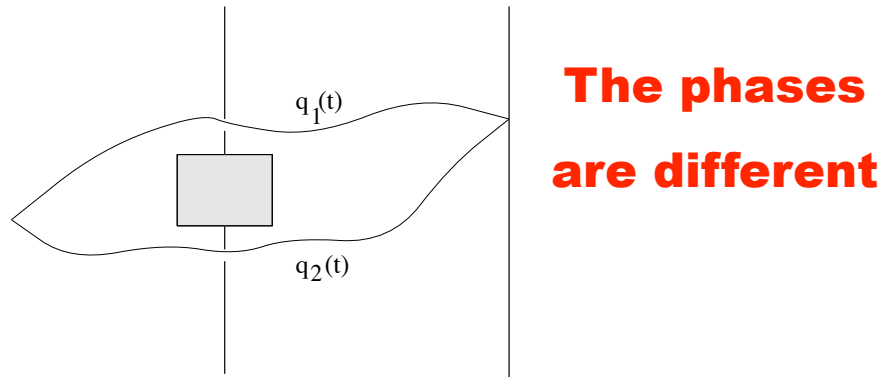


Figure 5: Two representative paths contributing to the amplitude for a given point on the screen.

How is this result affected if we add a magnetic field, \mathbf{B} ? We can describe this field by a vector potential, writing $\mathbf{B} = \nabla \times \mathbf{A}$. This affects the particle's motion by the following change in the Lagrangian:

$$L(\dot{\mathbf{q}}, \mathbf{q}) \rightarrow L'(\dot{\mathbf{q}}, \mathbf{q}) = L(\dot{\mathbf{q}}, \mathbf{q}) - \frac{e}{c} \mathbf{v} \cdot \mathbf{A}(\mathbf{q}).$$

Thus the action changes by

$$-\frac{e}{c} \int dt \mathbf{v} \cdot \mathbf{A}(\mathbf{q}) = -\frac{e}{c} \int dt \frac{d\mathbf{q}(t)}{dt} \cdot \mathbf{A}(\mathbf{q}(t)).$$

This integral is $\int d\mathbf{q} \cdot \mathbf{A}(\mathbf{q})$, the line integral of \mathbf{A} along the path taken by the particle. So including the effect of the magnetic field, the action of the first path is

$$S'[\mathbf{q}_1] = S[\mathbf{q}_1] - \frac{e}{c} \int_{\mathbf{q}_1(t)} d\mathbf{q} \cdot \mathbf{A}(\mathbf{q}),$$

and similarly for the second path.

Let us now look at the interference between the two paths, including the magnetic field.

$$\begin{aligned} e^{iS'[\mathbf{q}_1]/\hbar} + e^{iS'[\mathbf{q}_2]/\hbar} &= e^{iS'[\mathbf{q}_1]/\hbar} \left(1 + e^{i(S'[\mathbf{q}_2]-S'[\mathbf{q}_1])/ \hbar} \right) \\ &= e^{iS'[\mathbf{q}_1]/\hbar} \left(1 + e^{i\phi'_{12}} \right), \end{aligned} \tag{18}$$

CLASSIFICATION OF SOLVABLE FEYNMAN PATH INTEGRALS

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ABSTRACT

A systematic classification of Feynman path integrals in quantum mechanics is presented and a table of solvable path integrals is given which reflects the progress made during the last ten years or so, including, of course, the main contributions since the invention of the path integral by Feynman in 1942. An outline of the general theory is given. Explicit formulæ for the so-called basic path integrals are presented on which our general scheme to classify and calculate path integrals in quantum mechanics is based.

Table of Exactly Solvable Feynman Path Integrals

Quadratic Lagrangian	Radial Harmonic Oscillator	Pöschl-Teller Potential	Modified Pöschl-Teller Potential
Infinite square well	Liouville mechanics	Scarf potential	Reflectionless potential
Linear potential	Morse potential	Symmetric top	Rosen-Morse potential
Repelling oscillator	Uniform magnetic field	Magnetic top	Wood-Saxon potential
Forced oscillator	Motion in a section	Spheres	Hultén potential
Saddle point potential	Calogero model	Bispherical coordinates	Manning-Rosen potential
Uniform magnetic field	Aharonov-Bohm problems		Hyperbolic Scarf potential
Driven coupled oscillators	Coulomb potential		Pseudospheres
Two-time action (Polaron)	Coulomb-like potentials in polar and parabolic coordinates		Pseudo-bispherical coordinates
Second derivative Lagrangians	Nonrelativistic monopoles		Poincaré disc
Semi-classical expansion	Kaluza-Klein monopole		Hyperbolic Strip
Anharmonic oscillator	Poincaré plane		Hyperbolic spaces of rank one
	Hyperbolic space + magnetic field + potentials		Kepler problem on spheres, and on pseudospheres

Of course, in the case of general quantum mechanical problems, more than just one of the basic path integral solutions is required. However, such problems can be conveniently put into a hierarchy according to which of the basic path integral is

1998

C. Grosche F. Steiner

Handbook of Feynman Path Integrals



Springer

Preface

Our *Handbook of Feynman Path Integrals* appears just fifty years after Richard Feynman published his pioneering paper in 1948 entitled “Space-Time Approach to Non-Relativistic Quantum Mechanics”. As it is the case with many books, its origin goes back to a course first given by one of us (F.S.) on Feynman path integrals at the University of Hamburg during the summer semester of 1983. The other author was one of the students attending these lectures and who eventually decided to work on this subject for his diploma thesis. This was the starting point of our collaboration during the 1980s. At that time our main common interest was in the question of how to solve non-Gaussian path integrals (like the one for the hydrogen atom) and, more generally, path integrals in arbitrary curvilinear coordinates. It was in 1983, too, that one of us (F.S.) began to collect papers and preprints on path integrals, and to set up a comprehensive list of references on this subject. Eventually a systematic literature search was carried out (by C.G.). While we were working in various fields, above all in quantum chromodynamics, string theory, and quantum chaos, we conceived the idea of writing a *Handbook* on Feynman path integrals which would, on the one hand, serve the reader as a thorough *introduction* to the theory of path integrals, but would, on the other hand, also establish for the first time a comprehensive *table of Feynman path integrals* together with an extensive list of *references*. The whole enterprise was, however, delayed by various circumstances for several years. Here we put forward our *Handbook* to the gentle reader!

The book follows the general idea as originally conceived. Chapters 1–5 have the character of a *textbook* and give a self-contained, and up-to-date introduction to the theory of path integrals for those readers who have not yet studied path integrals, but have a good knowledge of the fundamentals of quantum mechanics as covered by standard courses in theoretical physics. Chapter 6 makes up the largest part of this Handbook and contains a rather complete *table of path integrals* in non-relativistic quantum mechanics, including supersymmetric quantum mechanics, and statistical mechanics. To each path integral listed in the table we attach a comprehensive list of *references* which altogether make up almost 1000 references. The *Introduction* in Chap. 1 is mainly of a historical nature and gives the reader some insight into the remarkable development of Feynman’s path integral approach. Since some of the historical facts are not so well known we thought it would be worthwhile to present them in Chap. 1.

Table 6.1. Applications of potential problems (examples)

Quadratic Lagrangian	Radial harmonic oscillator	Pöschl-Teller potential	Modified Pöschl-Teller pot.
Infinite square well	Coulomb potential	Scarf pots.	Reflectionless pot.
Linear potential	Morse potential	Symmetric top	Rosen-Morse pot.
Repelling oscillator	Uniform magnetic field	Magnetic top	Wood-Saxon pot.
Forced oscillator	Motion in a section	Higgs oscillator on spheres	Hultén pot.
Saddle point potential	Calogero model	Smorodinsky-Winternitz pot.	Manning-Rosen potential
Uniform magnetic field	Aharonov-Bohm potential		Hyperbolic Scarf potential
Driven coupled oscillators	Natanzon potential		Hyperbolic barrier potential
Two-time action (polaron)	Smorodinsky-Winternitz potentials		Hyperbolic spaces of rank one
Second derivative Lagrangians	Coulomb-like pots. in polar and parabolic coordinates		Kepler problem on (pseudo-) spheres
Semiclassical expansion	Non-relativistic monopoles		Natanzon potentials
Generating functional	Kaluza-Klein monopole		Hyperbolic strip
Moments formula	Poincaré plane + magnetic field + potentials		Higgs oscillator on pseudospheres
Effective potential	Dirac Coulomb problem		Hermitian spaces
Anharmonic oscillator	Anyons		Smorodinsky-Winternitz pots.

6 Table of Path Integrals

We present in the following *table of path integrals* exactly solvable path integrals according to the following classification scheme:

- 1) *General Formulæ*. This includes the different lattice definitions of path integrals on curved manifolds, transformation formulæ for canonical and time transformations, separable coordinate systems, and some perturbation methods.
- 2) *The General Quadratic Lagrangian*. Here we list the general formulæ for quadratic Lagrangians, including many explicit examples with electric and magnetic fields, couplings between oscillators in higher dimensions, two-time actions, some formulæ concerning the semi-classical approximation, trace formulæ, and, of course, the harmonic oscillator in its many appearances and modifications
- 3) *Discontinuous Potentials*. Here we state path integrals in half-spaces and boxes.
- 4) *The Radial Harmonic Oscillator*. This section includes Besselian type path integrals such as the Morse oscillator, motion in radial sectors, the Calogero model, and the general Besselian path integral, which is of the Natanzon type, cf. Table 6.3.
- 5) *The Pöschl–Teller Potential*. Path integrals related to the trigonometric version of the Pöschl–Teller potential are listed.
- 6) *The Modified Pöschl–Teller Potential*. This section contains path integrals related to the Rosen–Morse and Manning–Rosen potential, hyperbolic barriers, and the general Legendrian path integral, which is of the Natanzon type, cf. Table 6.3.
- 7) *Motion on Group Spaces and Homogeneous Spaces*. Path integrals for the quantum motion on homogeneous manifolds are listed, including some particular coordinate space representations, general expressions for path integrals on group spaces, and on spheres and hyperboloids.
- 8) *Coulomb Potentials*. Here we list all path integrals which are of the Coulomb type. They are related by means of a space-time transformation to Besselian path integrals.
- 9) *Magnetic Monopole and Anyon Systems*. In this section path integrals for monopoles, dyons, anyons, and applications to cosmology are cited.

- 10) *Motion in Hyperbolic Space.* Here we list path integrals for the quantum motion on hyperboloids. Some emphasis is on the hyperbolic plane, i.e., the Poincaré upper half-plane (Lobachevski space). Also the cases of magnetic fields are included, as well as the Higgs oscillator and the Kepler–Coulomb problem in spaces of constant curvature.
- 11) *Explicitly Time-Dependent Problems.* Here we list some general formulæ and specific examples of how to incorporate an explicit time dependence in the path integral. The general feature of this dependence is a “Galilean”-type modification of the usual potential problems.
- 12) *Point Interactions.* This section lists path integrals with point interactions, i.e., which are usually described by δ functions. General formulæ and some examples are presented. More general examples can be constructed by the interested reader by simply inserting some other path integral solution.
- 13) *Boundary Value Problems.* This section contains path integrals for the motion constrained by impenetrable walls and boxes with general boundary conditions. It generalizes Sect. 6.3, and includes the method of how to incorporate boundary conditions and absolute value problems from known unconstrained path integrals. Similarly to Sect. 6.12 general formulæ and some examples are presented.
- 14) *Coherent States.* Here the important coherent state path integral is given, together with several applications and generalizations to higher dimensions. Here we use $\hbar = 1$ throughout.
- 15) *Fermions.* Here the most important applications of the coherent state path integral are listed, i.e., the path integral formulation for fermions.
- 16) *Supersymmetric Quantum Mechanics.* Some supersymmetric path integral formulations and solutions are given.

In particular, the path integrals corresponding to solutions of the harmonic oscillator, respectively the general quadratic Lagrangian, are called *Gaussian path integrals* (section 6.2), those corresponding to the solutions of the radial harmonic oscillator are called *Besselian path integrals* (section 6.4), and those corresponding to the path integral solutions of the Pöschl–Teller and modified Pöschl–Teller potential are called *Legendrian path integrals* (Sect. 6.5 and 6.6), respectively. We call the *General Besselian* and *Legendrian path integral* solutions the *Basic path integrals*.

In the case of general quantum mechanical problems, more than just one of the basic path integral solutions is required. However, such problems can be conveniently put into a hierarchy according to which of the basic path integrals is the most important one for its solution. This classification scheme is listed in Table 6.1.

<http://users.physik.fu-berlin.de/~kleinert/kleinert/>

Hagen Kleinert

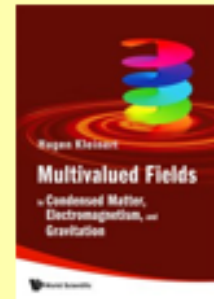
Professor Dr. Dr. h.c. mult.

Welcome to my Homepage!



receiving the **Max Born Prize with Medal 2008** in London
(on stage with with **Annemarie Kleinert**)

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8.1.1 Postulate I

The probability $P(b, a)$ of a particle moving from point a to point b is the square of the absolute value of a complex number, the transition function $K(b, a)$:

$$P(b, a) = |K(b, a)|^2 \quad (8.1)$$

8.1.2 Postulate II

The transition function $K(b, a)$ is given by the sum of a phase factor $e^{iS/\hbar}$, where S is the action, taken over all possible paths from a to b :

$$K(b, a) = \sum_{\text{paths}} k e^{iS/\hbar} \quad (8.2)$$

where the constant k can be determined by:

$$K(c, a) = \sum_{\text{paths}} K(c, b)K(b, a) \quad (8.3)$$

where we sum over all intermediate points b connecting a and c .

RECIPE STEPS

(1) CONSTRUCT ALL PATHS

(2) CALCULATE THE ACTION $S[x(t)]$ FOR EACH
PATH

$$S[x(t)] = \int_{t_1}^{t_2} L dt$$

↑
LAGRANGIAN

SUM
(3) (INTEGRATE) OVER ALL PATHS TO
OBTAIN THE PROPAGATOR

$$U(x_2, t_2; x_1, t_1) = A \sum_{\text{all paths}} e^{i S[x(t)]/\hbar}$$

(4) USE THE PROPAGATOR

$$\psi(x_2, t_2) = \int U(x_2, t_2; x_1, t_1) \psi(x_1, t_1) dx_1$$

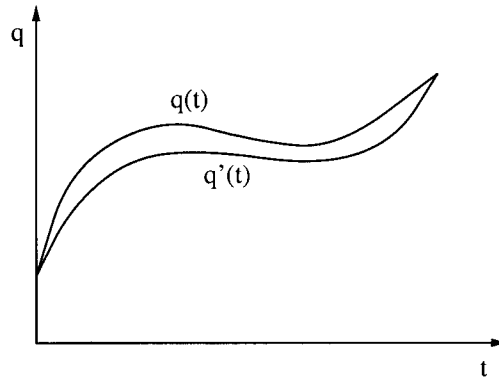


Figure 2: Two neighbouring paths.

However, this argument must be rethought for one exceptional path: that which extremizes the action, *i.e.*, the classical path, $q_c(t)$. For this path, $S[q_c + \eta] = S[q_c] + o(\eta^2)$. Thus the classical path and a very close neighbour will have actions which differ by much less than two randomly-chosen but equally close paths (Figure 3). This means that for fixed closeness

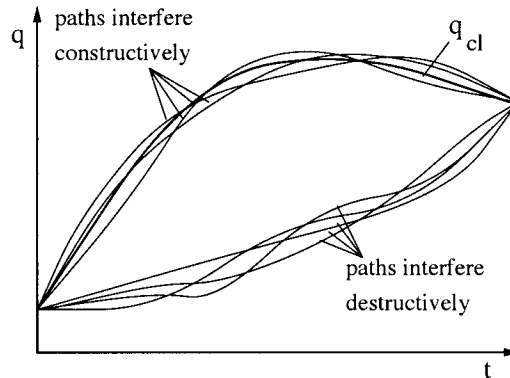
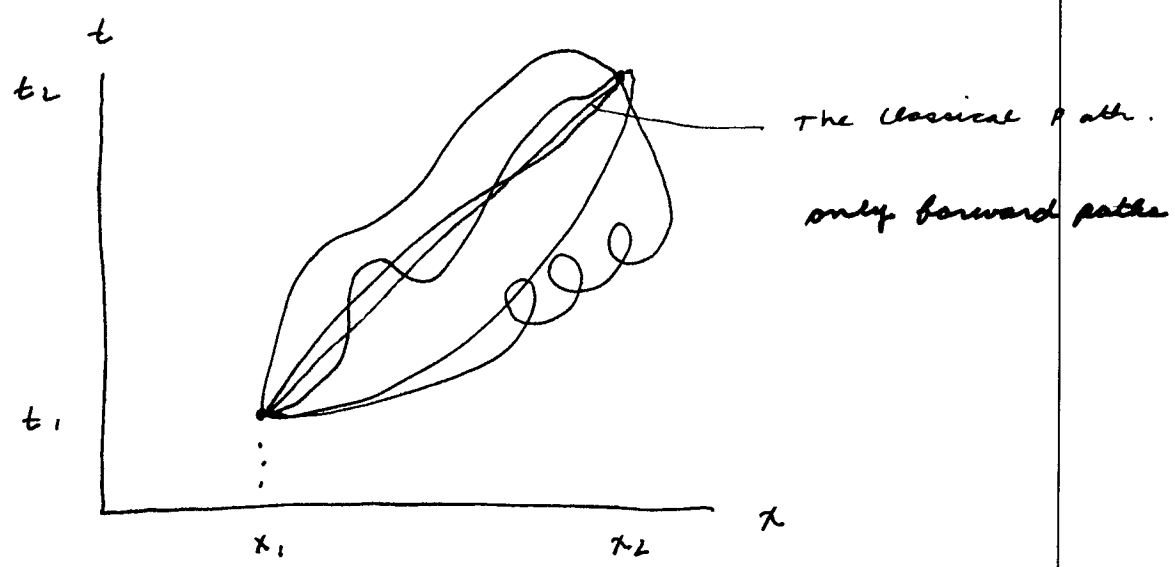


Figure 3: Paths near the classical path interfere constructively.

of two paths (I leave it as an exercise to make this precise!) and for fixed \hbar , paths near the classical path will on average interfere constructively (small phase difference) whereas for random paths the interference will be on average destructive.

