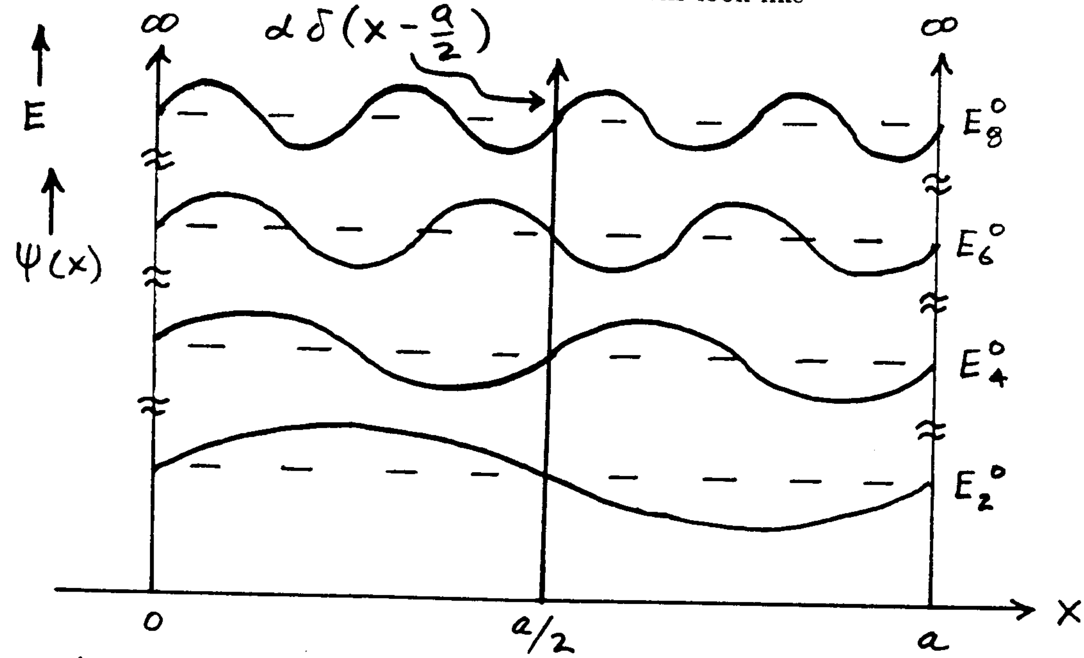
1. (c) Sketches of the first four even n wave functions look like



