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The mathematical formulation of quantum electrodynamics

15.1 Introduction

In 1950–51 Richard Feynman wrote two papers in which he described the mathematical formulation of his method in quantum electrodynamics. The first paper was entitled 'Mathematical formulation of the quantum theory of electromagnetic interaction'; it was received on 8 June 1950 and published in the *Physical Review*.[1] The second paper, entitled 'An operator calculus having applications in quantum electrodynamics', was received on 23 May 1951 and published in the October issue of the *Physical Review*.[2] Both of these papers were aimed at giving the mathematical basis of Feynman's methods, which had been published in the previous papers without giving proper derivations. In the 1950 paper Feynman used the methods of his Ph.D. thesis (see Chapter 6) and his 1948 paper in the *Reviews of Modern Physics* (see Chapter 10) to derive the basic equations and principles of his method. In the second paper,[2] Feynman gave further development of the formalism and discussed the new idea of how to proceed with noncommuting quantities. 'No results which are new are obtained in this way, but it does permit one to relate various formulas of operator algebra in quantum mechanics in a simpler manner than is often available. In particular, it is applied to quantum electrodynamics to permit an easier way to seeing relationships among the conventional formulation, that of Schwinger[3] and Tomonaga[4] and that of the author.'[5,6]

In these articles, besides deriving the old formulas, Feynman obtained many new relations and derivations, and invented new approaches to solve different mathematical problems in quantum mechanics and quantum field theory. But, in a strict sense, these two papers were not entirely mathematical, for in them one cannot find purely mathematical proof of theorems and formulas. On the contrary, at many places Feynman's statements required more careful analysis and proofs, and sometimes they were incorrect in a rigorous mathematical sense. For instance, in these articles Feynman paid no attention to the divergence problem at all, and left the dangerous points for future

investigation; still, these two papers were very constructive and quite formal mathematically. The main point was to describe all the steps that were needed to derive Feynman's rules and Feynman's approach to quantum electrodynamics from the path-integral method and to offer new ideas for obtaining a convenient general method with obvious physical applications.

15.2 Mathematical formulation of the quantum theory of electromagnetic interaction

In the paper with this title,[1] Feynman established the validity of the rules given in previous papers[5] for the solution of the problems in quantum electrodynamics. Starting with Fermi's formulation of the field as a set of harmonic oscillators, he integrated out the effect of the oscillators in the Lagrangian form of quantum mechanics. There resulted an expression for the effect of all virtual photons valid to all orders in $e^2/\hbar c$. Feynman showed that the evaluation of the expression as a power series in $e^2/\hbar c$ gives just the terms expected by the aforementioned rules.

In addition, Feynman established a relation between the amplitude for a given process in an arbitrary unquantized potential and in a quantum electrodynamical field. This relation permitted a simple general statement of the laws of quantum electrodynamics.

Feynman gave a description, in Lagrangian quantum mechanical form, of particles satisfying the Klein–Gordon equation. It involved the use of an extra parameter analogous to the proper time to describe the trajectory of the particle in four dimensions. In the special case of photons, Feynman discussed the problem of finding what real processes were implied by the formula for virtual processes.

By the Lagrangian form of quantum mechanics Feynman meant his path-integral method. He used the results of his Ph.D. thesis (see Chapter 6) and the 1948 paper in *Reviews of Modern Physics* (see Chapter 10) to give a detailed derivation of the rules, which he had proposed for calculations in quantum electrodynamics. He started from Fermi's assemblage of independent harmonic oscillators for the transverse part of the electromagnetic field, using the notations of Heitler.[7] The classical Lagrangian for the system of the electromagnetic field and the charged particles has the form

$$L = L_{\mathrm{p}} + L_{\mathrm{I}} + L_{\mathrm{C}} + L_{\mathrm{tr}}, \tag{15.1}$$

where L_{p} is the Lagrangian of the noninteracting particles (in the nonrelativistic case, $L_{\mathrm{p}} = \frac{1}{2}\sum_n m_n \dot{x}_n^2$ (see Section 6.2, where Feynman first used this form to illustrate his ideas); $L_{\mathrm{I}} = \sum_n e_n \dot{x}_n A_n^{\mathrm{tr}}(x_n)$ is the interaction Lagrangian, i.e. that part of the total Lagrangian which describes the interaction between the particles and the transverse electromagnetic field with potential $L_{\mathrm{C}} = -\frac{1}{2}\sum_n \sum_m e_n e_m / r_{nm}$ is the Coulomb Lagrangian, which describes the Coulomb

interaction connected with the longitudinal part of the electromagnetic field; and, finally, the Lagrangian of the free transverse electromagnetic field, i.e. the radiation field, is $L_{\mathrm{tr}}=\frac{1}{2}\sum_K\sum_r[(\dot{q}_K^{(r)})^2-k^2(q_K^{(r)})^2]$. Hence, the classical action of this system is

$$S=\int L\,dt=S_{\mathrm{p}}+\mathrm{S_I}+S_{\mathrm{C}}+S_{\mathrm{tr}}. \tag{15.2}$$

According to the principles of the path-integral method (see Chapters 6 and 10), the quantum amplitude for the transitions of the system from one quantum state to the other is the sum of terms $\exp[(i/\hbar)S]$ over all virtual paths in the classical configuration space of this system. In our case the system consists of two different parts: the classical electromagnetic field and the particles of matter. Using the technique which he had invented in his Ph.D. thesis for integrating out the oscillator variables (see Section 6.5 and Chapter 10), Feynman showed that in the present case the elimination of the oscillator degrees of freedom, i.e. the elimination of the transverse electromagnetic field, leads to the effective classical action (for the charged particles only) in a form $S_{\mathrm{p}}+R$, where

$$R=-\tfrac{1}{2}\sum_n\sum_m\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e_ne_m[1-\dot{x}_n(t)\dot{x}_m(s)]\,\delta_+\big((t-s)^2 - [x_n(t)-x_m(s)]^2\big)\,dt\,ds. \tag{15.3}$$

The real part of this term includes the δ-function instead of δ_+-function, and is just the Fokker classical action in the action-at-a-distance theory if the summation is performed only on the different values of n and m, i.e. if one includes the self-interaction of charged particles (see Chapter 5).

To obtain the transition amplitudes including the effect of the field, one must calculate the transition elements of $\exp(iR)$:

$$\langle\chi_{t''}|\exp(iR)|\psi_{t'}\rangle_{S_{\mathrm{p}}}$$

(see Chapters 6 and 10). Expanding the exponent in a Taylor series, we obtain the perturbation expansion of the transition amplitude in powers of $e^2/\hbar c$ (in the calculations, $\hbar=c=1$ is assumed) insofar as the expression (15.2) for R is proportional to this dimensionless interaction constant. Hence we have

$$\langle\psi_{t''}|\exp(iR)|\psi_{t'}\rangle_{S_{\mathrm{p}}}=\langle\psi t_{''}|1|\psi t_{t'}\rangle_{S_{\mathrm{p}}}+i\langle\psi_{t''}|R|\psi_{t'}\rangle_{S_{\mathrm{p}}}+\cdots,$$

and the calculation of the first-order term by the path integral method yields

$$\langle\psi_{t''}|R|\psi_{t'}\rangle_{S_{\mathrm{p}}}=-e^2\int dt\int ds\int\psi^*(x_{t''},t'')K(x_{t''},t'';x_{t'}t)\,\delta_+\big((t-s)^2 - (x_t-x_s)^2\big)K(x_t,t;x_s,s)K(x_s,s;x_{t'},t')\psi(x_{t'},t')\,d^3x_{t''}\,d^3x_t\,d^3x_s\,d^3x_{t'}, \tag{15.4}$$

which is just the starting formula for applying Feynman's method.

However, this derivation was only for the nonrelativistic particles. The corresponding extension of the formulas for the relativistic Dirac particles was derived by Feynman in a formal manner. Feynman did not succeed in deriving the Dirac equation from first principles by the path integral method, because the relativistic classical action did not lead directly to a spin-$\frac{1}{2}$ relativistic particle. For this purpose a proper generalization of the very classical mechanics of the relativistic particles is needed; namely, one has to involve the so-called Grassmann anticommuting variables, which are important ingredients of the modern supersymmetric theories. This approach has been successfully developed recently,[8] a long time after the fundamental work of Feynman. But one can find its roots already in the Feynman paper which we are discussing. The Grassmann variables can be presented as some matrices with suitable properties.

