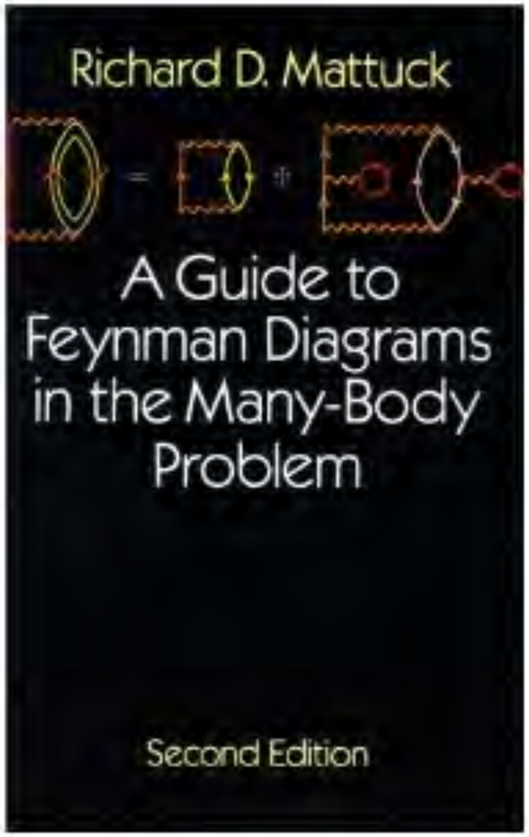

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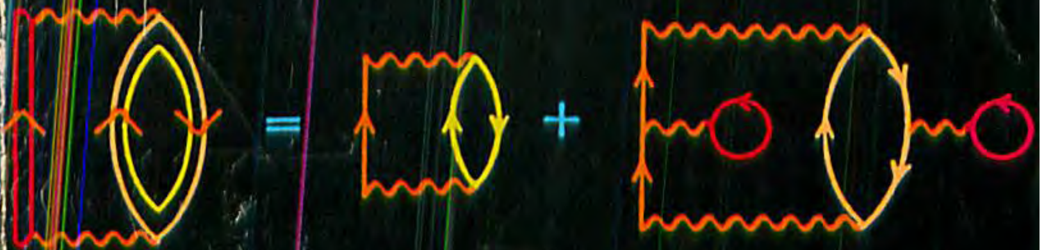

A Guide to Feynman Diagrams in the Many-Body Problem

Second Edition

Mattuck A Guide to Feynman Diagrams in the Many-Body Problem

Dover 0-486-67047-3

A Guide to Feynman Diagrams in the Many-Body Problem

Richard D. Mattuck

"(A) pedagogical jewel . . . Mattuck's fine sense of humor makes his book, which is a labor of love, a great delight to read."
—*Physics Today*

Among the most fertile areas of modern physics, many-body theory has produced a wealth of fundamental results in all areas of the discipline. Unfortunately the subject is notoriously difficult and, until the publication of this book, most treatments of the topic were inaccessible to the average experimenter or non-specialist theoretician.

The present work, by contrast, is well within the grasp of the nonexpert. It is intended primarily as a "self-study" book that introduces one aspect of many-body theory, i.e. the method of Feynman diagrams. The book also lends itself to use as a reference in courses on solid state and nuclear physics which make some use of the many-body techniques. And, finally, it can be used as a supplementary reference in a many-body course.

Chapters 1 through 6 provide an introduction to the major concepts of the field, among them Feynman diagrams, quasi-particles and vacuum amplitudes. Chapters 7 through 16 give basic coverage to topics ranging from Dyson's equation and the ladder approximation to Fermi systems at finite temperature and superconductivity. Appendixes summarize the Dirac formalism and include a rigorous derivation of the rules for diagrams. Problems are provided at the end of each chapter and solutions are given at the back of the book.

For this second edition, Dr. Mattuck, formerly of the H. C. Orsted Institute and the University of Copenhagen, added to many chapters a new section showing in mathematical detail how typical many-body calculations with Feynman diagrams are carried out. In addition, new exercises were included, some of which give the reader the opportunity to carry out simpler many-body calculations himself. A new chapter on the quantum field theory of phase transitions rounds out this unusually clear, helpful and informative guide to the physics of the many-body problem.

Unabridged Dover (1992) republication of the second edition published by McGraw-Hill International Book Company, New York, 1974. Prefaces. Appendices. Problems with solutions. References. Index. 247 black-and-white illustrations. xv + 429pp. 5⅜ × 8½. Paperbound.

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A guide to Feynman diagrams in the many-body problem

Richard D. Mattuck

SECOND EDITION

DOVER PUBLICATIONS, INC., *New York*

Preface to the second edition

I was delighted by the extreme reactions to the first edition of this book. One reviewer called it a 'pedagogical jewel ... useful as a crutch for poorly-prepared students', while another felt that it was primarily for people who were 'well-prepared and courageous'. The preface to the Russian edition referred to the pinball game analogy on p. 29 as some sort of world's record in popularization, but an English critic complained that the pinball picture was 'highly offensive' and 'had no place in a serious work of science'. A student told me that at his university, the book was known as 'Feynman Diagrams for Idiots', while other students felt that it was only for people exceptionally well-grounded in quantum mechanics. One critic stated that the 'possibilities for classroom use should be rather wide', but others claimed that the book was useless, since no detailed calculations were carried out in it.

In short, the first edition is too elementary and too advanced. Therefore, the purpose of the second edition is to make the book more advanced and more elementary. Toward this end, on the elementary side, a zeroth and first chapter have been added which are on the pre-kindergarten or nursery school level. This gives a view of the entire field based almost purely on pictures, cartoons, and virtual movies, with essentially no mathematics.

On the more advanced side, I have added to many chapters a new section showing in mathematical detail how typical many-body calculations with Feynman diagrams are carried out. For example, chapter 3 contains the detailed calculation of the energy and lifetime of an electron in an impure metal. In chapter 9, the single pair-bubble approximation is used to compute the quasi particle lifetime diagrammatically. The pair-bubble integrations are done in detail in chapter 10 and the results employed to obtain the form of the effective interaction in an electron gas, and the plasmon dispersion law in chapter 13. Chapter 14 contains the calculation of the finite temperature pair-bubble.

A number of new exercises have been added, some of which give the student the opportunity to carry out simpler many-body calculations himself. For example, Ex. 10.7 requires solving the K-matrix equation in ladder approximation, computing the integrals and showing that the hole lines give a negligible contribution in the low-density case.

A new chapter on the quantum field theory of phase transitions has been added. It includes, on the kindergarten level, an analysis of the 'staring crowd' transition (see p. 290) and on the more advanced level, the diagrammatic calculation of the magnetization and transition point for the ferromagnetic phase. There are also new chapters on the Kondo problem and on the renormalization group.

I have also written several new appendices. Appendix L reviews the analytic properties of propagators, which I make considerable use of at various points in the text. Appendix M shows the relation between the equation of motion and Feynman diagram methods for calculating the propagator. Appendix N gives the basic ideas of the 'reduced graph' method, used in connection with the Kondo problem.

In preparing the second edition, special thanks are due to Stud. Scient. Nikolai Nissen for pointing out better methods for carrying out many of the calculations, and for carefully reading and criticizing the new material.

I am also very grateful to my colleague Dr Ulf Larsen for the many fruitful and stimulating discussions of many-body theory we have had during the last five years, for his help in working out the chapter on the Kondo problem, and for weeding out many of the inaccuracies which had crept into the book.

I would like to thank Professor P. W. Atkins of Lincoln College, Oxford, for pointing out how the book could be modified to make it of more value to chemists.

I am much indebted to my cousin's son, David Lustbader, B.A., for his aid in improving chapter Zero, and to my own son, Allan, for help in pasting together the thousands of pieces of paper which were the raw material for the second edition. And I want to express my gratitude to my students, whose unending stream of questions forced me to replace fuzziness by clarity throughout the book.

And finally, a word of thanks to the many people who, by telling me how much they enjoyed the first edition (one wrote: 'Please allow me to express my gratitude for a ray of sunshine that you have cast into the windowless office of a second year graduate student in the form of your book on Feynman diagrams'), gave me the inspiration and fortitude to sweat my way through the production of the second edition.

Copenhagen, 1974

Preface to the first edition

This book is written for laymen, i.e., for experimental physicists and for those theoreticians who don't mind getting caught reading something easy.

Most laymen are aware that many-body theory is very much in vogue these days, and that it is producing a wealth of fundamental results in all fields of physics. Unfortunately, the subject is notoriously difficult, and the only previously available books on it are written on such a high level that they are completely inaccessible to the average experimenter or non-specialist theoretician.

The purpose of this book is to help bridge the pedagogical gap by providing an easy introduction to just one aspect of many-body theory, i.e., the method of Feynman diagrams. Since the word 'easy', along with its cousins 'elementary', 'introductory', or 'for five-year olds', has been applied to some pretty formidable physics literature in the past, I had better make clear how it is used here. It means first that, as far as I know, the present book is simpler than anything else which has been written in the modern many-body field. This establishes an upper bound on 'easy'. The lower bound is fixed by the system illustrated on p. 29. This is the classical example I have invented to introduce the main ideas of the subject. The whole first half of the book is derived essentially by analogy to this example.

Since this book does not fit into any of the usual categories, it may help to prevent misunderstanding if I state clearly what it is not. It is not a many-body 'textbook' in itself; it is simply an elementary introduction to the textbooks which already exist in the field. It does not prepare the student to plunge into the latest literature; it can only give him a glimpse of what this literature is about. It does not train students to do many-body calculations any more than a music-appreciation course trains students to compose music; it can, however, help them to grasp the elegance and significance of these calculations.