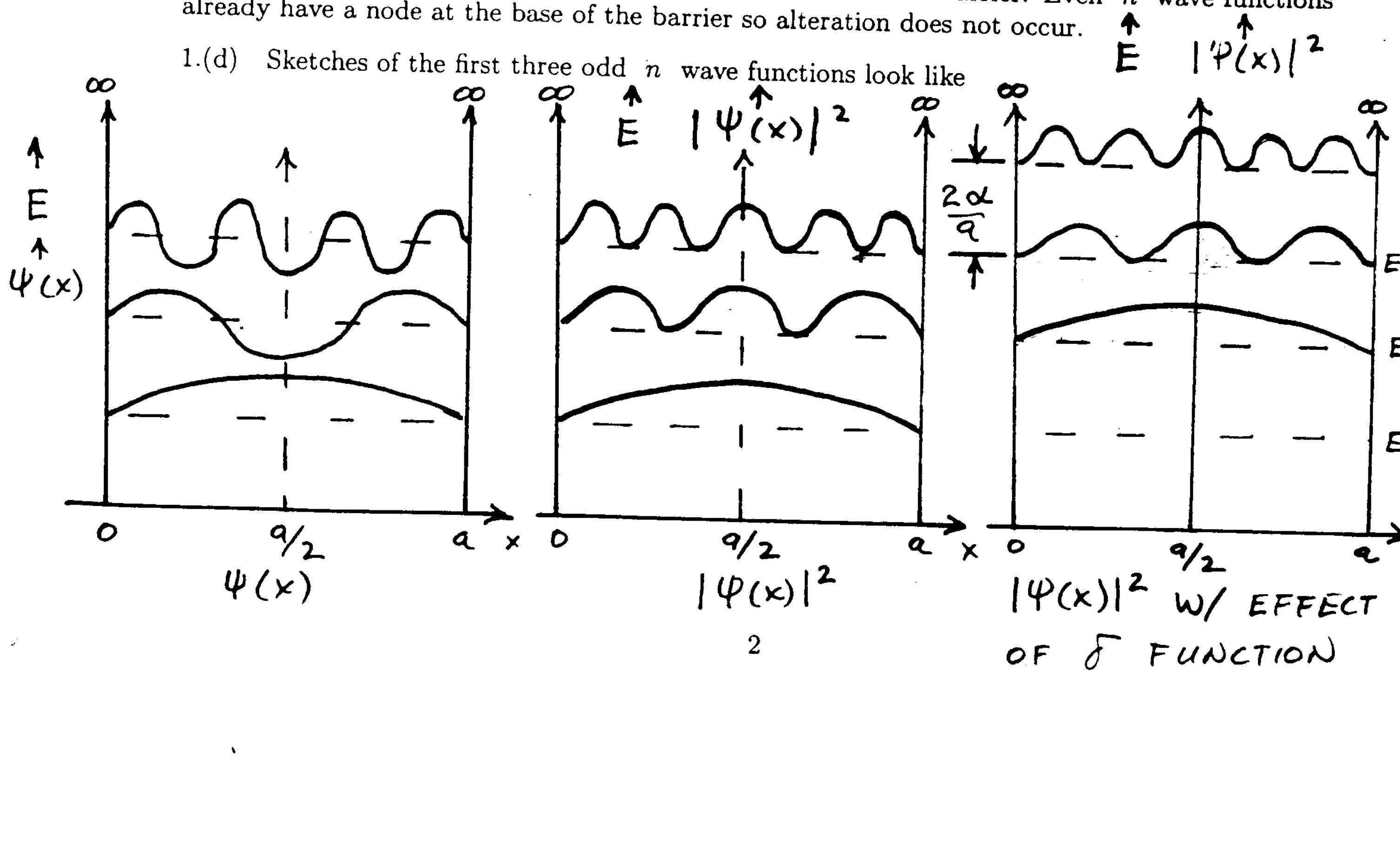
The perturbation

$$H_1 = \alpha\delta\left(x - \frac{a}{2}\right)$$

is a δ function in the center of the well, and is also illustrated in the sketches. The unperturbed wave functions for all even n wave functions have the value zero in the center of the well. The effect is $0 \cdot \text{anything} = 0$. The perturbation has nothing to perturb at that location so the entire wave function remains unperturbed. If the wave function is unperturbed, the energies are unperturbed.

Another view is the δ function is a barrier. An unperturbed wave function may be altered to go over the barrier, or it can have a node at the base of the barrier. Even n wave functions already have a node at the base of the barrier so alteration does not occur.

1.(d) Sketches of the first three odd n wave functions look like



The δ function is in a location relative to the odd n wave functions, and thus probability densities, where the wave functions are nonzero. If the δ function barrier were infinite, the wave function could not alter itself by going over so would have to adjust by having a zero value at the location of the δ function, *i.e.*, by having a node in the center. For our problem this is not the case, assuming $\alpha < 1$. The wave function is altered by the perturbation to go over the barrier. The effect is to raise the level of the energy by $2\alpha/a$ evenly all along the well. We may then, want to look at higher order corrections since we might expect more effect in the vicinity of the perturbation and less effect away from the perturbation. The first order, linear correction, however, describes a linear response, and the linear response described is the wave function goes over the barrier by raising itself to the top of the barrier and is otherwise unchanged.

1.(e) Since

$$E_n^1(x) = \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{a}x\right),$$

the maxima will occur at positions where the $\sin^2$ term is a maximum. For even n, or $n = 2, 4, 6, \ldots,$ this means