In Section 6 of his paper, entitled 'Extension to Dirac particles', Feynman gave the formal derivation of the relativistic formulas for spin-$\frac{1}{2}$ particles, considering the transition amplitude $(x_{t''}|x_{t'})$ for a Dirac particle as a 4×4 matrix (or $4^N \times 4^N$ matrix if we deal with N electrons). For such a matrix propagator, i.e. for the transition amplitude, the usual formulas from the path-integral method are still correct, and the path-integral-like representation of this amplitude was given by Feynman in the form

$$\langle \chi_{t''}|\exp(iR)|\psi_{t'}\rangle = \lim_{\varepsilon\to 0} \int \chi^*(x_{t''}^{(1)}, x_{t\sim}^{(2)}, \ldots)$$

$$\times \prod_n \left(\Phi_{p,n}^{(0)}\, d^3x_{t''}^{(n)}\, d^3x_{t''-\varepsilon}^{(n)} \ldots d^3x_t^{(n)} \right)$$

$$\times \exp(iR)\psi(x_{t'}^{(1)}, x_{t'}^{(2)}, \ldots), \tag{15.5}$$

where $\Phi_p = \prod_i^n (x_{i+1}|x_i)$ is a 4×4 matrix product for one-electron theory, or a $4^N \times 4^N$ matrix product if we deal with N electrons. The matrix expression $(x_i+1|x_i)$ is the propagator of the true Dirac particle, which is analogous to the expression $(x_i+1|x_i) = A^{-1}\exp[iS(x_{i+1}, x_i)]$ in Feynman's path-integral method (see Chapters 6 and 10); for a Dirac particle in an electromagnetic field it must be replaced by

$$(x_{i+1}|x_i)A = (x_{i+1}|x_i)^{(0)} \exp\{-i[SA_4(x_i, t_i) - (x_{i+1}-x_i)A(x_i, t_i)]\},$$

which Feynman verified directly from the Dirac equation.

A careful analysis of this procedure shows that it is equivalent to the modern one, using Grassmann's variables, and it may be considered as the first step in this direction.

Using the same procedure as in the nonrelativistic case, Feynman derived directly from equation (15.5) the formula

$$\langle\psi_{t''}|R|\psi_{t'}\rangle_{S_p} = -e^2 \int \psi^*(x_{t''}) K_0(x_{t''}, t''; x_t, t)\beta\alpha_\mu \times \delta_+\big((t-s)^2-(x_t-x_s)^2\big) K_0(x_t, t; x_s, s)\beta\alpha_\mu \times K_0(x_s, s; x'_{t'}, t')\beta\psi(x_{t''})\, d^3x_{t''}\, d^3x_t\, d^3x_s\, d^3x_{t'}\, dt\, ds. \quad (15.6)$$

Here one has to change the notation, $\beta\alpha_\mu = \gamma_\mu$ and $\psi^*\beta = \bar{\psi}$, and to replace the propagator K_0 by Feynman's K^+ to take into account Feynman's theory of positrons as particles moving backwards in time, instead of the Dirac hole theory to which the propagator K_0 corresponds. As a result of this, one obtains the fundamental formula of Feynman's approach to quantum electrodynamics,[9] which is the basis for the Feynman rules and diagrams in the lowest orders of perturbation theory.

In order to get his rules in the general case of arbitrary order in perturbation theory, Feynman derived the new equation for the quantum transition amplitude $T_{e^2}[B]$ in the external field B. He considered the dependence of this amplitude, $T_{e^2}[B]$, on the square of the charge, e^2, of the particles of matter, and he derived the new equation for the amplitude as

$$\frac{dT_{e^2}[B]}{d(e^2)} = \tfrac{1}{2}i \iint \frac{\delta^2 T_{e^2}[B]}{\delta B_\mu(1)\, \delta B_\mu(2)}\, \delta_+(s_{12}^2)\, d\tau_1\, d\tau_2. \quad (15.7)$$

Here $dT_{e^2}[B]/d(e^2)$ is the derivative of the amplitude $T_{e^2}[B]$ as a function of e^2, and $\delta T_{e^2}[B]/\delta B(x, t)$ is the corresponding functional derivative of the amplitude as a functional of the external field B; $s_{12}^2 = (x_{\mu,1}-x_{\mu,2})(x_{\mu,1}-x_{\mu,2})$ is the four-dimensional distance.

This formula permits one to derive in a simple manner the Taylor series of the amplitude as a function of the charge e^2, i.e. its perturbative expansion. It may also be considered as a differential-functional equation for the amplitude $T_{e^2}[B]$ with intial condition $T_0[B]$—the amplitude for the noninteracting particles with $e^2 = 0$, which was obtained by Feynman in his previous paper.[10]

Equation (15.7), and the corresponding formalism of functional derivation with respect to external fields, is very convenient for general considerations of the perturbation series in quantum field theory and is now generally used.

Using this powerful formalism, Feynman derived in Appendix C of this article the new equation for the relativistic electron; this generalization of the Dirac equation reads:

$$(i\nabla - m)\Phi_{e^2}[B, x] = B(x)\Phi_{e^2}[B, x] + ie^2\gamma_\mu \int \frac{\delta\Phi_{e^2}[B, x]}{\delta B_\mu(1)}\, \delta_+(s_{12}^2)\, d\tau_1. \quad (15.8)$$

Here $\nabla = \gamma_\mu\, \partial_\mu$, $B = \gamma_\mu B_\mu$, and $\Phi_{e^2}[B, x]$ is the Dirac four-component spinor as

a function of the charge square e^2 and as a functional of the external field B. Without the last term, equation (15.8) represents the usual Dirac equation. The new (last) term, included by Feynman, takes into account the self-interaction of the electron due to the quantum electrodynamical effects (the propagation of the electron with the emission and absorption of the virtual photon). Effects of vacuum polarization are left out. Equation (15.8) actually includes all the advances in the theory of the electron that were made after Dirac's original discovery of his famous equation.

In his 1950 paper Feynman gave one more formal mathematical trick, which continues to play a definite role in theoretical investigations up to now. This trick was connected with the extension of Feynman's theory to relativistic spin-0 particles, which obey the Klein–Gordon equation. Feynman invented an elegant trick to deal with such particles in his path-integral method. The formal obstacle in using the path-integral method directly for the Klein–Gordon equation was the fact that this equation is of second order with respect to time. This was necessary to ensure the relativistic invariance, but the path-integral method may be used without proper modification only for first-order evolution equations (see Chapter 10).

Starting from the Klein–Gordon equation

$$(i\,\partial/\partial x_\mu - A_\mu)^2\psi = m\psi \tag{15.9}$$

for the scalar relativistic field $\psi(x, t)$, Feynman proposed to describe the classical trajectory in space–time by giving the four variables $x_\mu(u)$ as functions of some fifth parameter u, rather than expressing x_1, x_2, and x_3 as functions of the time x_4. This parameter u may be considered as the proper time of the particle, which increases as we go along the trajectory, whether the trajectory is proceeding forward ($dx_4/du > 0$) or backward ($dx_4/du < 0$) in time. Instead of the function $\psi(x, t) = \psi(x)$ of four variables, one can introduce the new one, $\phi(x, u)$, a function of five variables, which obeys the new evolution equation of the first order with respect to u:

$$i\,\partial\phi/\partial u = \tfrac{1}{2}(i\partial/\partial x_\mu - A_\mu)^2\phi. \tag{15.10}$$

This equation is analogous to the Schrödinger equation, and one may use the path-integral method to solve it in the usual manner. Moreover, it turned out that the corresponding relativistically invariant classical action for the classical particles, which corresponds to equation (15.10), reads:

$$S = -\int_0^\infty [\tfrac{1}{2}(dx_\mu/du)^2 + (dx_\mu/du)A_\mu(x)]\,du. \tag{15.11}$$

One can see that this action functional is quadratic with respect to the velocities dx_μ/du. Hence, it leads to the solvable Gaussian path integral.

After the solution of equation (15.10) has been found, one can obtain the solution of the Klein–Gordon equation according to the formula

$$\psi(x) = \int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}im^2u\right)\phi(x, u)\, du, \tag{15.12}$$

i.e. by a simple Fourier transformation.

The analysis of this procedure according to the path-integral method, proposed by Feynman, showed that it is quite a general one, and led to the theory of path integral of constrained systems, which is a very important part of modern theoretical methods and has many applications in physics.