Thus heuristically, we conclude that if the problem is classical (action $\gg \hbar$), the most important contribution to the PI comes from the region around the path which extremizes the PI. In other words, the particle's motion is governed by the principle that the action is stationary. This, of course, is none other than the Principle of Least Action from which the Euler-Lagrange equations of classical mechanics are derived.

(1) CONSTRUCT ALL PATHS

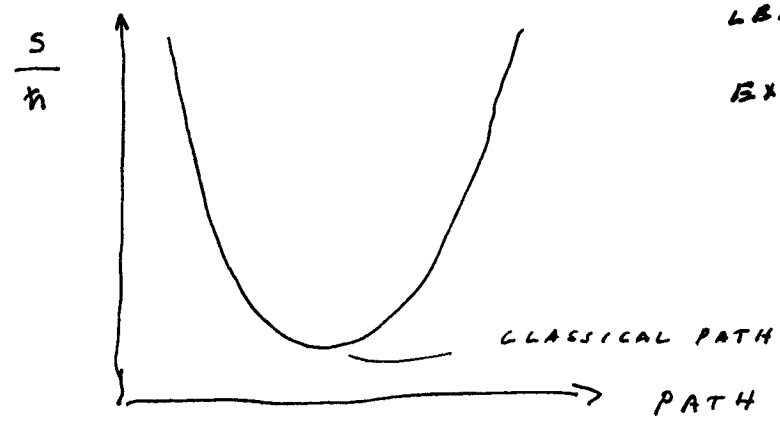


(2) CALCULATE THE ACTION

$$L(x, \dot{x}, t) = T - V$$

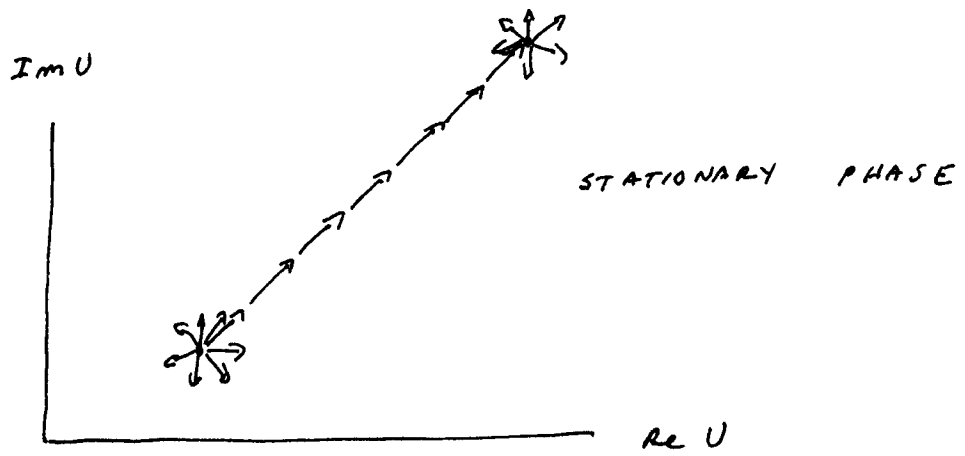
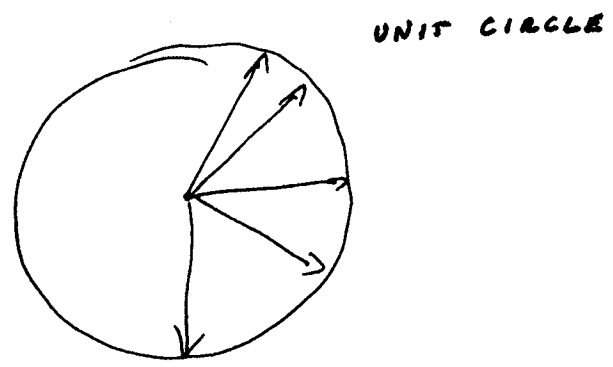
CLASSICAL PATH HAS MINIMUM ACTION

PHYSICS IS WHERE THE
LEAST ACTION IS
EXTREME ACTION IS



(3) CALCULATE THE PROPAGATOR

$$\sum_{\text{all paths}} e^{i S[x(t)]/\hbar}$$



PHASE COHERENCE

$$\frac{|S_{\text{CLASSICAL}} - S|}{\hbar} \leq \pi$$

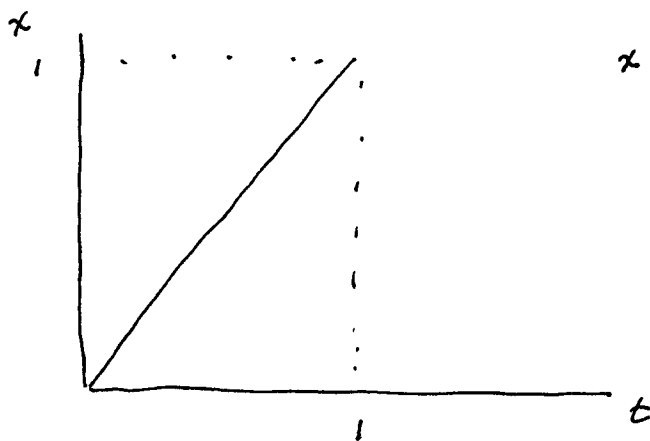
$$\Delta S \leq \pi \hbar$$

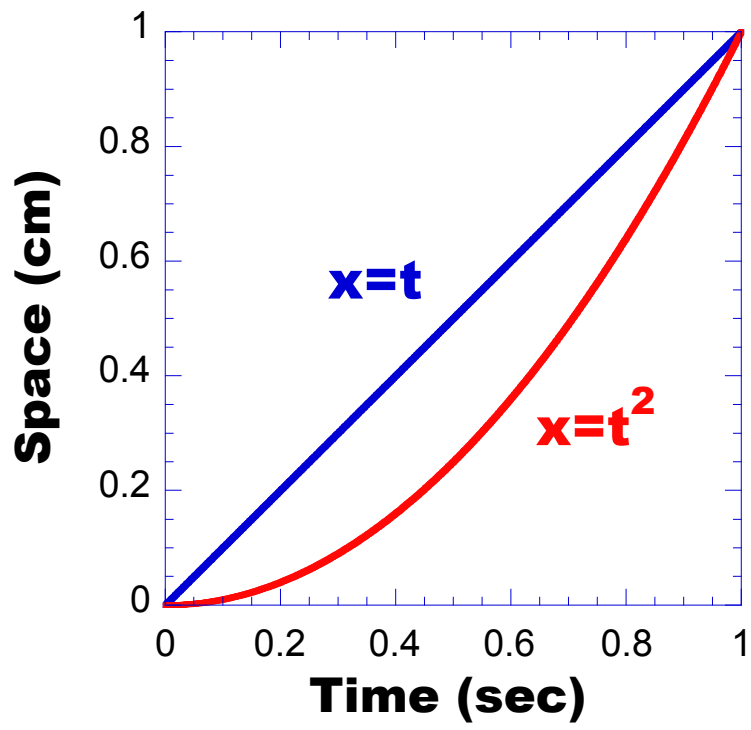
Q: Why do baseballs follow classical trajectories while electrons diffract, go thru both slits, etc.?

LAGRANGIAN for baseball and electron

$$L = T - V = \frac{1}{2} m v^2 - V(x) = \frac{1}{2} m v^2$$

THE CLASSICAL PATH





CALCULATE THE ACTION FOR THE CLASSICAL PATH

$$\begin{aligned}
 S_{CL} &= \int_0^1 L dt \\
 &= \int_0^1 \frac{1}{2} m v^2 dt \\
 &= \int_0^1 \frac{1}{2} m (1)^2 dt
 \end{aligned}$$

$$S_{CL} = \frac{1}{2} m$$

NOW LET'S DO A NON CLASSICAL PATH

$$x = t^2$$

$$v = \frac{dx}{dt} = 2t$$

$$S_{NC} = \int_0^1 L dt = \int_0^1 \frac{1}{2} m (2t)^2 dt = \frac{4}{3} \left(\frac{1}{2} m \right)$$

$$S_{NC} = \frac{4}{3} S_{CL}$$

How Much Does a Baseball Weigh?



What is its diameter?

How many stitches does it have?

[http://en.wikipedia.org/wiki/Baseball_\(ball\)](http://en.wikipedia.org/wiki/Baseball_(ball))

How much does a baseball weigh?

Mass of a Baseball

[The Physics Factbook™](#)

Edited by Glenn Elert -- Written by his students

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Bibliographic Entry	Result (w/surrounding text)	Standardized Result
Objectives of the Game . <i>Rules of Baseball</i> . Major League Baseball Enterprises, 1998.	"It shall weigh not less than five nor more than 5 ¼ ounces avoirdupois"	142–149 g
<i>The Cultural Encyclopedia of Baseball</i> . North Carolina: McFarland & Company, 1997.	"The final weight is required to be between 5 ounces and 5¼ ounces"	142–149 g
Adair, Robert K. <i>The Physics of Baseball</i> . New York, 1990.	"... are required to change the motion of the 5 1/8 ounce ball from a speed of 90 mph"	145 g
Author's measurement of a Wilson Baseball (Model A1001).	"142.54 g"	142.54 g

Baseball was invented around the last quarter of the 1800s. In the game of baseball, we all know that the ball is the main object in the game. A baseball is a small, round, and hard. The weight of a baseball must be between 5 and 5 ¼ ounces (142 to 149 grams) and its circumference from 9 to 9 ¼ inches (22.9 to 23.5 centimeters).

The formation of the ball begins with a ½ ounce (14 g), 2.9 inch (7.4 cm) diameter cork core. A layer of black rubber is then applied followed by a layer of red rubber each weighing ⅓ of an ounce (25 g). Afterwards, 121 yards (111 m) of blue-gray wool followed by 45 yards (41 m) of white wool yarn are added to the outside. The ball is then wrapped in cowhide covering held together by 216 stitches and some rubber cement. Red stitches are placed on the ball to allow pitchers to throw curve balls. Curve balls curve since the air resistance on the stitches is non-uniform.

From my experiment of weighing a baseball, the result came out to be 142.54 grams which fits within the accepted range of 142 to 149 grams.

Christina Lee -- 1999

Related pages in **The Physics Factbook**:

<http://hypertextbook.com/facts/1999/ChristinaLee.shtml>

THE DIFFERENCE IS THE MASS!

BASEBALL $m \sim 200g$

$$S_{CL} = \frac{1}{2} (200g) (1 \text{ cm/s})^2 (1s)$$

$$= 100 \text{ erg} \cdot \text{sec} \approx 10^{29} \hbar$$

$$\Delta S = \frac{1}{3} S_{CL} = 3 \times 10^{28} \hbar \gg \pi \hbar$$

SO, BASEBALL MUST STAY EXTREMELY CLOSE TO THE CLASSICAL PATH!

ELECTRON $m \sim 10^{-27} g$

$$S_{CL} = \frac{1}{2} (10^{-27}) (1)^2 (1)$$

$$= 5 \times 10^{-28} \text{ erg} \cdot \text{sec}$$

$$\Delta S \approx \frac{1}{2} \hbar < \pi \hbar$$

$$\Delta S = \frac{1}{3} S_{CL} = \frac{1}{6} \hbar \ll \pi \hbar$$

SO, ELECTRON WILL FOLLOW THE NON CLASSICAL PATH!

Likewise, the function $\chi^*(x, t)$ characterizes the experience, or, let us say, experiment to which the system is to be subjected. If a different region, r'' and different Lagrangian after t , were to give the same $\chi^*(x, t)$ via Eq. (16), as does region R'' , then no matter what the preparation, ψ , Eq. (14) says that the chance of finding the system in R'' is always the same as finding it in r'' . The two "experiments" R'' and r'' are equivalent, as they yield the same results. We shall say loosely that these experiments are to determine with what probability the system is in state χ . Actually, this terminology is poor. The system is really in state ψ . The reason we can associate a state with an experiment is, of course, that for an ideal experiment there turns out to be a unique state (whose wave function is $\chi(x, t)$) for which the experiment succeeds with certainty.

Thus, we can say: the probability that a system in state ψ will be found by an experiment whose characteristic state is χ (or, more loosely, the chance that a system in state ψ will appear to be in χ) is

$$\left| \int \chi^*(x, t) \psi(x, t) dx \right|^2. \quad (17)$$

These results agree, of course, with the principles of ordinary quantum mechanics. They are a consequence of the fact that the Lagrangian is a function of position, velocity, and time only.

6. THE WAVE EQUATION

To complete the proof of the equivalence with the ordinary formulation we shall have to show that the wave function defined in the previous section by Eq. (15) actually satisfies the Schroedinger wave equation. Actually, we shall only succeed in doing this when the Lagrangian L in (11) is a quadratic, but perhaps inhomogeneous, form in the velocities $\dot{x}(t)$. This is not a limitation, however, as it includes all the cases for which the Schroedinger equation has been verified by experiment.

The wave equation describes the development of the wave function with time. We may expect to approach it by noting that, for finite ϵ , Eq. (15) permits a simple recursive relation to be developed. Consider the appearance of Eq. (15) if

we were to compute ψ at the next instant of time:

$$\psi(x_{k+1}, t + \epsilon) = \int_{R_1} \exp \left[\frac{i}{\hbar} \sum_{i=-\infty}^k S(x_{i+1}, x_i) \right] \times \frac{dx_k dx_{k-1}}{A A} \dots \quad (15')$$

This is similar to (15) except for the integration over the additional variable x_k and the extra term in the sum in the exponent. This term means that the integral of (15') is the same as the integral of (15) except for the factor $(1/A) \exp(i/\hbar) S(x_{k+1}, x_k)$. Since this does not contain any of the variables x_i for i less than k , all of the integrations on dx_i up to dx_{k-1} can be performed with this factor left out. However, the result of these integrations is by (15) simply $\psi(x_k, t)$. Hence, we find from (15') the relation

$$\psi(x_{k+1}, t + \epsilon) = \int \exp \left[\frac{i}{\hbar} S(x_{k+1}, x_k) \right] \psi(x_k, t) dx_k / A. \quad (18)$$

This relation giving the development of ψ with time will be shown, for simple examples, with suitable choice of A , to be equivalent to Schroedinger's equation. Actually, Eq. (18) is not exact, but is only true in the limit $\epsilon \rightarrow 0$ and we shall derive the Schroedinger equation by assuming (18) is valid to first order in ϵ . The Eq. (18) need only be true for small ϵ to the first order in ϵ . For if we consider the factors in (15) which carry us over a finite interval of time, T , the number of factors is T/ϵ . If an error of order ϵ^2 is made in each, the resulting error will not accumulate beyond the order $\epsilon^2(T/\epsilon)$ or $T\epsilon$, which vanishes in the limit.

We shall illustrate the relation of (18) to Schroedinger's equation by applying it to the simple case of a particle moving in one dimension in a potential $V(x)$. Before we do this, however, we would like to discuss some approximations to the value $S(x_{i+1}, x_i)$ given in (11) which will be sufficient for expression (18).

The expression defined in (11) for $S(x_{i+1}, x_i)$ is difficult to calculate exactly for arbitrary ϵ from classical mechanics. Actually, it is only necessary that an approximate expression for $S(x_{i+1}, x_i)$ be

which may not be zero. The question is still more important in the coefficient of terms which are quadratic in the velocities. In these terms (19) and (20) are not sufficiently accurate representations of (11) in general. It is when the coefficients are constant that (19) or (20) can be substituted for (11). If an expression such as (19) is used, say for spherical coordinates, when it is not a valid approximation to (11), one obtains a Schroedinger equation in which the Hamiltonian operator has some of the momentum operators and coordinates in the wrong order. Equation (11) then resolves the ambiguity in the usual rule to replace p and q by the non-commuting quantities $(\hbar/i)(\partial/\partial q)$ and q in the classical Hamiltonian $H(p, q)$.

It is clear that the statement (11) is independent of the coordinate system. Therefore, to find the differential wave equation it gives in any coordinate system, the easiest procedure is first to find the equations in Cartesian coordinates and then to transform the coordinate system to the one desired. It suffices, therefore, to show the relation of the postulates and Schroedinger's equation in rectangular coordinates.

The derivation given here for one dimension can be extended directly to the case of three-dimensional Cartesian coordinates for any number, K , of particles interacting through potentials with one another, and in a magnetic field, described by a vector potential. The terms in the vector potential require completing the square in the exponent in the usual way for Gaussian integrals. The variable x must be replaced by the set $x^{(1)}$ to $x^{(3K)}$ where $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the coordinates of the first particle of mass m_1 , $x^{(4)}$, $x^{(5)}$, $x^{(6)}$ of the second of mass m_2 , etc. The symbol dx is replaced by $dx^{(1)}dx^{(2)}\dots dx^{(3K)}$, and the integration over dx is replaced by a $3K$ -fold integral. The constant A has, in this case, the value $A = (2\pi\hbar\epsilon i/m_1)^{\frac{1}{2}}(2\pi\hbar\epsilon i/m_2)^{\frac{1}{2}}\dots(2\pi\hbar\epsilon i/m_K)^{\frac{1}{2}}$. The Lagrangian is the classical Lagrangian for the same problem, and the Schroedinger equation resulting will be that which corresponds to the classical Hamiltonian, derived from this Lagrangian. The equations in any other coordinate system may be obtained by transformation. Since this includes all cases for which Schroedinger's equation has been checked with experiment, we may say our postulates are able to

describe what can be described by non-relativistic quantum mechanics, neglecting spin.

7. DISCUSSION OF THE WAVE EQUATION

The Classical Limit

This completes the demonstration of the equivalence of the new and old formulations. We should like to include in this section a few remarks about the important equation (18).