In short, it is not a text for the usual 'elementary course in many-body theory', because such a course would have as its purpose the bringing of beginners to the point where they would be able to do calculations and solve real problems in the field. It is rather intended primarily as a 'home study' book for non-specialists trying to get some idea of what Feynman diagrams in many-body physics are all about. In addition, it could serve as a reference in courses on solid state and nuclear physics which make some use of the many-body techniques. And, finally, it can be used (by those who like to start with something simple) as a supplementary reference in a many-body course.

Now a word about the organization of the book. Measured on a scale established by the other literature in the field, it is divided into three parts: kindergarten, elementary, and intermediate.

Chapters 1–6 constitute the kindergarten part. This provides an introduction to the major concepts of the field on a level somewhere between 'Donald Duck' and the 'American Journal of Physics'. The quantum diagram technique is developed by analogy to a transparent classical case. It is first applied in detail to trivial one-particle systems; this gives the reader a feeling for the method by showing him how it works on problems he can easily solve by elementary quantum mechanics. The many-body diagrams are presented using the same simple-minded approach. There is also a short introduction to second quantization, but this is optional, and no essential use of it is made in the first part of the book.

The kindergarten part may be read as a book in itself by people who just want to learn enough so they no longer tremble with awe when a many-body theoretician covers the blackboard with Feynman diagrams.

Chapters 7–16 constitute the elementary part. The topics here, ranging from second quantization to superconductivity, are standard for most of the other many-body books. But they are covered on a much lower, more restricted level. This means essentially that, first of all, the only physical properties of systems which I discuss are the energies of the ground and excited states, and that, secondly, there is no discussion of the analytic properties of propagators. I have instead concentrated exclusively on giving the reader a feeling for the diagrams themselves, their physical significance, and the various summation techniques for manipulating them. A novel feature of the pedagogical technique here is that all calculations are done completely diagrammatically up to the point where the diagram solution is translated into integrals; at this point, I simply state the numerical result and refer the reader to the appropriate book or paper for the details of the integrations.

The appendices A–J are the intermediate part of the book. They begin with a brief summary of Dirac formalism and include a more or less rigorous derivation of the rules for diagrams.

There are a few short exercises at the end of each chapter, and the answers to the exercises appear at the end of the book.

Note: Optional reading is enclosed in double brackets: ⟦ ⟧.

This book grew out of a series of lectures I gave to the Solid State Physics Study Group at the University of Copenhagen during 1962–5. Of the many people at the university who have aided me during this period, I wish especially to thank Professor H. Højgaard Jensen, both for giving me the

Contents

Chapter 0

The Many-Body Problem for Everybody

0.0 What the many-body problem is about

The many-body problem has attracted attention ever since the philosophers of old speculated over the question of how many angels could dance on the head of a pin. In the angel problem, as in all many-body problems, there are two essential ingredients. First of all, there have to be many bodies present—many angels, many electrons, many atoms, many molecules, many people, etc. Secondly, for there to be a problem, these bodies have to interact with each other. To see why this is so, suppose the bodies did not interact. Then each body would act independently of all the others, so that we could simply investigate the behaviour of each body separately. In other words, without interaction, instead of having one many-body problem, we would have many one-body problems. Thus, interactions are essential, and in fact the many-body problem may be defined as *the study of the effects of interaction between bodies on the behaviour of a many-body system*.

(It might be noted here, for the benefit of those interested in exact solutions, that there is an alternative formulation of the many-body problem, i.e., how many bodies are required before we have a problem? G. E. Brown points out that this can be answered by a look at history. In eighteenth-century Newtonian mechanics, the three-body problem was insoluble. With the birth of general relativity around 1910 and quantum electrodynamics in 1930, the two- and one-body problems became insoluble. And within modern quantum field theory, the problem of zero bodies (vacuum) is insoluble. So, if we are out after exact solutions, no bodies at all is already too many!)

The importance of the many-body problem derives from the fact that almost any real physical system one can think of is composed of a set of interacting particles. For example, nucleons in a nucleus interact by nuclear forces, electrons in an atom or metal interact by Coulomb forces, etc. Some examples are shown schematically in Fig. 0.1. Furthermore, it turns out that in the calculation of physical properties of such systems—for example, the energy levels of the atom, or magnetic susceptibility of the metal—interactions between particles play a very important role.

It should be clear from the variety of systems in Fig. 0.1 that the many-body problem is *not* a branch of solid state, or nuclear, or atomic physics, etc. It deals rather with *general* methods applicable to *all* many-body systems.

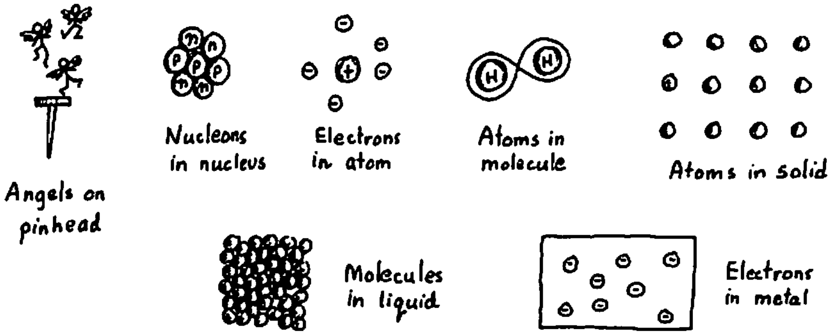


Fig. 0.1 *Some Many-body Systems*

The many-body problem is an extraordinarily difficult one because of the incredibly intricate motions of the particles in an interacting system. In Fig. 0.2 we contrast the simple behaviour of non-interacting particles with the complicated behaviour of interacting ones. Because of the complexity of the many-body problem, not much progress was made with it for a long time. In fact one of the preferred methods for solving the problem was simply to ignore it, i.e., pretend there were no interactions present. (Surprisingly enough, in some cases this 'method' produced good results anyway, and one of the great mysteries was how this could be possible!)

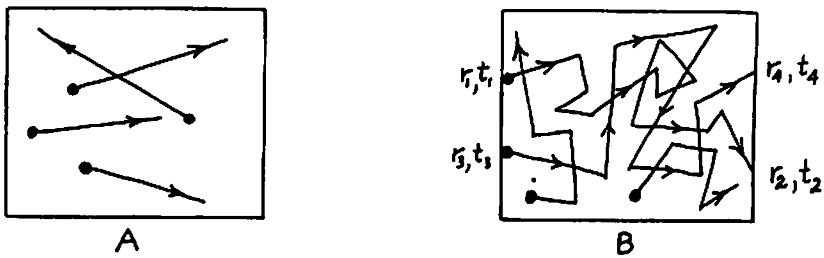


Fig. 0.2 A. *Non-interacting Particles*
B. *Interacting Particles*

Another of the early approaches to the problem, and one which is still used extensively today is the *canonical transformation* technique, described in appendix $\mathscr{A}$. This involves transforming the basic equations of the many-body system to a new set of coordinates in which the interaction term becomes small. Although considerable success has been achieved with this technique, it is not as systematic as one would like, and this sometimes makes it difficult to apply. It was this lack of a systematic method which kept the many-body field in its cradle well up into the 1950s.

The situation changed radically in 1956–7. In a series of pioneering papers, it was shown that the methods of *quantum field theory*, already famous for its success in elementary particle physics, provided a powerful, unified way of attacking the many-body problem. The new key opened many doors, and in rapid succession the idea was applied to nuclei, electrons in metals, ferromagnets, atoms, superconductors, plasmas, molecules—virtually everything in sight.

From that time on, much of the most exciting and fundamental research into the nature of matter has been based on the quantum field theory method. One of the things emerging from this research is a new simple picture of matter in which systems of interacting real particles are described in terms of approximately non-interacting fictitious bodies called 'quasi particles' and 'collective excitations'. Another thing is new results for calculated physical quantities which are in excellent agreement with experiment—for example, energy levels of light atoms, binding energy of nuclear matter, Fermi energy and effective electron mass in a variety of metals.

In this introductory chapter, we will give a physical picture of quasi particles and collective excitations. Then in the next chapter we show qualitatively how to describe quasi particles and calculate their properties by means of the quantum field theoretical technique known as the method of *Feynman diagrams*.

0.1. Simple example of non-interacting fictitious bodies

As mentioned at the beginning, one of nature's little surprises is that many-body systems often behave as if the bodies of which they are composed hardly interact at all! The reason for this is that the 'bodies' involved are not real but *fictitious*. That is, the system composed of *strongly* interacting *real* bodies acts *as if* it were composed of *weakly* interacting (or non-interacting) *fictitious* bodies. We consider now a very simple example of how this can occur.

Suppose we have two masses, m_1 and m_2 held together by a strong spring as shown in Fig. 0.3. That is, our system here consists of two strongly coupled real bodies. If this contraption is tossed up in a gravitational field, the motion of each body considered separately is very complicated because of the strong interaction (spring force) between the bodies.

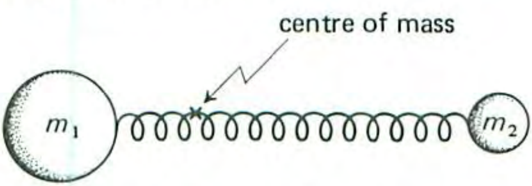


Fig. 0.3 *Two-body System*

However, we can break up the complicated motion into two independent simple motions: motion of the centre of mass and motion about the centre of mass. The centre of mass moves exactly as if it were an independent body of mass m_1+m_2, so it is one of the non-interacting fictitious bodies here. The other fictitious body is a body of mass $m_1 m_2/(m_1+m_2)$—the so-called 'reduced mass'—which moves independently relative to the centre of mass. Thus the system acts as if it were composed of two non-interacting fictitious bodies: the 'centre of mass body' and the 'reduced mass body'. (See appendix $\mathscr{A}$, eqs. ($\mathscr{A}$.11)–($\mathscr{A}$.14) for details.)

