Chapter 12

The yellow dress in his peripheral vision walked steadily toward him. "Likely going to the boutique at the corner of the block," came as a passing thought without him knowing why. Yet, as she approached more closely, she slowed perceptibly. And as she reached him, she stopped, facing him. Not knowing what he would see, he looked up feeling frumpled in the presence of style. Not a smile or a frown on her mouth or in her eyes, still it was obvious her attention was focused entirely on him. The moment fought awkwardly for words so he offered some. "Angular momentum?" fell from his lips without thought. "Edmonds," she replied without changing expression, and turning slowly, walked on without looking back[1]....

Total Angular Momentum

Quantum Mechanical Total Angular Momentum

Total angular momentum is the sum of orbital and spin angular momenta,

$$\vec{\mathcal{J}} = \vec{\mathcal{L}} + \vec{\mathcal{S}}, \tag{14-1}$$

where the vector notation simply serves to remind that there are three components in each operator. This is not an equation from classical mechanics. Recapping angular momentum developed to this point, orbital angular momentum operators are

$$\mathcal{L}^2 \text{ with eigenvalues of } l\big(l+1\big)\,\hbar^2, \text{ and } \mathcal{L}_z \text{ with eigenvalues of } m_l\hbar, \tag{14-2}$$

and $2l+1$ orbital angular momentum states. Spin angular momentum operators are

$$\mathcal{S}^2 \text{ with eigenvalues of } s\big(s+1\big)\,\hbar^2, \text{ and } \mathcal{S}_z \text{ with eigenvalues of } m_s\hbar. \tag{14-3}$$

and $2s+1$ spin angular momentum states. The reason these are the same except for the symbols used is the commutators of the components of all types of angular momentum are canonical. This is true of total angular momentum also, *i.e.*,

$$\big[\,\mathcal{J}_x,\,\mathcal{J}_y\,\big] = i\hbar\,\mathcal{J}_z, \qquad \big[\,\mathcal{J}_y,\,\mathcal{J}_z\,\big] = i\hbar\,\mathcal{J}_x, \qquad \text{and} \qquad \big[\,\mathcal{J}_z,\,\mathcal{J}_x\,\big] = i\hbar\,\mathcal{J}_y. \tag{14-4}$$

Knowing this, we anticipate the total angular momentum operators

$$\mathcal{J}^2 \text{ with eigenvalues of } j\big(j+1\big)\,\hbar^2, \text{ and } \mathcal{J}_z \text{ with eigenvalues of } m_j\hbar. \tag{14-5}$$

We will repeat enough of the ladder operator argument to support these eigenvalues for total angular momentum.

The point of the moment though, is $\vec{\mathcal{J}} = \vec{\mathcal{L}} + \vec{\mathcal{S}}$ is not a classical vector sum. We need $\big|\mathcal{J}\big|$, commonly denoted $\mathcal{J}$, such that the eigenvalue is $\sqrt{j(j+1)}\,\hbar$, and the component lengths must be $\sqrt{l(l+1)}\,\hbar$ and $\sqrt{s(s+1)}\,\hbar$. All three z–components are quantized also. Further, equation (14–1) means $\mathcal{J}_i = \mathcal{L}_i + \mathcal{S}_i$, where i is any component, x, y, or z. Any useful description

[1] Mickey Spillane *Physics in the Streets* (Publisher, Location, Year), page.

of a quantum mechanical system must satisfy all nine conditions. A classical vector sum need not satisfy the six conditions of quantization in length and quantization in the z–components.

Figure 14–1 is a semi–classical illustration of equation (14–1). The length of $\vec{\mathcal{J}}$, $\vec{\mathcal{L}}$, and $\vec{\mathcal{S}}$ are fixed, as are the z–components of each. The other two components are not fixed, however,

Figure 14 – 1. Semi – Classical Picture of Orbital and Spin Angular
Momenta Composing Total Angular Momentum

and each can be pictured as precessing such that total length and z–component are constant. In this picture, orbital, spin, and total angular momenta vectors all precess about the z–axis. Further, it would appear from the figure that spin and orbital magnetic moments will sum to the total magnetic moment. We will soon show this is a fact.

Total Angular Momentum Commutators

In addition to the commutation relations (14–4), it is necessary to consider the commutation relations between orbital and spin angular momentum. Orbital and spin angular momentum operators commute. They exist in different spaces, cannot interact with an object in another space, therefore they must commute, *i.e.*,

$$\left[\, \mathcal{L}_i,\, \mathcal{S}_j \,\right] = 0, \tag{14 – 6}$$

where i and j both represent all three components. This fact means that total angular momentum operators commute with orbital and spin angular momentum operators, or

$$\left[\, \mathcal{J}_i,\, \mathcal{L}_j \,\right] = \left[\, \mathcal{J}_i,\, \mathcal{S}_j \,\right] = 0, \tag{14 – 7}$$

for any set of i and j; x, y, or z.

Example 14–1: Show $\left[\, \mathcal{J}_x,\, \mathcal{L}_x \,\right] = 0$.

$$\begin{aligned} \left[\, \mathcal{J}_x,\, \mathcal{L}_x \,\right] &= \mathcal{J}_x\mathcal{L}_x - \mathcal{L}_x\mathcal{J}_x \\ &= \left(\mathcal{L}_x + \mathcal{S}_x\right)\mathcal{L}_x - \mathcal{L}_x\left(\mathcal{L}_x + \mathcal{S}_x\right) \\ &= \mathcal{L}_x^2 + \mathcal{S}_x\mathcal{L}_x - \mathcal{L}_x^2 - \mathcal{L}_x\mathcal{S}_x \\ &= \Big(\mathcal{L}_x^2 - \mathcal{L}_x^2\Big) + \Big(\mathcal{L}_x\mathcal{S}_x - \mathcal{L}_x\mathcal{S}_x\Big) = 0, \end{aligned}$$

where we have used the fact $\mathcal{L}_x$ and $\mathcal{S}_x$ commute to reverse the order of the operations in the last line. This procedure can be used to show any set of operators $\mathcal{J}_i$ and $\mathcal{L}_j$ or $\mathcal{S}_j$ commute.

Example 14–2: Show $\left[\,\mathcal{J}_x,\,\mathcal{J}_y\,\right] = i\hbar\,\mathcal{J}_z$ using component commutators.

Using $\left[\,\mathcal{A},\,\mathcal{B}\,\right] = -\left[\,\mathcal{B},\,\mathcal{A}\,\right]$, we can extend a commutation relation introduced in chapter 11,

$$\left[\,\mathcal{A}+\mathcal{B},\,\mathcal{C}\,\right] = \left[\,\mathcal{A},\,\mathcal{C}\,\right] + \left[\,\mathcal{B},\,\mathcal{C}\,\right],$$

$$\begin{aligned}
\Rightarrow\quad \left[\,\mathcal{A}+\mathcal{B},\,\mathcal{C}+\mathcal{D}\,\right] &= \left[\,\mathcal{A},\,\mathcal{C}+\mathcal{D}\,\right] + \left[\,\mathcal{B},\,\mathcal{C}+\mathcal{D}\,\right] \\
&= -\left[\,\mathcal{C}+\mathcal{D},\,\mathcal{A}\,\right] - \left[\,\mathcal{C}+\mathcal{D},\,\mathcal{B}\,\right] \\
&= -\left[\,\mathcal{C},\,\mathcal{A}\,\right] - \left[\,\mathcal{D},\,\mathcal{A}\,\right] - \left[\,\mathcal{C},\,\mathcal{B}\,\right] - \left[\,\mathcal{D},\,\mathcal{B}\,\right] \\
&= \left[\,\mathcal{A},\,\mathcal{C}\,\right] + \left[\,\mathcal{A},\,\mathcal{D}\,\right] + \left[\,\mathcal{B},\,\mathcal{C}\,\right] + \left[\,\mathcal{B},\,\mathcal{D}\,\right].
\end{aligned}$$

In terms of component operators,

$$\begin{aligned}
\left[\,\mathcal{J}_x,\,\mathcal{J}_y\,\right] &= \left[\,\mathcal{L}_x + \mathcal{S}_x\ \mathcal{L}_y + \mathcal{S}_y\,\right] \\
&= \left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] + \left[\,\mathcal{L}_x,\,\mathcal{S}_y\,\right] + \left[\,\mathcal{S}_x,\,\mathcal{L}_y\,\right] + \left[\,\mathcal{S}_x,\,\mathcal{S}_y\,\right] \\
&= i\hbar\mathcal{L}_z + 0 + 0 + i\hbar\mathcal{S}_z \\
&= i\hbar\big(\mathcal{L}_z + \mathcal{S}_z\big) \\
&= i\hbar\mathcal{J}_z.
\end{aligned}$$

Ladder Operator Arguments for J

So the components of total angular momentum share the same canonical commutation relations of orbital and spin angular momenta. This means the mathematics is the same with exception that the symbols are different, so we summarize vice duplicate the mathematics. We need to state the square of the total angular momentum operator commutes with all its components,

$$\left[\,\mathcal{J}^2,\,\mathcal{J}_i\,\right] = 0, \qquad (14-8)$$

where the subscript i indicates any of the three components x, y, or z. The square of total angular momentum, like orbital and spin angular momentum, is related to its components as

$$\mathcal{J}^2 = \mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2 \quad\Rightarrow\quad \mathcal{J}^2 - \mathcal{J}_z^2 = \mathcal{J}_x^2 + \mathcal{J}_y^2. \qquad (14-9)$$

The sum of the two components $\mathcal{J}_x^2 + \mathcal{J}_y^2$ would appear to factor

$$\big(\mathcal{J}_x + i\mathcal{J}_y\big)\big(\mathcal{J}_x - i\mathcal{J}_y\big).$$

Again, these operators do not commute, so this is not actually factoring. These are the raising and lowering operators for total angular momentum, specifically

$$\mathcal{J}_+ = \mathcal{J}_x + i\mathcal{J}_y, \qquad \text{and} \qquad \mathcal{J}_- = \mathcal{J}_x - i\mathcal{J}_y. \qquad (14-10)$$

Here

$$\big[\mathcal{J}^2,\,\mathcal{J}_\pm\big] = 0,$$

$$\big[\mathcal{J}_z,\,\mathcal{J}_\pm\big] = \pm\hbar\,\mathcal{J}_\pm.$$

The $\mathcal{J}^2$ and $\mathcal{J}_z$ operators will have different eigenvalues when they operate on the same basis vector, so there are two indices for each basis vector. The first index is the eigenvalue for $\mathcal{J}^2$, denoted α, and the second index is the eigenvalue for $\mathcal{J}_z$, denoted β. The form of the eigenvalue equations must be

$$\mathcal{J}^2 \big|\alpha,\, \beta> \,=\, \alpha \big|\alpha,\, \beta>, \qquad (14-11)$$

$$\mathcal{J}_z \big|\alpha,\, \beta> \,=\, \beta \big|\alpha,\, \beta> . \qquad (14-12)$$

Equations (14–11) and (14–12) are in total angular momentum state space which is the composition of orbital state space and spin state space. Using arguments similar to those of chapter 11 and chapter 13, $\mathcal{J}_\pm \big|\alpha,\, \beta>$ is an eigenvector of both $\mathcal{J}^2$ and $\mathcal{J}_z$, meaning

$$\mathcal{J}_z \big(\mathcal{J}_\pm \big|\alpha,\, \beta> \big) = \big(\beta \pm \hbar\big) \big(\mathcal{J}_\pm \big|\alpha,\, \beta> \big), \qquad (14-13)$$

and

$$\mathcal{J}^2 \big(\mathcal{J}_\pm \big|\alpha,\, \beta> \big) = \alpha \big(\mathcal{J}_\pm \big|\alpha,\, \beta> \big). \qquad (14-14)$$

The braket of eigenvectors and the operator $\mathcal{J}^2 - \mathcal{J}_z$ tells us

$$<\alpha,\, \beta \,\big|\, \mathcal{J}^2 - \mathcal{J}_z \,\big|\, \alpha,\, \beta> \,=\, \alpha - \beta^2 \geq 0 \quad \Rightarrow \quad \alpha \geq \beta^2,$$

so β is bounded by α. This means there is a maximum and minimum value of β for a given value of α, so the "ladder" has a top and a bottom. Operating with the raising operator on the eigenvector with a maximum value of β yields the zero vector, or more simply zero, so

$$\mathcal{J}_- \mathcal{J}_+ \big|\alpha,\, \beta_{\text{max}}> \,=\, 0 \quad \Rightarrow \quad \alpha = \beta_{\text{max}}^2 + \hbar \beta_{\text{max}}.$$