This equation gives the development of the wave function during a small time interval. It is easily interpreted physically as the expression of Huygens' principle for matter waves. In geometrical optics the rays in an inhomogeneous medium satisfy Fermat's principle of least *time*. We may state Huygens' principle in wave optics in this way: If the amplitude of the wave is known on a given surface, the amplitude at a near by point can be considered as a sum of contributions from all points of the surface. Each contribution is delayed in phase by an amount proportional to the *time* it would take the light to get from the surface to the point along the ray of least *time* of geometrical optics. We can consider (22) in an analogous manner starting with Hamilton's first principle of least *action* for classical or "geometrical" mechanics. If the amplitude of the wave ψ is known on a given "surface," in particular the "surface" consisting of all x at time t , its value at a particular nearby point at time $t + \epsilon$, is a sum of contributions from all points of the surface at t . Each contribution is delayed in phase by an amount proportional to the *action* it would require to get from the surface to the point along the path of least *action* of classical mechanics.¹⁶

Actually Huygens' principle is not correct in optics. It is replaced by Kirchoff's modification which requires that both the amplitude and its derivative must be known on the adjacent surface. This is a consequence of the fact that the wave equation in optics is second order in the time. The wave equation of quantum mechanics is first order in the time; therefore, Huygens' principle *is* correct for matter waves, action replacing time.

¹⁶ See in this connection the very interesting remarks of Schroedinger, Ann. d. Physik 79, 489 (1926).

The equation can also be compared mathematically to quantities appearing in the usual formulations. In Schroedinger's method the development of the wave function with time is given by

$$-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \mathbf{H}\psi, \tag{31}$$

which has the solution (for any ϵ if \mathbf{H} is time independent)

$$\psi(x, t+\epsilon) = \exp(-i\epsilon\mathbf{H}/\hbar)\psi(x, t). \tag{32}$$

Therefore, Eq. (18) expresses the operator $\exp(-i\epsilon\mathbf{H}/\hbar)$ by an approximate integral operator for small ϵ .

From the point of view of Heisenberg one considers the position at time t , for example, as an operator \mathbf{x} . The position \mathbf{x}' at a later time $t+\epsilon$ can be expressed in terms of that at time t by the operator equation

$$\mathbf{x}' = \exp(i\epsilon\mathbf{H}/\hbar)\mathbf{x}\exp-(i\epsilon\mathbf{H}/\hbar). \tag{33}$$

The transformation theory of Dirac allows us to consider the wave function at time $t+\epsilon$, $\psi(x', t+\epsilon)$, as representing a state in a representation in which \mathbf{x}' is diagonal, while $\psi(x, t)$ represents the same state in a representation in which \mathbf{x} is diagonal. They are, therefore, related through the transformation function $(x'|x)_\epsilon$ which relates these representations:

$$\psi(x', t+\epsilon) = \int (x'|x)_\epsilon \psi(x, t) dx.$$

Therefore, the content of Eq. (18) is to show that for small ϵ we can set

$$(x'|x)_\epsilon = (1/A) \exp(iS(x', x)/\hbar) \tag{34}$$

with $S(x', x)$ defined as in (11).

The close analogy between $(x'|x)_\epsilon$ and the quantity $\exp(iS(x', x)/\hbar)$ has been pointed out on several occasions by Dirac.¹ In fact, we now see that to sufficient approximations the two quantities may be taken to be proportional to each other. Dirac's remarks were the starting point of the present development. The points he makes concerning the passage to the classical limit $\hbar \rightarrow 0$ are very beautiful, and I may perhaps be excused for briefly reviewing them here.

First we note that the wave function at x'' at time t'' can be obtained from that at x' at time t' by

$$\begin{aligned} \psi(x'', t'') = & \text{Lim}_{\epsilon \rightarrow 0} \int \dots \int \\ & \times \exp\left[\frac{i}{\hbar} \sum_{i=0}^{j-1} S(x_{i+1}, x_i)\right] \\ & \times \psi(x', t') \frac{dx_0}{A} \frac{dx_1}{A} \dots \frac{dx_{j-1}}{A}, \end{aligned} \tag{35}$$

where we put $x_0 \equiv x'$ and $x_j \equiv x''$ where $j\epsilon = t'' - t'$ (between the times t' and t'' we assume no restriction is being put on the region of integration). This can be seen either by repeated applications of (18) or directly from Eq. (15). Now we ask, as $\hbar \rightarrow 0$ what values of the intermediate coordinates x_i contribute most strongly to the integral? These will be the values most likely to be found by experiment and therefore will determine, in the limit, the classical path. If \hbar is very small, the exponent will be a very rapidly varying function of any of its variables x_i . As x_i varies, the positive and negative contributions of the exponent nearly cancel. The region at which x_i contributes most strongly is that at which the phase of the exponent varies least rapidly with x_i (method of stationary phase). Call the sum in the exponent S ;

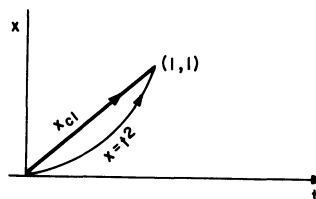
$$S = \sum_{i=0}^{j-1} S(x_{i+1}, x_i). \tag{36}$$

Then the classical orbit passes, approximately, through those points x_i at which the rate of change of S with x_i is small, or in the limit of small \hbar , zero, i.e., the classical orbit passes through the points at which $\partial S/\partial x_i = 0$ for all x_i . Taking the limit $\epsilon \rightarrow 0$, (36) becomes in view of (11)

$$S = \int_{t'}^{t''} L(\dot{x}(t), x(t)) dt. \tag{37}$$

We see then that the classical path is that for which the integral (37) suffers no first-order change on varying the path. This is Hamilton's principle and leads directly to the Lagrangian equations of motion.

Figure 8.3. Two possible paths connecting $(0, 0)$ and $(1, 1)$. The action on the classical path $x=t$ is $m/2$, while on the other, it is $2m/3$.



Consider another path

$$x = t^2 \quad (8.2.2)$$

which also links the two space-time points (Fig. 8.3.)

For a classical particle, of mass, say 1 g, the action changes by roughly $1.6 \times 10^{26} \hbar$, and the phase by roughly 1.6×10^{26} rad as we move from the classical path $x=t$ to the nonclassical path $x=t^2$. We may therefore completely ignore the nonclassical path. On the other hand, for an electron whose mass is $\simeq 10^{-27}$ g, $\delta S \simeq \hbar/6$ and the phase change is just around a sixth of a radian, which is well within the coherence range $\delta S/\hbar \lesssim \pi$. It is in such cases that assuming that the particle moves along a well-defined trajectory, $x_{cl}(t)$, leads to conflict with experiment.

**The normalized classical action
gives the exact propagator
for the free particle**

8.3. An Approximation to $U(t)$ for a Free Particle

Our previous discussions have indicated that, to an excellent approximation, we may ignore all but the classical path and its neighbors in calculating $U(t)$. Assuming that each of these paths contributes the same amount $\exp(iS_{cl}/\hbar)$, since S is stationary, we get

A' measures the number of coherent paths

$$U(t) = A' e^{iS_{cl}/\hbar} \quad (8.3.1)$$

where A' is some normalizing factor which “measures” the number of paths in the coherent range. Let us find $U(t)$ for a free particle in this approximation and compare the result with the exact result, Eq. (5.1.10).

The classical path for a free particle is just a straight line in the $x-t$ plane:

$$x_{cl}(t'') = x' + \frac{x-x'}{t-t'}(t''-t') \quad (8.3.2)$$

corresponding to motion with uniform velocity $v = (x-x')/(t-t')$. Since $\mathcal{L} = mv^2/2$ is a constant,

$$S_{cl} = \int_{t'}^{t''} \mathcal{L} dt'' = \frac{1}{2} m \frac{(x-x')^2}{t-t'}$$

so that

$$U(x, t; x', t') = A' \exp \left[\frac{im(x-x')^2}{2\hbar(t-t')} \right] \quad (8.3.3)$$

To find A' , we use the fact that as $t-t'$ tends to 0, U must tend to $\delta(x-x')$. Comparing Eq. (8.3.3) to the representation of the delta function encountered in Section 1.10 (see footnote on page 61),

Gaussian representation of Dirac delta fcn

$$\delta(x-x') \equiv \lim_{\Delta \rightarrow 0} \frac{1}{(\pi\Delta^2)^{1/2}} \exp \left[-\frac{(x-x')^2}{\Delta^2} \right]$$

(valid even if Δ is imaginary) we get

$$A' = \left[\frac{m}{2\pi\hbar i(t-t')} \right]^{1/2}$$

so that

Yields the exact free particle propagator!

$$U(x, t; x', 0) \equiv U(x, t; x') = \left(\frac{m}{2\pi\hbar i t} \right)^{1/2} \exp \left[\frac{im(x-x')^2}{2\hbar t} \right] \quad (8.3.4)$$

which is the exact answer! We have managed to get the exact answer by just computing the classical action! However, we will see in Section 8.6 that only for potentials of the form $V = a + bx + cx^2 + d\dot{x} + ex\dot{x}$ is it true that $U(t) = A(t) e^{iS_{cl}/\hbar}$. Furthermore, we can't generally find $A(t)$ using $U(x, 0; x') = \delta(x-x')$ since A can contain an arbitrary dimensionless function f such that $f \rightarrow 1$ as $t \rightarrow 0$. Here $f \equiv 1$ because we can't construct a nontrivial dimensionless f using just m , \hbar , and t (check this).

8.4. Path Integral Evaluation of the Free-Particle Propagator

Although our heuristic analysis yielded the exact free-particle propagator, we will now repeat the calculation without any approximation to illustrate path integration.

Consider $U(x_N, t_N; x_0, t_0)$. The peculiar labeling of the end points will be justified later. Our problem is to perform the path integral

$$\int_{x_0}^{x_N} e^{iS[x(t)]/\hbar} \mathcal{D}[x(t)] \quad (8.4.1)$$

where

$$\int_{x_0}^{x_N} \mathcal{D}[x(t)]$$

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of P is also an eigenstate of P^2 . So we feed the trial solution $|p\rangle$ into Eq. (5.1.3) and find

$$\frac{P^2}{2m}|p\rangle = E|p\rangle$$

or

$$\left(\frac{P^2}{2m} - E\right)|p\rangle = 0 \quad (5.1.4)$$

Since $|p\rangle$ is not a null vector, we find that the allowed values of p are

$$p = \pm(2mE)^{1/2} \quad (5.1.5)$$

In other words, there are two orthogonal eigenstates for each eigenvalue E :

$$|E, +\rangle = |p = (2mE)^{1/2}\rangle \quad (5.1.6)$$

$$|E, -\rangle = |p = -(2mE)^{1/2}\rangle \quad (5.1.7)$$

Thus, we find that to the eigenvalue E there corresponds a degenerate two-dimensional eigenspace, spanned by the above vectors. Physically this means that a particle of energy E can be moving to the right or to the left with momentum $|p| = (2mE)^{1/2}$. Now, you might say, “This is exactly what happens in classical mechanics. So what’s new?” What is new is the fact that the state

$$|E\rangle = \beta|p = (2mE)^{1/2}\rangle + \gamma|p = -(2mE)^{1/2}\rangle \quad (5.1.8)$$

is also an eigenstate of energy E and represents a *single* particle of energy E that can be caught moving either to the right or to the left with momentum $(2mE)^{1/2}$!

To construct the complete orthonormal eigenbasis of H , we must pick from each degenerate eigenspace any two orthonormal vectors. The obvious choice is given by the kets $|E, +\rangle$ and $|E, -\rangle$ themselves. In terms of the ideas discussed in the past, we are using the eigenvalue of a compatible variable P as an extra label within the space degenerate with respect to energy. Since P is a nondegenerate operator, the label p by itself is adequate. In other words, there is no need to call the state $|p, E = P^2/2m\rangle$, since the value of $E = E(p)$ follows, given p . We shall therefore drop this redundant label.

The propagator is then

$$\begin{aligned} U(t) &= \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-iE(p)t/\hbar} dp \\ &= \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-ip^2t/2m\hbar} dp \end{aligned} \quad (5.1.9)$$

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Exercise 5.1.1. Show that Eq. (5.1.9) may be rewritten as an integral over E and a sum over the \pm index as

$$U(t) = \sum_{\alpha = \pm} \int_0^{\infty} \left[\frac{m}{(2mE)^{1/2}} \right] |E, \alpha\rangle \langle E, \alpha| e^{-iEt/\hbar} dE$$

*Exercise 5.1.2.** By solving the eigenvalue equation (5.1.3) in the X basis, regain Eq. (5.1.8), i.e., show that the general solution of energy E is

$$\psi_E(x) = \beta \frac{\exp[i(2mE)^{1/2}x/\hbar]}{(2\pi\hbar)^{1/2}} + \gamma \frac{\exp[-i(2mE)^{1/2}x/\hbar]}{(2\pi\hbar)^{1/2}}$$

[The factor $(2\pi\hbar)^{-1/2}$ is arbitrary and may be absorbed into β and γ .] Though $\psi_E(x)$ will satisfy the equation even if $E < 0$, are these functions in the Hilbert space?

The propagator $U(t)$ can be evaluated explicitly in the X basis. We start with the matrix element

$$\begin{aligned} U(x, t; x') &\equiv \langle x|U(t)|x'\rangle = \int_{-\infty}^{\infty} \langle x|p\rangle \langle p|x'\rangle e^{-ip^2t/2m\hbar} dp \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ip(x-x')/\hbar} \cdot e^{-ip^2t/2m\hbar} dp \\ &= \left(\frac{m}{2\pi\hbar it} \right)^{1/2} e^{im(x-x')^2/2\hbar t} \end{aligned} \quad (5.1.10)$$

using the result from Appendix A.2 on Gaussian integrals. In terms of this propagator, any initial-value problem can be solved, since

$$\psi(x, t) = \int U(x, t; x') \psi(x', 0) dx' \quad (5.1.11)$$

Had we chosen the initial time to be t' rather than zero, we would have gotten

$$\psi(x, t) = \int U(x, t; x', t') \psi(x', t') dx' \quad (5.1.12)$$

where $U(x, t; x', t') = \langle x|U(t-t')|x'\rangle$, since U depends only on the time interval $t-t'$ and not the absolute values of t and t' . [Had there been a time-dependent potential such as $V(t) = V_0 e^{-\alpha t^2}$ in H , we could have told what absolute time it was by looking at $V(t)$. In the absence of anything defining an absolute time in the problem, only time differences have physical significance.] Whenever we set $t' = 0$, we will resort to our old convention and write $U(x, t; x', 0)$ as simply $U(x, t; x')$.

A nice physical interpretation may be given to $U(x, t; x', t')$ by considering a special case of Eq. (5.1.12). Suppose we started off with a particle localized at

**In the
position
basis**

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$x' = x'_0$, that is, with $\psi(x', t') = \delta(x' - x'_0)$. Then

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$$\psi(x, t) = U(x, t; x'_0, t') \quad (5.1.13)$$

In other words, the propagator (in the X basis) is the amplitude that a particle starting out at the space-time point (x'_0, t') ends with at the space-time point (x, t) . [It can obviously be given such an interpretation in any basis: $\langle \omega | U(t, t') | \omega' \rangle$ is the amplitude that a particle in the state $|\omega'\rangle$ at t' ends up with in the state $|\omega\rangle$ at t .] Equation (5.1.12) then tells us that the total amplitude for the particle's arrival at (x, t) is the sum of the contributions from all points x' with a weight proportional to the initial amplitude $\psi(x', t')$ that the particle was at x' at time t' . One also refers to $U(x, t; x'_0, t')$ as the "fate" of the delta function $\psi(x', t') = \delta(x' - x'_0)$.

Time Evolution of the Gaussian Packet

There is an unwritten law which says that the derivation of the free-particle propagator be followed by its application to the Gaussian packet. Let us follow this tradition.

Consider as the initial wave function the wave packet

t=0 packet
$$\psi(x', 0) = e^{ip_0 x' / \hbar} \frac{e^{-x'^2 / 2\Delta^2}}{(\pi\Delta^2)^{1/4}} \quad (5.1.14)$$

This packet has mean position $\langle X \rangle = 0$, with an uncertainty $\Delta X = \Delta / 2^{1/2}$, and mean momentum p_0 with uncertainty $\hbar / 2^{1/2} \Delta$. By combining Eqs. (5.1.10) and (5.1.12) we get

=> wavefunction
$$\psi(x, t) = \left[\pi^{1/2} \left(\Delta + \frac{i\hbar t}{m\Delta} \right) \right]^{-1/2} \cdot \exp \left[\frac{-(x - p_0 t / m)^2}{2\Delta^2 (1 + i\hbar t / m\Delta^2)} \right] \times \exp \left[\frac{ip_0}{\hbar} \left(x - \frac{p_0 t}{2m} \right) \right] \quad (5.1.15)$$

The corresponding probability density is

=> probability
$$P(x, t) = \frac{1}{\pi^{1/2} (\Delta^2 + \hbar^2 t^2 / m^2 \Delta^2)^{1/2}} \cdot \exp \left\{ \frac{-[x - (p_0/m)t]^2}{\Delta^2 + \hbar^2 t^2 / m^2 \Delta^2} \right\} \quad (5.1.16)$$

The main features of this result are as follows:

(1) The mean position of the particles is

$$\langle X \rangle = \frac{p_0 t}{m} = \frac{\langle P \rangle t}{m}$$

In other words, the classical relation $x = (p/m)t$ now holds between average quantities. This is just one of the consequences of the *Ehrenfest theorem* which states that the classical equations obeyed by dynamical variables will have counterparts in quantum mechanics as relations among expectation values. The theorem will be proved in the next chapter.

(2) The width of the packet grows as follows:

$$\Delta X(t) = \frac{\Delta(t)}{2^{1/2}} = \frac{\Delta}{2^{1/2}} \left(1 + \frac{\hbar^2 t^2}{m^2 \Delta^4} \right)^{1/2} \quad (5.1.17)$$

The increasing uncertainty in position is a reflection of the fact that any uncertainty in the initial velocity (that is to say, the momentum) will be reflected with passing time as a growing uncertainty in position. In the present case, since $\Delta V(0) = \Delta P(0)/m = \hbar/2^{1/2} m \Delta$, the uncertainty in X grows approximately as $\Delta X \simeq \hbar t / 2^{1/2} m \Delta$ which agrees with Eq. (5.1.17) for large times. Although we are able to understand the spreading of the wave packet in classical terms, the fact that the initial spread $\Delta V(0)$ is *unavoidable* (given that we wish to specify the position to an accuracy Δ) is a purely quantum mechanical feature.