0.2 Quasi particles and quasi horses

The above two-body example is easy enough to understand, but finding the weakly interacting fictitious bodies in a set of *many* strongly interacting real bodies is a bit harder. We consider first the fictitious bodies called 'quasi particles'. These arise from the fact that when a real particle moves through the system, it pushes or pulls on its neighbours and thus becomes surrounded by a 'cloud' of agitated particles similar to the dust cloud kicked up by a galloping horse in a western. The real particle plus its cloud is the quasi particle (Fig. 0.4).

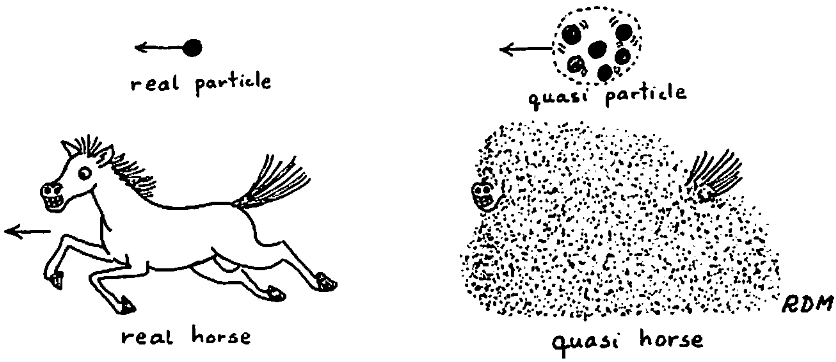


Fig. 0.4 *Quasi Particle Concept*

Just as the dust cloud hides the horse, the particle cloud 'shields' or 'screens' the real particles so that quasi particles interact only weakly with one another. The presence of the cloud also makes the properties of the quasi particle different from that of the real particle—it may have an '*effective mass*' different from the real mass, and a '*lifetime*'. These properties of quasi particles are directly observable experimentally.

It should be remarked that the quasi particle is in an excited energy level of the many-body system. Hence it is referred to as an '*elementary excitation*' of the system. (See appendix $\mathscr{A}$, §$\mathscr{A}$.2.) We now consider some examples of quasi particles.

1 Quasi ion in a classical liquid

Imagine that we have an electrolyte solution composed of an equal number of positive and negative ions moving about and colliding with each other as illustrated in Fig. 0.5. Let us focus our attention on a typical (+) ion in the

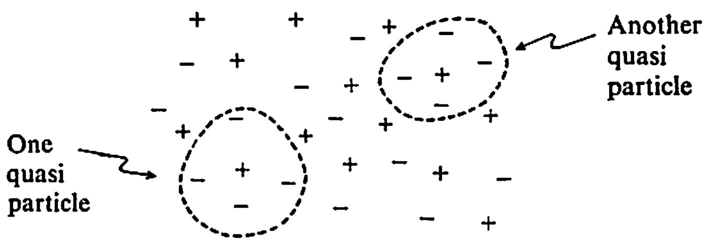


Fig. 0.5 *Quasi Particles in a Liquid of Positive and Negative Ions*

system. As this ion moves, on account of the strong Coulomb interaction, it will attract (−) ions to it. Some of these (−) ions will stick to the (+) for a while, then fall off due to collisions, then be replaced by other (−) ions, etc. Thus, on the average, because of the interaction, this typical (+) ion (and therefore every (+) ion) will be surrounded by a 'coat' or 'cloud' of (−) ions as shown in Fig. 0.5 inside the dotted lines. And of course each (−) ion will similarly have a coat of (+) ions. This coat of opposite charge will shield the ion's own charge so that its interaction with other similarly shielded ions will be much weaker than in the unshielded case. Thus the ions wearing their coats will act approximately independently of each other and constitute the quasi particles of this particular system. Many different types of systems of interacting particles may be described in this manner, and in general we have

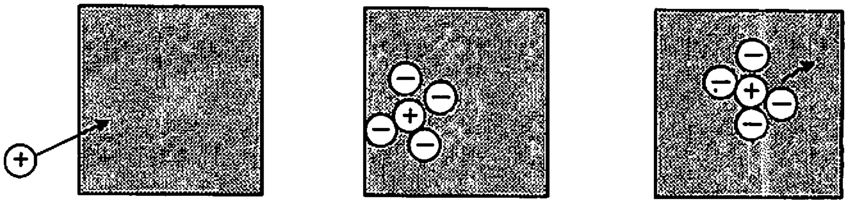
$$\text{real particle} + \begin{array}{c}\text{`coat' or `cloud'}\\ \text{of other particles}\end{array} = \text{quasi particle.} \tag{0.1}$$

Sometimes this same equation is stated in a more powerful terminology coming from quantum field theory:

$$\text{`bare' particle} + \begin{array}{c}\text{`clothing'}\\ \text{or `cloud'}\end{array} = \begin{array}{l}\text{`dressed' or `clothed'}\\ \text{or `physical' or}\\ \text{`renormalized' particle.}\end{array} \tag{0.2}$$

For example, in quantum electrodynamics a 'bare' electron interacting with a field of photons acquires a cloud of virtual photons around it, converting it into the 'dressed' electron. In a similar manner, the interaction between real particles is called the 'bare' interaction, while the weak interaction between quasi particles is referred to as the 'effective' or 'dressed' or 're-normalized' interaction.

It should be noted that each bare particle is simultaneously the 'core' of a quasi particle and a transient 'member' of the cloud of several other quasi particles. Therefore, if we try to visualize the whole system here as composed of quasi particles, we have to be careful, since each particle will have been counted more than once. For this reason, the quasi particle concept is valid only if one talks about a few quasi particles at a time, i.e., few in comparison with the total number of particles. In order to avoid this problem and concentrate attention on just a single quasi particle at a time, it is convenient to define quasi particles in terms of an experiment in which one adds an extra particle to the system, and observes the behaviour of this extra particle as it moves through the system. This is shown in Fig. 0.6 for a (+) ion.



(a) (b) (c)

Fig. 0.6 *Moving Quasi Ion.* (*a*) *Extra* (+) *Ion Shot into Liquid.* (*b*) (+) *Ion Acquires Cloud of* (−) *Ions, Turning it into Quasi Ion.* (*c*) *Quasi Ion Moves Through System*

With this intuitive picture in mind, it is possible to guess at some of the properties of quasi particles. First, because there is in general still a small interaction left between quasi particles, a quasi particle of momentum **p** will only keep this momentum for an average time τ_p. This can be understood from Figs. 0.6 and 0.5. If the quasi ion in Fig. 0.6 (*b*) has momentum **p**, it will propagate undisturbed an average time τ_p before undergoing a collision with another quasi ion in the system (that is, a quasi ion which *belongs* to the system, like those shown in Fig. 0.5, *not* one which we shoot into the system) which scatters it out of momentum state **p**. Hence

$$\text{quasi particles have a lifetime, } \tau_p. \tag{0.3}$$

The lifetime must be reasonably long for us to say that the quasi particle approximation is a good one. It can also be seen that because of the average coat of particles on its back, the quasi particle may have an 'effective' or 'renormalized' mass which is different from that of the bare particle. (The effective mass concept is not always applicable however.) This implies that free quasi particles (i.e., not in an externally applied field) have a new energy law

$$\epsilon' = \frac{p^2}{2m^*} \quad \text{instead of} \quad \epsilon = \frac{p^2}{2m} \tag{0.4}$$

where m^* is the effective mass. The difference

$$\epsilon_{\substack{\text{quasi} \\ \text{particle}}} - \epsilon_{\substack{\text{bare} \\ \text{particle}}} = \epsilon_{\text{self}} \tag{0.5}$$

is called the '*self-energy*' of the quasi particle. This comes from the interpretation that the bare particle interacts with the many-body system, creating the cloud, and the cloud in turn reacts back on the particle, disturbing its motion. Thus the particle is, in a sense, interacting with itself via the many-body system, and changing its own energy.

2 Quantum system: quasi electron in electron gas

The 'electron gas' is a simple model often used to describe many-body effects in metals. It consists of a box containing a large number of electrons interacting by means of the Coulomb force. In addition, there is a uniform, fixed, positive charge 'background' put into the box in order to keep the whole system electrically neutral. In the ground state, the electrons are spread out uniformly in the box, as shown schematically in Fig. 0.7.

Fig. 0.7 '*Electron Gas': Interacting Electrons Spread Out Uniformly in Box, plus Uniform, Fixed, Positive Charge Background*

Suppose now that we have a single, well-localized electron which we shoot into the electron gas (Fig. 0.8). Because of the repulsive Coulomb interaction between electrons, this extra electron repels other electrons away from it, so

Fig. 0.8 *Extra Electron Shot into Electron Gas*

The quasi electron moves or 'propagates' through the system as shown in Fig. 0.11.

We now notice that the positive hole cloud immediately around the extra electron partially shields the electron's own negative charge. Hence, if we have two quasi electrons as shown in Fig. 0.12, and these are far enough

Fig. 0.11 *Quasi Electron Propagates Through System*

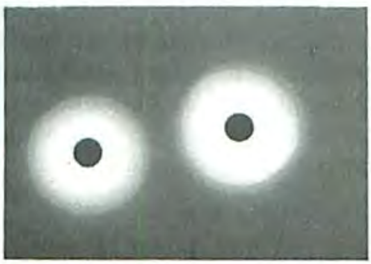


Fig. 0.12 *Two Quasi Electrons Interact only Weakly Because of Shielding*

apart so that their clouds do not overlap very much, then we see that because of the shielding the two quasi electrons will interact only weakly. That is, quasi electrons act nearly independently of one another. This is why metals generally behave as if their electrons were independent: it is not real electrons but rather quasi electrons we are looking at.