Similarly, operating with the lowering operator on the eigenvector with a minimum value of β yields the zero vector so

$$\mathcal{J}_+ \mathcal{J}_- \big|\alpha,\, \beta_{\text{min}}> \,=\, 0 \quad \Rightarrow \quad \alpha = \beta_{\text{min}}^2 - \hbar \beta_{\text{min}}.$$

This is true for any eigenvalue α, however, so we can equate these two equations and attain

$$\beta_{\text{max}} = -\beta_{\text{min}},$$

Since the eigenvector of $\mathcal{J}_z$, $\mathcal{J}_\pm \big|\, \alpha,\, \beta>$, is raised or lowered by $\hbar$, we assume the rungs of the ladder are separated by $\hbar$. If there are n steps between the bottom and top rungs of the ladder, there is a total separation of $n\hbar$ between the bottom and the top. From figure 14–2 we expect

$$\begin{aligned} 2\beta_{\text{max}} = n\hbar \quad \Rightarrow \quad \beta_{\text{max}} &= \frac{n\hbar}{2} \\ \Rightarrow \quad \alpha &= \beta_{\text{max}} \big(\beta_{\text{max}} + \hbar \big) \\ &= \frac{n\hbar}{2} \left(\frac{n\hbar}{2} + \hbar \right) \\ &= \hbar^2 \left(\frac{n}{2} \right) \left(\frac{n}{2} + 1 \right). \quad (14-15) \end{aligned}$$

If we let $j = n/2$,

$$\alpha = \hbar^2 j\big(j+1\big).$$

The fact $j = n/2$ vice just n is not consistent with the assumption that the rungs of the ladder are separated by $\hbar$. The rungs of the ladder are separated by $\hbar/2$ vice $\hbar$. The unit steps of $\hbar$ are due to increases/decreases in orbital angular momentum, and the increases/decreases in steps of $\hbar/2$ are due to increases/decreases in spin angular momentum. The z–component of total angular momentum is quantized in units of $\hbar/2$. We have brought to complete fruition the ladder operator argument introduced in chapter 11. Orbital and spin angular momentum are essentially subsets of total angular momentum.

The ladder operator argument is constructed from the eigenvalues of the $\mathcal{J}_z$. Since the step separation in the z–component of total angular momentum is $\hbar/2$, $\hbar/2$, must be an eigenvalue of $\mathcal{J}_z$. The ladder has maximum and minimum values of

$$\beta_{\text{min}} = -j\hbar, \qquad\qquad \beta_{\text{max}} = j\hbar,$$

where j is the total angular momentum quantum number. The eigenvalues of $\mathcal{J}_z$ are multiples of $\hbar/2$ ranging from $-j$ to j. The symbol conventionally used to denote the quantum number for which we have used a generic β is m_j, known as the **total magnetic moment quantum number**. The eigenvalue/eigenvector equation is

$$\mathcal{J}_z\big|\alpha,\, \beta > = m_j\hbar\big|\alpha,\, \beta > . \qquad (14-16)$$

Example 14–3: What are the possible results of a measurement of $\mathcal{J}^2$ for an electron?

A solitary electron is a spin 1/2 particle, so has $j = 1/2$. The orbital angular momentum is zero for a solitary electron, so spin is the only angular momentum extant in the system. The only possible result of a measurement of $\mathcal{J}^2$, then is

$$\hbar^2\, j\,\big(j+1\big) = \hbar^2\,\frac{1}{2}\,\left(\frac{1}{2}+1\right) = \hbar^2\,\frac{1}{2}\,\left(\frac{3}{2}\right) = \frac{3}{4}\hbar^2.$$

Example 14–4: What are the possible results of a measurement of $\mathcal{J}_z$ for an electron?

The possible results of a measurement of $\mathcal{S}_z$ are $-\hbar/2$ or $\hbar/2$ for a particle of spin 1/2. The reasoning is identical to example 13–10 because the only angular momentum extant in the system is intrinsic spin.

A comment is appropriate. Even though half integral values of $\hbar$ occur on the ladder, the ladder operators yield only integral differences in eigenvalues. The raising and lowering operators never guaranteed to take us everywhere, what they do is to take us to another eigenvalue. The physics of the situation is we must use an additional argument to attain the eigenvalues in between. The raising and lowering operators yield eigenvalues which are $\hbar$ apart. The reason for this is that for a given j, eigenvalues characterized by the quantum number m_j *are* $\hbar$ apart. If we start with a $\mathcal{J}_z$ eigenvalue which is half integral, say $m_j = 1/2$, operation with the raising operator will yield $m_j = 3/2$ and will bypass $m_j = 1$, because $m_j = 1$ is not a quantum number for the j which has $m_j = 1/2$. In other words, for a given system, all m_j are either integral or half integral.

Example 14–5: Show for a particle with total angular momentum characterized by j, there are $2j + 1$ possible eigenstates.

For a particle with a total angular momentum quantum number j, there is but one eigenvalue of $\mathcal{J}^2$ and the only quantity which can assume different values is m_j. The number of eigenstates is the number of different values m_j can assume. The allowed values of m_j range from $-j$ to j in integer steps. There are as many eigenstates which are negative as are positive. The quantum number j is the same as the maximum positive value as m_j. This explains the $2j$ term in $2j + 1$.

Next, is zero included? If zero is included, add one state to $2j$ for a total of $2j + 1$ possible states. Zero is not included if the total angular momentum quantum number is half integral; 1/2, 3/2, or 5/2 for instance. For instance, if $j = 5/2$, there are three possible positive values of m_j, namely $m_j = 5/2, 3/2,$ and $1/2$, and three negative values $m_j = -1/2, -3/2,$ and $-5/2$. We count six, but $2j = 2(5/2) = 5$, so adding 1 to $2j$ gives the correct number of possible eigenstates.

The total number of possible eigenstates for a particle of spin j is therefore $2j + 1$.

With equation (14–11) and total angular momentum/total magnetic moment quantum numbers, the eigenvalue/eigenvector equation for the square of total angular momentum is

$$\mathcal{J}^2 \big| j,\, m_j \big> = \hbar^2\, j\big(j+1\big) \big| j,\, m_j \big> . \qquad (14-17)$$

The eigenvalue/eigenvector equation for the z–component of total angular momentum is

$$\mathcal{J}_z \big| j,\, m_j \big> = m_j \hbar \big| j,\, m_j \big> . \qquad (14-18)$$

The magnitude of spin angular momentum is

$$\big|\, \mathcal{J}\, \big| = \sqrt{\mathcal{J}^2} = \hbar \sqrt{j(j+1)}. \qquad (14-19)$$

The normalized eigenvalue/eigenvector equations for the raising and lowering operators for spin are

$$\mathcal{J}_\pm \big| j,\, m_j \big> = \sqrt{j(j+1) - m_j(m_j \pm 1)}\, \hbar \big| j,\, m_j \pm 1 \big> . \qquad (14-20)$$

Example 14–6: Calculate $\mathcal{J}_+$ and $\mathcal{J}_-$ for all possible cases where $j = 1$.

The eigenstates of $j = 1$ are $|j,\, m_j> \rightarrow |1,\, m_j>$. Possible m_j are $1, 0, -1$, so possible eigenkets are $|1,\, 1>$, $|1,\, 0>$, and $|1, -1>$. There are three possible cases for each of the two operators, so we desire six total expressions. The three expressions for the raising operator are

$$\mathcal{J}_+ |1,\, 1> = \sqrt{1(1+1) - 1(1+1)}\, \hbar |1,\, 2> = 0|1,\, 2> = 0,$$

where we would disallow the eigenstate $|1,\, 2>$ on physical principals even if its coefficient did not vanish because $|1,\, 2>$ does not exist in a system with $j = 1$. Next

$$\mathcal{J}_+ |1,\, 0> = \sqrt{1(1+1) - 0(0+1)}\, \hbar |1,\, 1> = \sqrt{2}\, \hbar |1,\, 1>,$$

$$\mathcal{J}_+ |1, -1> = \sqrt{1(1+1) - -1(-1+1)}\, \hbar |1,\, 0> = \sqrt{2}\, \hbar |1,\, 0> .$$

The three expressions for the lowering operator are

$$\mathcal{J}_-|1,\,1\rangle = \sqrt{1(1+1)-1(1-1)}\,\hbar|1,\,0\rangle = \sqrt{2}\,\hbar|1,\,0\rangle,$$
$$\mathcal{J}_-|1,\,0\rangle = \sqrt{1(1+1)-0(0-1)}\,\hbar|1,-1\rangle = \sqrt{2}\,\hbar|1,-1\rangle,$$
$$\mathcal{J}_-|1,-1\rangle = \sqrt{1(1+1)--1(-1-1)}\,\hbar|1,-2\rangle = 0|1,-2\rangle = 0,$$

where again we would disallow $|1,-2\rangle$ on a physical basis even if the coefficient did not vanish because it is an eigenstate which is not in the system.

Matrix Forms of Total Angular Momentum Operators

The section is dominantly examples, The intent is to illustrate the use of equations (14–17) through (14–20), the raising and lowering operators, additional techniques with Dirac notation, and a method of forming matrix representations of operators. The matrix forms of components and the square of total angular momentum for each j are themselves useful results.

Example 14–7: Calculate the matrix form of $\mathcal{J}_z$ for $j = 1$.

Per example 14–6, the eigenstates of $j = 1$ are $|j,\,m_j\rangle \rightarrow |1,\,m_j\rangle$. Possible m_j are $1, 0, -1$, so possible eigenkets are $|1,\,1\rangle$, $|1,\,0\rangle$, and $|1,-1\rangle$. With this in mind, the matrix form of $\mathcal{J}_z$ is

$$\mathcal{J}_z = \begin{pmatrix} \langle 1,\,1|\,\mathcal{J}_z\,|1,\,1\rangle & \langle 1,\,1|\,\mathcal{J}_z\,|1,\,0\rangle & \langle 1,\,1|\,\mathcal{J}_z\,|1,-1\rangle \\ \langle 1,\,0|\,\mathcal{J}_z\,|1,\,1\rangle & \langle 1,\,0|\,\mathcal{J}_z\,|1,\,0\rangle & \langle 1,\,0|\,\mathcal{J}_z\,|1,-1\rangle \\ \langle 1,-1|\,\mathcal{J}_z\,|1,\,1\rangle & \langle 1,-1|\,\mathcal{J}_z\,|1,\,0\rangle & \langle 1,-1|\,\mathcal{J}_z\,|1,-1\rangle \end{pmatrix}.$$

Employing the eigenvalue/eigenvector equation $\mathcal{J}_z|j,\,m_j\rangle = m_j\hbar|j,\,m_j\rangle$, where the operator acts on the eigenket to the right within the braket,

$$\mathcal{J}_z = \begin{pmatrix} \langle 1,\,1|\,\hbar\,|1,\,1\rangle & \langle 1,\,1|\,0\,|1,\,0\rangle & \langle 1,\,1|-\hbar\,|1,-1\rangle \\ \langle 1,\,0|\,\hbar\,|1,\,1\rangle & \langle 1,\,0|\,0\,|1,\,0\rangle & \langle 1,\,0|-\hbar\,|1,-1\rangle \\ \langle 1,-1|\,\hbar\,|1,\,1\rangle & \langle 1,-1|\,0\,|1,\,0\rangle & \langle 1,-1|-\hbar\,|1,-1\rangle \end{pmatrix}$$
$$= \begin{pmatrix} \hbar\,\langle 1,\,1|1,\,1\rangle & 0 & -\hbar\,\langle 1,\,1|1,-1\rangle \\ \hbar\,\langle 1,\,0|1,\,1\rangle & 0 & -\hbar\,\langle 1,\,0|1,-1\rangle \\ \hbar\,\langle 1,-1|1,\,1\rangle & 0 & -\hbar\,\langle 1,-1|1,-1\rangle \end{pmatrix}$$

where the center column is zero because the eigenvalues of the center column are 0. All numbers, including $\hbar$ commute with the bra to the left so may be brought out of the braket. The result is the product of a number and an inner product. Since the inner product is also a number, all elements of the center column are zero times a number, which is zero.