15.3 An operator calculus having applications in quantum electrodynamics

Feynman's 1951 article with this title[2] was extremely rich in new mathematical ideas and inventions which he proposed to use in quantum theory, especially in quantum electrodynamics. He wrote: 'It is felt, in the face of daily experimental surprises for meson theory, that it may be worth while to spend one's time expressing electrodynamics in every physical and mathematical way possible. There may be some hope that a thorough understanding of electrodynamics might give a clue as to the structure of the more complete theory to which it is an approximation. This is one reason that this paper is published, even though it is little more than a mathematical re-expression of old material. A second reason is the desire to describe a mathematical method which may be useful in other fields.

'The mathematics is not completely satisfactory. No attempt has been made to maintain mathematical rigor. The excuse is not that it is expected that rigorous demonstrations can be easily supplied. Quite the contrary, it is believed that to put the present methods on a rigorous basis may be quite a difficult task, beyond the abilities of the author.'[11]

The main new mathematical idea in this article was the introduction and development of the new operator algebra which originated from Feynman's Ph.D. thesis (see Section 6.5) and from the 1948 *Reviews of Modern Physics* article (see Section 10.3).

It is well known that the principal mathematical difference between classical and quantum mechanics is that in the former all physical quantities are described as numerical functions of proper variables—coordinates and momenta of the system under consideration (see Section 6.2), while in quantum mechanics these quantities are described with corresponding operators in Hilbert space. These operators do not commute, i.e. the result of their action depends on the order in which they act. Usually this ordering, which is essential for the results, is described by the position in which the operators are written. Thus the product $\boldsymbol{AB}$ of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ means that one must first act by the operator $\boldsymbol{B}$ and then by operator $\boldsymbol{A}$. This product has a different action in comparison with the operator product $\boldsymbol{BA}$, which

means that one must first act by the operator A and then by the operator B. The impossibility of changing the places of the operators in quantum formulas without changing the very formulas is extremely inconvenient, and leads to great difficulties when one deals with quantum theory.

In Feynman's path-integral method there are no operators at all, but the same fundamental property of quantum theory reflects on the time ordering of the quantities in different formulas (see Section 10.3).

It seems quite natural to combine these two ways of describing the same property of the quantum variables, and to try to simplify the technique one uses in quantum theory. This is what Feynman did in his 1951 article.[2] He proposed to endow each operator A with an additional index s, writing the operator as A_s. Then the operator $A_s B_{s'}$ equals the usual operator product AB if $s > s'$, and to the usual operator product BA if $s < s'$. The order of the actual action of the operators no longer depends on their position on paper, so that the ordinary processes of analysis may be applied as though A_s and $B_{s'}$ were commuting variables. The indices s and s' are responsible for the right ordering of the action of the operators, since all mathematical manipulations are performed on the expressions we have to deal with.

For example, Feynman showed that the exponential function of the sum of two commuting operators α and β may be written in the form

$$\exp(\alpha+\beta)=\exp\left(\int_0^1 (\alpha_s+\beta_s)\,ds\right), \tag{15.13}$$

where one has to consider α and β formally as functions of the ordering parameter s.

For a commuting (classical) quantity we have the simple relation $\exp(a+b)=\exp(a)\cdot\exp(b)$, which is no longer true for the noncommuting operators α and β: $\exp(\alpha+\beta)\neq\exp(\alpha)\cdot\exp(\beta)$. But, after Feynman's invention is made use of, we have

$$\exp\left(\int_0^1 (\alpha_s+\beta_s)\,ds\right)=\exp\left(\int_0^1 \alpha_s\,ds\right)\cdot\exp\left(\int_0^1 \beta_s\,ds\right) \tag{15.14}$$

by analogy with the classical commutative case. As Feynman noted: 'Any operator function of $\alpha+\beta$ can, by replacing $\alpha+\beta$ by $\int_0^1 \alpha_s\,ds+\int_0^1 \beta_s\,ds$, be manipulated in a manifold of ways, many of which lead to useful formulas. (In a like manner, more complicated operator expressions can be rewritten using ordering indices. They may then be manipulated using all of the results of ordinary analysis.)'[1,2]

Feynman then added the presumption to use the subscript s as a formal argument of the corresponding operators, and to write down these operators as $A(s)$. This notation is convenient in the more general case in which the operator A actually depends explicitly on the order parameter. In this

case we should have strictly to write $A_s(s)$, but Feynman proposed to omit the subscript when no ambiguity will result from the change.

Another important remark was about the general sense of the mathematical expressions, which appeared in Feynman's new formalism. The expressions like $\int_0^1 \alpha_s \, ds$ are considered not as usual integrals, which have to be calculated, but as some functional. Hence the complete expression like (15.13) is a functional of the argument functions $\alpha(s)$, $\beta(s)$, etc. 'With each such functional we are endeavoring to associate an operator. The operator depends on the functional in a complex way (the operator is a functional of a functional) so that, for example, the operator corresponding to the product of two functionals is not (in general) the simple product of the operator corresponding to the separate factors. (The corresponding statement equating the sum of the functionals and the sum of the corresponding operators is true, however.) Hence, we can consider the most complex expressions involving a number of operators M, N, as described by functionals $F[M(s), N(s), \ldots]$ of the argument functions $M(s), N(s), \ldots (M_s, N_s, \ldots)$. For each functional we are to find the corresponding operator in some simple form; that is, we wish to disentangle the functional. One fact we know is that any analytic rearrangement may be performed which leaves the value of the functional unchanged for arbitrary $M(s), N(s), \ldots$ considered as ordinary numerical functions. Besides, there are a few special operations which we may perform on $F[M(s), N(s), \ldots]$, to disentangle the expressions, which are valid only because the functional does represent an operator according to our rules. These special operations (such as extracting an exponential factor) are, of course, proper to the new calculus; and our powers of analysis in this field will increase as we develop more of them.'[13]

The first example of this new operation, disentangling of the operator functionals, which Feynman invented, was the new proof of the basic formula of the time-dependent perturbation theory:

$$\exp(\alpha+\beta)=\exp\alpha+\int_0^1 \exp[(1-s)\alpha]\cdot\beta\exp(s\alpha)\,ds+\cdots. \tag{15.15}$$

Another very useful result was the generalization of the well-known theorem which says that, if we replace the operators $M(s), N(s), \ldots$ in the functional $F[M(s), N(s), \ldots]$ by $M'(s)=U^{-1}M(s)U$, $N'(s)=U^{-1}N(s)U, \ldots$, where U is some constant operator, then

$$F[M'(s), N'(s), \ldots]=U^{-1}F[M(s), N(s), \ldots]U. \tag{15.16}$$

This formula is a result of a special kind of disentangling of operators. Feynman generalized it for the case when U depends explicitly on the ordering parameter $U=U(s)$, and it obeys the following operator differential equation:

$$dU(s)/ds=P(s)U(s). \tag{15.17}$$

In this general case the corresponding formula, derived by Feynman, reads:

$$F[M'(s), N'(s), \ldots] = U^{-1}(1)F[M(s), N(s), \ldots]\exp\left(\int_0^1 P(s)\,ds\right)U(0), \tag{15.18}$$

where $M'(s) = U^{-1}(s)M(s)U(s)$, $N'(s) = U^{-1}(s)N(s)U(s)$, $\ldots$; $U(\mathrm{O})$ is the operator $U(s)$ for $s = \mathrm{O}$, and $U(1)$ is the operator $U(s)$ for $s = 1$. The formula (15.16) is obviously the special case of equation (15.18) and follows from it when $P(s)$ equals zero, i.e. when U is a constant operator, since from equation (15.17) one obtains

$$U(s) = \exp\left(\int_0^s P(s')\,ds'\right)U(0). \tag{15.19}$$

Feynman called the procedure, described by the formula (15.18), a disentangling of an exponential factor.

15.4 Applications in quantum mechanics and quantum electrodynamics

In his 1951 article[2] Feynman demonstrated many physical applications of the new operator calculus, which he had invented and developed. In the applications, the ordering parameter is usually the time t, and Feynman's new formalism became the time-ordered operator calculus. Thus, one can speak of the time-ordered exponent of an operator, etc.