If the particle in question were macroscopic, say of mass 1 g, and we wished to fix its initial position to within a proton width, which is approximately 10^{-13} cm, the uncertainty in velocity would be

$$\Delta V(0) \simeq \frac{\hbar}{2^{1/2} m \Delta} \simeq 10^{-14} \text{ cm/sec}$$

It would be over 300,000 years before the uncertainty $\Delta(t)$ grew to 1 millimeter! We may therefore treat a macroscopic particle classically for any reasonable length of time. This and similar questions will be taken up in greater detail in the next chapter.

Exercise 5.1.3 (Another Way to Do the Gaussian Problem). We have seen that there exists another formula for $U(t)$, namely, $U(t) = e^{-iHt/\hbar}$. For a free particle this becomes

$$U(t) = \exp \left[\frac{i}{\hbar} \left(\frac{\hbar^2 t}{2m} \frac{d^2}{dx^2} \right) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar t}{2m} \right)^n \frac{d^{2n}}{dx^{2n}} \quad (5.1.18)$$

Consider the initial state in Eq. (5.1.14) with $p_0 = 0$, and set $\Delta = 1$, $t' = 0$:

$$\psi(x, 0) = \frac{e^{-x^2/2}}{(\pi)^{1/4}}$$

Find $\psi(x, t)$ using Eq. (5.1.18) above and compare with Eq. (5.1.15).

Hints: (1) Write $\psi(x, 0)$ as a power series:

$$\psi(x, 0) = (\pi)^{-1/4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n! (2)^n}$$

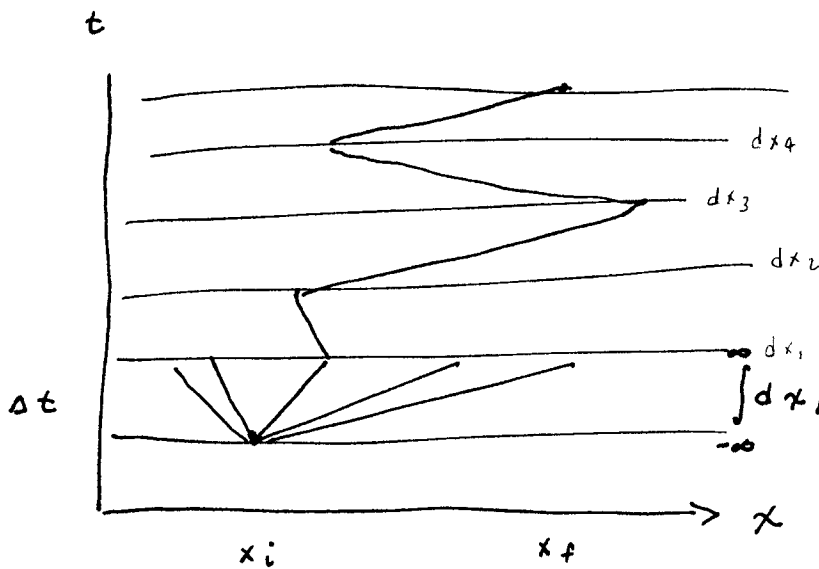
HOW TO DO THE PATH INTEGRAL?

$$A \sum_{\text{all paths}} e^{i S[x(t)] / \hbar}$$

$$A \int_{x_i}^{x_f} e^{i S[x(t)] / \hbar} D[x(t)]$$

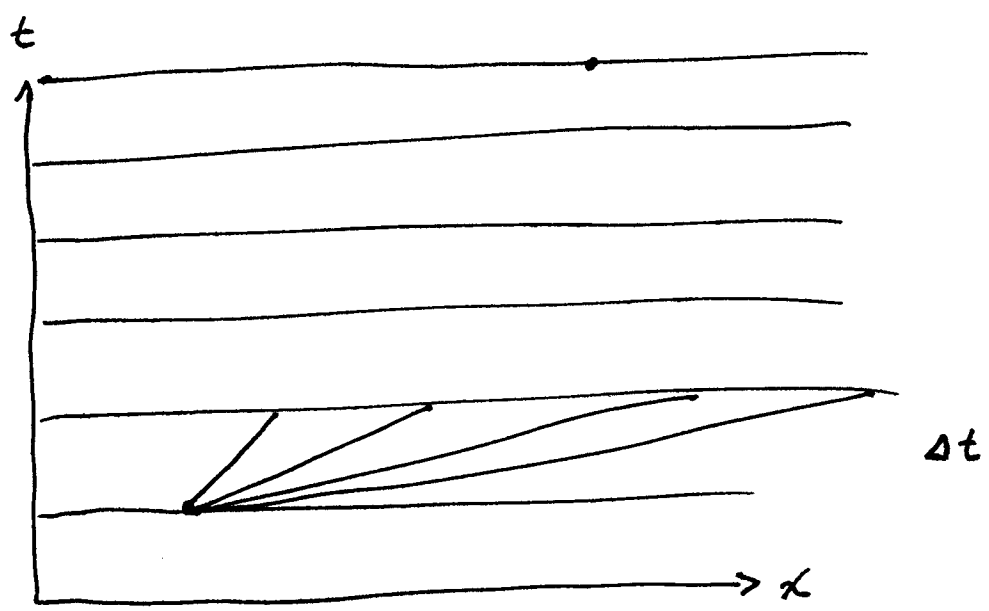
FUNCTIONAL INTEGRAL

$S[x(t)]$ IS A FUNCTIONAL



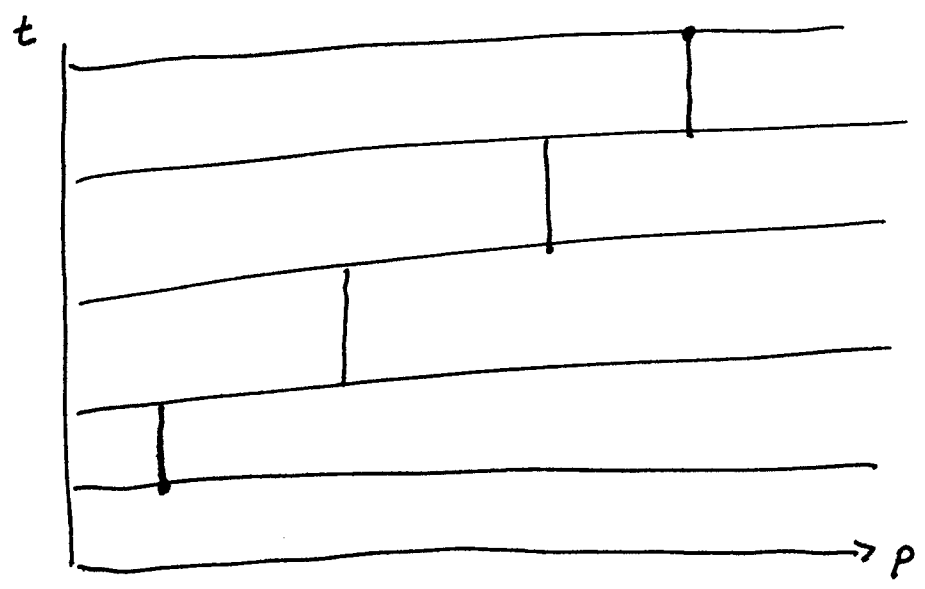
$$\lim_{\Delta t \rightarrow 0}$$

NRQM \Rightarrow INFINITE VELOCITIES



$$\text{PATH INTEGRAL} = \lim_{\Delta t \rightarrow 0} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots$$

MOMENTUM SPACE

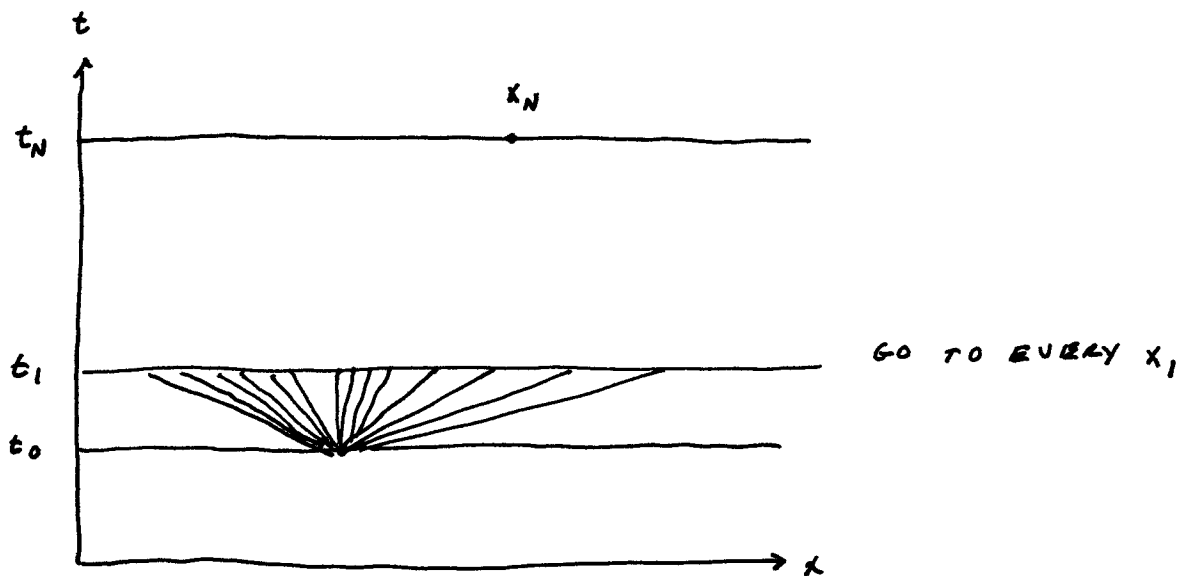


13 1/4" 500 SHEETS FULLER'S SQUARE
 12 3/8" 10 SHEETS FULLER'S SQUARE
 12 1/8" 20 SHEETS FULLER'S SQUARE
 12 3/16" 20 SHEETS FULLER'S SQUARE
 12 1/8" 20 SHEETS FULLER'S SQUARE



Model 1012 P

TIME SLICING



CONTINUOUS ACTION

$$S = \int_{t_0}^{t_N} L(t) dt = \int_{t_0}^{t_N} \frac{1}{2} m v^2 dt$$

DISCRETE APPROXIMATION

$$x_i = x(t_i)$$

$$S = \sum_{i=0}^{N-1} \frac{1}{2} m \left(\frac{x_{i+1} - x_i}{\epsilon} \right)^2 \epsilon$$

PATH INTEGRAL

$$U(x_N, t_N; x_0, t_0) = \int_{x_0}^{x_N} e^{i S[x(t)]/\hbar} \mathcal{D}[x(t)]$$

$$= \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0}} A \int_{-\infty}^{\infty} \int \dots \int e^{\frac{i}{\hbar} \frac{m}{2} \sum_{i=0}^N (x_{i+1} - x_i)^2 / \epsilon} dx_1, \dots, dx_{N-1}$$

BECAUSE THE INTEGRAND IS GAUSSIAN,
WE CAN DO THE INTEGRALS!

DO THE dx_1 INTEGRATION

$$\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar} \left[(x_2 - x_1)^2 + (x_1 - x_0)^2 \right] \right] dx_1 \sim e^{-i(x_2 - x_0)^2 / 2\hbar}$$

DO THE dx_2 INTEGRATION

$$\int dx_2 \sim e^{-i(x_3 - x_0)^2 / 3\hbar}$$

⋮

$$\sim e^{-i(x_N - x_0)^2 / N\hbar}$$

$$\exp\left[\frac{i m (x_N - x_0)^2}{2 \hbar N \epsilon} \right]$$

$$N \rightarrow \infty$$

$$\epsilon \rightarrow 0$$

$$N\epsilon \rightarrow t_N - t_0$$

so that

$$U(x, t; x', t') = A' \exp \left[\frac{im(x-x')^2}{2\hbar(t-t')} \right] \quad (8.3.3)$$

To find A' , we use the fact that as $t-t'$ tends to 0, U must tend to $\delta(x-x')$. Comparing Eq. (8.3.3) to the representation of the delta function encountered in Section 1.10 (see footnote on page 61),

$$\delta(x-x') \equiv \lim_{\Delta \rightarrow 0} \frac{1}{(\pi\Delta^2)^{1/2}} \exp \left[-\frac{(x-x')^2}{\Delta^2} \right]$$

(valid even if Δ is imaginary) we get

$$A' = \left[\frac{m}{2\pi\hbar i(t-t')} \right]^{1/2}$$

so that

$$U(x, t; x', 0) \equiv U(x, t; x') = \left(\frac{m}{2\pi\hbar i t} \right)^{1/2} \exp \left[\frac{im(x-x')^2}{2\hbar t} \right] \quad (8.3.4)$$

which is the exact answer! We have managed to get the exact answer by just computing the classical action! However, we will see in Section 8.6 that only for potentials of the form $V = a + bx + cx^2 + d\dot{x} + ex\dot{x}$ is it true that $U(t) = A(t) e^{iS_{cl}/\hbar}$. Furthermore, we can't generally find $A(t)$ using $U(x, 0; x') = \delta(x-x')$ since A can contain an arbitrary dimensionless function f such that $f \rightarrow 1$ as $t \rightarrow 0$. Here $f \equiv 1$ because we can't construct a nontrivial dimensionless f using just m , \hbar , and t (check this).

8.4. Path Integral Evaluation of the Free-Particle Propagator

Although our heuristic analysis yielded the exact free-particle propagator, we will now repeat the calculation without any approximation to illustrate path integration.

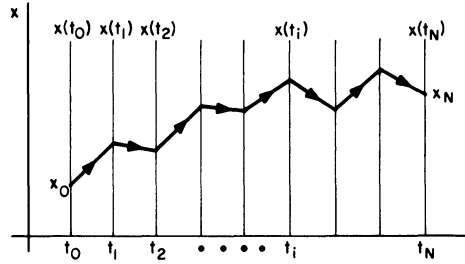
Consider $U(x_N, t_N; x_0, t_0)$. The peculiar labeling of the end points will be justified later. Our problem is to perform the path integral

$$\int_{x_0}^{x_N} e^{iS[x(t)]/\hbar} \mathcal{D}[x(t)] \quad (8.4.1)$$

where

$$\int_{x_0}^{x_N} \mathcal{D}[x(t)]$$

Figure 8.4. The discrete approximation to a path $x(t)$. Each path is specified by $N-1$ numbers $x(t_1), \dots, x(t_{N-1})$. To sum over paths we must integrate each x_i from $-\infty$ to $+\infty$. Once all integrations are done, we can take the limit $N \rightarrow \infty$.



is a symbolic way of saying “integrate over all paths connecting x_0 and x_N (in the interval t_0 and t_N).” Now, a path $x(t)$ is fully specified by an infinity of numbers $x(t_0), \dots, x(t), \dots, x(t_N)$, namely, the values of the function $x(t)$ at every point t in the interval t_0 to t_N . To sum over all paths we must integrate over all possible values of these infinite variables, except of course $x(t_0)$ and $x(t_N)$, which will be kept fixed at x_0 and x_N , respectively. To tackle this problem, we follow the idea that was used in Section 1.10: we trade the function $x(t)$ for a discrete approximation which agrees with $x(t)$ at the $N+1$ points $t_n = t_0 + n\varepsilon$, $n=0, \dots, N$, where $\varepsilon = (t_N - t_0)/N$. In this approximation each path is specified by $N+1$ numbers $x(t_0), x(t_1), \dots, x(t_N)$. The gaps in the discrete function are interpolated by straight lines. One such path is shown in Fig. 8.4. We hope that if we take the limit $N \rightarrow \infty$ at the end we will get a result that is insensitive to these approximations.† Now that the paths have been discretized, we must also do the same to the action integral. We replace the continuous path definition

replace the continuous action

$$S = \int_{t_0}^{t_N} \mathcal{L}(t) dt = \int_{t_0}^{t_N} \frac{1}{2} m \dot{x}^2 dt$$

by a discrete version

$$S = \sum_{i=0}^{N-1} \frac{m}{2} \left(\frac{x_{i+1} - x_i}{\varepsilon} \right)^2 \varepsilon \quad (8.4.2)$$

where $x_i = x(t_i)$. We wish to calculate

$$\begin{aligned} U(x_N, t_N; x_0, t_0) &= \int_{x_0}^{x_N} \exp\{iS[x(t)]/\hbar\} \mathcal{D}[x(t)] \\ &= \lim_{\substack{N \rightarrow \infty \\ \varepsilon \rightarrow 0}} A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left[\frac{i}{\hbar} \frac{m}{2} \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^2}{\varepsilon} \right] \\ &\quad \times dx_1 \cdots dx_{N-1} \end{aligned} \quad (8.4.3)$$

† We expect that the abrupt changes in velocity at the points $t_0 + n\varepsilon$ that arise due to our approximation will not matter because \mathcal{L} does not depend on the acceleration or higher derivatives.

It is implicit in the above that x_0 and x_N have the values we have chosen at the outset. The factor A in the front is to be chosen at the end such that we get the correct scale for U when the limit $N \rightarrow \infty$ is taken.

Let us first switch to the variables

simple

change variables

$$y_i = \left(\frac{m}{2\hbar\varepsilon}\right)^{1/2} x_i$$

Gaussian

We then want

$$\lim_{N \rightarrow \infty} A' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-\sum_{i=0}^{N-1} \frac{(y_{i+1} - y_i)^2}{i}\right] dy_1 \cdots dy_{N-1} \quad (8.4.4)$$

where

integrate over $y_0, y_1, y_2, \dots, y_n$

$$A' = A \left(\frac{2\hbar\varepsilon}{m}\right)^{(N-1)/2}$$

Although the multiple integral looks formidable, it is not. Let us begin by doing the y_1 integration. Considering just the part of the integrand that involves y_1 , we get

first integral

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{i} [(y_2 - y_1)^2 + (y_1 - y_0)^2]\right\} dy_1 = \left(\frac{i\pi}{2}\right)^{1/2} e^{-(y_2 - y_0)^2/2i} \quad (8.4.5)$$

y_1 goes away!