The remaining non–zero elements are $\hbar$ times an inner product. When $\hbar$ is factored out of the matrix, we are left with elements which are inner products. These are inner products in an orthonormal basis, so

$$\langle 1,\,i|1,\,j\rangle = \delta_{i,j},$$

and writing the orthonormality condition for each non–vanishing element,

$$\mathcal{J}_z = \hbar\begin{pmatrix} \langle 1,\,1|1,\,1\rangle = 1 & 0 & \langle 1,\,1|1,-1\rangle = 0 \\ \langle 1,\,0|1,\,1\rangle = 0 & 0 & \langle 1,\,0|1,-1\rangle = 0 \\ \langle 1,-1|1,\,1\rangle = 0 & 0 & \langle 1,-1|1,-1\rangle = 1 \end{pmatrix}$$
$$= \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is the matrix form of $\mathcal{J}_z$ for $j = 1$.

There are a couple of points of instruction before we proceed. First, compare the matrix form of $\mathcal{J}_z$ for $j = 1$ to $\mathcal{L}_z$ of chapter 11. They are identical. The reason is that if spin is not considered, or if $l =$ anything and $s = 0$, the total angular momentum operator $\vec{\mathcal{J}} = \vec{\mathcal{L}} + \vec{\mathcal{S}}$ reduces to the orbital angular momentum operator. In particular, if $l = 1$ and $s = 0$, we have a case where $j = 1$. A word of caution — adding orbital and spin quantum numbers is not usually simple addition. Per introductory comments, the quantum numbers represent quantized observables, and the observables are the things which must make sense. The quantum numbers are just indices used to uniquely identify eigenstates.

Secondly, notice that since $j = 1$ in the above example, its use in each ket is redundant. Just as for spin where we uniquely identify spin up $|+\rangle$ vice $|\frac{1}{2}, \frac{1}{2}\rangle$ or $|\frac{1}{2}, +\rangle$, we need only carry the index that can vary. If $j = 1$ is understood, $|1\rangle$ uniquely identifies the state $|1, 1\rangle$, and $|0\rangle$ uniquely identifies the state $|1, 0\rangle$. We need not explicitly write all of the eigenvalues of the complete set of commuting observables to identify the eigenstate, the quantum numbers are sufficient to uniquely identify the eigenstate. This is an economy which we have already employed. And since one of the quantum numbers is determined by stating $j = 1$, we need not write that index in the ket or bra.

Example 14–8: Calculate the matrix form of $\mathcal{J}_x$ for $j = 1$.

The raising and lowering operators are both composed of $\mathcal{J}_x$ and $\mathcal{J}_y$. The strategy will be to attain expressions of $\mathcal{J}_+$ and $\mathcal{J}_-$ using equation (14–20), then substitute $\mathcal{J}_x$ and $\mathcal{J}_y$, and solve for $\mathcal{J}_x$. We have the expressions of $\mathcal{J}_+$ and $\mathcal{J}_-$ for all six possibilities for $j = 1$ from example 14–6. We select two of these,

$$\mathcal{J}_+|-1\rangle = \Big(\mathcal{J}_x + i\mathcal{J}_y\Big)|-1\rangle = \sqrt{2}\,\hbar|0\rangle,$$

$$\mathcal{J}_-|-1\rangle = \Big(\mathcal{J}_x - i\mathcal{J}_y\Big)|-1\rangle = 0,$$

because when we add them we get

$$2\mathcal{J}_x|-1\rangle = \sqrt{2}\,\hbar|0\rangle \quad \Rightarrow \quad \mathcal{J}_x|-1\rangle = \frac{\hbar}{\sqrt{2}}|0\rangle\ .$$

When we add

$$\mathcal{J}_+|1\rangle = \Big(\mathcal{J}_x + i\mathcal{J}_y\Big)|1\rangle = 0,$$

$$\mathcal{J}_-|1\rangle = \Big(\mathcal{J}_x - i\mathcal{J}_y\Big)|1\rangle = \sqrt{2}\,\hbar|0\rangle,$$

we get

$$2\mathcal{J}_x|1\rangle = \sqrt{2}\,\hbar|0\rangle \quad \Rightarrow \quad \mathcal{J}_x|1\rangle = \frac{\hbar}{\sqrt{2}}|0\rangle\ .$$

Lastly, when we add

$$\mathcal{J}_+|0\rangle = \Big(\mathcal{J}_x + i\mathcal{J}_y\Big)|0\rangle = \sqrt{2}\,\hbar|1\rangle,$$

$$\mathcal{J}_-|0\rangle = \Big(\mathcal{J}_x - i\mathcal{J}_y\Big)|0\rangle = \sqrt{2}\,\hbar|-1\rangle,$$

we attain

$$2\mathcal{J}_x|0\rangle = \sqrt{2}\,\hbar\Big(|1\rangle + |-1\rangle\Big) \quad \Rightarrow \quad \mathcal{J}_x|0\rangle = \frac{\hbar}{\sqrt{2}}\Big(|1\rangle + |-1\rangle\Big).$$

We now have an expression for $\mathcal{J}_x$ operating on each possible eigenstate. This is key in the formation of the matrix, which is

$$\mathcal{J}_x = \begin{pmatrix} <1|\,\mathcal{J}_x\,|1> & <1|\,\mathcal{J}_x\,|0> & <1|\,\mathcal{J}_x\,|-1> \\ <0|\,\mathcal{J}_x\,|1> & <0|\,\mathcal{J}_x\,|0> & <0|\,\mathcal{J}_x\,|-1> \\ <-1|\,\mathcal{J}_x\,|1> & <-1|\,\mathcal{J}_x\,|0> & <-1|\,\mathcal{J}_x\,|-1> \end{pmatrix}.$$

The operation of $\mathcal{J}_x$ on the ket in the braket leaves

$$\mathcal{J}_x = \begin{pmatrix} <1|\,\frac{\hbar}{\sqrt{2}}\,|0> & <1|\,\frac{\hbar}{\sqrt{2}}\,(|1>+|-1>) & <1|\,\frac{\hbar}{\sqrt{2}}\,|0> \\ <0|\,\frac{\hbar}{\sqrt{2}}\,|0> & <0|\,\frac{\hbar}{\sqrt{2}}\,(|1>+|-1>) & <0|\,\frac{\hbar}{\sqrt{2}}\,|0> \\ <-1|\,\frac{\hbar}{\sqrt{2}}\,|0> & <-1|\,\frac{\hbar}{\sqrt{2}}\,(|1>+|-1>) & <-1|\,\frac{\hbar}{\sqrt{2}}\,|0> \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}}\begin{pmatrix} <1|0> & <1|(|1>+|-1>) & <1|0> \\ <0|0> & <0|(|1>+|-1>) & <0|0> \\ <-1|0> & <-1|(|1>+|-1>) & <-1|0> \end{pmatrix},$$

since $\hbar/\sqrt{2}$ commutes with the bra and can be factored out of the elements of the matrix as a coefficient. Then because of the orthonormality of eigenstates,

$$\mathcal{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1+0 & 0 \\ 1 & 0+0 & 1 \\ 0 & 0+1 & 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which is the desired result. The matrix is also identical to $\mathcal{L}_x$ for orbital angular momentum discussed in chapter 11.

Using procedures similar to example 14–8,

$$\mathcal{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

for $j = 1$, which is the same as $\mathcal{L}_y$ of chapter 11. Were we to square all the component matrices and add them, we would find

$$\mathcal{J}^2 = \mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2 = 2\hbar^2\mathcal{I},$$

per the calculation done in example 11–8.

These are matrices we have seen. We need some examples calculating matrix operators we have not yet seen.

Example 14–9: Calculate $\mathcal{J}_x$ for $j = 3/2$.

Again, the strategy will be to attain expressions of $\mathcal{J}_+$ and $\mathcal{J}_-$ using equation (14–20), substitute $\mathcal{J}_x$ and $\mathcal{J}_y$, and solve for $\mathcal{J}_x$. For $j = 3/2$, there are $2\left(\frac{3}{2}\right) + 1 = 4$ possible eigenstates. For $\mathcal{J}_+$ acting on each of these, where we denote $\big|\,\frac{3}{2},\, m_j> \;\rightarrow\; |m_j>$,

$$\mathcal{J}_+\Big|\frac{3}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}+1\right)}\,\hbar\Big|\frac{5}{2}> = \sqrt{\frac{15}{4} - \frac{15}{4}}\,\hbar\Big|\frac{5}{2}> = 0\Big|\frac{5}{2}> = 0,$$

$$\mathcal{J}_+\Big|\frac{1}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)}\,\hbar\Big|\frac{3}{2}> = \sqrt{\frac{15}{4} - \frac{3}{4}}\,\hbar\Big|\frac{3}{2}> = \sqrt{3}\,\hbar\Big|\frac{3}{2}>,$$

$$\mathcal{J}_+\Big| -\frac{1}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - -\frac{1}{2}\left(-\frac{1}{2}+1\right)}\,\hbar\Big|\frac{1}{2}> = \sqrt{\frac{15}{4} + \frac{1}{4}}\,\hbar\Big|\frac{1}{2}> = 2\,\hbar\Big|\frac{1}{2}>,$$

$$\mathcal{J}_+\Big| -\frac{3}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - -\frac{3}{2}\left(-\frac{3}{2}+1\right)}\,\hbar\Big| -\frac{1}{2}> = \sqrt{\frac{15}{4} - \frac{3}{4}}\,\hbar\Big| -\frac{1}{2}> = \sqrt{3}\,\hbar\Big| -\frac{1}{2}>,$$

$$\mathcal{J}_-\Big|\frac{3}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}-1\right)}\,\hbar\Big|\frac{1}{2}> = \sqrt{\frac{15}{4} - \frac{3}{4}}\,\hbar\Big|\frac{1}{2}> = \sqrt{3}\,\hbar\Big|\frac{1}{2}>,$$

$$\mathcal{J}_-\Big|\frac{1}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\,\hbar\Big| -\frac{1}{2}> = \sqrt{\frac{15}{4} + \frac{1}{4}}\,\hbar\Big| -\frac{1}{2}> = 2\,\hbar\Big| -\frac{1}{2}>,$$

$$\mathcal{J}_-\Big| -\frac{1}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - -\frac{1}{2}\left(-\frac{1}{2}-1\right)}\,\hbar\Big| -\frac{3}{2}> = \sqrt{\frac{15}{4} - \frac{3}{4}}\,\hbar\Big| -\frac{3}{2}> = \sqrt{3}\,\hbar\Big| -\frac{3}{2}>,$$

$$\mathcal{J}_-\Big| -\frac{3}{2}> = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - -\frac{3}{2}\left(-\frac{3}{2}-1\right)}\,\hbar\Big| -\frac{5}{2}> = \sqrt{\frac{15}{4} - \frac{15}{4}}\,\hbar\Big| -\frac{5}{2}> = 0.$$

We have the expressions of $\mathcal{J}_+$ and $\mathcal{J}_-$ for all eight possibilities for $j = 3/2$. Then adding,

$$\begin{aligned} \mathcal{J}_+\Big|\frac{3}{2}> &= \Big(\mathcal{J}_x + i\mathcal{J}_y\Big)\Big|\frac{3}{2}> = 0, \\ \mathcal{J}_-\Big|\frac{3}{2}> &= \Big(\mathcal{J}_x - i\mathcal{J}_y\Big)\Big|\frac{3}{2}> = \sqrt{3}\,\hbar\Big|\frac{1}{2}>, \end{aligned}$$

$$\Rightarrow \quad 2\mathcal{J}_x\Big|\frac{3}{2}> = \sqrt{3}\,\hbar\Big|\frac{1}{2}> \quad \Rightarrow \quad \mathcal{J}_x\Big|\frac{3}{2}> = \frac{\sqrt{3}}{2}\hbar\Big|\frac{1}{2}> .$$