3 *Single electron in a metal*

Actually, the simplest quantum example of the quasi particle idea occurs not in a true many-body system, but rather in a system containing one particle moving in an external potential, i.e., a conduction electron in a metal. In a perfect metal the positive ions form a regular periodic lattice (we ignore lattice vibrations for the moment) so that the electron moves in a periodic force field due to the attractive Coulomb interaction between the ions and the electron (see Fig. 0.13a). In an imperfect metal, the periodicity is spoiled by the presence of a more or less random distribution of some impurity ions in the lattice, or the presence of some displaced ions (Fig. 0.13b).

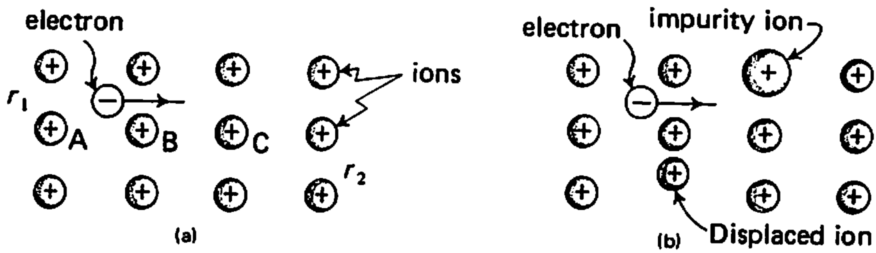


Fig. 0.13 (*a*) *Conduction Electron in Perfect Metal.* (*b*) *Imperfect Metal*

Since the lattice here is assumed fixed, there is no 'moving cloud' of lattice ions following the electron. Nevertheless, it turns out that even these stationary lattice ions are capable of 'clothing' the electron, and we find that for a perfect lattice, there is an effective mass, m^*, and an infinite lifetime. Addition of imperfections causes the lifetime to become finite.

4 *Quasi nucleon*

Despite powerful short-range forces between nucleons in a nucleus, they behave in many respects as if they were independent of each other, as is indicated by the success of the nuclear shell model. The nearly independent particles here are not the nucleons themselves, but the nucleons each surrounded by a cloud of other nucleons, i.e., the quasi nucleons.

5 *Bogoliubov quasi particles* ('*bogolons*')

These are the elementary excitations in a superconductor. We include them here since they are called quasi particles, but actually their structure is quite different from the 'particle plus cloud' picture described above. They consist of a linear combination of an electron in state $(+k, \uparrow)$ and a 'hole' in $(-k, \downarrow)$.

0.3 Collective excitations

As we have seen, the quasi particle consists of the original real, individual particle, plus a cloud of disturbed neighbours. It behaves very much like an individual particle, except that it has an effective mass and a lifetime. But there also exist other kinds of fictitious particles in many-body systems, i.e., 'collective excitations'. These do not centre around individual particles, but instead involve collective, wavelike motion of *all* the particles in the system simultaneously. Here are some examples:

1 *Plasmons*

If a thin metal foil is bombarded with high energy electrons, it is possible to set up sinusoidal oscillations in the density of the electron gas in the foil. This is known as a 'plasma wave', and it has a frequency ω_p and a wavelength λ_p (see Fig. 0.14a). The plasma wave may be visualized as built up of 'holes' in the low-density regions and extra electrons in the high-density regions as shown in Fig. 0.14(*b*). Just as light waves are quantized into units having energy $E=\hbar\omega$ called photons, plasma waves are quantized into units with energy $E_p=\hbar\omega_p$ called *plasmons*.

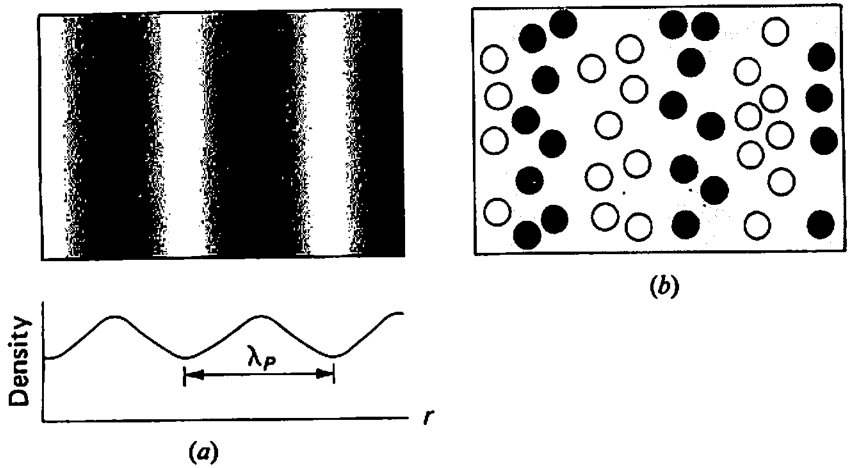


Fig. 0.14 (*a*) *Plasma Wave in Electron Gas.* (*b*) *Particle-hole Picture of Plasma Wave*

2 *Phonons*

Sound waves are sinusoidal oscillations in the crystal lattice of a solid. They are quantized into collective excitations called 'phonons'. (See appendix 𝒜.)

3 *Magnons*

In ferromagnets there are regular fluctuations in the density of spin angular momentum known as 'spin waves'. The collective excitation here is the spin wave quantum known as the 'magnon'.

4 *Nuclear quanta*

In nuclei, one finds various vibrational and rotational motions; the associated quanta are the collective excitations in this case.

In the next chapter, we will describe in a very qualitative way how to find the properties of quasi particles and collective excitations by means of 'propagators' and 'Feynman diagrams'.

Further reading

Appendix 𝒜

Patterson (1964).

Pines (1963), chap. 1.

Chapter 1

Feynman Diagrams, or how to Solve the Many-Body Problem by means of Pictures

1.1 Propagators—the heroes of the many-body problem

We have seen that many-body systems consisting of strongly interacting real particles can often be described as if they were composed of weakly interacting fictitious particles: quasi particles and collective excitations. The question now is, how can we calculate the properties of these fictitious particles—for example, the effective mass and lifetime of quasi particles? There are various ways of doing this (see appendix $\mathscr{A}$) but the hero roles in the treatment of the many-body problem are played by quantum field theoretical quantities known as *Green's functions* or *propagators*. These are essentially a generalization of the ordinary, familiar undergraduate Green's function. They come in all sizes and shapes—one particle, two particle, no particle, advanced, retarded, causal, zero temperature, finite temperature—an assortment to suit every situation and taste.

There are three reasons for the immense popularity propagators are enjoying these days. First of all, they yield in a direct way the most important physical properties of the system. Secondly, they have a simple physical interpretation. Thirdly, they can be calculated in a way which is highly systematic and 'automatic' and which appeals to one's physical intuition.

The idea behind the propagator method is this: the detailed description of a many-body system requires in the classical case the position of each particle as a function of time, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, ..., $\mathbf{r}_N(t)$, or in the quantum case, the time-dependent wave function of the whole system, $\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$. A glance at Fig. 0.2B shows that this is an extremely complicated business. Fortunately, it turns out that in order to find the important physical properties of a system it is not necessary to know the detailed behaviour of each particle in the system, but rather just the *average* behaviour of one or two typical particles. The quantities which describe this average behaviour are the *one-particle propagator* and *two-particle propagator* respectively, and physical properties may be calculated directly from them.

Consider the one-particle propagator first. It is defined as follows: We put a particle into the interacting system at point r_1 at time t_1 and let it move through the system colliding with the other particles for a while (i.e., let it 'propagate' through the system). Then the one-particle propagator is the probability (or in quantum systems, the probability *amplitude*—see §3.1) that the particle will be observed at the point r_2 at time t_2. (Note that instead of putting the particle in at a definite point, it is sometimes more convenient to put it in with definite momentum, say p_1, and observe it later with momentum p_2.) The single-particle propagator yields directly the energies and lifetimes of quasi particles. It also gives the momentum distribution, spin and particle density and can be used to calculate the ground state energy.

Similarly, the two-particle propagator is the probability amplitude for observing one particle at r_2, t_2 and another at r_4, t_4 if one was put into the system at r_1, t_1 and another at r_3, t_3 (see Fig. 0.2B). This also has a wide variety of talents, giving directly the energies and lifetimes of collective excitations, as well as the magnetic susceptibility, electrical conductivity, and a host of other non-equilibrium properties.

There is also another useful quantity, the 'no-particle propagator' or so-called 'vacuum amplitude' defined thus: We put no particle into the system at time t_1, let the particles in the system interact with each other from t_1 to t_2, then ask for the probability amplitude that no particles emerge from the system at time t_2. This may be used to calculate the ground state energy and the grand partition function, from which all equilibrium properties of the system may be determined.

1.2 Calculating propagators by Feynman diagrams: the drunken man propagator

There are two different methods available for calculating propagators. One is to solve the chain of differential equations they satisfy—this method is discussed briefly in appendix M. The other is to expand the propagator in an infinite series and evaluate the series approximately. This can be carried out in a general, systematic, and picturesque way with the aid of *Feynman diagrams*.

Just to get an idea of what these diagrams are, consider the following simple example (see Fig. 1.1). A man who has had too much to drink, leaves a party at point 1 and on the way to his home at point 2, he can stop off at one or more bars—Alice's Bar (A), Bardot Bar (B), Club Six Bar (C), ..., etc. He can wind up either at his own home 2, or at any one of his friends' apartments, 3, 4, etc. We ask for the probability, $P(2, 1)$, that he gets home. This probability, which is just the propagator here (with time omitted for simplicity), is the sum of the probabilities for all the different ways he can propagate from 1 to 2 interacting with the various bars.

The first way he can propagate is 'freely' from 1 to 2, i.e., without stopping at a bar. Call the probability for this free propagation $P_0(2, 1)$.