Consider next the integration over y_2 . Bringing in the part of the integrand involving y_2 and combining it with the result above we compute next

second integral

$$\left(\frac{i\pi}{2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-(y_3 - y_2)^2/i} \cdot e^{-(y_2 - y_0)^2/2i} dy_2 = \left(\frac{i\pi}{2}\right)^{1/2} e^{-(2y_3^2 + y_0^2)/2i} \left(\frac{2\pi i}{3}\right)^{1/2} e^{(y_0 + 2y_3)^2/6i}$$

y_2 goes away!

$$= \left[\frac{(i\pi)^2}{3}\right]^{1/2} e^{-(y_3 - y_0)^2/3i} \quad (8.4.6)$$

By comparing this result to the one from the y_1 integration, we deduce the pattern: if we carry out this process $N - 1$ times so as to evaluate the integral in Eq. (8.4.4), it will become

$$\frac{(i\pi)^{(N-1)/2}}{N^{1/2}} e^{-(y_N - y_0)^2/Ni}$$

only y_n and y_0 remain!

or

$$\frac{(i\pi)^{(N-1)/2}}{N^{1/2}} e^{-m(x_N - x_0)^2/2\hbar\epsilon Ni}$$

Bringing in the factor $A(2\hbar\epsilon/m)^{(N-1)/2}$ from up front, we get

$$U = A \left(\frac{2\pi\hbar\epsilon i}{m} \right)^{N/2} \left(\frac{m}{2\pi\hbar i N \epsilon} \right)^{1/2} \exp \left[\frac{im(x_N - x_0)^2}{2\hbar N \epsilon} \right]$$

If we now let $N \rightarrow \infty$, $\epsilon \rightarrow 0$, $N\epsilon \rightarrow t_N - t_0$, we get the right answer provided

$$A = \left[\frac{2\pi\hbar\epsilon i}{m} \right]^{-N/2} \equiv B^{-N} \quad (8.4.7)$$

It is conventional to associate a factor $1/B$ with each of the $N-1$ integrations and the remaining factor $1/B$ with the overall process. In other words, we have just learnt that the precise meaning of the statement “integrate over all paths” is

$$\int \mathcal{D}[x(t)] = \lim_{\substack{\epsilon \rightarrow 0 \\ N \rightarrow \infty}} \frac{1}{B} \int_{-\infty}^{\infty} \int \dots \int_{-\infty}^{\infty} \frac{dx_1}{B} \cdot \frac{dx_2}{B} \dots \frac{dx_{N-1}}{B}$$

where

**Normalization
Factor**

$$B = \left(\frac{2\pi\hbar\epsilon i}{m} \right)^{1/2} \quad (8.4.8)$$

8.5. Equivalence to the Schrödinger Equation

The relation between the Schrödinger and Feynman formalisms is quite similar to that between the Newtonian and the least action formalisms of mechanics, in that the former approach is local in time and deals with time evolution over infinitesimal periods while the latter is global and deals directly with propagation over finite times.

In the Schrödinger formalism, the change in the state vector $|\psi\rangle$ over an infinitesimal time ϵ is

$$|\psi(\epsilon)\rangle - |\psi(0)\rangle = \frac{-i\epsilon}{\hbar} H |\psi(0)\rangle \quad (8.5.1)$$

which becomes in the X basis

$$\psi(x, \epsilon) - \psi(x, 0) = \frac{-i\epsilon}{\hbar} \left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, 0) \right] \psi(x, 0) \quad (8.5.2)$$

**To a mathematician,
it is obvious that:**

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

**Gauss was a
mathematician.**

Gauss 1828



Gauss 1840



Gauss 1855



A.2. Gaussian Integrals

We discuss here all the Gaussian integrals that we will need. Consider

$$I_0(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx, \quad \alpha > 0 \quad (\text{A.2.1})$$

This integral cannot be evaluated by conventional methods. The trick is to consider

$$I_0^2(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\alpha y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx dy$$

Switching to polar coordinates in the x - y plane,

$$\begin{aligned} I_0^2(\alpha) &= \int_0^{\infty} \int_0^{2\pi} e^{-\alpha \rho^2} \rho d\rho d\phi \\ &= \pi/\alpha \end{aligned}$$

Therefore

$$I_0(\alpha) = (\pi/\alpha)^{1/2} \quad (\text{A.2.2})$$

By differentiating with respect to α we can get all the integrals of the form

$$I_{2n}(\alpha) = \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx$$

For example,

$$\begin{aligned} I_2(\alpha) &= \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{\partial}{\partial \alpha} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \\ &= -\frac{\partial}{\partial \alpha} I_0(\alpha) = \frac{1}{2\alpha} \left(\frac{\pi}{\alpha}\right)^{1/2} \end{aligned} \quad (\text{A.2.3})$$

The integrals $I_{2n+1}(\alpha)$ vanish because these are integrals of odd functions over an even interval $-\infty$ to $+\infty$. Equations (A.2.2) and (A.2.3) are valid even if α is purely imaginary.

Consider next

$$I_0(\alpha, \beta) = \int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x} dx \quad (\text{A.2.4})$$

By completing the square on the exponent, we get

$$I_0(\alpha, \beta) = e^{\beta^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(x-\beta/2\alpha)^2} dx = e^{\beta^2/4\alpha} \left(\frac{\pi}{\alpha}\right)^{1/2} \quad (\text{A.2.5})$$

These results are valid even if α and β are complex, provided $\text{Re } \alpha > 0$. Finally, by applying to both sides of the equation

$$\int_0^{\infty} e^{-ar} dr = \frac{1}{a}$$

the operator $(-d/d\alpha)^n$, we obtain

$$\int_0^{\infty} r^n e^{-ar} dr = \frac{n!}{a^{n+1}}$$

Consider this integral with $\alpha = 1$ and n replaced by $z-1$, where z is an arbitrary complex number. This defines the *gamma function* $\Gamma(z)$

$$\Gamma(z) = \int_0^{\infty} r^{z-1} e^{-r} dr$$

For real, positive and integral z ,

$$\Gamma(z) = (z-1)!$$

A.3. Complex Numbers

A complex variable z can be written in terms of two real variables x and y , and $i = (-1)^{1/2}$, as

$$z = x + iy \quad (\text{A.3.1})$$

Its *complex conjugate* z^* is defined to be

$$z^* = x - iy \quad (\text{A.3.2})$$

One may invert these two equations to express the *real and imaginary parts*, x and y , as

$$x = \frac{1}{2}(z + z^*), \quad y = (z - z^*)/2i \quad (\text{A.3.3})$$

Laplace was once asked by Alexander von Humboldt (a German scientist who, in addition to writing an immensely popular work on science called Kosmos, was also responsible for making mountain climbing a popular sport) who the great mathematician in Germany was. Without hesitation, Laplace said, "Pfaff". "Pfaff?" Humboldt said. "What about Gauss?" "Gauss is the greatest mathematician in the world" was Laplace's reply.

Laplace



Humboldt



Pfaff



LE PFAFF.

Laplace went in state to beg Napoleon to accept a copy of his work, and the following account of the interview is well authenticated, and so characteristic of all the parties concerned that I quote it in full.

Lagrange had told Napoleon that the book contained no mention of the name of God; Napoleon, who was fond of putting embarrassing questions, received it with the remark, "M. Laplace, they tell me you have written this large book on the system of the universe, and have never once mentioned its Creator."

Laplace, who, though the most supple of politicians, was as stiff as a martyr on every point of his philosophy, drew himself up and answered bluntly, "I had no need of that hypothesis."

Lagrange exclaimed "But, it is a fine hypothesis; it explains many things."

T R A I T É
D E
M É C A N I Q U E C É L E S T E ,

P A R P . S . L A P L A C E ,

**Membre de l'Institut national de France , et du Bureau
des Longitudes.**

T O M E P R E M I E R .

D E L ' I M P R I M E R I E D E C R A P E L E T .

A P A R I S ,

**Chez J. B. M. DUPRAT, Libraire pour les Mathématiques ,
quai des Augustins.**

A N V I I .

Laplace



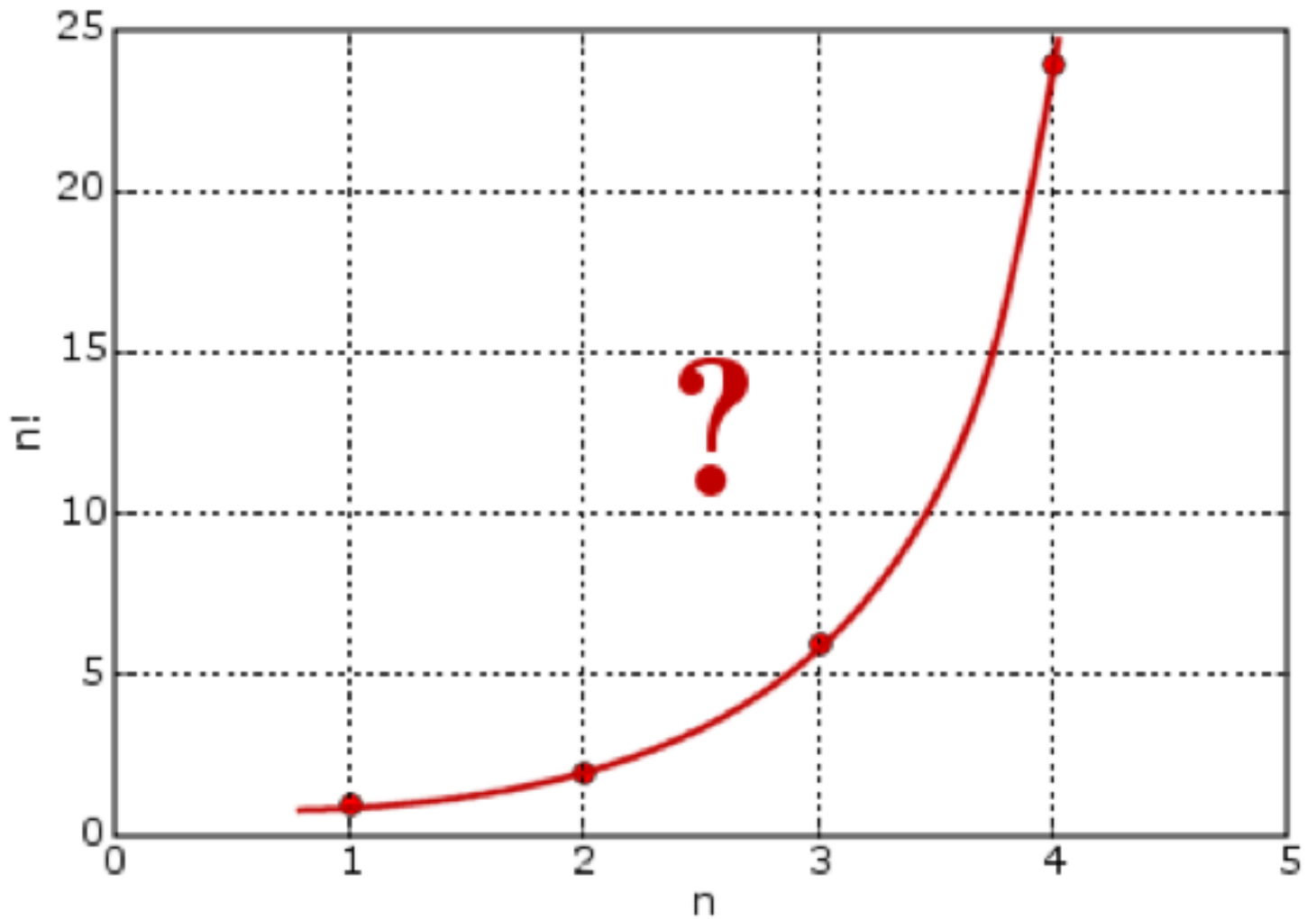
Napoleon 1812



Lagrange



How to Generalize the Factorial to Non-Integer Arguments ...



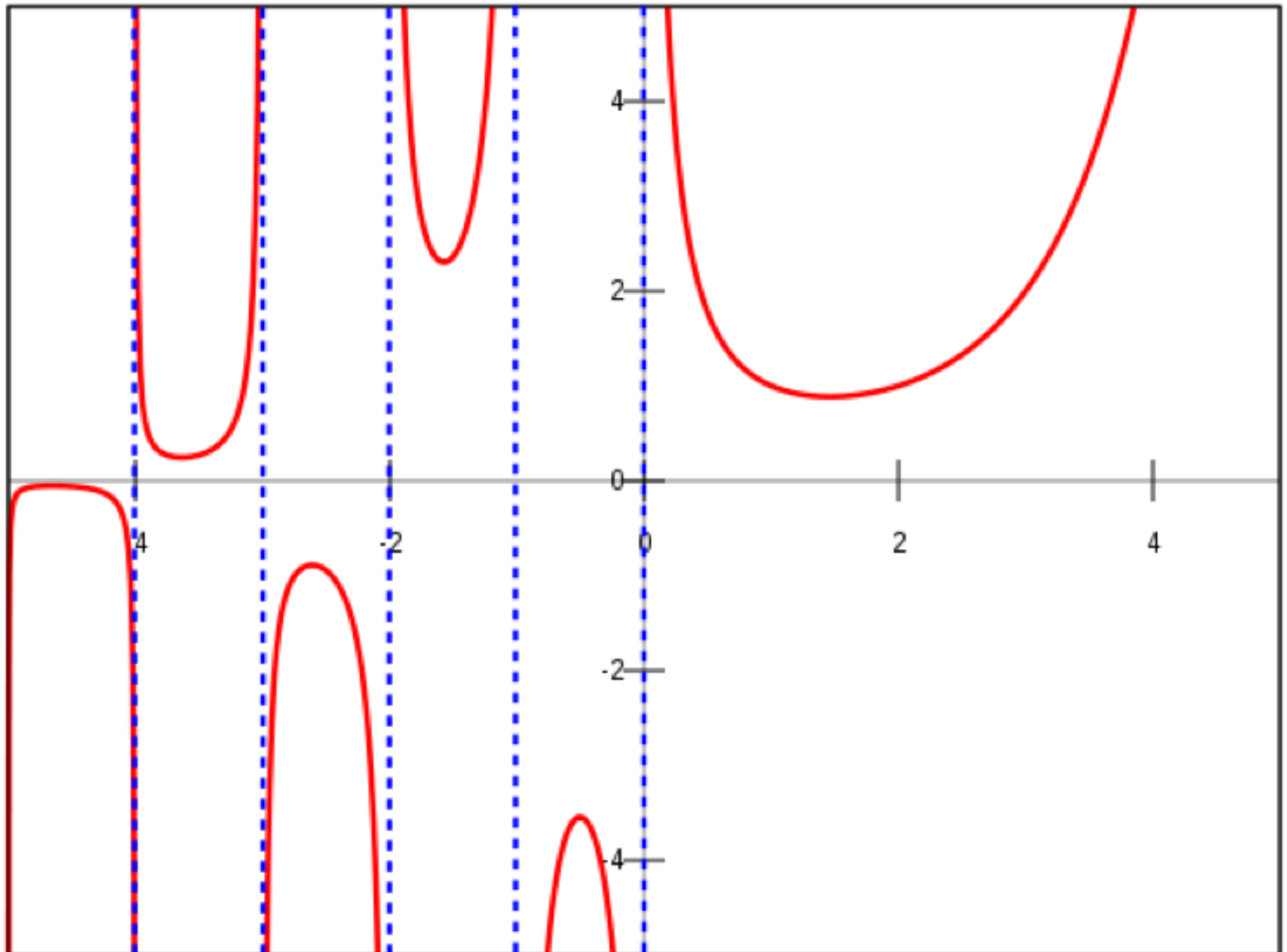
Introduction

The gamma function can be thought of as the natural way to generalize the concept of the factorial to non-integer arguments.