The first example of the applications of this new calculus was the derivation of the expressions for the quantum evolution operator $\Omega(t_1, t_2)$. This unitary operator may be written as $\Omega(t_1, t_2) = \exp[(-i(t_2 - t_1)H]$, if the quantum Hamiltonian operator does not depend on the time t. Feynman wrote this operator in the form of a time-ordered exponential: $\Omega(t_1, t_2) = \exp(-i\int_{t_1}^{t_2} H_t\,dt)$. In the case of an explicitly time-dependent Hamiltonian $H(t)$, we have to replace H_t by $H_t(t)$, but using the convention described in the previous section Feynman wrote the formula for the operator in the form

$$\Omega(t_1, t_2) = \exp\left(-i\int_{t_1}^{t_2} H(t)\,dt\right). \tag{15.20}$$

If the Hamiltonian is written as a sum of some zero-order Hamiltonian $H_0(t)$ and some perturbation $U(t)$, i.e. $H(t) = H_0(t) + U(\mathrm{t})$, we can immediately use the formula (15.15), and as a result of disentangling of the operators in the time-ordered exponential (15.20) we will get the perturbative expansion of the corresponding evolution operator $\Omega^U(t_1, t_2)$ in powers of the perturbation $U(t)$. Hence the first term in this expansion will be

$$-i\int_{t_1}^{t_2}\Omega^{(0)}(t_2, t)U(t)\Omega^{(0)}(t, t_1)\,dt, \tag{15.21}$$

where $\Omega^{(0)}(t_1, t_2)$ is the evolution operator of the unperturbed problem, with the Hamiltonian $H_0(t)$. If $K(x_2, t_2; x_1, t_1)$ is the coordinate integral kernel of the operator $\Omega(t_1, t_2)$, this formula reads:

$$-i\int_{t_1}^{t_2}K^{(0)}(x_2, t_2; x, t)U(x, t)K^{(0)}(x, t; x_1, t_1)\,dx\,dt. \tag{15.22}$$

This is exactly the result which Feynman had used in his previous papers on quantum electrodynamics; there it was derived by the path-integral method, which leads to the same formula.

The next application of the new calculus was the derivation of the method of treating the complex physical system by parts, which Feynman had invented in his Ph.D. thesis using path-integrals. Now the method had been generalized and the derivation was independent of the path-integral method. Feynman considered the system consisting of two separate parts (a) and (b) with Hamiltonians $H^{(a)}$ and $H^{(b)}$. These two parts do interact with each other, and the interaction term in the Hamiltonian is $U[x^{(a)}, x^{(b)}]$. Hence, the entire Hamiltonian of the composite system is $H^{(a)}+H^{(b)}+U[x^{(a)}, x^{(b)}]$. Disentangling the time-ordered exponent in the matrix element m of the evolution operator of the system, where

$$m=\langle\psi_2\phi_2|\exp\Bigg(-i\int_t^{t_2}H^{(a)}(t)\,dt-i\int_t^{t_2}H^{(b)}(t)\,dt$$
$$-i\int_t^{t_2}U[x^{(a)}(t), x^{(b)}(t)]\,dt\Bigg)|\psi_1\phi_1\rangle. \tag{15.23}$$

Feynman showed that this element may be represented in the form

$$m=\langle\phi_2|\exp\Bigg(-i\int_t^{t_2}H^{(b)}(t)\,dt\Bigg)T^{(a)}[x^{(b)}(t)]|\phi_1\rangle, \tag{15.24}$$

where

$$T^{(a)}[x^{(b)}(t)]=\langle\psi_2|\exp\Bigg(-i\int_t^{t_2}H^{(a)}(t)\,dt-i\int_t^{t_2}U[x^{(a)}(t), x^{(b)}(t)]\,dt\Bigg)|\psi_1\rangle \tag{15.25}$$

is the matrix element for the system (a) alone, considering that in the interaction potential $U[x^{(a)}, x^{(b)}]$ all operators referring to (b) are arbitrary numerical functions of t.

Feynman concluded that in this way we can analyze one part of the pair of interacting systems without having yet analyzed the other. The influence of a

on b is completely contained in the operator functional $T^{(a)}[x^{(b)}(t)]$. Feynman thought that this separation may be useful in the analysis of the theory of measurement and of quantum statistical mechanics,[14] which makes that form useful in analyzing quantum properties of the electromagnetic field. Feynman expected therefore that with the present operator notation it should be equally easy to make this analysis. That this is indeed true is shown by an example later on. Since this, the main advantage of the Lagrangian form, can be so easily managed with the new notation for operators, this may take the place of the Lagrangian form in many applications. It is in some ways a more powerful and general form than the Lagrangian. It is not restricted to the nonrelativistic mechanics in any way. A possible advantage of the other form at present might be a slight increase in *anschaulichkeit* offered for the interpretation of nonrelativistic quantum mechanics.[15]

The third application of the new operator calculus was the introduction of the interaction representation for the evolution operator. This is the most convenient representation to perform real calculations in the time-dependent perturbation theory, and it gives the simplest form of this theory. Starting from the expression

$$\Omega^U(t_1, t_2)=\exp\left(-i\int_{t_1}^{t_2} H^{(0)}(t)\,dt\right)\exp\left(-i\int_{t_1}^{t_2} U(t)\,dt\right) \quad (15.26a)$$

for the evolution operator of the system with Hamiltonian $H(t)=H_0(t)+U(t)$, and using the formula (15.18), Feynman derived the interaction representation of the evolution operator in the form

$$\Omega^U(t_1, t_2)=S(t_2)\exp\left(-i\int_{t_1}^{t_2} U'(t)\,dt\right)S^{-1}(t_1), \quad (15.26b)$$

where $U'(s)=S^{-1}(t)U(t)S(t)$ is the perturbation in the interaction representation, and

$$S(t)=\exp\left(-i\int_0^t H^{(0)}(t)\,dt\right) \quad (15.27)$$

is the evolution operator of the nonperturbed system. Hence the time-dependent perturbation theory simply comes to evaluating

$$\Omega'(t_1, t_2)=\exp\left(-i\int_{t_1}^{t_2} U'(t)\,dt\right) \quad (15.28)$$

by expansion in power series of the exponential in this formula.

Feynman then applied the new formulas to calculate matrix elements of the evolution operator of the system coupled to a harmonic oscillator. Here he obtained the old results reached by the path-integral method in his earlier papers.[16,14,1]

The next important application of the new technique was in quantum electrodynamics, where the evolution operator was written down in the form

$$\exp\left(-i\int_0^t [H_{\mathrm{m}}(t)+H_{\mathrm{f}}(t)+H_{\mathrm{i}}(t)]\,dt\right). \tag{15.29}$$

Here $H_{\mathrm{m}}(t)$ is the Hamiltonian of the electron–positron field, $H_{\mathrm{f}}(t)$ is that of the electromagnetic field, and $H_{\mathrm{i}}(t)$ represents the interaction of these fields. Disentangling the exponential factor

$$\exp\left(-i\int_0^t [H_{\mathrm{m}}(t)+H_{\mathrm{f}}(t)]\,dt\right),$$

Feynman arrived at the interaction representation, and the problem was reduced to the analysis of the operator $\exp(-i\int_0^t H_{\mathrm{i}}'(t)\,dt)$. Omitting the prime for the interaction representation, and considering the case for $t_1\to-\infty$, $t_2\to+\infty$, Feynman simplified the problem to the study of the operator

$$S=\exp\left(-i\int_{-\infty}^{+\infty} j_\mu(x,t)A_\mu(x,t)\,d^3x\,dt\right). \tag{15.30}$$

This operator is just the Heisenberg S-matrix in the interaction representation. Feynman also introduced the evolution operator in the interaction representation in the form

$$\Omega(\tau)=\exp\left(-i\int_{-\infty}^{\tau} j_\mu(x,t)A_\mu(x,t)\,d^3x\,dt\right) \tag{15.31}$$

and then derived the differential equation for this operator:

$$i\,d\Omega(\tau)/d\tau=\left(\int j_\mu(x,\tau)A_\mu(x,\tau)\,d^3x\right)\Omega(\tau). \tag{15.32}$$

This is the Schrödinger equation for quantum electrodynamics written in the interaction representation.

Feynman noted that 'the apparent lack of covariance implied by time to define the differential equation can be removed by analyzing S in a slightly different manner, suggested by Tomonaga and Schwinger.[17,18]

The integration over the variables x in equation (15.30) may be replaced by integration on some three-dimensional space-like surface in four-dimensional space–time. Then one has to consider the evolution operator as a functional of the surface σ: $\Omega=\Omega(\sigma)$. The evaluation of the variational derivative of this functional with respect to the surface element of the surface leads directly to the Tomonaga–Schwinger equation:

$$\frac{\delta\Omega(\sigma)}{\delta\sigma(x,t)}=-ij_\mu(x,t)A_\mu(x,t)\Omega(\tau). \tag{15.33}$$

Thus one can see the relation between Feynman's approach and that of Tomonaga and Schwinger.