The second way he can propagate is to go freely from 1 to bar A (the probability for this is $P_0(A, 1)$), then stop off at bar A for a drink (call the probability for this $P(A)$), then go freely from A to 2 (probability$=P_0(2,A)$). Assume for simplicity that the three processes here are independent. Then the total probability for this second way is the product of the probabilities for each process taken separately, i.e., $P_0(A,1)\times P(A)\times P_0(2,A)$. (This is like the case in coin-tossing: since each toss is independent, the probability of first tossing a head, then a tail, equals the probability of tossing a head times the probability of tossing a tail.)

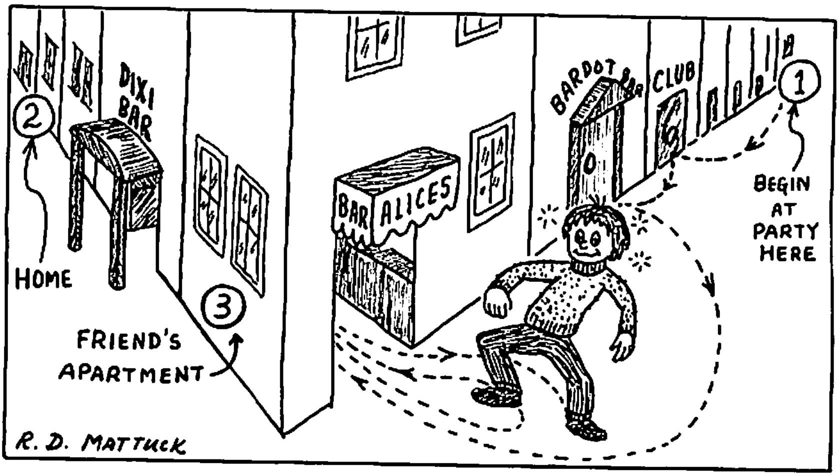


Fig. 1.1 *Propagation of Drunken Man*

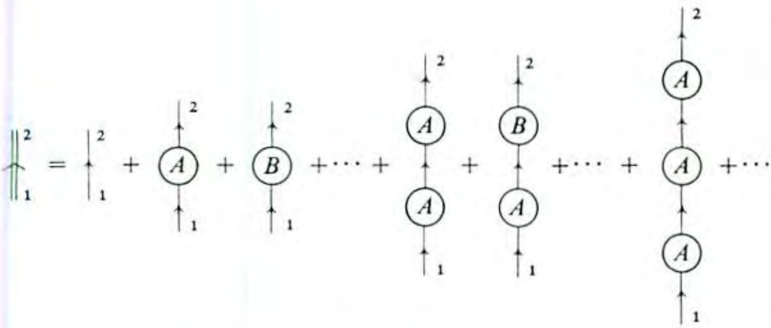
(Reproduced with the kind permission of *The Encyclopedia of Physics*)

The third way he can propagate is from 1 to B to 2, with probability $P_0(B,1)P(B)P_0(2,B)$. Or he could go from 1 to C to 2, etc., or from 1 to A to B to 2, or from 1 to A, come out of A, go back into A, then go to 2, and so on. The total probability, $P(2,1)$ is then given by the sum of the probabilities for each way, i.e., the infinite series:

$$P(2,1) = P_0(2,1)+P_0(A,1)P(A)P_0(2,A)+P_0(B,1)P(B)P_0(2,B)+\cdots$$
$$+P_0(A,1)P(A)P_0(B,A)P(B)P_0(2,B)+\cdots. \quad (1.1)$$

This is an example of a 'perturbation series', since each interaction with a bar 'perturbs' the free propagation of the drunken man.

Now, such a series is a complicated thing to look at. To make it easier to read, we follow the journal 'Classic Comics' where difficult literary classics are translated into picture form. Let us make a 'picture dictionary' to associate diagrams with the various probabilities as in Table 1.1. Using this dictionary, series (1.1) can be drawn thus:



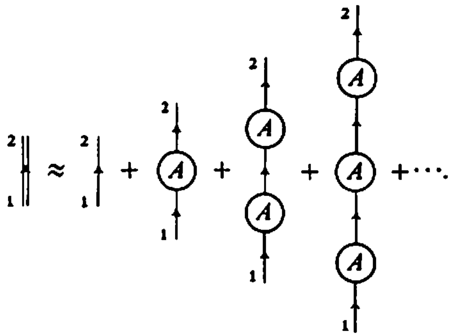
(1.2)

Since, by dictionary Table 1.1, each diagram element stands for a factor, series (1.2) is completely equivalent to (1.1). However it has the great advantage that it also reveals the physical meaning of the series, giving us a 'map' which helps us to keep track of all the sequences of interactions with bars which the drunken man can have in going from 1 to 2.

Table 1.1 *Diagram dictionary for drunken man propagator*

Word	Picture	Meaning
$P(2,1)$		probability of propagation from 1 to 2
$P_0(s,r)$		probability of free propagation from r to s
$P(X)$		probability of stopping off at bar X for a drink

The series may be evaluated approximately by selecting the most important types of terms in it and summing them to infinity. This is called *partial summation*. For example, suppose the man is in love with Alice, so that $P(A)$ is large, and all the other $P(X)$'s are small. Then Alice's bar diagrams will dominate, and the series (1.2) may be approximated by a sum over just repeated interactions with Alice's Bar:



(1.3)

Using the above dictionary, this can be translated into functions:

$$P(2,1) \approx P_0(2,1)+P_0(A,1)P(A)P_0(2,A)+ \\ +P_0(A,1)P(A)P_0(A,A)P(A)P_0(2,A)+\cdots. \quad (1.4)$$

Assume for simplicity that all $P_0(s,r)$ are equal to the same number, c, i.e., $P_0(2,1)=P_0(2,A)=P_0(A,1)=P_0(A,A)=c$. Then series (1.4) becomes

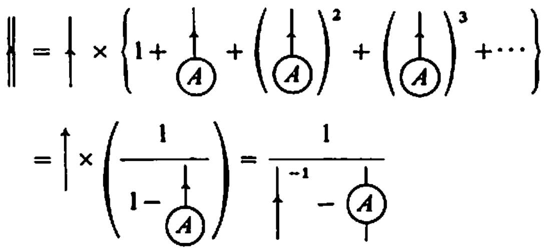
$$P(2,1) = c+c^2P(A)+c^3P^2(A)+\cdots \\ = c\{1+cP(A)+[cP(A)]^2+[cP(A)]^3+\cdots\}. \quad (1.5)$$

The series in brackets is geometric and can be summed exactly to yield $1/(1-cP(A))$, so that

$$P(2,1) = c\times\left(\frac{1}{1-cP(A)}\right) = \frac{1}{c^{-1}-P(A)} \quad (1.6)$$

which is the solution for the propagator in this case.

Note that since each diagram element stands for a factor, we could have done calculation (1.5), (1.6) completely diagrammatically:

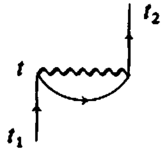


(1.7)

The partial summation method is extremely useful in dealing with the strong interactions between particles in the many-body problem, and it is the basic method which will be used throughout this book.

as in (1.11):

open oyster

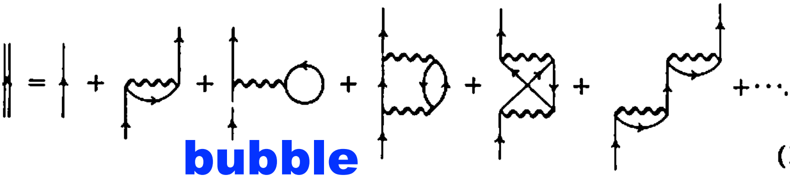


(1.11)

'Open oyster' diagram
(closed oyster is in (1.19))

The diagrams in (1.8)–(1.11) are called *Feynman diagrams* after their inventor, Richard P. Feynman who employed them in his Nobel prize-winning work on quantum electrodynamics. They are used extensively in elementary particle physics.

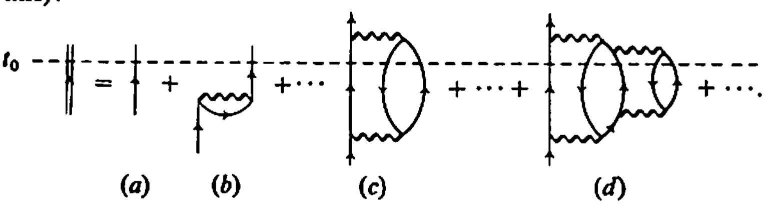
The total single particle propagator is the sum of the amplitudes for all possible ways the particle can propagate through the system. This will include the above processes, repetitions of them, plus an infinite number of others. Thus we find



(1.12)

(Note: the interpretation of the 'bubble' diagram, just after the open oyster, will be discussed in chapter 4.)