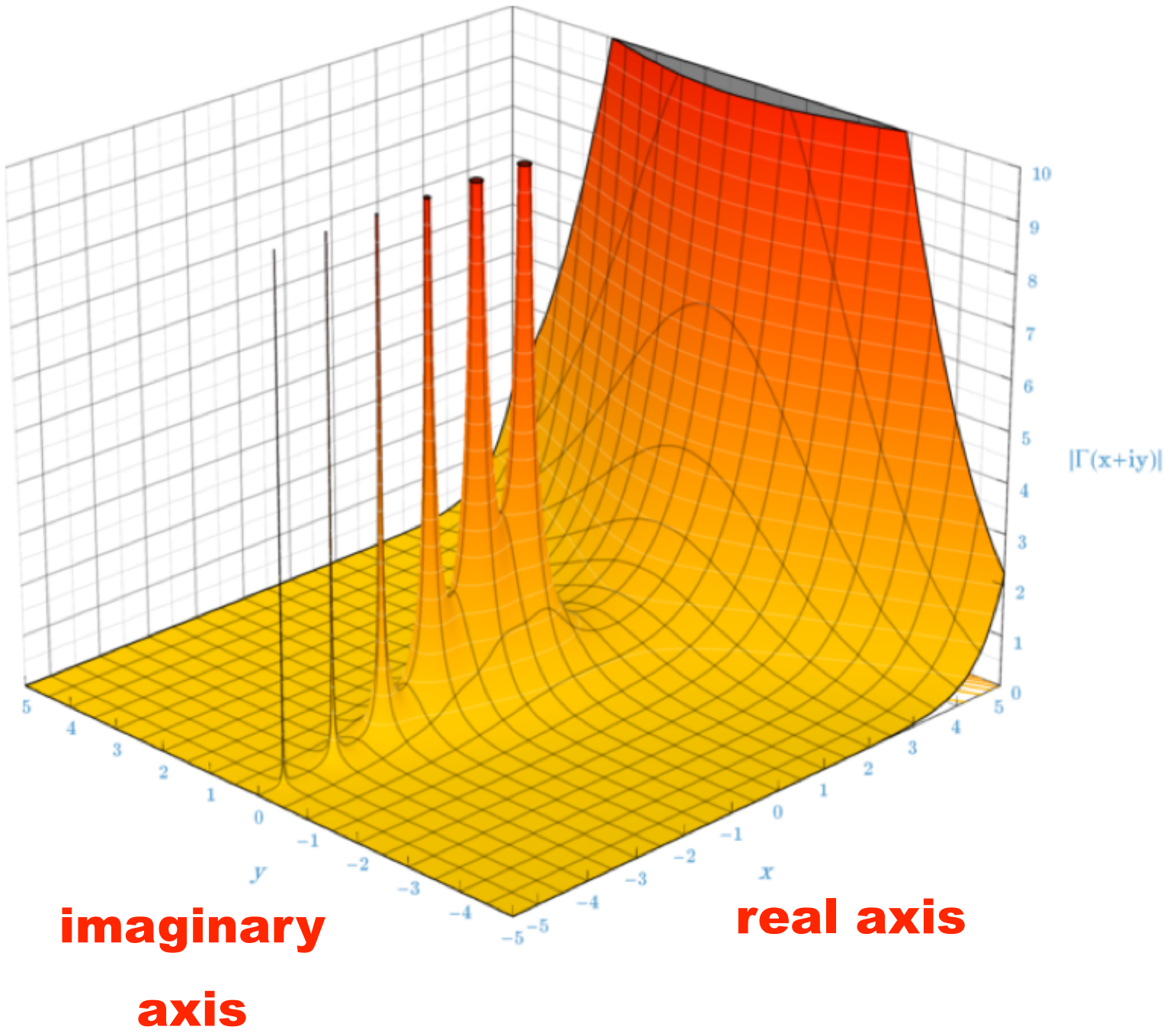
Leonhard Euler came up with a formula for such a generalization in 1729. At around the same time, James Stirling independently arrived at a different formula, but was unable to show that it always converged. In 1900, Charles Hermite showed that the formula given by Stirling does work, and that it defines the same function as Euler's.

The Gamma Function on the Real Axis

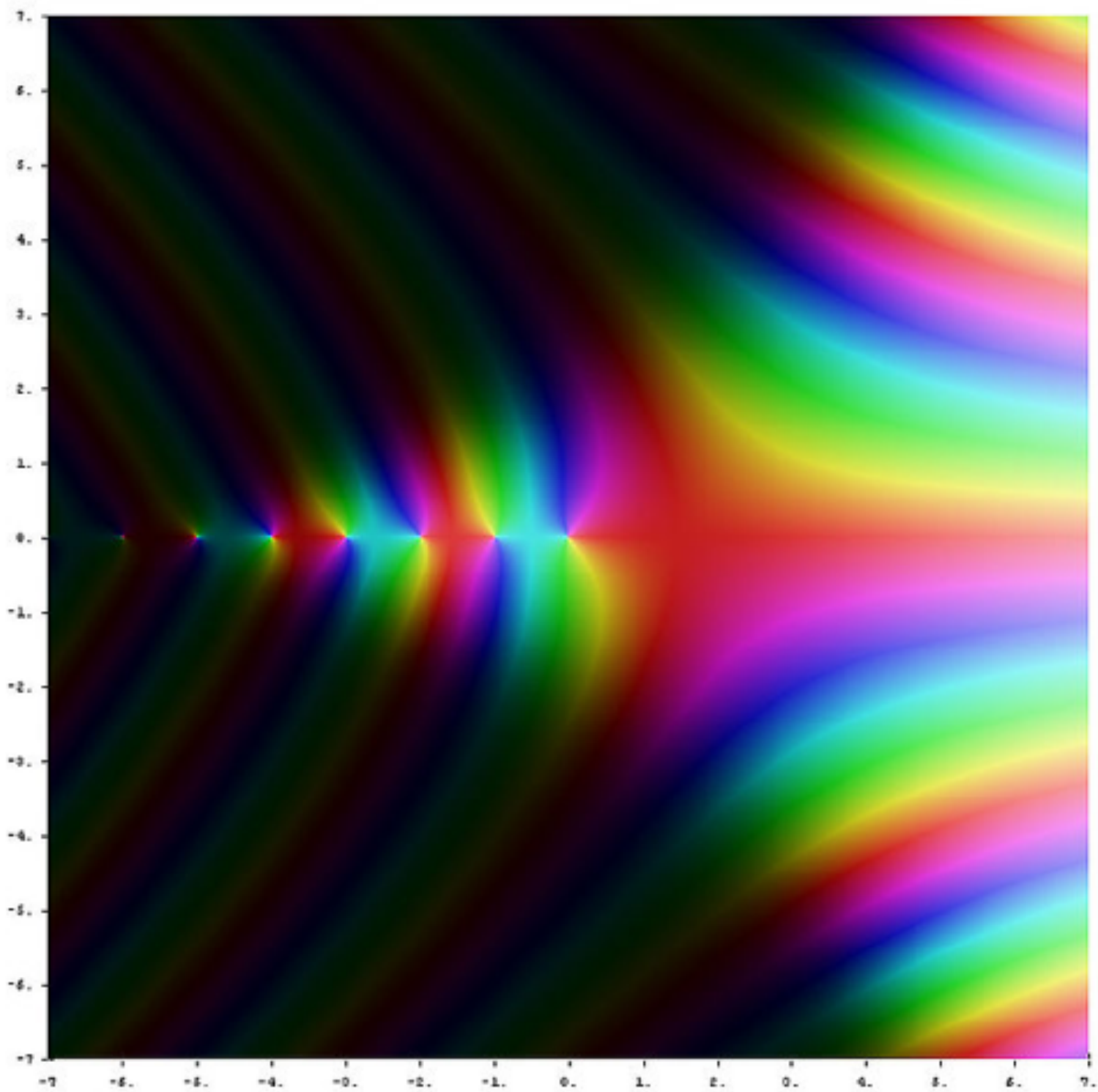
Gamma function



Magnitude of Complex Gamma

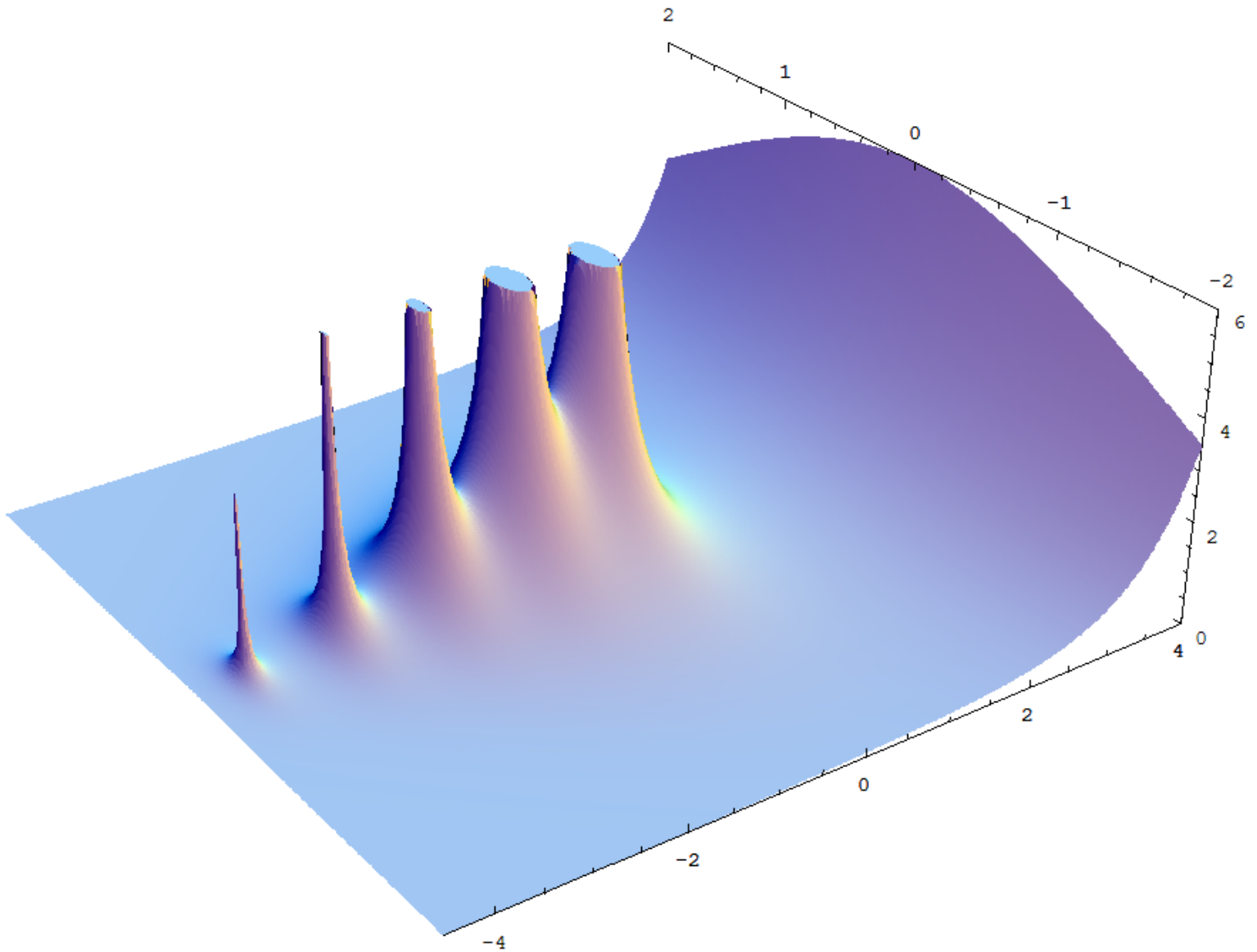


Magnitude and Phase of Complex Gamma

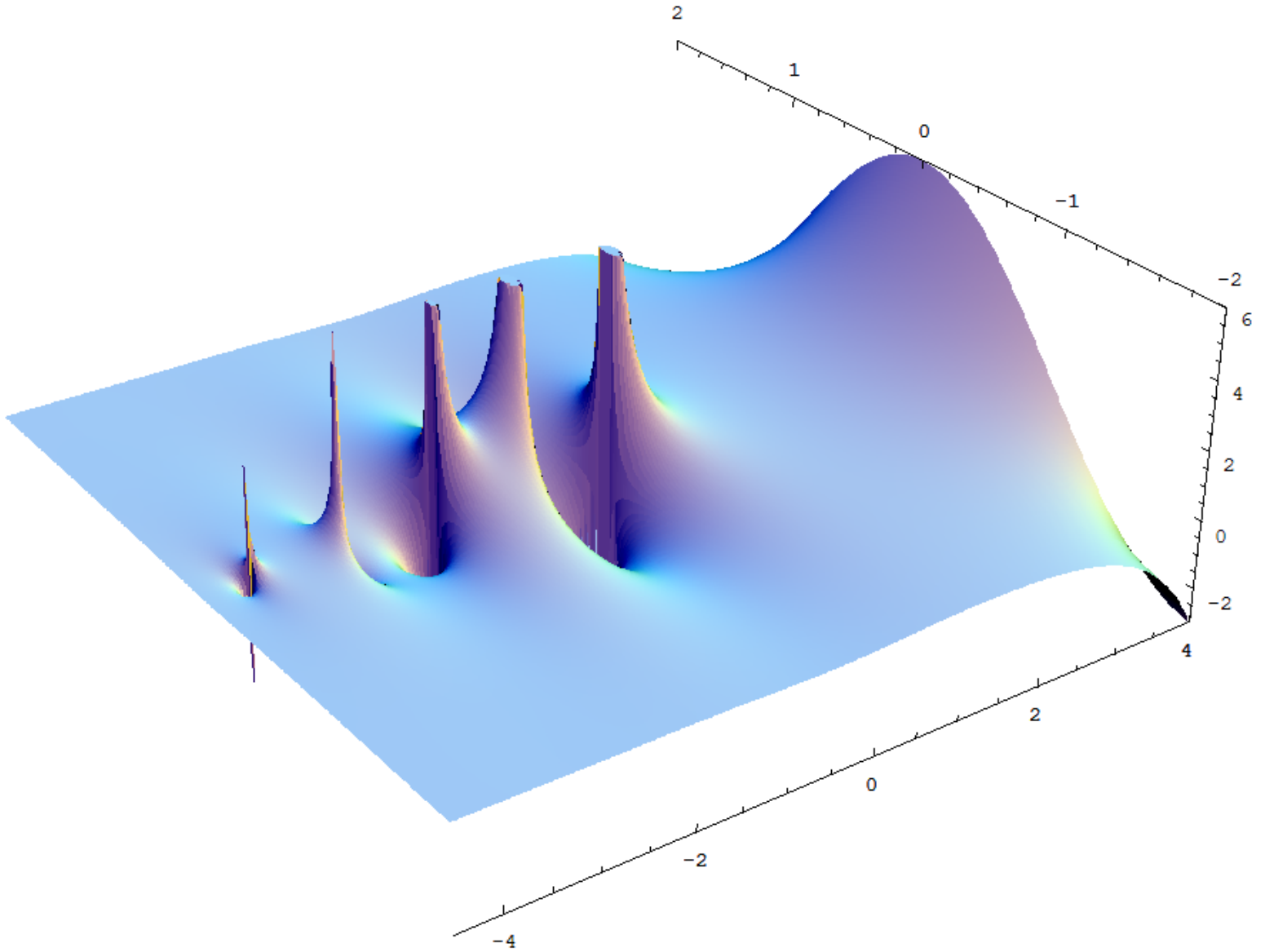


real axis

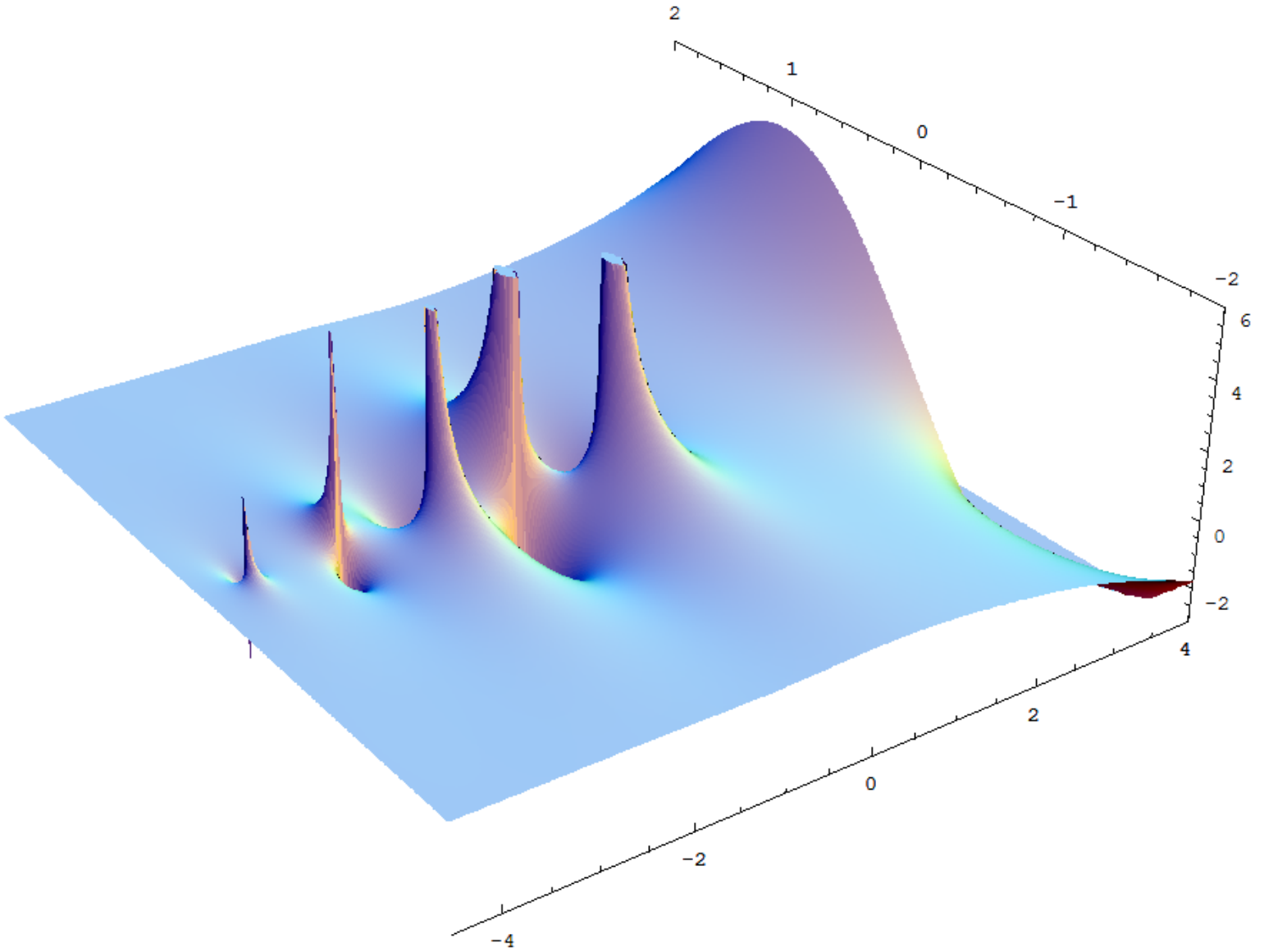
Magnitude of the Gamma Function



Real Part of the Gamma Function



Imaginary Part of the Gamma Function



We will work to first order in ε and therefore to second order in η [see Eq. (8.5.6) above]. We expand

$$\begin{aligned}\psi(x + \eta, 0) &= \psi(x, 0) + \eta \frac{\partial \psi}{\partial x} + \frac{\eta^2}{2} \frac{\partial^2 \psi}{\partial x^2} + \dots \\ \exp\left[-\left(\frac{i}{\hbar}\right)\varepsilon V\left(x + \frac{\eta}{2}, 0\right)\right] &= 1 - \frac{i\varepsilon}{\hbar} V\left(x + \frac{\eta}{2}, 0\right) + \dots \\ &= 1 - \frac{i\varepsilon}{\hbar} V(x, 0) + \dots\end{aligned}$$

since terms of order $\eta\varepsilon$ are to be neglected. Equation (8.5.7) now becomes

$$\begin{aligned}\psi(x, \varepsilon) &= \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right) \left[\psi(x, 0) - \frac{i\varepsilon}{\hbar} V(x, 0)\psi(x, 0)\right. \\ &\quad \left.+ \eta \frac{\partial \psi}{\partial x} + \frac{\eta^2}{2} \frac{\partial^2 \psi}{\partial x^2}\right] d\eta\end{aligned}$$

Consulting the list of Gaussian integrals in Appendix A.2, we get

$$\begin{aligned}\psi(x, \varepsilon) &= \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} \left[\psi(x, 0) \left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{1/2} - \frac{\hbar\varepsilon}{2im} \left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{1/2} \frac{\partial^2 \psi}{\partial x^2}\right. \\ &\quad \left.- \frac{i\varepsilon}{\hbar} \left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{1/2} V(x, 0)\psi(x, 0)\right]\end{aligned}$$

or

$$\psi(x, \varepsilon) - \psi(x, 0) = \frac{-i\varepsilon}{\hbar} \left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, 0)\right] \psi(x, 0) \quad (8.5.8)$$

which agrees with the Schrödinger prediction, Eq. (8.5.1).