15.5 Dyson's elaboration of the theories of Feynman, Schwinger, and Tomonaga

A detailed comparison between the different approaches to quantum electrodynamics proposed by Feynman, by Tomonaga, and by Schwinger was given by Freeman Dyson.[19] He stressed that 'the advantages of the Feynman theory are simplicity and ease of application, while those of Tomonaga–Schwinger are generality and theoretical completeness.'[20] At that time, when much of Feynman's theory was still unpublished, Dyson (who knew about Feynman's work from personal discussions with him) undertook to investigate the relations between these three approaches, which at first seemed to be completely different from each other, and to establish their equivalence. Dyson started from the Schrödinger equation in quantum field theory. One may consider the classical field as a mechanical system with some energy density in the volume d^3x around the point x in three-dimensional space, given by the Hamiltonian density $H(x)$. Then the Hamiltonian of this system will be $\int H(x)\, d^3x$. If one replaces the classical Hamiltonian by the quantum operator $H(x)$, the Schrödinger equation will be the following generalization of equation (15.32):

$$i\hbar\, \partial\Phi/\partial t = \left(\int H(x)\, d^3x \right) \Phi.$$

In this form of the field equation the relativistic invariance is not transparent. In order to reach a relativistic invariant form, Tomonaga used Dirac's many-time formalism in field theory and then developed his version of quantum electrodynamics.[21] Schwinger used in his approach,[22] as a leading idea, another invention of Dirac's, namely Dirac's formalism of canonical transformations. But both Tomonaga and Schwinger started from a Schrödinger-like equation for the quantum state $\phi(\sigma)$ in the relativistically invariant form

$$i \frac{\delta\Phi(\sigma)}{\delta\sigma(x)} = H(x)\Phi(\sigma).$$

It is equivalent to equation (15.33).

The Hamiltonian may be represented in the form $H(x) = H_0(x) + H_1(x)$, where $H_0(x)$ is the energy density of the free electromagnetic and electron fields, and $H_1(x)$ is that of the interaction of these fields. We can make a canonical transformation, $\phi(\sigma) = T(\sigma)\psi(\sigma)$, of the interaction representation by the operator $T(\sigma)$, defined as

$$T(\sigma)=\exp\left(-i\int t(x)H_0(x)\,d^3x\right).$$

The Schrödinger-like equation in this representation reads:

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)}=H_1(x)\Psi(\sigma),$$

where $H_1(x,t)=T(\sigma)^{-1}H_1(x)T(\sigma)=H_1(x)$ is the time-dependent interaction Hamiltonian in the interaction representation $[x=(x,t)]$.

The term $H_1(x)$ is a sum of the Hamiltonian $H_i(x)$ of the interaction between electron–positron and photon fields and the Hamiltonian $H_e(x)$ of the interaction with the external electromagnetic field. The elimination of the radiation field oscillators in Feynman's theory corresponds in Schwinger's theory to a canonical transformation $\Psi=\Omega(\sigma)\Theta(\sigma)$. The operator $\Omega(\sigma)$ satisfies the equation

$$i\frac{\delta\Omega(\sigma)}{\delta\sigma(x)}=H_i(x)\Omega(\sigma),$$

analogous to equation (15.33). For quantum electrodynamics, the time-dependent form of the interaction Hamiltonian is $H_i(x,t)=j_\mu(x,t)A_\mu(x,t)$. Hence we have arrived at the same equation as (15.33).

The solution of the equation for $\Psi(\sigma)$ may be written in the form

$$\Psi(\sigma)=\Omega(\sigma)\Theta,$$

where the state vector Θ is constant if the external field vanishes; otherwise it obeys the equation

$$i\frac{\delta\Theta(\sigma)}{\delta\sigma(x)}=H_T(x)\Theta(\sigma).$$

Here $H_T(x)=\Omega(\sigma)^{-1}H_i(x)\Omega(\sigma)$ represents the interaction of the charged particle with the external field, taking into account radiative corrections. This equation describes the deviation of the state vector of the single charged particle in an external field from the constant state vector of the free particle.

The operator $\Omega(\infty)$, as we know, is just the Heisenberg S-matrix operator, which describes all transitions from some initial state Θ_{in} to corresponding final state Θ_{out}. If the system with state vector Θ undergoes no transitions with the passage of time, the state vector Θ is called 'steady'. Hence, the steady vector obeys the equation $\Omega(\infty)\Theta=\Theta$.

The definition of the state vectors $\Theta(\sigma)$ via the state vectors $\Psi(\sigma)$ is unsymmetrical between past and future, and a new type of state vector, denoted by Θ', may be defined by the equation $\Theta'(\sigma)=\Omega(\infty)\Theta(\sigma)$. Since $\Omega(\infty)$ is a unitary operator, independent of σ, the state vectors Θ and Θ' are the same

vector in different coordinate systems. For the steady states they are simply identical.

As we know, to avoid the divergence in quantum electrodynamics, one must perform the renormalization of the mass of the bare electron–positron field $\psi(x)$. The corresponding renormalized Hamiltonian will be denoted as $H_I(x)=H_i(x)-\delta m\bar{\psi}(x)\psi(x)$, δm being the (infinite) mass renormalization constant.

Now we are ready to explain the relation between the Feynman theory and the Schwinger–Tomonaga theory, as discovered by Dyson.[19] 'The Schwinger theory works directly with equations [for Θ and H_T], the aim being to calculate the matrix of the "effective external potential energy" H_T between states, specified by their state vectors $[\Theta]$.'[23]

The $\Omega(\sigma)$ operator is given by the formula

$$\Omega(\sigma)=1+(-i)\int_{-\infty}^{\sigma} H(x_1)\,dx_1 + (-i)^2\int_{-\infty}^{\sigma} dx_1\int_{-\infty}^{\sigma(x)_1} H(x_1)H(x_2)\,dx_2+\cdots,$$

and the interaction Hamiltonian with radiative corrections is

$$H_T(x_0)=\sum_{n=0}^{\infty}(i)^n\int_{-\infty}^{\sigma} dx_1\int_{-\infty}^{\sigma(x_1)} dx_2\ldots\int_{-\infty}^{\sigma(x_n)} dx_n \times[H_I(x_n),[\ldots,[H_I(x_2),[H_I(x_1),H_I(x_0)]]\ldots]].$$

The repeated commutators are typical of Schwinger's theory, and their calculation is quite long and difficult. Nevertheless Schwinger was able to find all basic quantum electrodynamical effects up to second order in perturbation theory.

As Dyson noted, 'In the Feynman theory the basic principle is to preserve symmetry between past and future. Therefore, the matrix elements of the operator H_T are evaluated in a "mixed representation"; the matrix elements are calculated between an initial state specified by its state vector $[\Theta_1]$ and a final state specified by its state vector $[\Theta'_2]$.'[23]

The matrix element of H_T between two such states in the Schwinger representation is

$$\Theta_2^* H_T\Theta_1=\Theta'_2\Omega(\infty)H_T\Theta_1,$$

and, therefore, in Feynman's theory the operator H_T is replaced by the new operator $H_F(x)=\Omega(\infty)H_T(x)$. The corresponding formula for this operator is

$$H_F(x_0)=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dx_1\ldots\int_{-\infty}^{\infty} dx_n P[H_e(x_0)H_I(x_1)\ldots H_I(x_n)].$$

Here the letter P denotes the symmetrization with respect to all indices of the variables x_i. This formula is simpler than Schwinger's formula for the operator H_T. In addition, the operator $H_F(x)$ now appears to be a function only of the variable x, instead of being a functional of the surface σ.