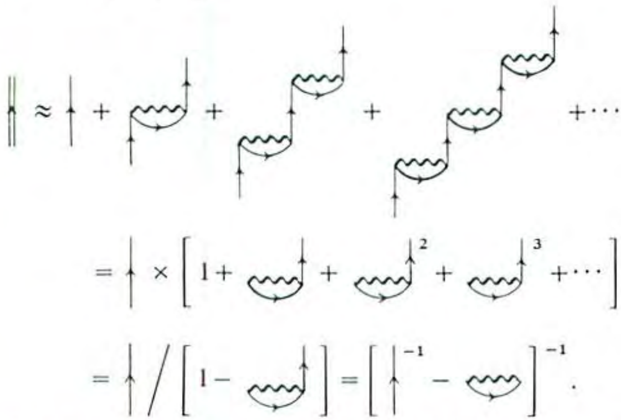
We can see the direct connection between the one-particle propagator and the quasi particle by looking at all the diagrams at a particular time t_0 (dashed line):



(1.13)

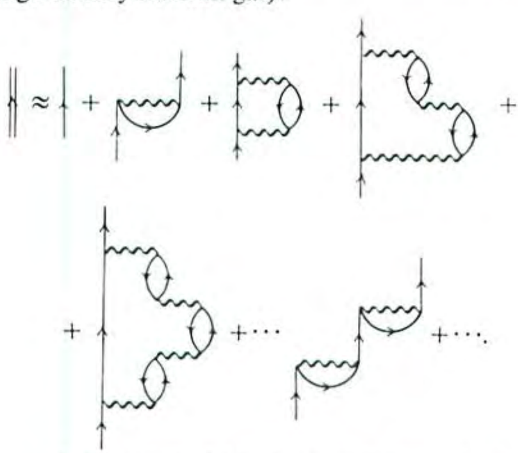
At t_0, we see that various situations may exist: there may be just the bare particle (*a*), or there may exist two particles plus one hole created by the second-order sequence (*c*), or three particles plus two holes in (*d*), etc. That is, the diagrams show all the configurations of particles and holes which may be kicked up by the bare particle as it churns through the many-body system. If we now compare with the picture of the quasi particle in Fig. 0.10, we see that *the diagrams reveal the content of the ever-changing cloud of particles and holes surrounding the bare particle and converting it into a quasi particle.*

Just as in the drunken man case, the propagator here may be calculated approximately by doing a partial sum. For example, we can sum over all diagrams containing repeated open oyster parts since they constitute a geometric series (cf. (1.7)):



(1.14)

For the electron gas, this is the 'Hartree-Fock' approximation. We can also include 'ring' diagrams in the sum, i.e., diagrams in which the self-energy parts are composed of rings of particle–hole pair bubbles (these are the most important in a high-density electron gas):



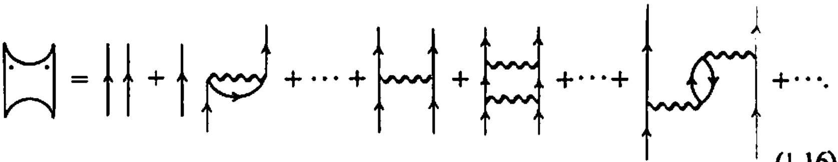
(1.15)

This sum can be carried out and yields the so-called 'random phase approximation' or 'RPA', which is extremely useful in analysing the properties of metals.

Note that the essential thing involved in the above partial sums is the structure or *topology* of the diagrams, i.e., how the various lines are connected to each other. Thus we could sum (1.14) because each diagram consisted of single lines connecting the same repeated part. This diagram topology is the key to the quantum field theoretical method in the many-body problem.

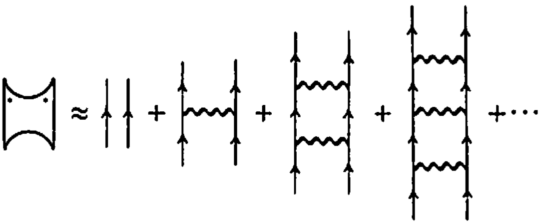
1.5 The two-particle propagator and the particle-hole propagator

The two-particle propagator is the sum over the probability amplitudes for all the ways two particles can enter the system, interact with each other and with the particles in the system, then emerge again. The diagram series for it is (note that the dots on the diagram for the two-particle propagator show the points at which directed lines emerge):



(1.16)

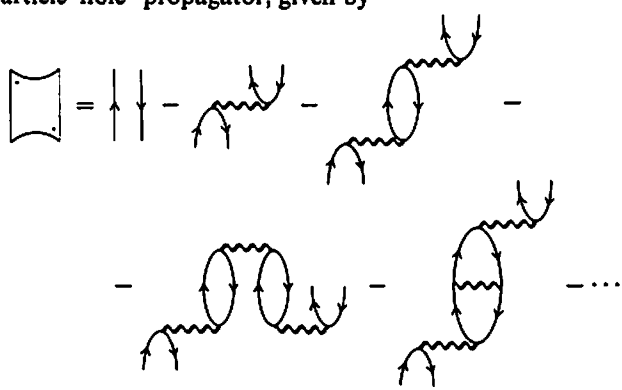
A partial sum over all 'ladder' diagrams here:



(1.17)

is called 'ladder' approximation, and is very useful in describing nuclear matter, and low-density systems.

The 'particle–hole' propagator, given by



(1.18)

may be used to find the energy and lifetime of collective excitations, e.g. plasmons.

1.6 The no-particle propagator ('vacuum amplitude')

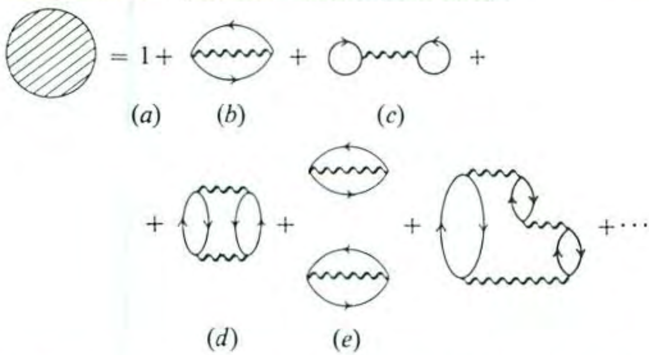
The ground state energy of a many-body system may be obtained directly from the no-particle propagator, or 'vacuum amplitude'. This is the sum of amplitudes for all the ways the system can begin at time t_1 with no extra or lifted-out particles, or holes in it (this is the undisturbed or 'Fermi vacuum' state), have its particles interact with each other, and wind up at t_2 with no extra or lifted-out particles, or holes. The simplest process is where nothing at all happens—the system just sits there. A first-order process occurs in which two particles change places with each other as shown in the following diagram

oyster

(1.19)

'Oyster' diagram

A more complicated process is shown in Fig. 1.4. The vacuum amplitude may thus be represented by the following diagram series:



(1.20)

where '1' is for the nothing-at-all process and (*d*) is the picture for Fig. 1.4. (The 'double bubble' diagram, (*c*), is discussed in chapter 5.)

The vacuum amplitude series gives us a vivid picture of the ground state of the many-body system as a sort of 'virtual witches' brew', constantly seething, with particles and holes boiling up, bubbling, and colliding, as in Fig. 1.5.

In conclusion, we see that Feynman diagrams have many appealing features, besides their utility as a calculational tool. One thing which was already pointed out in §1.2 is the fact that they show directly the physical meaning of the perturbation term they represent. Another thing is that they reveal at a glance the structure of very complicated approximations by showing which sets of diagrams have been summed over. In this way, they have introduced a new language into physics, and one often sees phrases like 'ladder approximation' or 'ring approximation' even in articles in which no diagrams appear. And finally, one cannot be immune to the Klee-like charm of the diagrams. Includ-

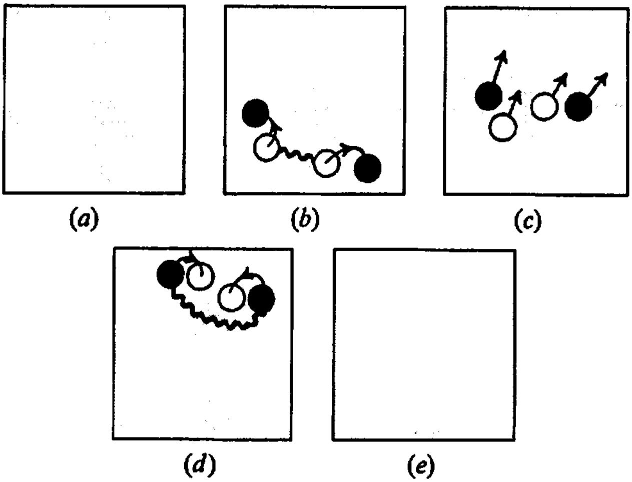


Fig. 1.4 *Virtual Movie of Second-order Vacuum Amplitude Process*
(*a*) Vacuum.
(*b*) At time t_1 interaction between two particles in system causes two particles to be lifted out, forming two holes.
(*c*) The two particle–hole pairs propagate freely through the system.
(*d*) Both pairs annihilated at time t'.
(*e*) Vacuum.

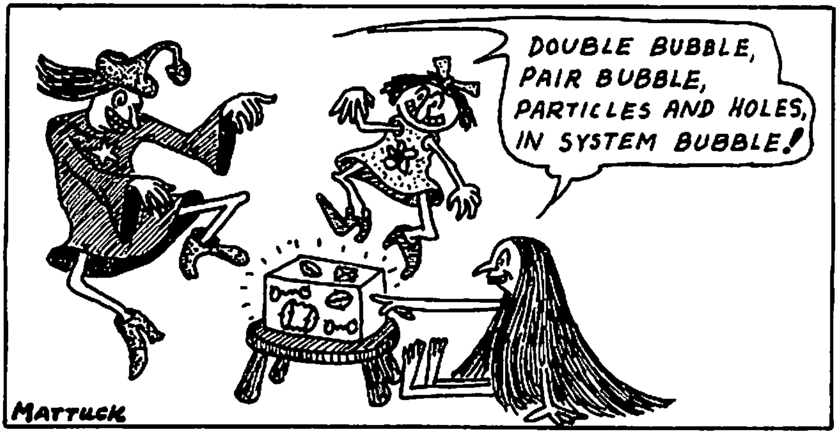


Fig. 1.5 *Modern View of a Many-body System in its Ground State*

ing in their ranks, in addition to the above, such characters as the 'necklace', the 'potato' and the 'tadpole', plus infinite numbers yet unnamed, they constitute what might indeed be called 'perturbation theory in comic-book form.'

Chapter 2: Classical Quasi Particles and the Pinball Propagator

then τ may be identified as the quasi particle lifetime; it clearly must be fairly large if the quasi particle picture is to be useful. Thus, if we calculate P and find that it shows the above behaviour, then the system is describable in terms of quasi particles and their lifetime and effective mass may be determined.