8.6. Potentials of the Form $V = a + bx + cx^2 + dx + ex\ddagger$

We wish to compute

$$U(x, t; x') = \int_{x'}^x e^{iS[x(t'')]/\hbar} \mathcal{D}[x(t'')] \quad (8.6.1)$$

‡ This section may be omitted without loss of continuity.

General Potential

L depends on x and dx/dt (x-dot)

Let us write every path as

$$x(t'') = x_{cl}(t'') + y(t'') \quad \mathbf{x} \quad (8.6.2)$$

It follows that

$$\dot{x}(t'') = \dot{x}_{cl}(t'') + \dot{y}(t'') \quad \mathbf{x\text{-dot}} \quad (8.6.3)$$

Since all the paths agree at the end points, $y(0) = y(t) = 0$. When we slice up the time into N parts, we have for intermediate integration variables

$$x_i \equiv x(t_i'') = x_{cl}(t_i'') + y(t_i'') \equiv x_{cl}(t_i'') + y_i$$

Since $x_{cl}(t_i'')$ is just some constant at t_i'' ,

$$dx_i = dy_i$$

and

change variables

$$\int_{x'}^x \mathcal{D}[x(t'')] = \int_0^0 \mathcal{D}[y(t'')] \quad (8.6.4)$$

so that Eq. (8.6.1) becomes

$$U(x, t; x') = \int_0^0 \exp \left\{ \frac{i}{\hbar} S[x_{cl}(t'') + y(t'')] \right\} \mathcal{D}[y(t'')] \quad (8.6.5)$$

The next step is to expand the functional S in a Taylor series about x_{cl} :

expand S in a Taylor series

$$\begin{aligned} S[x_{cl} + y] &= \int_0^t \mathcal{L}(x_{cl} + y, \dot{x}_{cl} + \dot{y}) dt'' \\ &\equiv \int_0^t \left[\mathcal{L}(x_{cl}, \dot{x}_{cl}) + \left(\frac{\partial \mathcal{L}}{\partial x} \Big|_{x_{cl}} y + \frac{\partial \mathcal{L}}{\partial \dot{x}} \Big|_{x_{cl}} \dot{y} \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\partial^2 \mathcal{L}}{\partial x^2} \Big|_{x_{cl}} y^2 + 2 \frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} \Big|_{x_{cl}} y \dot{y} + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} \Big|_{x_{cl}} \dot{y}^2 \right) \right] dt'' \end{aligned} \quad (8.6.6)$$

The series terminates here since \mathcal{L} is a quadratic polynomial.

The first piece $\mathcal{L}(x_{cl}, \dot{x}_{cl})$ integrates to give $S[x_{cl}] \equiv S_{cl}$. The second piece, linear in y and \dot{y} , vanishes due to the classical equation of motion. In the last piece, if we recall

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - a - bx - cx^2 - d\dot{x} - ex\dot{x} \quad (8.6.7)$$

we get

$$\frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial x^2} = -c \quad (8.6.8)$$

$$\frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} = -e \quad (8.6.9)$$

$$\frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} = m \quad (8.6.10)$$

Consequently Eq. (8.6.5) becomes

$$U(x, t; x') = \exp\left(\frac{iS_{cl}}{\hbar}\right) \int_0^0 \exp\left[\frac{i}{\hbar} \int_0^t \left(\frac{1}{2} m \dot{y}^2 - cy^2 - ey\dot{y}\right) dt''\right] \times \mathcal{D}[y(t'')] \quad (8.6.11)$$

Since the path integral has no memory of x_{cl} , it can only depend on t . So

only need to find $A(t)$
$$U(x, t; x') = e^{iS_{cl}/\hbar} A(t) \quad (8.6.12)$$

where $A(t)$ is some unknown function of t . Now if we were doing the free-particle problem, we would get Eq. (8.6.11) with $c=e=0$. In this case we know that [see Eq. (8.3.4)]

for the free particle
$$A(t) = \left(\frac{m}{2\pi\hbar it}\right)^{1/2} \quad (8.6.13)$$

same for linear potential

Since the coefficient b does not figure in Eq. (8.6.11), it follows that the same value of $A(t)$ corresponds to the linear potential $V=a+bx$ as well. For the harmonic oscillator, $c=\frac{1}{2}m\omega^2$, and we have to do the integral

for the harmonic oscillator, need to do this integral

$$A(t) = \int_0^0 \exp\left[i/\hbar \int_0^t \frac{1}{2} m (\dot{y}^2 - \omega^2 y^2)\right] dt'' \mathcal{D}[y(t'')] \quad (8.6.14)$$

The evaluation of this integral is discussed in the book by Feynman and Hibbs referred to at the end of this section. Note that even if the factor $A(t)$ in $\psi(x, t)$ is not known, we can extract all the probabilistic information at time t .

Notice the ease with which the Feynman formalism yields the full propagator in these cases. Consider in particular the horrendous alternative of finding the eigenfunctions of the Hamiltonian and constructing from them the harmonic oscillator propagator.

The path integral method may be extended to three dimensions without any major qualitative differences. In particular, the form of U in Eq. (8.6.12) is valid for potentials that are at most quadratic in the coordinates and the velocities. An

interesting problem in this class is that of a particle in a uniform magnetic field. For further details on the subject of path integral quantum mechanics, see R. P. Feynman and A. R. Hibbs, *Path Integrals and Quantum Mechanics*, McGraw-Hill (1965), and Chapter 21.

Exercise 8.6.1. * Verify that

$$U(x, t; x', 0) = A(t) \exp(iS_{cl}/\hbar), \quad A(t) = \left(\frac{m}{2\pi\hbar it} \right)^{1/2}$$

agrees with the exact result, Eq. (5.4.31), for $V(x) = -fx$. Hint: Start with $x_{cl}(t'') = x_0 + v_0 t'' + \frac{1}{2}(f/m)t''^2$ and find the constants x_0 and v_0 from the requirement that $x_{cl}(0) = x'$ and $x_{cl}(t) = x$.

Exercise 8.6.2. Show that for the harmonic oscillator with

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$U(x, t; x') = A(t) \exp \left\{ \frac{i m \omega}{2 \hbar \sin \omega t} [(x^2 + x'^2) \cos \omega t - 2 x x'] \right\}$$

where $A(t)$ is an unknown function. (Recall Exercise 2.8.7.)

Exercise 8.6.3. We know that given the eigenfunctions and the eigenvalues we can construct the propagator:

In 441 we needed:

$$U(x, t; x', t') = \sum_n \psi_n(x)^* \psi_n(x') e^{-iE_n(t-t')/\hbar}$$

eigenfunctions

(8.6.15)

eigenvalues

Consider the reverse process (since the path integral approach gives U directly), for the case of the oscillator.

(1) Set $x = x' = t' = 0$. Assume that $A(t) = (m\omega/2\pi i \hbar \sin \omega t)^{1/2}$ for the oscillator. By expanding both sides of Eq. (8.6.15), you should find that $E = \hbar\omega/2, 5\hbar\omega/2, 9\hbar\omega/2, \dots$, etc. What happened to the levels in between?

(2) (Optional). Now consider the extraction of the eigenfunctions. Let $x = x'$ and $t' = 0$. Find $E_0, E_1, |\psi_0(x)|^2$, and $|\psi_1(x)|^2$ by expanding in powers of $a = \exp(i\omega t)$.

Exercise 8.6.4. * Recall the derivation of the Schrödinger equation (8.5.8) starting from Eq. (8.5.4). Note that although we chose the argument of V to be the midpoint $x + x'/2$, it did not matter very much: any choice $x + a\eta$, (where $\eta = x' - x$) for $0 \leq a \leq 1$ would have given the same result since the difference between the choices is of order $\eta \varepsilon \simeq \varepsilon^{3/2}$. All this was thanks to the factor ε multiplying V in Eq. (8.5.4) and the fact that $|\eta| \simeq \varepsilon^{1/2}$, as per Eq. (8.6.5).

PHASE SPACE PATH INTEGRAL

$$\int e^{iS/\hbar} \mathcal{D}[p(t)] \mathcal{D}[x(t)]$$

$$L = p \dot{x} - H(x, p)$$



$$2T - (T + V)$$

$$T - V$$

$$S = \int_{t_1}^{t_2} L dt$$

WHEN $H \propto p^2$ AND x AND p SEPARATE

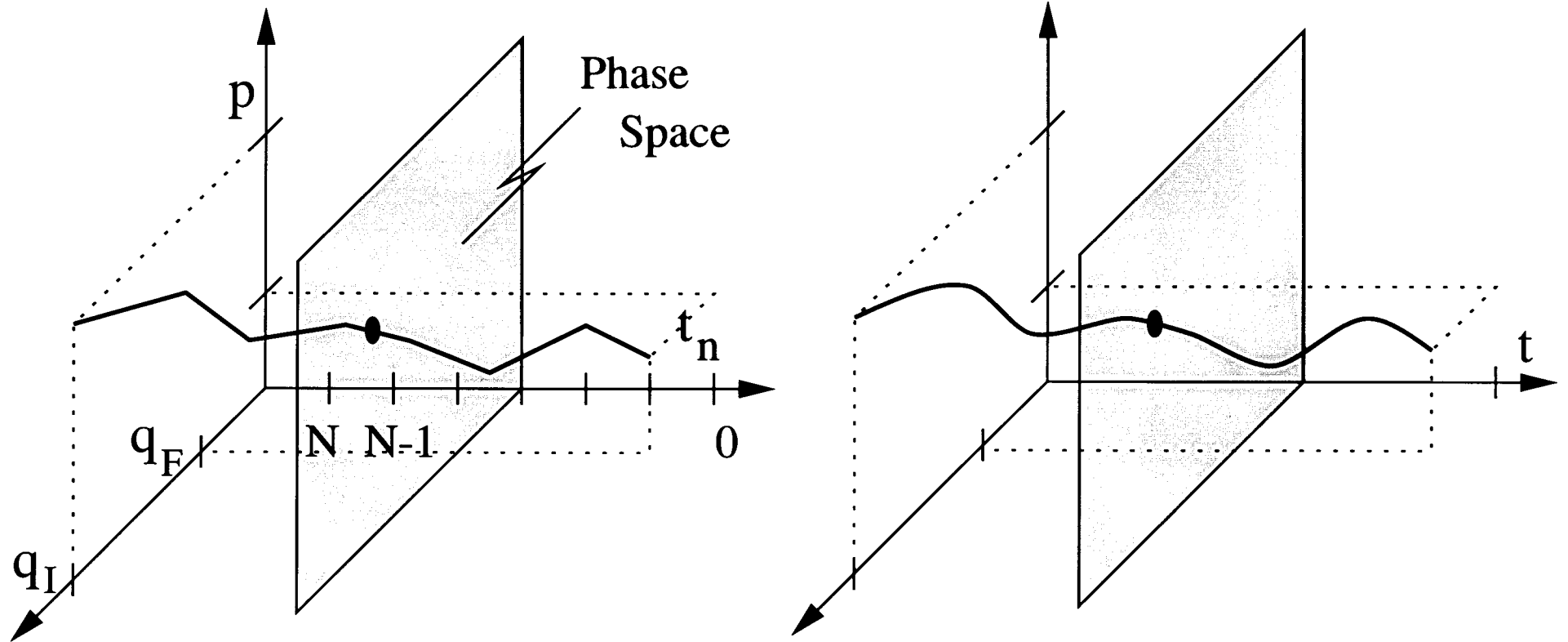
DO $\int \mathcal{D}[p(t)] \Rightarrow$ position space

QUANTUM
GRAVITY

path integral

$$\int \mathcal{D}[g_{\mu\nu}] \overset{e^{iS_g/\hbar}}{\uparrow} \overset{\text{everything}}{\mathcal{D}}[\text{fields}] e^{iS_m/\hbar}$$

Phase space path integral $D[p(t)] D[x(t)]$



Path Integrals: Part II

In this chapter we return to path integrals for a more detailed and advanced treatment. The tools described here are so widely used in so many branches of physics, that it makes sense to include them in a book such as this. This chapter will be different from the earlier ones in that it will try to introduce you to a variety of new topics without giving all the derivations in same detail as before. It also has a list of references to help you pursue any topic that attracts you. The list is not exhaustive and consists mostly of pedagogical reviews or books. From the references these references contain, you can pursue any given topic in greater depth. All this will facilitate the transition from course work to research.

In Chapter 8 the path integral formula for the propagator was simply postulated and shown to lead to the same results as the operator methods either by direct evaluation of the propagator (in the free particle case) or by showing once and for all that the Schrödinger equation followed from the path integral prescription for computing the time evolution.

We begin this chapter by doing the reverse: we start with the operator Hamiltonian $H = P^2/2m + V$ and derive the propagator for it as a path integral. We shall see that there are many types of path integrals one can derive. We will discuss

- The configuration space path integral, discussed in Chapter 8.
- The phase space path integral.
- The coherent state path integral.

You will see that the existence of many path integrals is tied to the existence of many resolutions of the identity, i.e., to the existence of many bases.

Following this we will discuss two applications: to the Quantum Hall Effect (QHE) and a recent development called the Berry Phase.

We then turn to imaginary time quantum mechanics and its relation to statistical mechanics (classical and quantum) as well the calculation of tunneling amplitudes by a semiclassical approximation. You will learn about instantons, the transfer matrix formulation, and so on.

Finally, we discuss path integrals for two problems with no classical limit: a spin Hamiltonian and a fermionic oscillator.

21.1. Derivation of the Path Integral

Let us assume that the Hamiltonian is time-independent and has the form

$$H = \frac{P^2}{2m} + V(X) \quad (21.1.1)$$

The propagator is defined by **$U(t) = \exp(-i H t / \hbar)$**

$$U(xt; x'0) \equiv U(x, x', t) = \langle x | \exp\left(-\frac{i}{\hbar} H t\right) | x' \rangle \quad (21.1.2)$$

It was stated in Chapter 8 that U may be written as a sum over paths going from $(x'0)$ to (xt) . We will now see how this comes about.

First, it is evident that we may write

factor into N steps
$$\exp\left(-\frac{i}{\hbar} H t\right) = \left[\exp\left(-\frac{i}{\hbar} H \frac{t}{N}\right) \right]^N \quad (21.1.3)$$

for any N . This merely states that $U(t)$, the propagator for a time t , is the product of N propagators $U(t/N)$. Let us define

$$\varepsilon = \frac{t}{N} \quad (21.1.4)$$

and consider the limit $N \rightarrow \infty$. Now we can write

$$\exp\left(-\frac{i\varepsilon}{\hbar} (P^2/2m + V(X))\right) \simeq \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) \quad (21.1.5)$$

momentum position

because of the fact that

$$e^A e^B = e^{A+B+1/2[A,B]+\dots} \quad (21.1.6)$$

which allows us to drop the commutator shown (and other higher-order nested commutators not shown) on the grounds that they are proportional to higher powers of ε which is going to 0. While all this is fine if A and B are finite dimensional matrices with finite matrix elements, it is clearly more delicate for operators in Hilbert space which could have large or even singular matrix elements. We will simply assume that in the limit $\varepsilon \rightarrow 0$ the \simeq sign in Eq. (21.1.5) will become the equality sign for the purpose of computing any reasonable physical quantity.

position and momentum do not commute, but ...

N steps

So we have to compute

$$\underbrace{\langle x | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) \cdot \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) \cdots | x' \rangle}_{N \text{ times}} \quad (21.1.7)$$

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PATH
INTEGRALS:
PART II

The next step is to introduce the resolution of the identity:

resolution of the identity

$$I = \int_{-\infty}^{\infty} dx |x\rangle \langle x| \quad (21.1.8)$$

between every two adjacent factors of $U(t/N)$. Let us illustrate the outcome by considering $N=3$. We find (upon renaming x, x' as x_3, x_0 for reasons that will be clear soon)

$$\begin{aligned} U(x_3, x_0, t) &= \int \prod_{n=1}^2 dx_n \langle x_3 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_2 \rangle \\ &\quad \times \langle x_2 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_1 \rangle \\ &\quad \times \langle x_1 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_0 \rangle \end{aligned} \quad (21.1.9)$$

Consider now the evaluation of the matrix element

$$\langle x_n | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_{n-1} \rangle \quad (21.1.10)$$

When the rightmost exponential operates on the ket to its right, the operator X gets replaced by the eigenvalue x_{n-1} . Thus,

$$\begin{aligned} &\langle x_n | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_{n-1} \rangle \\ &= \langle x_n | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | x_{n-1} \rangle \exp\left(-\frac{i\varepsilon}{\hbar} V(x_{n-1})\right) \end{aligned} \quad (21.1.11)$$

Consider now the remaining matrix element. It is simply the free particle propagator from x_{n-1} to x_n in time ε . We know what it is [say from Eq. (5.1.10)] or the following exercise

$$\langle x_n | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | x_{n-1} \rangle = \left[\frac{m}{2\pi i \hbar \varepsilon} \right]^{1/2} \exp\left[\frac{im(x_n - x_{n-1})^2}{2\hbar \varepsilon} \right] \quad (21.1.12)$$

Exercise 21.1.1. Derive the above result independently of Eq. (5.1.10) by introducing a resolution of the identity in terms of momentum states between the exponential operator and the position eigenket in the left-hand side of Eq. (21.1.12). That is, use

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} |p\rangle\langle p| \quad (21.1.13)$$

where the plane wave states have a wave function given by

$$\langle x|p\rangle = e^{ipx/\hbar} \quad (21.1.14)$$

which explains the measure for the p integration.