Adding equations where the eigenstate of the raising and lowering operator are identical, the other three cases are

$$\begin{aligned} \mathcal{J}_+\Big|\frac{1}{2}> &= \Big(\mathcal{J}_x + i\mathcal{J}_y\Big)\Big|\frac{1}{2}> = \sqrt{3}\,\hbar\Big|\frac{3}{2}>, \\ \mathcal{J}_-\Big|\frac{1}{2}> &= \Big(\mathcal{J}_x - i\mathcal{J}_y\Big)\Big|\frac{1}{2}> = 2\,\hbar\Big| -\frac{1}{2}>, \end{aligned}$$

$$\Rightarrow \quad 2\mathcal{J}_x\Big|\frac{1}{2}> = \hbar\left(\sqrt{3}\Big|\frac{3}{2}> + 2\Big| -\frac{1}{2}>\right) \quad \Rightarrow \quad \mathcal{J}_x\Big|\frac{1}{2}> = \hbar\left(\frac{\sqrt{3}}{2}\Big|\frac{3}{2}> + \Big| -\frac{1}{2}>\right),$$

$$\begin{aligned} \mathcal{J}_+\Big| -\frac{1}{2}> &= \Big(\mathcal{J}_x + i\mathcal{J}_y\Big)\Big| -\frac{1}{2}> = 2\,\hbar\Big|\frac{1}{2}>, \\ \mathcal{J}_-\Big| -\frac{1}{2}> &= \Big(\mathcal{J}_x - i\mathcal{J}_y\Big)\Big| -\frac{1}{2}> = \sqrt{3}\,\hbar\Big| -\frac{3}{2}>, \end{aligned}$$

$$\Rightarrow \quad 2\mathcal{J}_x\Big| -\frac{1}{2}> = \hbar\left(2\Big|\frac{1}{2}> + \sqrt{3}\Big| -\frac{3}{2}>\right) \quad \Rightarrow \quad \mathcal{J}_x\Big| -\frac{1}{2}> = \hbar\left(\Big|\frac{1}{2}> + \frac{\sqrt{3}}{2}\Big| -\frac{3}{2}>\right),$$

$$\mathcal{J}_+\Big| -\frac{3}{2}> \;= \Big(\mathcal{J}_x + i\mathcal{J}_y\Big)\Big| -\frac{3}{2}> = \sqrt{3}\,\hbar\Big| -\frac{1}{2}>,$$

$$\mathcal{J}_-\Big| -\frac{3}{2}> \;= \Big(\mathcal{J}_x - i\mathcal{J}_y\Big)\Big| -\frac{3}{2}> = 0,$$

$$\Rightarrow \quad 2\mathcal{J}_x\Big| -\frac{3}{2}> = \sqrt{3}\,\hbar\Big| -\frac{1}{2}> \quad \Rightarrow \quad \mathcal{J}_x\Big| -\frac{3}{2}> = \frac{\sqrt{3}}{2}\hbar\Big| -\frac{1}{2}> .$$

We are ready to form $\mathcal{J}_x$, which is

$$\mathcal{J}_x = \begin{pmatrix} <\frac{3}{2}|\,\mathcal{J}_x\,|\frac{3}{2}> & <\frac{3}{2}|\,\mathcal{J}_x\,|\frac{1}{2}> & <\frac{3}{2}|\,\mathcal{J}_x\,|-\frac{1}{2}> & <\frac{3}{2}|\,\mathcal{J}_x\,|-\frac{3}{2}> \\ <\frac{1}{2}|\,\mathcal{J}_x\,|\frac{3}{2}> & <\frac{1}{2}|\,\mathcal{J}_x\,|\frac{1}{2}> & <\frac{1}{2}|\,\mathcal{J}_x\,|-\frac{1}{2}> & <\frac{1}{2}|\,\mathcal{J}_x\,|-\frac{3}{2}> \\ <-\frac{1}{2}|\,\mathcal{J}_x\,|\frac{3}{2}> & <-\frac{1}{2}|\,\mathcal{J}_x\,|\frac{1}{2}> & <-\frac{1}{2}|\,\mathcal{J}_x\,|-\frac{1}{2}> & <-\frac{1}{2}|\,\mathcal{J}_x\,|-\frac{3}{2}> \\ <-\frac{3}{2}|\,\mathcal{J}_x\,|\frac{3}{2}> & <-\frac{3}{2}|\,\mathcal{J}_x\,|\frac{1}{2}> & <-\frac{3}{2}|\,\mathcal{J}_x\,|-\frac{1}{2}> & <-\frac{3}{2}|\,\mathcal{J}_x\,|-\frac{3}{2}> \end{pmatrix}$$

$$= \begin{pmatrix} <\frac{3}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|\frac{1}{2}> & <\frac{3}{2}|\,\hbar(\frac{\sqrt{3}}{2}\,|\frac{3}{2}> + |-\frac{1}{2}>) & <\frac{3}{2}|\,\hbar(|\frac{1}{2}> + \frac{\sqrt{3}}{2}\,|-\frac{3}{2}>) & <\frac{3}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|-\frac{1}{2}> \\ <\frac{1}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|\frac{1}{2}> & <\frac{1}{2}|\,\hbar(\frac{\sqrt{3}}{2}\,|\frac{3}{2}> + |-\frac{1}{2}>) & <\frac{1}{2}|\,\hbar(|\frac{1}{2}> + \frac{\sqrt{3}}{2}\,|-\frac{3}{2}>) & <\frac{1}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|-\frac{1}{2}> \\ <-\frac{1}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|\frac{1}{2}> & <-\frac{1}{2}|\,\hbar(\frac{\sqrt{3}}{2}\,|\frac{3}{2}> + |-\frac{1}{2}>) & <-\frac{1}{2}|\,\hbar(|\frac{1}{2}> + \frac{\sqrt{3}}{2}\,|-\frac{3}{2}>) & <-\frac{1}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|-\frac{1}{2}> \\ <-\frac{3}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|\frac{1}{2}> & <-\frac{3}{2}|\,\hbar(\frac{\sqrt{3}}{2}\,|\frac{3}{2}> + |-\frac{1}{2}>) & <-\frac{3}{2}|\,\hbar(|\frac{1}{2}> + \frac{\sqrt{3}}{2}\,|-\frac{3}{2}>) & <-\frac{3}{2}|\,\hbar\frac{\sqrt{3}}{2}\,|-\frac{1}{2}> \end{pmatrix}$$

$$= \hbar \begin{pmatrix} \frac{\sqrt{3}}{2} <\frac{3}{2}|\frac{1}{2}> & \frac{\sqrt{3}}{2} <\frac{3}{2}|\frac{3}{2}> + <\frac{3}{2}|-\frac{1}{2}> & <\frac{3}{2}|\frac{1}{2}> + \frac{\sqrt{3}}{2} <\frac{3}{2}|-\frac{3}{2}> & \frac{\sqrt{3}}{2} <\frac{3}{2}|-\frac{1}{2}> \\ \frac{\sqrt{3}}{2} <\frac{1}{2}|\frac{1}{2}> & \frac{\sqrt{3}}{2} <\frac{1}{2}|\frac{3}{2}> + <\frac{1}{2}|-\frac{1}{2}> & <\frac{1}{2}|\frac{1}{2}> + \frac{\sqrt{3}}{2} <\frac{1}{2}|-\frac{3}{2}> & \frac{\sqrt{3}}{2} <\frac{1}{2}|-\frac{1}{2}> \\ \frac{\sqrt{3}}{2} <-\frac{1}{2}|\frac{1}{2}> & \frac{\sqrt{3}}{2} <-\frac{1}{2}|\frac{3}{2}> + <-\frac{1}{2}|-\frac{1}{2}> & <-\frac{1}{2}|\frac{1}{2}> + \frac{\sqrt{3}}{2} <-\frac{1}{2}|-\frac{3}{2}> & \frac{\sqrt{3}}{2} <-\frac{1}{2}|-\frac{1}{2}> \\ \frac{\sqrt{3}}{2} <-\frac{3}{2}|\frac{1}{2}> & \frac{\sqrt{3}}{2} <-\frac{3}{2}|\frac{3}{2}> + <-\frac{3}{2}|-\frac{1}{2}> & <-\frac{3}{2}|\frac{1}{2}> + \frac{\sqrt{3}}{2} <-\frac{3}{2}|-\frac{3}{2}> & \frac{\sqrt{3}}{2} <-\frac{3}{2}|-\frac{1}{2}> \end{pmatrix},$$

and when we consider the orthonormality of eigenstates, this is

$$= \hbar \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} + 0 & 0 + 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 + 0 & 1 + 0 & 0 \\ 0 & 0 + 1 & 0 + 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 + 0 & 0 + \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{J}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

Were we to follow similar procedures except subtract vice add the expressions for identical eigenstates of the raising and lowering operators, we would attain expressions for $\mathcal{J}_y$, and would find

$$\mathcal{J}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{3}\,i & 0 & 0 \\ \sqrt{3}\,i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}\,i \\ 0 & 0 & \sqrt{3}\,i & 0 \end{pmatrix}$$

Example 14–10: Calculate $\mathcal{J}_z$ for $j = 3/2$.

$$\mathcal{J}_z = \begin{pmatrix} <\frac{3}{2}|\,\mathcal{J}_z\,|\frac{3}{2}> & <\frac{3}{2}|\,\mathcal{J}_z\,|\frac{1}{2}> & <\frac{3}{2}|\,\mathcal{J}_z\,|-\frac{1}{2}> & <\frac{3}{2}|\,\mathcal{J}_z\,|-\frac{3}{2}> \\ <\frac{1}{2}|\,\mathcal{J}_z\,|\frac{3}{2}> & <\frac{1}{2}|\,\mathcal{J}_z\,|\frac{1}{2}> & <\frac{1}{2}|\,\mathcal{J}_z\,|-\frac{1}{2}> & <\frac{1}{2}|\,\mathcal{J}_z\,|-\frac{3}{2}> \\ <-\frac{1}{2}|\,\mathcal{J}_z\,|\frac{3}{2}> & <-\frac{1}{2}|\,\mathcal{J}_z\,|\frac{1}{2}> & <-\frac{1}{2}|\,\mathcal{J}_z\,|-\frac{1}{2}> & <-\frac{1}{2}|\,\mathcal{J}_z\,|-\frac{3}{2}> \\ <-\frac{3}{2}|\,\mathcal{J}_z\,|\frac{3}{2}> & <-\frac{3}{2}|\,\mathcal{J}_z\,|\frac{1}{2}> & <-\frac{3}{2}|\,\mathcal{J}_z\,|-\frac{1}{2}> & <-\frac{3}{2}|\,\mathcal{J}_z\,|-\frac{3}{2}> \end{pmatrix}$$

and having seen the orthonormality condition applied repeatedly, we do a bit of mental gymnastics, since we can see which elements will have identical eigenstates which will eventually be evaluted as 1, and which elements have eigenstates which differ, and will eventually be evaluated as 0, so

$$\mathcal{J}_z = \begin{pmatrix} <\frac{3}{2}|\,\frac{3}{2}\hbar\,|\frac{3}{2}> & 0 & 0 & 0 \\ 0 & <\frac{1}{2}|\,\frac{1}{2}\hbar\,|\frac{1}{2}> & 0 & 0 \\ 0 & 0 & <-\frac{1}{2}|\,-\frac{1}{2}\hbar\,|-\frac{1}{2}> & 0 \\ 0 & 0 & 0 & <-\frac{3}{2}|\,\frac{3}{2}\hbar\,|-\frac{3}{2}> \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{J}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Example 14–11: Show the eigenvalues of the matrix form of $\mathcal{J}_z$ for $j = 3/2$ are consistent with the eigenvalue/eigenvector equation for $\mathcal{J}_z$ for $j = 3/2$.

The eigenvalues of the diagonal matrix are the elements on the diagonal, times the coefficient $\hbar/2$, so the eigenvalues are $3\hbar/2$, $\hbar/2$, $-\hbar/2$, and $-3\hbar/2$. The eigenvalue/eigenvector equation for $\mathcal{J}_z$ is

$$\mathcal{J}_z\big|j,\, m> = m\hbar\,\big|j,\, m>,$$

where the eigenvalue is $m\hbar$. For $j = 3/2$, the possible values of m are $3/2$, $1/2$, $-1/2$ and $-3/2$, so the eigenvalues are $3\hbar/2$, $\hbar/2$, $-\hbar/2$, and $-3\hbar/2$, and are identical to the eigenvalues from the matrix form of the operator, as they must be.

Example 14–12: Calculate the matrix form of $\mathcal{J}^2$ for $j = 3/2$.