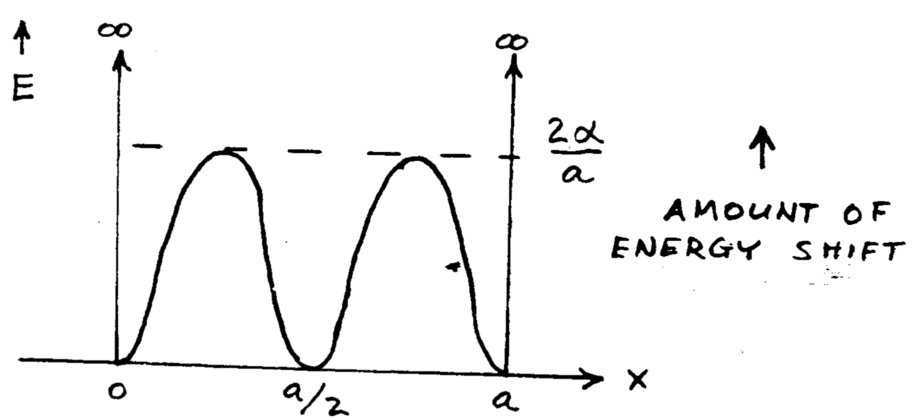
$$\frac{n\pi}{a}x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = \frac{k\pi}{2}, \quad k = 1, 3, 5, \ldots, (2n-1)$$

$$\Rightarrow \quad x = \frac{ka}{2n}, \quad k = 1, 3, 5, \ldots, (2n-1) \quad \text{for even } n.$$

A sketch of variation as a function of position for $n = 2$ is simply a sketch of

$$E_n^1(x) = \frac{2\alpha}{a}\sin^2\left(\frac{2\pi}{a}x\right),$$

which looks like



1.(f) Similar rationale is applied to find minima for odd n, or $n = 1, 3, 5, \ldots$. The minima will occur at positions where the sine is zero. This means

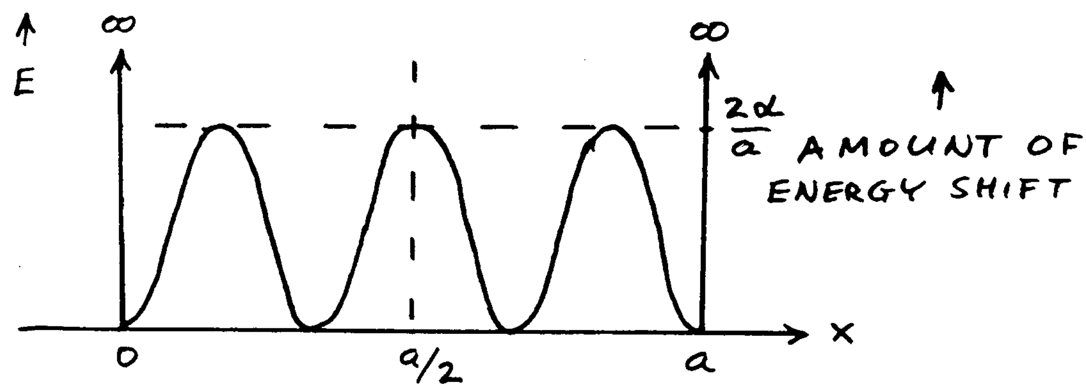
$$\frac{n\pi}{z}x = 0, \pi, 2\pi, 3\pi, \ldots = k\pi, \quad k = 0, 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad x = \frac{ka}{n}, \quad k = 0, 1, 2, 3, \ldots, n \quad \text{for odd } n.$$

A sketch of variation as a function of position for $n = 3$ is a sketch of

$$E_n^1(x) = \frac{2\alpha}{a} \sin^2\left(\frac{3\pi}{a}x\right),$$

which looks like



Minima occur at both walls, $a/3$, and $2a/3$.

GRIFFITHS 6.4

(a) Using the potential, wave functions, and perturbation of the last problem, the position space integrals are

$$<m|H_1|n> = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{m\pi x}{a}\right) \alpha\delta\left(x - \frac{a}{2}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) dx$$

$$\Rightarrow \quad <m|H_1|n> = \frac{2\alpha}{a} \int_0^a \sin\left(\frac{m\pi x}{a}\right) \delta\left(x - \frac{a}{2}\right) \sin\left(\frac{n\pi x}{a}\right) dx.$$

(b) This integral is evaluated by substituting the value of x which makes the argument of the δ function zero for the independent variable x in the integrand so

$$<m|H_1|n> = \frac{2\alpha}{a} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right).$$

$$E^1_{fs}\left(n=2,\ j=\frac{3}{2}\right)=\frac{\left(13.6\ \text{eV}/2^2\right)^2}{2\left(0.511\times10^6\ \frac{\text{eV}}{\text{c}^2}\right)\text{c}^2}\left(3-\frac{4\cdot 2}{\frac{3}{2}+\frac{1}{2}}\right)=\frac{11.56\ \text{eV}}{2\left(0.511\times10^6\right)}(3-4)$$

$$\Rightarrow\quad E^1_{fs}\left(n=2,\ j=\frac{3}{2}\right)=-1.13\times10^{-5}\ \text{eV}.$$

(c) For the $n=3$ state, there are three calculations using the same approach.