Resuming our derivation, we now have

$$\begin{aligned} & \langle x_n | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) |x_{n-1}\rangle \\ &= \left[\frac{m}{2\pi i\hbar\varepsilon}\right]^{1/2} \exp\left[\frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon}\right] \exp\left(-\frac{i\varepsilon}{\hbar} V(x_{n-1})\right) \end{aligned} \quad (21.1.15)$$

Collecting all such factors (there are just two more in this case with $N=3$), we can readily see that for general N

$$\begin{aligned} U(x_N, x_0, t) &= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \left[\int \prod_{n=i}^{N-1} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} dx_n \right] \\ &\times \exp\left[\sum_{n=1}^N \frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right] \end{aligned} \quad (21.1.16)$$

If we drop the V terms we see that this is in exact agreement with the free particle path integral of Chapter 8. For example, the measure for integration has exactly N factors of B^{-1} as per Eq. (8.4.8), of which $N-1$ accompany the x -integrals. With the V term, the integrand is just the discretized version of $\exp(iS/\hbar)$:

$$\begin{aligned} & \exp\left[\sum_{n=1}^N \frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right] \\ &= \exp\frac{i}{\hbar} \varepsilon \sum_{n=1}^N \left[\frac{m(x_n - x_{n-1})^2}{2\varepsilon^2} - V(x_{n-1}) \right] \end{aligned} \quad (21.1.17)$$

We can go back to the continuum notation and write all this as follows:

$$U(x, x', t) = \int [\mathcal{D}x] \exp \left[\frac{i}{\hbar} \int_0^t \mathcal{L}(x, \dot{x}) dt \right] \quad (21.1.18)$$

where

configuration space path integral

$$\int [\mathcal{D}x] = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{1/2} \int \left[\prod_{n=i}^{N-1} \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{1/2} dx_n \right] \quad (21.1.19)$$

The continuum notation is really a schematic for the discretized version that preceded it, and we need the latter to define what one means by the path integral. It is easy to make many mistakes if one forgets this. In particular, there is no reason to believe that replacing differences by derivatives is always legitimate. For example, in this problem, in a time ε , the variable being integrated over typically changes by $\mathcal{O}(\varepsilon^{1/2})$ and not $\mathcal{O}(\varepsilon)$, as explained in the discussion before Eq. (8.5.6). The works in the Bibliography at the end of this chapter discuss some of the subtleties. The continuum version is, however, very useful to bear in mind since it exposes some aspects of the theory that would not be so transparent otherwise. It is also very useful for getting the picture at the semiclassical level and for finding whatever connection there is between the macroscopic world of smooth paths and the quantum world. We will take up some examples later.

The path integral derived above is called the *Configuration Space* path integral or simply the path integral. We now consider another one. Let us go back to

$$\underbrace{\langle x_N | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) \cdot \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) \cdot \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) \dots |x_0\rangle}_{N \text{ times}} \quad (21.1.20)$$

Let us now introduce resolutions of the identity between *every exponential* and the next. We need two versions

**insert two
resolutions
of the identity**

$$I = \int_{-\infty}^{\infty} dx |x\rangle \langle x| \quad (21.1.21)$$

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} |p\rangle \langle p| \quad (21.1.22)$$

where the plane wave states have a wave function given by

$$\langle x | p \rangle = e^{ipx/\hbar} \quad (21.1.23)$$

Let us first set $N=3$ and insert three resolutions of the identity in terms of p -states and two in terms of x -states with x and p resolutions alternating. This gives us

$$\begin{aligned}
 U(x_3, x_0, t) = & \int [\mathcal{D}p\mathcal{D}x] \langle x_3 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | p_3 \rangle \\
 & \times \langle p_3 | \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_2 \rangle \langle x_2 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | p_2 \rangle \\
 & \times \langle p_2 | \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_1 \rangle \langle x_1 | \exp\left(-\frac{i\varepsilon}{2m\hbar} P^2\right) | p_1 \rangle \\
 & \times \langle p_1 | \exp\left(-\frac{i\varepsilon}{\hbar} V(X)\right) | x_0 \rangle
 \end{aligned} \tag{21.1.24}$$

where

$$\int [\mathcal{D}p\mathcal{D}x] = \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2N-1 \text{ times}} \prod_{n=1}^N \frac{dp_n}{2\pi\hbar} \prod_{n=1}^{N-1} dx_n \tag{21.1.25}$$

Evaluating all the matrix elements of the exponential operators is trivial since each operator can act on the eigenstate to its right and get replaced by the eigenvalue. Collecting all the factors (a strongly recommended exercise for you) we obtain

$$U(x, x', t) = \int [\mathcal{D}p\mathcal{D}X] \exp \left[\sum_{i=1}^N \left(\frac{-i\varepsilon}{2m\hbar} p_n^2 + \frac{i}{\hbar} p_n (x_n - x_{n-1}) - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right) \right] \tag{21.1.26}$$

This formula derived for $N=3$ is obviously true for any N . In the limit $N \rightarrow \infty$, i.e., $\varepsilon \rightarrow 0$, we write schematically in continuous time (upon multiplying and dividing the middle term by ε), the following continuum version:

phase space path integral

$$U(x, x', t) = \int [\mathcal{D}p\mathcal{D}x] \exp \left[\frac{i}{\hbar} \int_0^t [p\dot{x} - \mathcal{H}(x, p)] dt \right] \tag{21.1.27}$$

where $\mathcal{H} = p^2/2m + V(x)$ and $(x(t), p(t))$ are now written as functions of a continuous variable t . This is the *Phase Space Path Integral* for the propagator. The continuum version is very pretty [with the Lagrangian in the exponent, but expressed in terms of (x, p)] but is only a schematic for the discretized version preceding it.

In our problem, since p enters the Hamiltonian quadratically, it is possible to integrate out all the N variables p_n . Going back to the discretized form, we isolate

if $H \sim p^2$, then recover configuration space path integral
if not, then must use the phase space path integral

the part that depends on just p 's and do the integrals:

$$\begin{aligned} & \prod_1^N \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp \left[\left(\frac{-i\varepsilon}{2m\hbar} p_n^2 + \frac{i}{\hbar} p_n (x_n - x_{n-1}) \right) \right] \\ &= \prod_1^N \left(\frac{m}{2\pi i\hbar\varepsilon} \right)^{1/2} \exp \left[\frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} \right] \end{aligned} \quad (21.1.28)$$

If we now bring in the x -integrals we find that this gives us exactly the configuration space path integral, as it should.

Note that if p does not enter the Hamiltonian in a separable quadratic way, it will not be possible to integrate it out and get a path integral over just x , in that we do not know how to do non-Gaussian integrals. In that case we can only write down the phase space path integral.

We now turn to two applications that deal with the path integrals just discussed.

The Landau Levels

We now discuss a problem that is of great theoretical interest in the study of QHE (see Girvin and Prange). We now explore some aspects of it, not all having to do with functional integrals. Consider a particle of mass μ and charge q in the x - y plane with a uniform magnetic field B along the z -axis. This is a problem we discussed in Exercise (12.3.8). Using a vector potential

$$\mathbf{A} = \frac{B}{2} (-y\mathbf{i} + x\mathbf{j}) \quad (21.1.29)$$

we obtained a Hamiltonian

$$H = \frac{[P_x + qYB/2c]^2}{2\mu} + \frac{[P_y - qXB/2c]^2}{2\mu} \quad (21.1.30)$$

You were asked to verify that

$$Q = \frac{(cP_x + qYB/2)}{qB} \quad P = (P_y - qBX/2c) \quad (21.1.31)$$

were canonical variables with $[Q, P] = i\hbar$. It followed that H was given by the formula

$$H = \frac{P^2}{2\mu} + \frac{1}{2} \mu \omega_0^2 Q^2 \quad (21.1.32)$$

and had a harmonic oscillator spectrum with spacing $\hbar\omega_0$, where

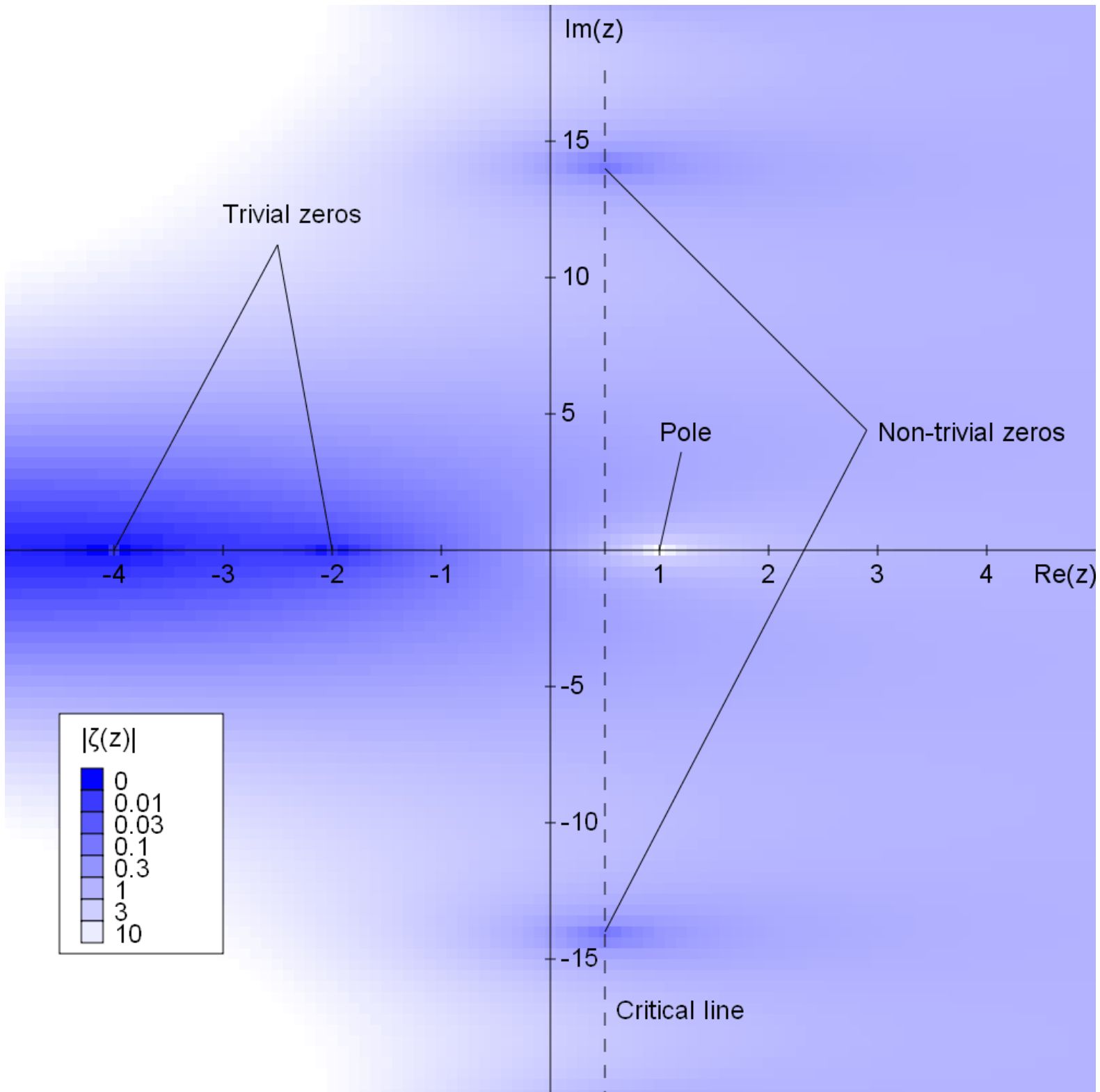
$$\omega_0 = qB/\mu c \quad (21.1.33)$$

Riemann zeta function

$$\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

**generalizes this
sum to complex n**

Riemann Hypothesis



**All non-trivial zeros are
on the critical line**

Why should you care?

Millennium Prize Problems

P versus NP problem

Hodge conjecture

Poincaré conjecture

Riemann hypothesis

Yang–Mills existence and mass gap

Navier–Stokes existence and smoothness

Birch and Swinnerton-Dyer conjecture

V • T • E

\$ 1,000,000

boring reasons

Why does the Clay Foundation care?

"distribution of prime numbers"

Riemann Hypothesis

Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g., 2, 3, 5, 7, etc. Such numbers are called *prime numbers*, and they play an important role, both in pure mathematics and its applications. The distribution of such prime numbers among all natural numbers does not follow any regular pattern, however the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function

$$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \dots$$

called the *Riemann Zeta function*. The Riemann hypothesis asserts that all *interesting* solutions of the equation

$$\zeta(s) = 0$$

lie on a certain vertical straight line. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers.

In the opinion of many mathematicians, the Riemann hypothesis, and its extension to general classes of L-functions, is probably the most important open problem in pure mathematics today.

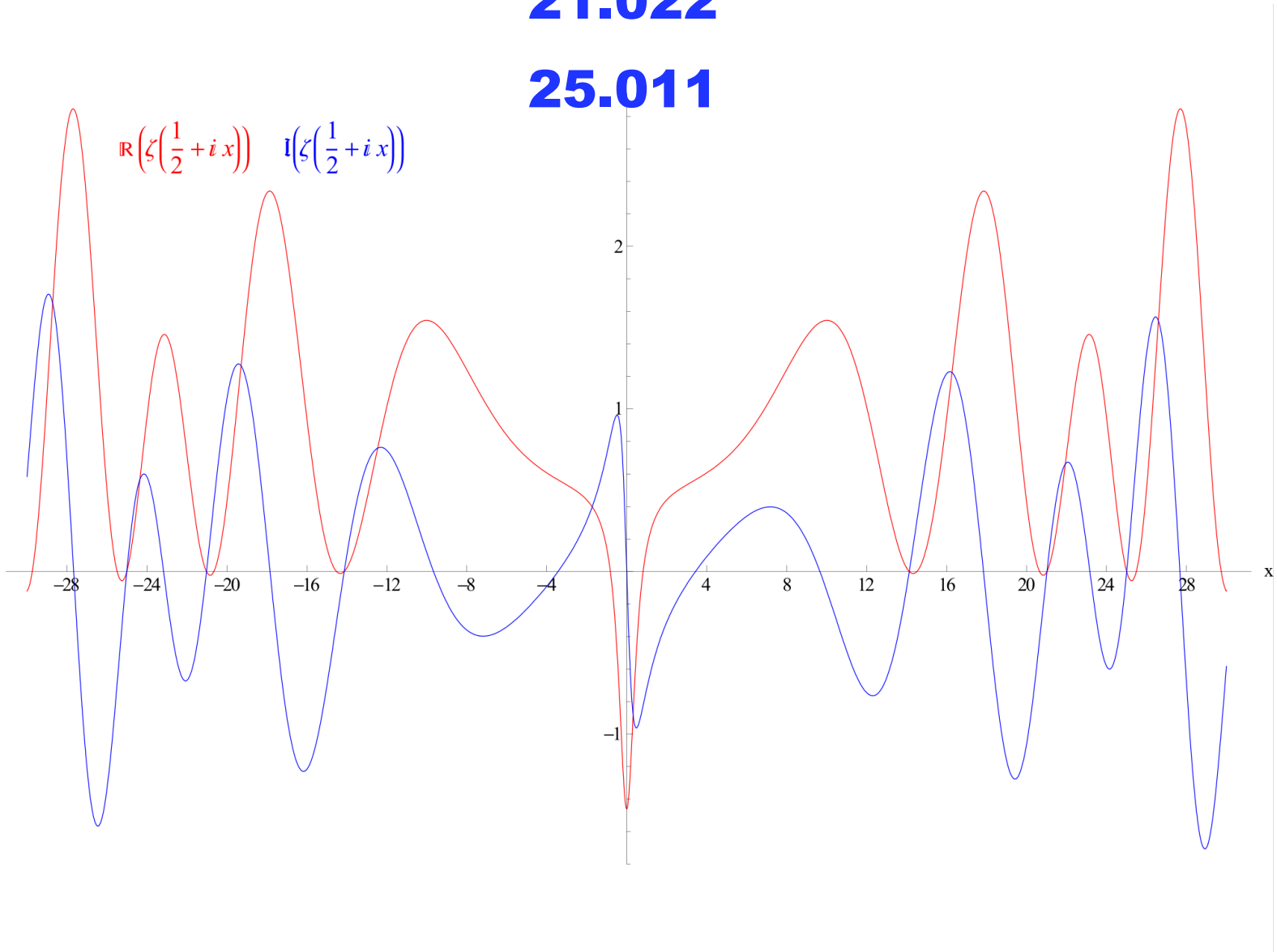
First three non-trivial zeros

14.135

21.022

25.011

$$\Re\left(\zeta\left(\frac{1}{2}+ix\right)\right) \quad \Im\left(\zeta\left(\frac{1}{2}+ix\right)\right)$$



<http://www.claymath.org/>

http://en.wikipedia.org/wiki/Riemann_hypothesis

<http://mathworld.wolfram.com/HilbertsProblems.html>

<http://mathworld.wolfram.com/SmalesProblems.html>