From the previous two examples, we have $\mathcal{J}_x$, $\mathcal{J}_y$, and $\mathcal{J}_z$, and attain the desired result using

$$\mathcal{J}^2 = \mathcal{J}_x^2 + \mathcal{J}_y^2 + \mathcal{J}_z^2.$$

The squares of the components are

$$\mathcal{J}_x^2 = \mathcal{J}_x\mathcal{J}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & \sqrt{3} & 0 \\ 0 & 3+4 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 4+3 & 0 \\ 0 & \sqrt{3} & 0 & 3 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & \sqrt{3} & 0 \\ 0 & 7 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 7 & 0 \\ 0 & \sqrt{3} & 0 & 3 \end{pmatrix},$$

$$\mathcal{J}_y^2 = \mathcal{J}_y\mathcal{J}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{3}\,i & 0 & 0 \\ \sqrt{3}\,i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}\,i \\ 0 & 0 & \sqrt{3}\,i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{3}\,i & 0 & 0 \\ \sqrt{3}\,i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}\,i \\ 0 & 0 & \sqrt{3}\,i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & -\sqrt{3} & 0 \\ 0 & 3+4 & 0 & -\sqrt{3} \\ -\sqrt{3} & 0 & 4+3 & 0 \\ 0 & -\sqrt{3} & 0 & 3 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & -\sqrt{3} & 0 \\ 0 & 7 & 0 & -\sqrt{3} \\ -\sqrt{3} & 0 & 7 & 0 \\ 0 & -\sqrt{3} & 0 & 3 \end{pmatrix},$$

$$\mathcal{J}_z^2 = \mathcal{J}_z\mathcal{J}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}.$$

The sum of the squares of these components is

$$\mathcal{J}^2 = \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & \sqrt{3} & 0 \\ 0 & 7 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 7 & 0 \\ 0 & \sqrt{3} & 0 & 3 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & -\sqrt{3} & 0 \\ 0 & 7 & 0 & -\sqrt{3} \\ -\sqrt{3} & 0 & 7 & 0 \\ 0 & -\sqrt{3} & 0 & 3 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 3+3+9 & 0 & \sqrt{3}-\sqrt{3} & 0 \\ 0 & 7+7+1 & 0 & \sqrt{3}-\sqrt{3} \\ \sqrt{3}-\sqrt{3} & 0 & 7+7+1 & 0 \\ 0 & \sqrt{3}-\sqrt{3} & 0 & 3+3+9 \end{pmatrix} = \frac{15}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{15}{4}\hbar^2\mathcal{I}.$$

Example 14–13: Show the eigenvalues of the matrix form of $\mathcal{J}^2$ for $j = 3/2$ are consistent with the eigenvalue/eigenvector equation for $\mathcal{J}^2$ for $j = 3/2$.

The eigenvalue/eigenvector equation for $\mathcal{J}^2$ is

$$\mathcal{J}^2 |j,\, m> = j(j+1)\hbar^2 |j,\, m>,$$

where the eigenvalue of $\mathcal{J}^2$ is $j(j+1)\hbar^2$. For $j = 3/2$, the single possible eigenvalue is

$$j(j+1)\hbar^2 = \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2 = \frac{15}{4}\hbar^2,$$

which is consistent with a matrix of the form $15\hbar^2\mathcal{I}/4$, which has the single possible eigenvalue $15\hbar^2/4$.

The Problem of Combining Two Angular Momentum States

First, we want to clarify the problem and describe the form of the answer. The title to this section could be "The Problem of Combining Two Total Angular Momentum States," but the adjective total is understood at this point. If orbital or spin angular momentum is being addressed, it should be specified as a subset of total angular momentum.

The substance of this and the following sections is commonly called the **addition of angular momentum**. A better description is the one used; combination of angular momentum states of two or more particles. We have two individual eigenstates and want an eigenstate for the combination. In classical vector addition, the resultant must be the vector sum of the two component angular momenta. Also, the z–components must sum to that of the resultant. These constraints must be satisfied in a quantum mechanical system. A quantum mechanical system has additional constraints, the "length" and "length" of the z–component of an angular momentum "vector" is quantized. A semi–classical description is seen in figure 14–3. This could be a classical figure, other than the quantum mechanical length must be $\mathcal{J} = \sqrt{j(j+1)}\,\hbar$, in all three cases. As indicated in the figure, a resultant can be composed of an infinite number of classical vectors.

Quantum mechanically, there are a limited number of possible combinations because of quantization of "length."

Figure 14 – 2. A Semi – Classical Picture of Combination of Angular Momentum

For simplicity, we will address only two particles, though three or more particles can be addressed by a similar conceptual development.

"Addition of angular momentum" means we want to find the angular momentum eigenstates of the system or possible angular momentum eigenstates of the system given the angular momentum eigenstates or possible angular momentum of the two particles composing the system. We are looking for eigenstates where

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}_1 + \vec{\mathcal{J}}_2.$$

The first step in solving this problem is to realize each of the three angular momenta have components which are canonical, *i.e.*,

$$\left[\, \mathcal{J}_{1_i},\, \mathcal{J}_{1_j} \,\right] = i\hbar \mathcal{J}_{1_k}, \qquad \left[\, \mathcal{J}_{2_i},\, \mathcal{J}_{2_j} \,\right] = i\hbar \mathcal{J}_{2_k}, \qquad \text{and} \qquad \left[\, \mathcal{J}_i,\, \mathcal{J}_j \,\right] = i\hbar \mathcal{J}_k.$$

Each of the three angular momenta satisfies the eigenvalue/eigenvector equations,

$$\mathcal{J}_1^2 \big| j_1,\, m_1> = j_1\big(j_1+1\big)\hbar^2\big| j_1,\, m_1>, \qquad \mathcal{J}_2^2 \big| j_2,\, m_2> = j_2\big(j_2+1\big)\hbar^2\big| j_2,\, m_2>,$$

and

$$\mathcal{J}\big| j,\, m> = j\big(j+1\big)\hbar^2\big| j,\, m>,$$

and

$$\mathcal{J}_{1_z}\big| j_1,\, m_1> = m_1\hbar\big| j_1,\, m_1>, \quad \mathcal{J}_{2_z}\big| j_2,\, m_2> = m_2\hbar\big| j_2,\, m_2>, \quad \text{and} \quad \mathcal{J}_z\big| j,\, m> = m\hbar\big| j,\, m> .$$

Part of the reason to write these explicitly is to point out the notation and the difference in eigenstates. The eigenstates are denoted differently because they exist in different subspaces. Each particle and the combined system have there own total angular momentum and total magnetic moment quantum numbers. The eigenstates of the first particle are $|j_1,\, m_1>$, which is a complete set of eigenstates for the first particle. The eigenstates of the second particle are $|j_2,\, m_2>$, which is a complete set of eigenstates for the second particle. And the eigenstates of the combined system are $|j,\, m>$, which is a complete set of eigenstates for the system. The eigenstates of the combined system are described by the direct product of the components, $|j_1,\, m_1\!> \bigotimes |j_2,\, m_2\!>$, and since there are four quantum numbers, we would write this as

$$|j_1,\, m_1;\, j_2,\, m_2\!> = |j_1,\, m_1\!> \bigotimes |j_2,\, m_2\!>, \qquad (14-21)$$

which describes the subspace in which the desired eigenstates exist. This is the subspace in which the eigenstates $|j,\, m\!>$ exist. Once in the space where $|j_1,\, m_1;\, j_2,\, m_2\!>$ describe the eigenstates, the transition to $|j,\, m\!>$ is a change of basis.

Some authors use two consecutive kets vice a ket with four indices, *i.e.*,

$$|\, j_1,\, m_1\!>|\, j_2,\, m_2\!> = |\, j_1,\, m_1;\, j_2,\, m_2\!> .$$

Realize this is just a different method of specifying a combined eigenstate.

Since there are $2j_1+1$ possible eigenstates for a particle with angular momentum quantum number j_1, and there are $2j_2+1$ possible eigenstates for a particle with angular momentum quantum number j_2, there are a total of

$$\big(2j_1+1\big)\big(2j_2+1\big)$$

possible eigenstates for the combined system. There are $\big(2j_1+1\big)\big(2j_2+1\big)$ possible values of $|j_1,\, m_1;\, j_2,\, m_2\!>$ or $|j,\, m\!>$.

Besides conceptual development and notation, equation (14–21) highlights the fact the eigenstates of the two particles exist in different subspaces. That the eigenstates of the two particles exist in different subspaces means they commute, *i.e.*,

$$\big[\, \mathcal{J}_1,\, \mathcal{J}_2\, \big] = 0. \qquad (14-22)$$

Further, since they exist in different subspaces, operators of the first particle have no effect on the second, and operators of the second particle have no effect on the first. The following two examples use Dirac notation to symbolically "prove" equations (14–22) and (14–21), though the argument is completely dependent upon the fact the angular momentum states exist in different subspaces.

Example 14–14: Show $\big[\, \mathcal{J}_{1_x},\, \mathcal{J}_{2_y}\, \big] = 0.$

This example is a partial proof of equation (14–22). If the component operators were for the same particle, we would expect them to be canonical, *i.e.*, the commutator would be $i\hbar$ times the z–component operator of that particle. The subscripts indicate operators of different particles, and therefore different subspaces though, so the operators of particle one have no effect on the eigenkets of particle two, and vice versa. Symbolically,

$$\begin{aligned}
\mathcal{J}_1^2\big|j_2,\, m_2\!> &= \big(\mathcal{J}_{1_x}^2 + \mathcal{J}_{1_y}^2 + \mathcal{J}_{1_z}^2\big)\big|j_2,\, m_2\!> \\
&= \mathcal{J}_{1_x}^2\big|j_2,\, m_2\!> + \mathcal{J}_{1_y}^2\big|j_2,\, m_2\!> + \mathcal{J}_{1_z}^2\big|j_2,\, m_2\!> \\
&= \mathcal{J}_{1_x}\mathcal{J}_{1_x}\big|j_2,\, m_2\!> + \mathcal{J}_{1_y}\mathcal{J}_{1_y}\big|j_2,\, m_2\!> + \mathcal{J}_{1_z}\mathcal{J}_{1_z}\big|j_2,\, m_2\!> \\
&= \mathcal{J}_{1_x}\big(0\big) + \mathcal{J}_{1_y}\big(0\big) + \mathcal{J}_{1_z}\big(0\big) = 0+0+0 = 0.
\end{aligned}$$

Similarly, $\mathcal{J}_2^2\big|j_1,\, m_1\!>$.

There are two cases, the commutator can act on an eigenstate of particle one or particle two. If it acts on an eigenket of particle one,

$$\begin{aligned}\left[\, \mathcal{J}_{1_x},\, \mathcal{J}_{2_y} \,\right] \big| j_1,\, m_1 \!> &= \mathcal{J}_{1_x}\, \mathcal{J}_{2_y} \big| j_1,\, m_1 \!> - \mathcal{J}_{2_y}\, \mathcal{J}_{1_x} \big| j_1,\, m_1 \!> \\ &= \mathcal{J}_{1_x} \big(0\big) - \mathcal{J}_{2_y}\, \alpha \big| j_1,\, m_1 \!> \\ &= 0 - \alpha \mathcal{J}_{2_y} \big| j_1,\, m_1 \!> = 0 - \alpha\big(0\big) = 0 - 0 = 0,\end{aligned}$$

where α is a generic eigenvalue. Using similar procedures, there is an identical result for the second case,

$$\left[\, \mathcal{J}_{1_x},\, \mathcal{J}_{2_y} \,\right] \big| j_2,\, m_2 \!> = 0,$$

therefore, $\left[\, \mathcal{J}_{1_x},\, \mathcal{J}_{2_y} \,\right] = 0$. Equation (14–22) follows by repeating the same argument for the other sets of components and adding all the zeros. However it is expressed, equation (14–22) is a consequence of the fact the individual particle operators and eigenstates exist in different subspaces.

Example 14–15: Show the components of the combined angular momentum operators are canonical.