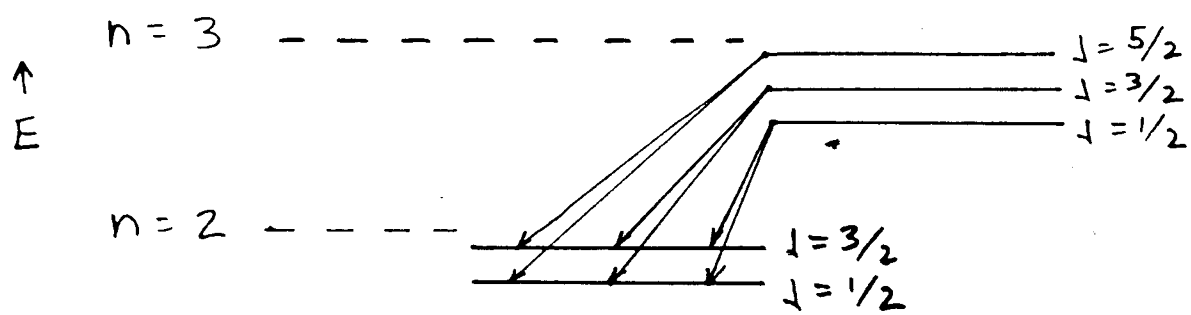
$$E^1_{fs}\left(n=3,\ j=\frac{1}{2}\right)=\frac{\left(13.6\ \text{eV}/3^2\right)^2}{2\left(0.511\times10^6\ \frac{\text{eV}}{\text{c}^2}\right)\text{c}^2}\left(3-\frac{4\cdot 3}{\frac{1}{2}+\frac{1}{2}}\right)=\frac{2.283\ \text{eV}}{2\left(0.511\times10^6\right)}(3-12)$$

$$\Rightarrow\quad E^1_{fs}\left(n=3,\ j=\frac{1}{2}\right)=-2.01\times10^{-5}\ \text{eV}.$$

$$E^1_{fs}\left(n=3,\ j=\frac{3}{2}\right)=2.234\times10^{-6}\left(3-\frac{4\cdot 3}{\frac{3}{2}+\frac{1}{2}}\right)\ \text{eV}=-6.70\times10^{-6}\ \text{eV}.$$