2.3 Calculation of the propagator by means of diagrams

The actual calculation of the propagator P is quite complicated, but it is easy to illustrate all the principles involved with the aid of a simple analogue example in which the many-body system is replaced by a set of fixed scattering centres. (The system considered here is essentially the same as the drunken man case in chapter 1, but it will be treated in much more detail.)

The example involves the particle accelerator in Fig. 2.3. A pinball is injected at the point $\mathbf{r}_1$, at time t_1 and propagates through the system, being scattered at the various centres. We ask for the probability $P(\mathbf{r}_2, t_2, \mathbf{r}_1, t_1)$ that the particle reaches the point $\mathbf{r}_2$ at time t_2.

The scattering mechanism is assumed to be such that (1) if the pinball strikes the shaded circle at animal A, then there is probability $P(A)$ that it is scattered and $1-P(A)$ that it will go straight through without scattering, (2) the probability distribution of pinball paths and velocities after scattering at A must be independent of the pinball path and velocity before scattering—that is, the pinball loses its 'memory' of how it got to A.

(There are many ways in which the above properties can be approximately realized. For example, the shaded circle could be a round peg which is pushed up so that it protrudes above the playing board surface a fraction $P(A)$ of the time, and is pulled in so that it is flush with the surface (hence cannot scatter the pinball) the rest of the time. Or we could have an immovable peg (i.e., always protruding) within the shaded circle, having a diameter such that the ratio of the peg diameter to that of the circle$=P(A)$. The loss of memory could be achieved by attaching a 'shuffling' device to each peg—like for example rapidly rotating spokes. The choice of method and the 'Rube Goldberg' details are, however, left as an exercise to the reader. They are of no importance for our discussion!)

For the sake of simplicity, let us leave time out of the argument to begin with, and consider just $P(\mathbf{r}_2, \mathbf{r}_1)$; this is the probability that if the particle begins at $\mathbf{r}_2$ it will finish at $\mathbf{r}_2$ regardless of the time. From the definition of probability, $P(\mathbf{r}_2, \mathbf{r}_1)$ is the sum of the probabilities for all the different ways the particle can go through the machine which begin at $\mathbf{r}_1$ and wind up at $\mathbf{r}_2$. For example, it could go 'directly' from $\mathbf{r}_1$ to $\mathbf{r}_2$ (i.e., without being scattered on the way) or it could go from $\mathbf{r}_1$ to the giraffe, be scattered off the giraffe and fall to $\mathbf{r}_2$. Or it could scatter from the giraffe to the monkey to $\mathbf{r}_2$. Or it could scatter twice on the giraffe before falling to $\mathbf{r}_2$. And so on.

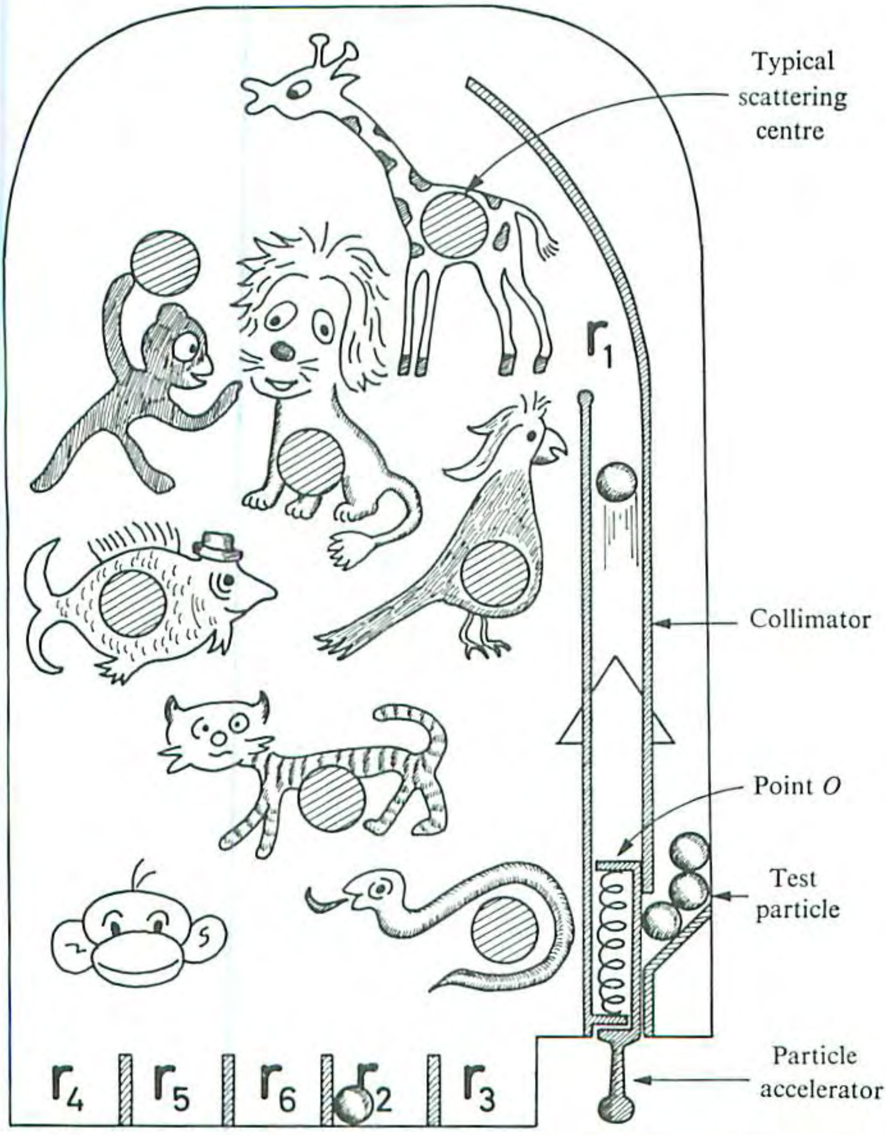


Fig. 2.3 *Classical Analogue Machine to Illustrate the Single-particle Propagator*

We first calculate the probability that the pinball will follow any particular path through the system. Let $P_0(\mathbf{r}_j, \mathbf{r}_i)$ = probability that if the pinball leaves $\mathbf{r}_i$ then it travels to $\mathbf{r}_j$ without being scattered by an animal en route ('free propagator'). The simplest path the pinball can follow is from $\mathbf{r}_1$ to $\mathbf{r}_2$ without scattering; this has probability $P_0(\mathbf{r}_2, \mathbf{r}_1)$. Another path is from $\mathbf{r}_1$ to the giraffe at $\mathbf{r}_G$ (probability $= P_0(\mathbf{r}_G, \mathbf{r}_1)$), scattering at the giraffe (probability $= P(G)$), then from $\mathbf{r}_G$ to $\mathbf{r}_2$ (probability $= P_0(\mathbf{r}_2, \mathbf{r}_G)$). Because the pinball loses its memory after the scattering at $\mathbf{r}_G$, these probabilities are independent of each other, and the joint probability for the whole path is just the product of the probabilities for each part of the path:

$$P\{(\mathbf{r}_1 \to \mathbf{r}_G), (\text{scattered at } \mathbf{r}_G), (\mathbf{r}_G \to \mathbf{r}_2)\} = P_0(\mathbf{r}_G, \mathbf{r}_1) P(G) P_0(\mathbf{r}_2, \mathbf{r}_G). \quad (2.12)$$

(Note that a process in which the pinball goes from $\mathbf{r}_1$ to $\mathbf{r}_G$, is not scattered at $\mathbf{r}_G$, and continues to $\mathbf{r}_2$, is not included in (2.12), but in the free propagator, $P_0(\mathbf{r}_2, \mathbf{r}_1)$.) The probabilities for the other paths are calculated in a similar fashion.

The total probability, $P(\mathbf{r}_2, \mathbf{r}_1)$, is just the sum of the probabilities for the various paths. Thus we find

$$\begin{aligned} P(\mathbf{r}_2, \mathbf{r}_1) = {} & P_0(\mathbf{r}_2, \mathbf{r}_1) + P_0(\mathbf{r}_G, \mathbf{r}_1) P(G) P_0(\mathbf{r}_2, \mathbf{r}_G) + P_0(\mathbf{r}_M, \mathbf{r}_1) P(M) P_0(\mathbf{r}_2, \mathbf{r}_M) + \\ & + P_0(\mathbf{r}_G, \mathbf{r}_1) P(G) P_0(\mathbf{r}_G, \mathbf{r}_G) P(G) P_0(\mathbf{r}_2, \mathbf{r}_G) + \cdots \end{aligned} \quad (2.13)$$

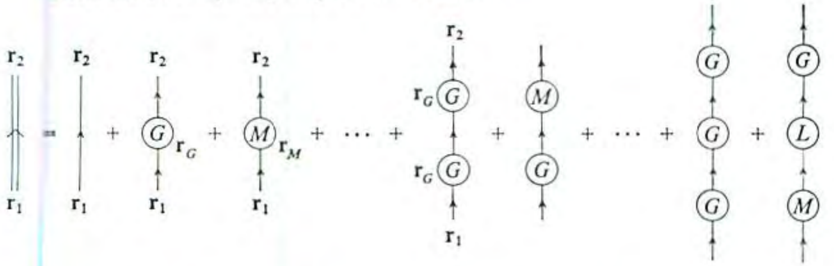
where G = giraffe, M = monkey, etc. What we have here is evidently just a perturbation expansion of the propagator, in which the $P(A)$'s play the same sort of role that the matrix elements of the perturbation, V_{kl}, play in quantum mechanical perturbation expansions.

In order to make series (2.13) easier to interpret, we draw a 'picture dictionary' to associate diagrams with the various probabilities as shown in Table 2.1.