One commutator of the components of the combined angular momentum operators is

$$\left[\, \mathcal{J}_x,\, \mathcal{J}_y \,\right] = \left[\, \mathcal{J}_{1_x} + \mathcal{J}_{2_x},\, \mathcal{J}_{1_y} + \mathcal{J}_{2_y} \,\right],$$

and using the development of example 14–2 and the fact operators of different particles exist in different subspaces so have no effect on one another, this is

$$\begin{aligned}\left[\, \mathcal{J}_x,\, \mathcal{J}_y \,\right] &= \left[\, \mathcal{J}_{1_x},\, \mathcal{J}_{1_y} \,\right] + \left[\, \mathcal{J}_{1_x},\, \mathcal{J}_{2_y} \,\right] + \left[\, \mathcal{J}_{2_x},\, \mathcal{J}_{1_y} \,\right] + \left[\, \mathcal{J}_{2_x},\, \mathcal{J}_{2_y} \,\right] \\ &= i\hbar \mathcal{J}_{1_z} + 0 + 0 + i\hbar \mathcal{J}_{2_z} \\ &= i\hbar \big(\mathcal{J}_{1_z} + \mathcal{J}_{2_z} \big) = i\hbar \mathcal{J}_z.\end{aligned}$$

Other components are addressed similarly, so

$$\left[\, \mathcal{J}_i,\, \mathcal{J}_j \,\right] = i\hbar \mathcal{J}_k.$$

Clebsch–Gordan Coefficients

The fact a set of kets spans a space is expressed in the completeness relation,

$$\sum_i \big| i \!><\! i \big| = \mathcal{I}.$$

If the kets are eigenstates of angular momentum we can write this

$$\sum_{j,\, m} \big| j,\, m \!><\! j,\, m \big| = \mathcal{I},$$

or

$$\sum_{j_1,\, m_1,\, j_2,\, m_2} \big| j_1,\, m_1;\, j_2,\, m_2 \!><\! j_1,\, m_1;\, j_2,\, m_2 \big| = \mathcal{I}.$$

The last two equations are both statements in the space of the combined angular momentum of the two particles. They are, therefore, equivalent statements except they are expressed in different bases. We want $|j,\, m\rangle$ because it uses the quantum numbers of the combined angular momenta, though on occasion it may be useful to go in the other direction. A change of basis is a unitary transformation. Here, this is accomplished by multiplication by the identity, which is the matrix equivalent of multiplication by one.

$$\begin{aligned} |j,\, m\rangle &= \mathcal{I}\big|j,\, m\rangle \\ &= \sum_{j_1,\, m_1,\, j_2,\, m_2} \big|j_1,\, m_1;\, j_2,\, m_2\rangle\langle j_1,\, m_1;\, j_2,\, m_2\big|j,\, m\rangle \\ &= \sum_{j_1,\, m_1,\, j_2,\, m_2} \langle j_1,\, m_1;\, j_2,\, m_2\big|j,\, m\rangle\, \big|j_1,\, m_1;\, j_2,\, m_2\rangle \end{aligned}$$

where $\langle j_1,\, m_1;\, j_2,\, m_2\big|j,\, m\rangle$ are known as the **Clebsch–Gordan coefficients**. There are a variety of different names and notations used for these. Clebsch–Gordan coefficients is the most popular name, but they are called Wigner, vector addition, and vector coupling coefficients by some authors. The symbol C is often used to denote them, and it will often have a variety of quantum numbers as superscripts, subscripts, or arguments, such as $C^{j_1,\, j_2,\, j}_{m_1,\, m_2,\, m}$, or $C(j_1,\, m_1,\, j_2,\, m_2,\, j,\, m)$. We will use just C, and the quantum state will be apparent from the eigenstates for which it is a coefficient. If a Clebsch–Gordan coefficient is used apart from eigenstates, superscripts and subscripts are practical.

The Clebsch–Gordan coefficient, $\langle j_1,\, m_1;\, j_2,\, m_2\big|j,\, m\rangle$, is an inner product which is a number that can be complex. There is a choice of phase in defining them, and the dominant convention is to choose the phase such that they are real numbers.

If you find the eigenstate(s) $|j,\, m\rangle$ in terms of a linear combination of the $|j_1,\, m_1;\, j_2,\, m_2\rangle$, which are just a specification of the component eigenstates, you have "added" the angular momenta. The problem of combination or addition of angular momenta reduces to attaining a linear combination of appropriate eigenstates each with an appropriate coefficient. These coefficients are the Clebsch–Gordan coefficients.

A conventional assumption is that j_1 and j_2 do not vary for a given problem. In other words, we assume we know the angular momenta of the two particles which compose the system. The problem becomes simpler and more practical if you know $j_1 = 3/2$ and $j_2 = 2$, or $j_1 = 1$ and $j_2 = 1/2$ for instance. If the j_i's are fixed, the summation for the change of basis is simplified to

$$|j,\, m\rangle = \sum_{m_1,\, m_2} \langle j_1,\, m_1;\, j_2,\, m_2\big|j,\, m\rangle\, \big|j_1,\, m_1;\, j_2,\, m_2\rangle,$$

since we cannot sum over an index that does not vary.

We are going to change the notation slightly in light of this assumption. Clebsch–Gordan coefficients are commonly compiled in tables such as table 14–1. Each subsection represents a portion of a matrix. These are subsections for various possibilities of j_1 and j_2, $j_1 = 3/2$ and $j_2 = 2$, or $j_1 = 1$ and $j_2 = 1/2$ for instance. The indices for vertical columns are generally j and m. The indices for horizontal columns are generally m_1 and m_2, where j_1 and j_2 are understood because they are fixed for that portion of the table. If we write the two j_i indices first, as $|j_1,\, j_2,\, m_1,\, m_2\rangle$, it is easier to put the quantum numbers where they belong. Keeping track of indices can become formidable, so the change in the order of the quantum numbers is worthwhile for the purposes of organization.

We will construct two of the lower order subsections of the table 14–1. We intend to summarize theory, practice using tools developed to this point, and develop competency in using table 14–1 in doing so.

Example 14–16: Show $<j_1, \, j_2, \, m_1, \, m_2 | j, \, m> = 0$ unless $m = m_1 + m_2$.

Since $\mathcal{J}_z = \mathcal{J}_{1_z} + \mathcal{J}_{2_z}$,

$$\mathcal{J}_z | j, \, m> = \big(\mathcal{J}_{1_z} + \mathcal{J}_{2_z}\big) \sum_{m_1, \, m_2} <j_1, \, j_2, \, m_1, \, m_2 | j, \, m> \, |j_1, \, j_2, \, m_1, \, m_2>$$

where the indices of the summation indicate j_1 and j_2 are fixed. The inner product $< j_1, \, j_2, \, m_1, \, m_2 \big| j, \, m >$ is the Clebsch–Gordan coefficient which we will write as C, so the above equation becomes

$$\begin{aligned} \mathcal{J}_z | j, \, m> &= \big(\mathcal{J}_{1_z} + \mathcal{J}_{2_z}\big) \sum_{m_1, \, m_2} C \big| j_1, \, j_2, \, m_1, \, m_2> \\ &= \sum_{m_1, \, m_2} C \big(\mathcal{J}_{1_z} + \mathcal{J}_{2_z}\big) \big| j_1, \, j_2, \, m_1, \, m_2> \\ &= \sum_{m_1, \, m_2} C \big(\mathcal{J}_{1_z} \big| j_1, \, j_2, \, m_1, \, m_2> + \mathcal{J}_{2_z} \big| j_1, \, j_2, \, m_1, \, m_2> \big) \end{aligned}$$

$$\Rightarrow \quad m\hbar \big| j, \, m> = \sum_{m_1, \, m_2} C \big(m_1 \hbar \big| j_1, \, j_2, \, m_1, \, m_2> + m_2 \hbar \big| j_1, \, j_2, \, m_1, \, m_2> \big)$$

since $\mathcal{J}_{1_z}$ has no effect on the indices of the second particle, and $\mathcal{J}_{2_z}$ has no effect on the indices describing the first particle. Then

$$m\hbar \big| j, \, m> = \hbar \sum_{m_1, \, m_2} C \big(m_1 + m_2 \big) \big| j_1, \, j_2, \, m_1, \, m_2>$$

$$\Rightarrow \quad m \big| j, \, m> = \sum_{m_1, \, m_2} C \big(m_1 + m_2 \big) \big| j_1, \, j_2, \, m_1, \, m_2> .$$

But

$$|j, \, m> = \sum_{m_1, \, m_2} <j_1, \, j_2, \, m_1, \, m_2 \big| j, \, m> \, |j_1, \, j_2, \, m_1, \, m_2>$$

so

$$m \sum_{m_1, \, m_2} <j_1, \, j_2, \, m_1, \, m_2 \big| j, \, m> \, |j_1, \, j_2, \, m_1, \, m_2> = \sum_{m_1, \, m_2} C \big(m_1 + m_2 \big) \big| j_1, \, j_2, \, m_1, \, m_2>$$

$$\Rightarrow \quad \sum_{m_1, \, m_2} C \, m \, \big| j_1, \, j_2, \, m_1, \, m_2> \; - \sum_{m_1, \, m_2} C \big(m_1 + m_2 \big) \big| j_1, \, j_2, \, m_1, \, m_2> = 0$$

$$\Rightarrow \quad \sum_{m_1, \, m_2} C \big(m - m_1 - m_2 \big) \big| j_1, \, j_2, \, m_1, \, m_2> = 0.$$

Since the $|j_1, \, j_2, \, m_1, \, m_2>$ must be linearly independent, each coefficient must vanish individually, *i.e.*,

$$C \big(m - m_1 - m_2 \big) = 0.$$

Therefore, either $m - m_1 - m_2 = 0 \quad \Rightarrow \quad m = m_1 + m_2$, or $C = 0$.

Combination of Angular Momenta for Spin 1/2 Particles

The first system we examine is the simplest and possibly the most important, the system of two spin 1/2 particles where orbital angular momentum is zero. In other words, $j_1 = j_2 = 1/2$. The tools needed to derive the $j_1 = j_2 = 1/2$ subsection of table 14–1 are knowledge of the relationships between angular momentum and total magnetic moment quantum numbers, the normalization condition, the lowering operator, and the orthonormality condition. Since these have been previously developed, we present the derivation in the next four examples using these tools of calculation.

Example 14–17: What are the possible eigenstates of two particles with spin 1/2 for each eigenbasis?

For $j_1 = j_2 = 1/2$,

$$\big| j_1,\, j_2,\, m_1,\, m_2 \big> = \Big| \frac{1}{2},\, \frac{1}{2},\, m_1,\, m_2 \big> \,.$$

Since j_1 and j_2 do not change for this problem, we can write just

$$\big| j_1,\, j_2,\, m_1,\, m_2 \big> = \big| m_{,}\, m_2 \big> \,.$$

Remembering $|\, m \,| \le j$ for each particle, and eigenstates of m_i are separated by integral values of $\hbar$, the possible eigenstates of each particle are

$$\Big| \frac{1}{2} \big>, \qquad \Big| -\frac{1}{2} \big> \,.$$

As in chapter 13, we can refer to these as spin up and spin down. The possible eigenstates of each particle can also be denoted

$$|+\big>, \qquad |-\big> \,.$$

Simply substituting these into $|j_1, j_2, m_1, m_2\big> = \big| m_{,} m_2 \big>$, the possible eigenstates of the combined system are

$$\Big| \frac{1}{2},\, \frac{1}{2} \big>, \qquad \Big| \frac{1}{2},\, -\frac{1}{2} \big>, \qquad \Big| -\frac{1}{2},\, \frac{1}{2} \big>, \qquad \Big| -\frac{1}{2},\, -\frac{1}{2} \big>,$$

or

$$|+,\, +\big>, \qquad |+,\, -\big>, \qquad |-,\, +\big>, \qquad |-,\, -\big> \,.$$

The possible eigenstates of the $|\, j,\, m\big>$ are easier. Since $|\, m \,| \le j$ and $j_{\text{max}} = 1/2 + 1/2 = 1$, and the eigenvalues of $\mathcal{J}_z$ are separated by integral values of $\hbar$, the possible eigenstates in the $|\, j,\, m\big>$ basis are

$$\big| 1,\, 1 \big>, \qquad \big| 1,\, 0 \big>, \qquad \big| 1,\, -1 \big>, \qquad \text{and} \qquad \big| 0,\, 0 \big> \,.$$

It is important to know that there are

$$(2j_1 + 1)(2j_2 + 1) = \left(2 \left(\frac{1}{2} \right) + 1 \right) \left(2 \left(\frac{1}{2} \right) + 1 \right) = (2)(2) = 4$$

possible eigenstates in either basis, and that is apparent from example 14–17.

Example 14–18: Derive the Clebsch–Gordan coefficient for a system with two particles with spin up.

For two spin up particles, each with eigenstate $|+>$,

$$j = j_1 + j_2 = \frac{1}{2} + \frac{1}{2} = 1$$

is the maximum value of total angular momentum. The maximum value of the total magnetic moment for the system is

$$m_{\max} = j = 1.$$

For two spin up particles, this is the only possibility, so one eigenstate in the $|j, m>$ basis is $|1, 1>$. This corresponds to $|+, +>$ in the $|j_1, j_2, m_1, m_2>$ basis. There are no other possibilities to attain this $|j, m>$ state. Therefore

$$|j, m> = \sum_{m_1, m_2} C\big|m_1, m_2>$$

$$\Rightarrow \quad \big|1, 1> = C\big|+, +> .$$

Normalizing

$$1 = <1, 1\big|1, 1> = <+, +\big|C^* C\big|+, +> = C^2 <+, +\big|+, +> = C$$

because of the orthonormality of eigenstates, the innner product of identical eigenstates is one, and thus the Clebsch–Gordan coefficient is 1.