Replacing the operator H_e with the unit matrix, instead of the more complicated formula in Schwinger's theory, one can obtain from here the following formula for the Heisenberg S-matrix in the Feynman theory:

$$\Omega(\infty) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n \, P[H_I(x_1) \ldots H_I(x_n)].$$

Both equations (15.32) and (15.33) define the operator (15.30), which can be written down in the invariant form

$$S = \exp\left(-i \int j_\mu(1) A_\mu(1) \, d\tau_1 \right). \tag{15.34}$$

The matrix taken between states in which the field is empty of photons initially and finally was written by Feynman in the form

$$S_{00} = \exp\left(-\tfrac{1}{2} i e^2 \iint j_\mu(1) j_\mu(2) \, \delta_+(s_{12}^2) \, d\tau_1 \, d\tau_2 \right). \tag{15.35}$$

Then, disentangling the exponent of the photon operators A in the general case, Feynman arrived at the following expression for the operator S:

$$\begin{aligned} S = & \exp\left(-i \int j_\mu(1) A^-_{\mu + \infty}(1) \, d\tau_1 \right) \\ & \times \exp\left(-\tfrac{1}{2} i e^2 \iint j_\mu(1) j_\mu(2) \, \delta_+(s_{12}^2) \, d\tau_1 \, d\tau_2 \right) \\ & \times \exp\left(-i \int j_\mu(2) A^+_{\mu - \infty}(2) \, d\tau_2 \right). \end{aligned} \tag{15.36}$$

Here the $A^+_{\mu t}$ operator annihilates photons, and the $A^-_{\mu t}$ operator creates them at the instant t. In the expression (15.36) these operators are completely disentangled and one can very easily calculate the matrix elements.

Finally, Feynman noted: 'Taken between states empty of photons the result is S_{00} of equation (15.35), for the annihilation operator A^+ on the state of zero photons is zero, and the creation operator A^- has zero amplitude for leaving a state without photons. If there is one photon present initially and we ask that no photon remain, we shall have to annihilate it and create none, so that A^- and A^+ exponentials are expanded in power series, we must take only the term linear in A^+ and independent of A^-. This is equivalent to a first-order action of potential . . . in perturbation. The corresponding rules for higher numbers

of real photons are readily derived from equation [(15.36)]. In this way we have completed an independent deduction of all the main formal results in quantum electrodynamics, by use of the operator notation.'[24]

15.6 The Dirac equation

Feynman described the applications of the new operator calculus to the Dirac equation in three sections and one appendix of his 1951 article. First, Feynman derived by his new method the basic results of his 1949 article on the theory of positrons.[25] He considered the electron in the external nonquantized electromagnetic field $B=\gamma_\mu B_\mu$, first omitting the contribution of the closed loop diagrams.[26] The solution of the Dirac equation with a source function F,

$$(i\nabla - B - m)\Psi = F, \tag{15.37}$$

was written in the form

$$\Psi = (i\nabla - B - m)^{-1} F, \tag{15.38}$$

with the help of the inverse operator $(i\nabla - B - m)^{-1}$, which may be interpreted in the definite sense implied by the limit of the operator when mass m has a vanishing negative imaginary part. From the definition of the propagator $K_+^B(2, 1)$ in the 1949 article,[25] we can write

$$K_+^B(2, 1) = (i\nabla - B - m)^{-1} i\,\delta(2, 1). \tag{15.39}$$

Then the operator identity

$$(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - \cdots, \tag{15.40}$$

written in the space-coordinate representation for $A=(i\nabla - m)$ and B simply leads to the perturbation expansion of the propagator $K_+^B(2, 1)$,

$$K_+^B(2, 1) = K_+(2, 1) - i\int K_+(2, 3)B(3)K_+(3, 1)\,d\tau_3$$

$$- \iint K_+(2, 4)B(4)K_+(4, 3)B(3)K_+(3, 1)\,d\tau_3\,d\tau_4 + \cdots. \tag{15.41}$$

The contribution of the closed loops is the factor $C_v = \exp(-L)$ (see Section 13.6, equation (13.28)), where L is not very easily defined directly by the operators. But the first-order change on changing the potential B to $B+\Delta B$ is

$$\Delta L = \text{trace}\{[(i\nabla - B - m)^{-1} - (i\nabla - m)^{-1}]\Delta B\}, \tag{15.42}$$

where the 'trace' means the diagonal integral in coordinates. It should be noted that this approach of Feynman's is the basis of the modern description of closed-loop effects in quantum field theory.

In the next section of his article, Feynman gave a new representation of the inverse operator $(i\nabla - B - m)^{-1}$, which makes it possible to write down the formula directly for the quantity L, instead of equation (15.42). This representation was based on the formula $\int_0^\infty \exp(iWx)\, dW = i/x$ (or rather $\lim_{\varepsilon \to 0}[i/(x+i\varepsilon)]$), which leads to the representation

$$(i\nabla - B - m)^{-1} = \int_0^\infty \exp[i(i\nabla - B - m)W]\, dW, \tag{15.43}$$

or, involving the ordering operator w, to the new representation

$$(i\nabla - B - m)^{-1} = \int_0^\infty \exp\left(i \int_0^W [i(i\nabla(w) - B(w)]\, dw \right) \exp(-imW)\, dW. \tag{15.44}$$

Starting from this equation Feynman derived the following formula for L:

$$L = \int_0^\infty \operatorname{trace} \left\{ \exp\left(i \int_0^W [i(i\nabla(w) - B(w)]\, dW \right) \right\} \exp(-imW) \frac{dW}{W}, \tag{15.45}$$

and gave the regularization procedure for this divergent quantity.

Another important result, derived from representation (15.44), was the formula for the expectation value R for a single charge between photon-free states:

$$R = \int_0^\infty \exp\left(-\int_0^W \nabla(w)\, dw \right) \exp\left(-\tfrac{1}{2} ie^2 \int_0^{w'} \int_0^{w''} \gamma_\mu(w')\gamma_\mu(w'')\, \delta_+(s^2_{w'w''}) \right) \times \exp(-imW)\, dW. \tag{15.46}$$

This expression gives a description of a Dirac electron interacting with itself. An extra factor $\exp[-i\int_0^W B(w)\, dw]$ will describe such an electron in an external field with potential B. Feynman also gave a proper generalization of this result for a many-particle system, which gives the possibility also to calculate the effects of closed loops.

Thus the new operator calculus was shown to lead to a complete description of all quantum electrodynamical effects, and to give new powerful tools for solving the problems in quantum mechanics and quantum field theory.

In his 1951 article Feynman took one more step in the investigation of the Dirac equation. By analogy with the case of the Klein–Gordon equation (15.9), he proposed to introduce a fifth variable w for the Dirac equation, and to consider the spinor wave function $\Phi(x, w)$, which obeys the equation

$$-i\, \partial\Phi/\partial w = (i\nabla - B)\Phi, \tag{15.47}$$

analogous to equation (15.10). The solution of the original Dirac equation

may be obtained from the solution of equation (15.47) according to the formula

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi(x, w) \exp(-imw)\, dw, \tag{15.48}$$

which is analogous to equation (15.2).

Another version of this idea was developed in Appendix D of this article. It was closer to the ideas of Fock[27] and Nambu.[28] Now the Dirac equation, $(i\nabla - B)\Psi = m\Psi$, was written in the form

$$[(i\,\partial/\partial x_\mu) - B_\mu]^2\Psi - \tfrac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\Psi = m\Psi, \tag{15.49}$$

which is analogous to the form (15.10) of the Klein–Gordon equation. Here the matrices $\sigma_{\mu\nu} = \frac{1}{2}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ are 4×4 matrices, analogous to those of Pauli, and $F_{\mu\nu} = \partial B_\nu/\partial \mathrm{x}_\mu - \partial B_\mu/\partial x_\nu$ is the field tensor. Adding the fifth parameter u in the spinor wave function $\Psi(x, u)$, Feynman wrote down the equation for $\Phi(x, u)$ in the form

$$i\,\partial\Phi/\partial u = \tfrac{1}{2}[(i\,\partial/\partial x_\mu) - B_\mu]^2\Phi - \tfrac{1}{4}\sigma_{\mu\nu}F_{\mu\nu}\Phi. \tag{15.50}$$

It can then be analyzed by the Lagrangian method. The final result is the sum over all trajectories $x_\mu(u)$ of the hypercomplex amplitude

$$\exp\left(-i\int_0^{u_0} [\tfrac{1}{2}(dx_\mu/du)^2 + (dx_\mu/du)B_\mu(x) - \tfrac{1}{4}\sigma_{\mu\nu}(u)F_{\mu\nu}(x(u))]\, du\right). \tag{15.51}$$

We can see how close Feynman came to the current ideas of how to treat the Dirac equation via the path-integral method using hypercomplex numbers. The further development of this idea was successful, and led to the application of the Grassmann variables for this purpose, as a special kind of hypercomplex numbers.