Table 2.1 *Diagram dictionary for the pinball propagator*

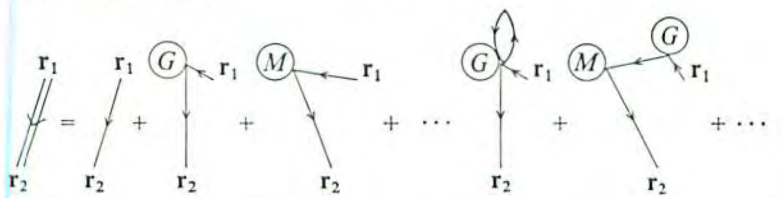
Word	Picture
$P(\mathbf{r}_j, \mathbf{r}_i)$	
$P_0(\mathbf{r}_j, \mathbf{r}_i)$	
$P(A)$	

Then the series (2.13) may be drawn thus:



(2.14)

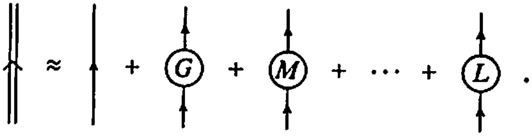
Equations (2.14) and (2.13) are of course completely equivalent to each other, being in one-to-one correspondence by the dictionary Table 2.1. But the picture has the advantage of revealing the physical meaning of (2.13), showing directly the particle shooting out from $\mathbf{r}_1$, undergoing various sequences of collisions and coming finally to $\mathbf{r}_2$. It presents in a vivid and systematic way the total probability as the sum of the probabilities associated with all the possible paths or 'histories' the particle can have as it goes through the system. Note that it is possible to interpret the $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_G$, ... on the diagrams as being points in real space if we just re-draw the diagrams so the points lie as in Fig. 2.3 thus:



(2.15)

It is important to observe that in terms of diagrams, 'the sum of the probabilities for all the different ways the particle can go from $\mathbf{r}_1$ to $\mathbf{r}_2$, interacting with the various scatterers' may be translated into 'the sum of all possible different diagrams which can be built up out of labelled circles connected by directed lines, beginning at $\mathbf{r}_1$ and terminating at $\mathbf{r}_2$'. This is because there is just one diagram corresponding to each physical path through the system.

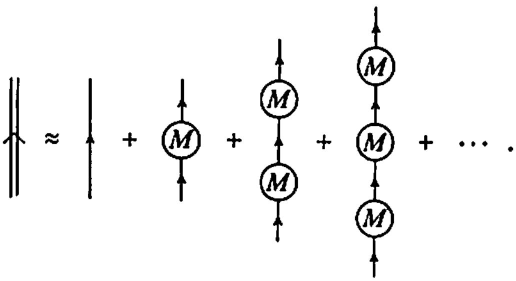
How can this series be evaluated? If we assume that the P_0's are large, say $\sim\frac{1}{2}$ or so, and the various interaction $P(A)$'s are small, say $\sim\frac{1}{10}$, then the higher order diagrams (i.e., terms; note that by order here we mean the total number of interactions) will give successively smaller contributions, and just as in ordinary perturbation theory, we can get an approximate solution by simply summing the series up through the first- or second-order terms. Thus, the zeroth-order approximation would be just the unperturbed case where the particle propagates freely from $\mathbf{r}_1$ to $\mathbf{r}_2$. When we add the possibility of a perturbing (scattering) interaction with the various animals just once each, we get the first-order approximation



(2.16)

Allowing two interactions gives the second-order approximation and so on. If, on the other hand, one or more of the interaction terms $P(A)$ is large (i.e., strong scattering at A) this method is not practical, since the series converges too slowly, and the summation must be carried out to extremely high orders to give a good result.

However, there is another kind of approximation we can make in this strong interaction case, an approximation that does not stop at second order, but instead sums over diagrams to infinite order. Suppose, for example, that only P (monkey) is large and all the other $P(A)$'s are small. Then the monkey diagrams will dominate, and the series may be approximated by the sum over just repeated monkeys, thus:



(2.17)

Translating each element of the diagrams into the appropriate probability, it is easy to write down the corresponding series:

$$P(\mathbf{r}_2,\mathbf{r}_1) \approx P_0(\mathbf{r}_2,\mathbf{r}_1)+P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)+ \\ +P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_M,\mathbf{r}_M)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)+\cdots. \tag{2.18}$$

And now we notice that this infinite series is easily summed, since it is just a geometric progression:

$$\begin{aligned} P(\mathbf{r}_2,\mathbf{r}_1) &\approx P_0(\mathbf{r}_2,\mathbf{r}_1)+P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)\times \\ &\quad\times[1+P(M)P_0(\mathbf{r}_M,\mathbf{r}_M)+P(M)^2P_0(\mathbf{r}_M,\mathbf{r}_M)^2+\cdots] \\ &= P_0(\mathbf{r}_2,\mathbf{r}_1)+\frac{P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)}{1-P(M)P_0(\mathbf{r}_M,\mathbf{r}_M)}. \end{aligned} \tag{2.19}$$

Thus, we have obtained an approximate solution for the propagator $P(\mathbf{r}_2,\mathbf{r}_1)$ which is valid in the strong interaction case.

Chapter 3

Quantum Quasi Particles and the Quantum Pinball Propagator

3.1 The quantum mechanical propagator

In this chapter we are going to solve the simplest existing example of a quantum field theoretical problem. We call it the 'quantum pinball game' since it is the precise quantum analogue of the classical pinball machine just discussed, and in fact gives rise to a diagrammatic series having exactly the same form as (2.25). It is a sub-trivial problem, one which can be solved in a microsecond by elementary quantum mechanics. It takes a little longer to do by diagrams, but like its classical cousin in Fig. 2.3 has the great merit of illustrating all the basic principles without immersing the reader in a morass of mathematics. At the end of the chapter, the diagram method is applied to a non-trivial problem, i.e., finding the energy and lifetime of an electron propagating through a set of randomly distributed scattering centres (e.g., impurity atoms in a metal).

The fundamental difference between the classical propagator, P, and the quantum propagator, G, is that P is a probability, whereas G is a probability *amplitude*, with corresponding probability given by $|G|^2$ $(= G^* G)$. Thus in the classical case, the total probability for propagation from point 1 to point 2 is just the sum of the probabilities for each propagation process taken separately:

$$P(2,1)_{\text{classical}} = P(\text{process I}) + P(\text{process II}) + \cdots.$$

But in the quantum case, the total probability *amplitude* is the sum of the probability *amplitudes* for each process taken separately

$$G(2,1) = G(\text{process I}) + G(\text{process II}) + \cdots$$

so that the corresponding probability is given by

$$P(2,1)_{\text{quantum}} = G^* G = \underbrace{|G(\text{I})|^2}_{P(\text{I})} + \underbrace{|G(\text{II})|^2}_{P(\text{II})} + \underbrace{G(\text{I})^* G(\text{II}) + G(\text{II})^* G(\text{I})}_{\text{interference terms}} + \cdots.$$

Because of the characteristic 'interference terms', the quantum probability is not just the sum of the probabilities for the individual processes, in contrast to the classical case.

A familiar example of this is the decay of an atom, molecule, or nucleus from a state i to a state f by means of photon emission. Suppose the atom can either decay directly: $i \rightarrow f$, or via the intermediate state m: $i \rightarrow m \rightarrow f$. Then we have (call A the probability amplitude):

$$\begin{aligned} P(i \rightarrow f) &= A^* A = |A(i \rightarrow f) + A(i \rightarrow m \rightarrow f)|^2 \\ &= |A(i \rightarrow f)|^2 + |A(i \rightarrow m \rightarrow f)|^2 + A^*(i \rightarrow f)\, A(i \rightarrow m \rightarrow f) \\ &\quad + A^*(i \rightarrow m \rightarrow f)\, A\,(i \rightarrow f), \end{aligned}$$

which shows the interference between processes $i \rightarrow f$ and $i \rightarrow m \rightarrow f$. (See also Feynman (1965), pp. 19, 20.)

Let us begin by defining the quantum propagator in general, then show what it looks like in the case of free particles and quasi particles. The quantum analogue of the classical propagator is (assuming that the Hamiltonian is time-independent, so that the propagator depends only on time differences):

$$iG(\mathbf{r}_2, \mathbf{r}_1, t_2 - t_1)_{t_2 > t_1} = iG^+(\mathbf{r}_2, \mathbf{r}_1, t_2 - t_1)$$

= probability amplitude that if at time t_1 we add a particle at point $\mathbf{r}_1$ to the interacting system in its ground state, then at time t_2 the system will be in its ground state with an added particle at $\mathbf{r}_2$. (3.1)

The i factor is purely for decoration (a matter of convention) and the + superscript denotes $t_2 > t_1$. (The precise meaning of the word 'add' here is discussed in detail in §9.2.) The probability corresponding to the amplitude (3.1) is

$$P(\mathbf{r}_2, \mathbf{r}_1, t_2 - t_1) = G^+(\mathbf{r}_2, \mathbf{r}_1, t_2 - t_1)^*\, G^+(\mathbf{r}_2, \mathbf{r}_1, t_2 - t_1).$$

Note that it is not necessarily the 'same' particle which is observed at t_2, since this has no meaning in the systems of identical particles with which we shall generally deal. Note also that a more precise way of saying that the particle is 'at point $\mathbf{r}_1$' is to say that it is 'in the position eigenstate $\delta(\mathbf{r} - \mathbf{r}_1)$'.

The quantity G^+ defined in (3.1) is called a 'retarded' propagator (or Green's function). By definition, it is equal to zero for $t_2 \leqslant t_1$. There is also an 'advanced' propagator, G^-, which is finite for $t_2 \leqslant t_1$; this will be discussed in chapter 4. (See appendix L for other types of retarded and advanced propagators.)

It is actually more convenient to work with an equivalent definition of G in terms of arbitrary single-particle eigenstates, $\phi_k(\mathbf{r})$, instead of position eigen-