Example 14–19: Derive the Clebsch–Gordan coefficient for $|j, m> = |1, 0>$.

We can attain $|1, 0>$ from $|1, 1>$ by using the lowering operator on both sides of $|1, 1> = |+, +>$. The strategy is to use the eigenvalue/eigenvector equation on each side of the equation, so we need to keep track of the m_i's, and thus it is more practical to write the m_i's, vice $+$ and $-$ to specify the kets. Realizing $\mathcal{J}_- = \mathcal{J}_{1_-} + \mathcal{J}_{2_-}$, in general

$$\mathcal{J}_-\big|j, m> = \big(\mathcal{J}_{1_-} + \mathcal{J}_{2_-}\big)\big|m_1, m_2>$$

which for this problem is

$$\begin{aligned} \mathcal{J}_-\big|1, 1> &= \big(\mathcal{J}_{1_-} + \mathcal{J}_{2_-}\big)\Big|\frac{1}{2}, \frac{1}{2}> \\ &= \mathcal{J}_{1_-}\Big|\frac{1}{2}, \frac{1}{2}> + \mathcal{J}_{2_-}\Big|\frac{1}{2}, \frac{1}{2}> \end{aligned}$$

$$\Rightarrow \quad \sqrt{j(j+1) - m(m-1)}\,\hbar\big|1, 1-1>$$

$$= \sqrt{j_1(j_1+1) - m_1(m_1-1)}\,\hbar\Big|\frac{1}{2} - 1, \frac{1}{2}> + \sqrt{j_2(j_2+1) - m_2(m_2-1)}\,\hbar\Big|\frac{1}{2}, \frac{1}{2} - 1>$$

where $\mathcal{J}_{1_-}$ acts only on m_1 and $\mathcal{J}_{2_-}$ acts only on m_2. Explicitly

$$\Rightarrow \quad \sqrt{1(1+1) - 1(1-1)}\,\hbar\big|1, 0>$$

$$= \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\,\hbar\Big|-\frac{1}{2},\,\frac{1}{2}> + \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\,\hbar\Big|\,\frac{1}{2},\,-\frac{1}{2}>$$

$$\Rightarrow \quad \sqrt{2-0}\,\big|\,1,\,0> = \sqrt{\frac{3}{4}+\frac{1}{4}}\Big|-\frac{1}{2},\,\frac{1}{2}> + \sqrt{\frac{3}{4}+\frac{1}{4}}\Big|\,\frac{1}{2},\,-\frac{1}{2}>$$

$$\Rightarrow \quad \sqrt{2}\,\big|\,1,\,0> = \Big|-\frac{1}{2},\,\frac{1}{2}> + \Big|\,\frac{1}{2},\,-\frac{1}{2}>$$

$$\Rightarrow \quad \big|\,1,\,0> = \frac{1}{\sqrt{2}}\Big|-\frac{1}{2},\,\frac{1}{2}> + \frac{1}{\sqrt{2}}\Big|\,\frac{1}{2},\,-\frac{1}{2}>\,.$$

Using the "spin up/spin down" notation, this is

$$\big|\,1,\,0> = \frac{1}{\sqrt{2}}\Big|\,-,\,+> + \frac{1}{\sqrt{2}}\Big|\,+,\,->\,.$$

Here we have a linear combination of two eigenstates of the other basis.

Using the lowering operator on $|\,j,\,m> = |\,1,\,0>$ in the form $\mathcal{J}_- = \mathcal{J}_{1_-} + \mathcal{J}_{2_-}$, we conclude

$$\big|\,1,\,-1> = \Big|\,-\frac{1}{2},\,-\frac{1}{2}> \qquad \text{or} \qquad \big|\,1,\,-1> = \big|\,-,\,->\,.$$

The Clebsch–Gordan coefficient here is 1.

Example 14–20: What are the Clebsch–Gordan coefficients for the state $|\,j,\,m> = |\,0,\,0\,>$?

There are two ways to attain $j = 0$. These are $m_1+m_2 = 1/2-1/2$ or $m_1+m_2 = -1/2+1/2$, so $|\,0,\,0\,>$ is a linear combination of $|\,+,\,->$ and $|\,-,\,+>$. This linear combination must be orthonormal to the other three eigenstates, including $|\,1,\,0\,>$ which is itself a linear combination of these eigenstates. We can write the orthonormality condition for these two eigenstates

$$0 = <1,\,0\,|\,0,\,0> = \left(<-,\,+\,|\frac{1}{\sqrt{2}} + <+,\,-\,|\frac{1}{\sqrt{2}}\right)\left(a\big|\,+,\,-> + b\big|\,-,\,+>\right)$$

where a and b are the numbers we seek. Multiplying these

$$0 = \frac{1}{\sqrt{2}}\Big(a<-,+\cancel{|}+,-> \;+\; b<-,+\big|-,+> + a<+,-\big|+,-> + b<+,-\cancel{|}-,+>\Big),$$

where inner products with eigenstates which are not identical are zero so are struck. The other two inner products are one, so

$$a + b = 0 \qquad \Rightarrow \qquad a = -b,$$

which means

$$\big|\,0,\,0> = \; a\big|+,-> - \, a\big|-,+>\,.$$

Normalizing,

$$1 = <0,0\big|0,0> = \Big(<+,-\big|a^* - <-,+\big|a^*\Big)\Big(a\big|+,-> - a\big|-,+>\Big)$$

$$\Rightarrow \quad <+,-\big|a^*a\big|+,-> - <+,-\big|a^*a\big|-,+> - <-,+\big|a^*a\big|+,-> + <-,+\big|a^*a\big|-,+> = 1$$

$$\Rightarrow \quad |\,a\,|^2 \Big(<+,-|+,-> - \cancel{<+,-|-,+>} - \cancel{<-,+|+,->} + <-,+|-,+> \Big) = 1$$

where inner products are either zero or one, so

$$2|\,a\,|^2 = 1 \quad \Rightarrow \quad a = \frac{1}{\sqrt{2}}$$

and

$$|\,0,\,0> = \frac{1}{\sqrt{2}}|+,-> - \frac{1}{\sqrt{2}}|-,+>,$$

or

$$|\,0,\,0> = \frac{1}{\sqrt{2}}\Big|\frac{1}{2},-\frac{1}{2}> - \frac{1}{\sqrt{2}}\Big|-\frac{1}{2},\frac{1}{2}>,$$

where the m_i's are denoted specifically.

We could complete the entire matrix where $<j_1,\, j_2,\, m_1,\, m_2\,|\,j,\, m>$ are the elements. We have attained all the non–zero elements, have derived enough matrices in this chapter to illustrate their construction, so will jump to the matrix without deriving all the zeros. The $j_1 = j_2 = 1/2$ portion of table 14–1 is a condensation of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we write the matrix in tabular form, and denote indices of different quantum numbers, this is

		j	$=$	1	1	0	1
		m	$=$	1	0	0	-1
j_1	j_2	m_1	m_2				
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1

Since $j_1 = j_2 = 1/2$ for the entire table, we need not write them. This is the way Schiff[2] writes these. Most others use something like table 14–1. To read table 14–1, you need to know that the numbers at the top of each rectangle are j and m, the numbers at the left of each rectangle are m_1 and m_2, the numbers in the rectangle which is their intersection are the Clebsch–Gordan coefficients, there is a radical over each Clebsch–Gordan coefficient, and the negative signs on some coefficients go outside the radical. ...Other than that, they are straight forward....

We will shortly derive the portion of table 14–1 for $j_1 = 1$ and $j_2 = 1/2$. This is most easily addressed in a different subspace than what we have done for $j_1 = 1/2$ and $j_2 = 1/2$. Realize both cases are in a subspace of a Hilbert space. This is true of other cases where $j_1 =$ anything and $j_2 =$ anything. What we have done without calling attention to it is to take a small portion

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[2] Schiff *Quantum Mechanics* (McGraw–Hill, New York, 1968), 3rd ed., p 218.

of a bigger problem and solve the bite sized portion. This is a primary use of subspaces. It is not critical you consciously grasp the implications of working in a subspace of a Hilbert space, but to be aware that bite sized problems are often part of a larger scheme can aid understanding.

Combination of Angular Momenta for Two Particles of Angular Momentum 1 and 1/2

We are going to parallel the derivation of the last section for a two particle system where individual angular momenta quantum numbers are $j = 1$ and $j = 1/2$. Because of the similarities to the last section, some of the development is abbreviated.

Example 14–21: How many combined eigenstates are possible for a two particle system where individual angular momenta quantum numbers are $j = 1$ and $j = 1/2$?

There are

$$\big(2j_1 + 1\big)\big(2j_2 + 1\big) = \big(2\,(1) + 1\big)\Big(2\left(\frac{1}{2}\right) + 1\Big) = (3)(2) = 6$$

possible eigenstates in a basis of combined states.

Example 14–22: What are j_{max} and m_{max} in the $|\,j.\,m>$ basis?

$$j_{\text{max}} = j_{1_{\text{max}}} + j_{2_{\text{max}}} = 1 + \frac{1}{2} = \frac{3}{2},$$

and since $|\,m\,| \le j$,

$$m_{\text{max}} = j_{\text{max}} = \frac{3}{2}.$$

This leads to the conclusion that one state in the $|\,j.\,m>$ basis is $\big|\,\frac{3}{2},\,\frac{3}{2}>$. There is but one way to attain this state, and that is when the maximum values of m_i are present for each j_i. In a calculation essentially the same as example 14–18, we find the Clebsch–Gordan coefficient is one, so

$$\Big|\,\frac{3}{2},\,\frac{3}{2}> = \Big|\,1,\,\frac{1}{2},\,1,\,\frac{1}{2}> \qquad \text{or} \qquad \Big|\,\frac{3}{2},\,\frac{3}{2}> = \Big|\,1,\,\frac{1}{2}>$$

where the last eigenstate must be understood to be $|\,m_1,\,m_2>$. You can see from the last equation that you need to be clear about the meaning of the indices. It is impossible to tell whether either ket in the last equation is in the $|\,j.\,m>$ basis or the $|\,m_1,\,m_2>$ basis from the context.

Example 14–22: Derive the Clebsch–Gordan coefficients for $|\,j,\,m> = \big|\,\frac{3}{2},\,\frac{1}{2}>$.

Similar to example 14–19, we can act on both sides of the eigenstate equation $\big|\,\frac{3}{2},\,\frac{3}{2}> = \big|\,1,\,\frac{1}{2},\,1,\,\frac{1}{2}>$ with the lowering operator, where we will use only the last two indices to denote $|\,m_1,\,m_2\,>$, but will express the final answer using all four indices $|\,j_1,\,j_2,\,m_1,\,m_2\,>$. The lowering operator acting on

$$\Big|\,\frac{3}{2},\,\frac{3}{2}> = \Big|\,1,\,\frac{1}{2}>$$

is

$$\begin{aligned} \mathcal{J}_-\Big|\,\frac{3}{2},\,\frac{3}{2}> &= \big(\mathcal{J}_{1_-} + \mathcal{J}_{2_-}\big)\Big|\,1,\,\frac{1}{2}> \\ &= \mathcal{J}_{1_-}\Big|\,1,\,\frac{1}{2}> + \mathcal{J}_{2_-}\Big|\,1,\,\frac{1}{2}> \end{aligned}$$

$$\Rightarrow \quad \sqrt{j(j+1)-m(m-1)}\,\hbar \Big|\, \frac{3}{2},\, \frac{3}{2}-1>$$

$$= \sqrt{j_1(j_1+1)-m_1(m_1-1)}\,\hbar \Big|\, 1-1,\, \frac{1}{2}> \; + \; \sqrt{j_2(j_2+1)-m_2(m_2-1)}\,\hbar \Big|\, 1,\, \frac{1}{2}-1>$$

where $\mathcal{J}_{1-}$ acts only on m_1 and $\mathcal{J}_{2-}$ acts only on m_2. On the right side of the equation, we use $j_1 = 1$ and $j_2 = 1/2$, though neither is explicitly stated in the $|\, m_1,\, m_2>$ notation,

$$\Rightarrow \quad \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}-1\right)}\,\hbar \Big|\, \frac{3}{2},\, \frac{1}{2}>$$

$$= \sqrt{1(1+1)-1(1-1)}\,\hbar \Big|\, 0,\, \frac{1}{2}> \; + \; \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\,\hbar \Big|\, 1,\, -\frac{1}{2}>$$

$$\Rightarrow \quad \sqrt{\frac{15}{4}-\frac{3}{4}} \Big|\, \frac{3}{2},\, \frac{1}{2}> = \sqrt{2-0} \Big|\, 0,\, \frac{1}{2}> \; + \; \sqrt{\frac{3}{4}+\frac{1}{4}} \Big|\, 1,\, -\frac{1}{2}>$$

$$\Rightarrow \quad \sqrt{3} \Big|\, \frac{3}{2},\, \frac{1}{2}> = \sqrt{2} \Big|\, 0,\, \frac{1}{2}> \; + \Big|\, 1,\, -\frac{1}{2}>$$

$$\Rightarrow \quad \Big|\, \frac{3}{2},\, \frac{1}{2}> = \sqrt{\frac{2}{3}} \Big|\, 0,\, \frac{1}{2}> \; + \; \sqrt{\frac{1}{3}} \Big|\, 1,\, -\frac{1}{2}>,$$

or

$$\Big|\, \frac{3}{2},\, \frac{1}{2}> = \sqrt{\frac{2}{3}} \Big|\, 1,\, \frac{1}{2},\, 0,\, \frac{1}{2}> \; + \; \sqrt{\frac{1}{3}} \Big|\, 1,\, \frac{1}{2},\, 1,\, -\frac{1}{2}> .$$

Example 14–23: Derive the Clebsch–Gordan coefficients for $|\, j,\, m> = \big|\, \frac{3}{2},\, -\frac{1}{2}>$.