Another suggestion in this direction was given in Appendix B, after the operator (15.46) for self-action was disentangled and represented, using $C_\mu(W) = \int_0^W \gamma_\mu(w)\, dw$ and $\dot{C}_\mu(w) = \gamma_\mu(w)$, in the form

$$R = \int_0^{\infty} \exp[ip_{\mu,\infty}C_\mu(w)]\exp\left(-\tfrac{1}{2}ie^2\int_0^{w'}\int_0^{w''} \dot{C}_\mu(w')\dot{C}(w'')\right.$$

$$\left.\times\,\delta_+[\{\dot{C}_\mu(w') - \dot{C}_\mu(w'')\}]^2\, dw'\, dw''\right)\exp(-imW)\, dW. \tag{15.52}$$

Feynman noted: 'Not much has been done with this expression. (It is suggestive that perhaps coordinates and the space–time they represent may in

some future theory be replaced completely by an analysis of ordered quantities in some hypercomplex algebra).'[29]

15.7 Functional integration and the path-integral method

In the Appendices A, B, and C of his 1951 paper, Feynman desribed a new development and generalization of the functional methods and especially of his path-integral method. He wrote: 'In this Appendix [A,B,C] an attempt will be made to discuss some properties of ordered operators and of functionals in a somewhat general way. Almost certainly many of the equations will be incorrect in their general form. This is especially true of those involving the Fourier transform in function space. However, it is expected that they are correct in special cases in which the formulas have been applied in the main part of this paper. Therefore, at least at first, when new results using these methods are derived, care should be taken to check the final result in some independent way. It is analogous to using power series expansions, or Fourier transforms, in a calculation in a situation in which the conditions for the validity of the power expansions or of the transform have not been checked, or are not known to be satisfied. The physicist is very familiar with such a situation and usually satisfied with it, especially since he is confident that he can tell if the answer is physically reasonable. But mathematicians may be completely repelled by the liberties taken here. The liberties are taken not because the mathematical problems are considered unimportant. On the contrary, this appendix is written to encourage the study of these forms from a mathematical standpoint. In the meantime, just as a poet has license from the rules of grammar and pronunciation, we should like to ask for "physicists' license" from the rules of mathematics in order to express what we wish to say in as simple a manner as possible.' [30]

These words of Feynman indicate the spirit of his mathematical investigations, which were far from being mathematically rigorous, but very, very constructive.

As far as it is clear what the linear combination and the exponent of an arbitrary operator is, and, on the other hand, in the analysis of commuting variables, more complicated functions may be represented as a superposition of the exponents, using Fourier transformation, Feynman proposed to extend this idea to the case of the functionals for which the operators are defined. 'Thus the superposition rule permits a wide increase in the class of functionals for which we have defined the operators. In fact, with some mathematical license, we have defined the operator for any functional. We wish to imagine that any functional can be represented as a superposition of exponential ones in a manner analogous to the representation of an arbitrary function as a superposition of exponential functions. Thus, we expect to be able to write for

any functional $F[M(s)]$ (the true mathematical restrictions are completely unknown to me)

$$F[M(s)] = \int \exp\left(i \int_0^1 \mu(s)M(s)\, ds\right) \mathscr{F}[\mu(s)] \mathscr{D}\mu(s), \tag{15.53}$$

where $\mathscr{F}[\mu(s)]$ is a new (complex) functional, the functional transform of $F[M(s)]$, and $\int \ldots \mathscr{D}\mu(s)$ represents (some kind of an) integration over the space of functions $\mu(s)$. For simplicity, we take the case of just a one-argument function $M(s)$. If $F[M(s)]$ is given, $\mathscr{F}$ can be determined perhaps from

$$\mathscr{F}[\mu(s)] = \int \exp\left(-i \int_0^1 \mu(s)M(s)\, ds\right) F[M(s)] \mathscr{D}M(s), \tag{15.54}$$

with suitable normalization. Then, if $\mathscr{F}$ is known, we define the operator $F[M(s)]$ as

$$F[M(s)] = \int \exp\left(i \int_0^1 \mu(s)M(s)\, ds\right) \mathscr{F}[\mu(s)] \mathscr{D}\mu(s), \tag{15.55}$$

where $\mu(s)$ is a numerical function. Since we have already defined the operator $\exp(i \int_0^1 \mu(s)M(s)\, ds)$ (by some proper differential equation), we now simply require superimposition of such operators for various $\mu(s)$. The extension to functionals of several variables is evident.'[30]

This presents Feynman's idea of how to define the functional of an operator argument, using the functionals of different functions. Unfortunately, the further development of this idea showed that it is impossible in general, in a strict mathematical sense, to carry out the operations which Feynman had proposed to use only in quite restricted cases. Especially, it is hard to define proper integration in the space of functions in general (see Section 10.41).

In Appendix B Feynman used this method in the very important case of functionals of the type $F[\boldsymbol{P}(s), \boldsymbol{Q}(s)]$, where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are quantum mechanical momentum and position operators, which obey the canonical commutation relation

$$\boldsymbol{PQ} - \boldsymbol{QP} = -i. \tag{15.56}$$

Using Fourier transformations (15.53) and (15.54) of the functionals and representation (15.55) of the functional of operator arguments, combined with his disentangling procedure, Feynman showed that the matrix elements of the functional $F[\boldsymbol{P}(s), \boldsymbol{Q}(s)]$ between arbitrary states, with wave functions $g(Q)$ for the initial state and $f(Q)$ for the final state, may be written in a form

$$(g^* F f) = \iint g^*(q_1) \exp\left(i \int p(s)\dot{q}(s)\, ds\right) F[p(s)q(s)] \mathscr{D}p(s) \mathscr{D}q(s) f(q_0)\, dq_0\, dq_1. \tag{15.57}$$

Here $p(s)$ and $q(s)$ are the usual functions, which give the path in phase space of the classical system, being in correspondence to the quantum one. In the formula (15.57) all operators have been eliminated. It represents the matrix elements of the ordered operator $F[\boldsymbol{P}(s), \boldsymbol{Q}(s)]$ as a path-integral of the corresponding functional $F[P(s)\ Q(s)]$ on the phase space of the classical system. Here the path integral on the phase space of the classical system was first introduced by Feynman. Its rigorous definition is still an open problem.

Feynman immediately applied this result to the case of the operator functional $S=\exp(-i\int_0^T \boldsymbol{H}(t)\,dt)$, where $\boldsymbol{H}(t)$ is the quantum Hamiltonian of the system. This operator functional is just the evolution operator of the quantum system. For example, for the Hamiltonian $\boldsymbol{H}(t)=(1/2m)\boldsymbol{P}^2+V(\boldsymbol{Q}, t)$, according to the formula (15.57) we have the matrix elements

$$(g^*Ff)=$$

$$\iint g^*(q_1)\exp\left(i\int_0^T p(t)\dot{q}(t)\,dt - i\int_0^T (1/2m)p(t)^2\,dt - i\int_0^T V(q(t), t)\,dt\right)$$

$$\times f(q_0)\mathscr{D}p(t)\mathscr{D}q(t)\,dq_0\,dq_1. \quad (15.58)$$

The integral on the functions $P(t)$ can be easily done, because the functional in the exponent in (15.58) depends quadratically on $P(t)$. The corresponding technique was developed by Feynman in the following Appendix C. The result is

$$(g^*Sf)=\iint g^*(q_T)\exp\left(+i\int_0^T [\tfrac{1}{2}m\dot{q}(t)^2 - V(q(t), t)]\,dt\right) f(q_0)\mathscr{D}q(t)\,dq_0\,dq_T. \quad (15.59)$$

That is, the transition amplitude from point q_0 at $t=0$ to q_T at $t=T$ is the integral over all trajectories connecting these points of $\exp(\int_0^T L[\dot{q}(t), q(t)]\,dt)$, L being the Lagrangian for this problem. 'This is the fundamental theorem on which the interpretation of [Feynman's path-integral formulation of quantum mechanics] is based.'[31] Thus, starting from his operator calculus Feynman derived his old path-integral method.