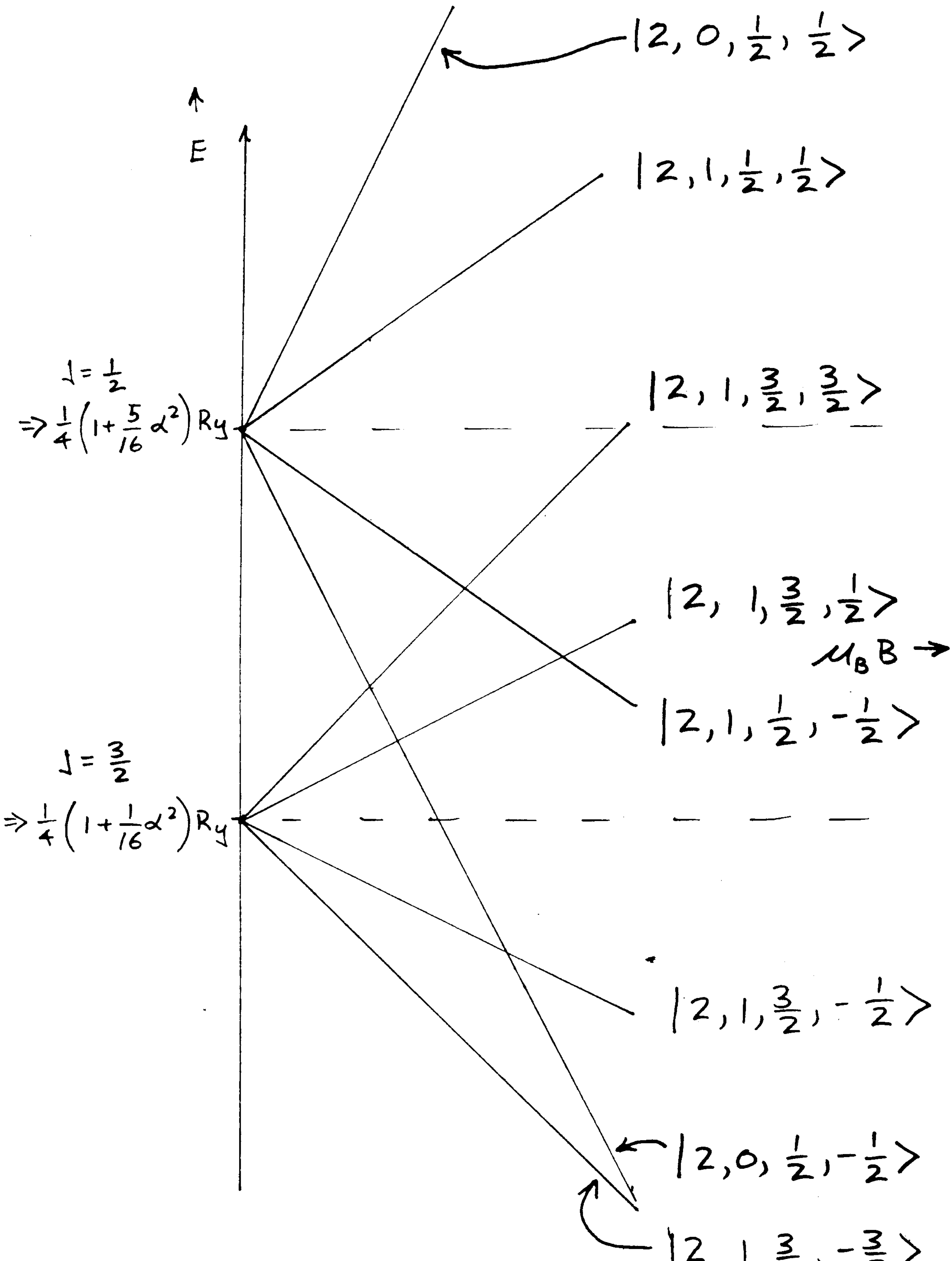
$$E^1_{fs}\left(n=3,\ j=\frac{5}{2}\right)=2.234\times10^{-6}\left(3-\frac{4\cdot 3}{\frac{5}{2}+\frac{1}{2}}\right)\ \text{eV}=-2.23\times10^{-6}\ \text{eV}.$$

(d) There are six possible transitions from state $n=3$ to $n=2$. A diagram which illustrates transitions may be helpful.



To calculate the energy differences reflected by spectral line splitting in the fine structure we will use Griffiths' equation 6.66,

$$E_{nj}=-\frac{13.6}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)\right]\ \text{eV}\quad\text{where}\quad\alpha=\frac{1}{137.036}\quad\text{and}\quad\Delta E=E_{3j_i}-E_3-(E_{3j_k}-E_2)$$



E
$|2,0,\frac{1}{2},\frac{1}{2}\rangle$
$|2,1,\frac{1}{2},\frac{1}{2}\rangle$
$J=\frac{1}{2}$
$\Rightarrow \frac{1}{4}\left(1+\frac{5}{16}\alpha^2\right)Ry$
$|2,1,\frac{3}{2},\frac{3}{2}\rangle$
$|2,1,\frac{3}{2},\frac{1}{2}\rangle$
$\mu_B B$
$|2,1,\frac{1}{2},-\frac{1}{2}\rangle$
$J=\frac{3}{2}$
$\Rightarrow \frac{1}{4}\left(1+\frac{1}{16}\alpha^2\right)Ry$
$|2,1,\frac{3}{2},-\frac{1}{2}\rangle$
$|2,0,\frac{1}{2},-\frac{1}{2}\rangle$
$|2,1,\frac{3}{2},-\frac{3}{2}\rangle$