We are going to do one more because there are a few details in this reduction for which illustration may be useful. Employing the lowering operator,

$$\mathcal{J}_{-} \Big|\, \frac{3}{2},\, \frac{1}{2}> \; = \left(\mathcal{J}_{1-} + \mathcal{J}_{2-}\right) \left(\sqrt{\frac{1}{3}} \Big|\, 1,\, -\frac{1}{2}> + \sqrt{\frac{2}{3}} \Big|\, 0,\, \frac{1}{2}> \right)$$

$$= \mathcal{J}_{1-} \sqrt{\frac{1}{3}} \Big|\, 1,\, -\frac{1}{2}> + \mathcal{J}_{1-} \sqrt{\frac{2}{3}} \Big|\, 0,\, \frac{1}{2}> + \mathcal{J}_{2-} \sqrt{\frac{1}{3}} \Big|\, 1,\, -\frac{1}{2}> + \mathcal{J}_{2-} \sqrt{\frac{2}{3}} \Big|\, 0,\, \frac{1}{2}> .$$

Notice the coefficients from example 14–23 are retained. Notice also that four terms are created when the individual lowering operators are distributed. As the j_i to be added get larger, the number of terms becomes greater. Though each of the individual calculations is not difficult, the total number of calculations and indices of which to keep track becomes significant for $j_1 = 2$ and $j_2 = 2$, for instance. Economy becomes more important. The last equation becomes

$$\mathcal{J}_{-} \Big|\, \frac{3}{2},\, \frac{1}{2}> = \sqrt{\frac{1}{3}} \mathcal{J}_{1-} \Big|\, 1,\, -\frac{1}{2}> + \sqrt{\frac{2}{3}} \mathcal{J}_{1-} \Big|\, 0,\, \frac{1}{2}> + \sqrt{\frac{1}{3}} \mathcal{J}_{2-} \Big|\, 1,\, -\frac{1}{2}> + \sqrt{\frac{2}{3}} \mathcal{J}_{2-} \Big|\, 0,\, \frac{1}{2}> .$$

Since we have seen all the radicals formed and evaluated numerous times, this becomes

$$2 \Big|\, \frac{3}{2},\, -\frac{1}{2}> \; = \sqrt{\frac{1}{3}} \sqrt{2} \Big|\, 0,\, -\frac{1}{2}> + \sqrt{\frac{2}{3}} \sqrt{2} \Big|\, -1,\, \frac{1}{2}> + \sqrt{\frac{1}{3}} \sqrt{0} \not\Big|\, 1,\, -\frac{3}{2}> \; + \; \sqrt{\frac{2}{3}} \sqrt{1} \Big|\, 0,\, -\frac{1}{2}>$$

$$\Rightarrow \quad 2 \Big|\, \frac{3}{2},\, -\frac{1}{2}> \; = \sqrt{\frac{2}{3}} \Big|\, 0,\, -\frac{1}{2}> + 2 \sqrt{\frac{1}{3}} \Big|\, -1,\, \frac{1}{2}> + \sqrt{\frac{2}{3}} \Big|\, 0,\, -\frac{1}{2}>$$

$$\Rightarrow \quad 2 \Big|\, \frac{3}{2},\, -\frac{1}{2}> \; = 2 \sqrt{\frac{2}{3}} \Big|\, 0,\, -\frac{1}{2}> + 2 \sqrt{\frac{1}{3}} \Big|\, -1,\, \frac{1}{2}>$$

where the same state in the same basis is added algebraically. Therefore

$$\left|\frac{3}{2},\,-\frac{1}{2}\right> = \sqrt{\frac{2}{3}}\left|0,\,-\frac{1}{2}\right> + \sqrt{\frac{1}{3}}\left|-1,\,\frac{1}{2}\right>,$$

or

$$\left|\frac{3}{2},\,-\frac{1}{2}\right> = \sqrt{\frac{2}{3}}\left|1,\,\frac{1}{2},\,0,\,-\frac{1}{2}\right> + \sqrt{\frac{1}{3}}\left|1,\,\frac{1}{2},\,-1,\,\frac{1}{2}\right>$$

when all four indices are used.

We could attain

$$\left|\frac{3}{2},\,-\frac{3}{2}\right> = \left|1,\,\frac{1}{2},\,-1,\,-\frac{1}{2}\right>$$

by an additional application of the lowering operators. That gives us four of the six possible eigenstates in the combined system. We attain

$$\left|\frac{1}{2},\,\frac{1}{2}\right> = \sqrt{\frac{2}{3}}\left|1,\,\frac{1}{2},\,1,\,-\frac{1}{2}\right> - \sqrt{\frac{1}{3}}\left|1,\,\frac{1}{2},\,0,\,\frac{1}{2}\right>$$

from the orthogonality condition and then normalize it. Applying the lowering operator to this yields the sixth state which is

$$\left|\frac{1}{2},\,-\frac{1}{2}\right> = \sqrt{\frac{1}{3}}\left|1,\,\frac{1}{2},\,0,\,-\frac{1}{2}\right> - \sqrt{\frac{2}{3}}\left|1,\,\frac{1}{2},\,-1,\,\frac{1}{2}\right>.$$

The associated matrix, which spans the entire subspace but represents only a miniscule portion of the Hilbert space, is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In tabular form,

j	$=$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
m	$=$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
m_1	m_2						
1	$\frac{1}{2}$	1	0	0	0	0	0
1	$-\frac{1}{2}$	0	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	0	0	0
0	$\frac{1}{2}$	0	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0	0	0
0	$-\frac{1}{2}$	0	0	0	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	0
-1	$\frac{1}{2}$	0	0	0	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	0
-1	$-\frac{1}{2}$	0	0	0	0	0	1

which is condensed to form the 1 X 1/2 subsection of table 14–1.

Combination of Other Angular Momenta States

This section provides some specific examples of how table 14–1 is read and used.

Example 14–24: What is $|\, j,\, m\!> \,=\, |\, 2,\, 0\!>$ for a two particle system in terms of a linear combination of two individual particle states where both particles have $j = 1$?

Going to the $1 \text{ X } 1$ subsection of table 14–1, the first coefficient under the $j = 2,\ m = 0$ column is $\frac{1}{6}$ which means $\sqrt{\frac{1}{6}}$, and the row to the immediate left is $|\, m_1,\, m_2\!> \,=\, |\, 1,\, -1\!>$. The second coefficient under the $j = 2,\ m = 0$ column is also $\frac{2}{3} \Rightarrow \sqrt{\frac{2}{3}}$, and sliding to the left, this is the coefficient of $|\, m_1,\, m_2\!> \,=\, |\, 0,\, 0\!>$. The third element in this column is $\frac{1}{6} \Rightarrow \sqrt{\frac{1}{6}}$, and $\Rightarrow |\, m_1,\, m_2\!> \,=\, |\, -1,\, 1\!>$. The combined eigenstate in terms of individual eigenstate quantum numbers is

$$|\, 2,\, 0\!> \,=\, \sqrt{\frac{1}{6}}\,|\, 1,\, -1\!> + \sqrt{\frac{2}{3}}\,|\, 0,\, 0\!> + \sqrt{\frac{1}{6}}\,|\, -1,\, 1\!>$$

or including all four indices,

$$|\, 2,\, 0\!> \,=\, \sqrt{\frac{1}{6}}\,|\, 1,\, 1,\, 1,\, -1\!> + \sqrt{\frac{2}{3}}\,|\, 1,\, 1,\, 0,\, 0\!> + \sqrt{\frac{1}{6}}\,|\, 1,\, 1,\, -1,\, 1\!> .$$

Example 14–25: What is the probability of measuring $\mathcal{J}_{1_z} = \hbar$ for the state of example 14–24?

The statement $\mathcal{J}_{1_z} = \hbar \Rightarrow m_1 = 1$. There is but one component of $\big|\, 2,\, 0\!>$ for which $m_1 = 1$, which is $|\, 1,\, -1\!>$. If we measure $\mathcal{J}_{1_z} = \hbar$, then we have sampled $|\, 1,\, -1\!>$. The probability of measuring this state by postulate is

$$\begin{aligned} P(m_1 = 1) &= \Big|\, <\!1,\, -1\big| \left(\sqrt{\frac{1}{6}}\,|\, 1,\, -1\!> + \sqrt{\frac{2}{3}}\,|\, 0,\, 0\!> + \sqrt{\frac{1}{6}}\,|\, -1,\, 1\!> \right) \Big|^2 \\ &= \Big|\, <\!1,\, -1\big| \sqrt{\frac{1}{6}}\,|\, 1,\, -1\!> \Big|^2 \end{aligned}$$

because the inner product of $<\!1,\, -1|$ with the other two states is zero, so

$$P(m_1 = 1) = \frac{1}{6}\Big|\, <\!1,\, -1\big|\, 1,\, -1\!> \Big|^2 = \frac{1}{6}.$$

By the way, how important is it that we know the composition of the system? If we know $|\, j,\, m\!> \,=\, |\, 2,\, 0\!>$ without knowing $j_1 = j_2 = 1$, we cannot answer the question of example 14–25. Look at table 14–1; $|\, j,\, m\!> \,=\, |\, 2,\, 0\!>$ appears in the $1 \text{ X } 1$, $\frac{3}{2} \text{ X } \frac{1}{2}$, $2 \text{ X } 1$, $\frac{3}{2} \text{ X } \frac{3}{2}$, and $2 \text{ X } 2$ subsections. And there are many more subsections not shown that span a Hilbert space. It is critical if we actually want to answer questions like those asked in the previous two examples that we know the composition.

Example 14–26: What is the linear combination of $|\, j,\, m\!>$ for $|\, j_1,\, j_2,\, m_1,\, m_2\!> \,=\, |\, 2,\, 1,\, 1,\, 0\!>$?

The tables work in both directions. The state $|\,2,\,1,\,1,\,0>$ tells us $j_1 = 2$ and $j_2 = 1$. From the 2 X 1 subsection of table 14–1, we find the row where $|\,m_1,\,m_2> = |\,1,\,0>$. It says

$$|\,2,\,1,\,1,\,0> = \sqrt{\frac{8}{15}}\,|\,3,\,1> + \sqrt{\frac{1}{6}}\,|\,2,\,1> - \sqrt{\frac{3}{10}}\,|\,1,\,1> .$$

Example 14–27: What is the probability of measuring $\mathcal{J}^2 = 12\hbar^2$ for the state of example 14–26?

The statement $\mathcal{J}^2 = 12\hbar^2$ means $j = 3$, because the eigenvalues of $\mathcal{J}^2$ are $j(j+1)\hbar^2$. There is but one state with $j = 3$ in the linear combination of $\big|\,2,\,1,\,1,\,0>$ which is $|\,3,\,1>$. We know which inner products will be zero or one, so the probability of measuring $12\hbar$ reduces to the square of the appropriate coefficient, so

$$P\big(j(j+1) = 12\big) = \frac{8}{15}.$$