In Appendix C of the 1951 article,[2] Feynman developed the technique for the calculation of the so-called Gaussian functional integrals, which is the basis of such types of calculations in field theory even up to now. Practically, this is the only working method to apply functional integration in field theory, and by now it has been highly modified and developed. In a large number of problems the operators appear in exponentials only up to the second degree. For this reason, it is handy to have available a formula for the integration of Gaussian functionals. We can define a Gaussian functional $G[y(s)]$, of one function $y(s)$, as one of the form $G[y(s)]=\exp\, iE[y(s)]$, with $E[y(s)]$ quadratic. Thus, we have

$$E[y(s)] = \tfrac{1}{2}\int_0^1 \int_0^1 A(t, s)y(t)y(s)\,dt\,ds + \int_0^1 B(s)y(s)\,ds, \tag{15.60}$$

where $A(t, s)$ and $B(s)$ are functionals, independent of y (that is, Gaussian if the second functional derivative of ln G is independent of y).

For this type of functional Feynman obtained the explicit results for functional integrals

$$\int \exp\{iE[y(s)]\}\mathscr{D}y(s) = I[A, B], \tag{15.61a}$$

$$\int \exp\{iE[y(s)]\}y(t)\mathscr{D}y(s) = \tilde{y}(t)I[A, B], \tag{15.61b}$$

$$\int \exp\{iE[y(s)]\}y(t)y(t')\mathscr{D}y(s) = [\tilde{y}(t)\tilde{y}(t') + iN(t, t')]I[A, B], \tag{15.61c}$$

and so on for higher powers of y. Here, in equations (15.61a–c) all terms on the right-hand sides were explicitly determined by coefficients $A(t, s)$ and $B(s)$ of the quadratic form (15.60). The function $\tilde{y}(t)$ is the solution of the integral equation

$$\int_0^1 A(t, s)\tilde{y}(s)\,ds = -B(t),$$

so $\tilde{y}(t) = \int_0^1 N(s, t)B(s)\,ds$, and it gives the extremum of the functional $E[y(s)]$; $N(s, t)$ is the kernel of the inverse integral operator, which corresponds to the integral operator defined by the kernel $A(t, s)$. Finally,

$$I[A, B] = G[\tilde{y}(s)]I[A, 0].$$

Feynman used these formulas to rederive some basic relations in quantum mechanics and quantum electrodynamics. Feynman's formulas for the Gaussian functional integrals have been proved to be strictly correct. They have been generalized and justified for different types of functional integrals, including the ones on Grassmann's noncommuting variables. These formulas have provided a convenient basis for many original investigations in quantum field theory, in statistical physics, in probability theory, and in other domains of physics and mathematics.

15.8 Some concluding remarks

Feynman's two articles of 1950 and 1951 which we have treated in this chapter were the final ones in the sequence of his fundamental articles on quantum electrodynamics. Feynman described these papers as follows. 'I had invented a

new mathematical method [the operator calculus] for dealing with operators according to a parameter which, to this day, I feel is a great invention, and which nobody uses for anything; nobody pays any attention to it. Some day it will be recognized as an important invention. I still think it is something very important, just as important as I felt when I first wrote it.

'I had used it to formulate quantum electrodynamics. I invented it to do that. It was in fact the mathematical formulation that I expressed at the Pocono Conference—that was this crazy language. Dates don't mean anything. It was published in 1951, but it had all been invented by 1948. I called it the operator calculus.

'I published it at that time because, after I had given the rules, and proved that they were the same as the other things [of Schwinger and Tomonaga], it was important [to show it formally]. Dyson had already given a proof. People don't bother to read my proof because it's too elaborate and funny, odd notations and path integrals, etc., but I had to do it in my own way for my own purposes. My paper on the 'Mathematical formulation of the quantum theory of electromagnetic interaction' was a rather unnecessary paper, because Dyson had done it in some way, and all I wanted to say was how I did it. But the other paper, on the operator calculus[2] was not completely empty; I felt it was important. In the years since I had invented it I had accumulated a whole lot of debris. For instance, I had noticed certain ways of representing spin-0 particles with path integrals, I had the operator calculus, and a whole lot of other things which I did not know where to put. Most of it was, of course, the operator calculus, but in the various appendices I included a whole variety of other things. With this paper I disgorged myself of all the things I had thought about in the context of quantum electrodynamics; this was an entire backlog of valuable things. I still think that the central item, the operator calculus, was an important invention.

'With this paper I had completed the project on quantum electrodynamics. I didn't have anything else remaining that required publishing. In these two papers,[1,2] I put everything that I had done and thought should be published on the subject. And that was the end of my published work on this field.'[32]

Feynman was never completely satisfied with his work on quantum electrodynamics, especially the renormalization procedure. As he recalled: '. . . I had answered Bethe's original request for a way of cutting it off so that all shenanigans with subtraction will be straightened out. In other words, it was obvious to me that it was renormalizable in the manner that Bethe proposed it would be, that by changing the mass and the charge, we would have expressed everything in terms of experimental mass and charge, all the results would be finite and one wouldn't need all these shenanigans.

'While I believed at the time that I wasn't finished, that I would find a satisfactory way of cutting off, I knew that the way I had cut it off destroyed unitarity, for instance, temporarily. Only the limit was presumably OK. I believed that there was a way of cutting the theory off somehow which

wouldn't destroy anything, that wasn't artificial, or just a mathematical trick. It could be solved. With the finite cut-off the theory would also be sound by itself. And that the real theory was the limit of that, and that was the way it should be. So I delayed a little bit my publication. I was urged to publish it anyway, and you'll find it in that paper[25] [The theory of positrons, 1949] a kind of apology, that I apologized for publishing it before I had straightened this little thing out, this minor problem! And thank God I did, because it has never been straightened out to this day. This minor problem! I thought that perhaps Schwinger had straightened it out because he had a great reputation, but when I thought about what he did, I realized that he had in his work the same problem in another form. And I mentioned it in that paper and explained it. I knew that we both hadn't solved it. And that we hadn't really solved quantum electrodynamics in the sense of finding a sound theory that the limit of which is electrodynamics; we couldn't prove that the electrodynamics we were writing was self-consistent and possibly not unitary, possibly incorrect, and very likely is so, but it's never been resolved. It has never been proved one way or the other whether electrodynamics renormalizes a consistent theory.' [33]

As we see, up to the very end Feynman had not changed the opinion which he had expressed in his Nobel lecture: 'Therefore, I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug. I am, of course, not sure of that.' [34]

In October 1961, Feynman would be invited to give a report at the twelfth Solvay Conference on the general theme of quantum field theory to celebrate the fiftieth anniversary of the famous 1911 Solvay Conference on radiation theory and quanta, at which Einstein, Planck, and Sommerfeld had presented their wonderful results on quantum theory. On this occasion, it was the same subject: the quantum theory of the interaction of light and matter. A large group of people, including Bethe, Weisskopf, Schwinger, and Dyson, were invited to attend this conference. Feynman was invited to give a report on the (then) current status of quantum electrodynamics.

Upon receiving the invitation to attend the Solvay Conference, Feynman felt rather honored, 'because these conferences always had a reputation. I remember when Bethe was invited sometime back [1948], he took very seriously the writing of his report for the conference. So, like father, like son: when I was invited to the Solvay Conference, I had to write a report and I took it very seriously, just as Bethe had done, imitating him in a way. In my report I discussed the subject as if I were addressing those people who had been present in 1911, telling them how far the problems which they had considered had evolved.' [32]

At the 1962 Solvay Conference, Feynman would declare: 'Considerable evidence for the general validity of QED is, of course, provided by the enormous variety of ordinary phenomena which, under rough calculation, are

seen to be consistent with it. The superfluidity of helium and the superconductivity of metals having been recently explained, there are to my knowledge no phenomena occurring under known conditions where quantum electrodynamics should provide an explanation and where at least a qualitative explanation in these terms has not been found. The search for discrepancies has turned from looking for gross deviations in complex situations to looking for large discrepancies at very high energies, or by looking for tiny deviations from the theory in very simple, but very accurately measured, situations. [No discrepancies have been found.]' [35]

Would the basic ideas of quantum electrodynamics be violated by a future theory? Feynman's answer was yes. Does there exist a tendency for that? 'No. I don't see any tendencies, though I think it would be violated. I am just guessing. Would it be violated at higher energies? Probably yes. I don't see any tendencies. I can't say that a thing is right if I think it will be violated.' [32]

Feynman would enjoy his visit to Brussels, where not only all his friends in the field—including great personalities like Dirac, Heisenberg, and Wigner—were present, but he would also meet the king and queen of Belgium. His preoccupation with the problems of quantum electrodynamics had been over for quite some time, but he would continue to make use of the physical conceptions and mathematical techniques he had pioneered in this field.

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