Total Angular Momentum: Solved Problems

1. Complete the missing steps in Griffith's derivation of the singlet and triplet state for the addition of two spin 1/2 particles, namely: given two spin 1/2 particles, show that

(a) $S_- \mid 1, 0> = \sqrt{2}\hbar \mid 1, -1>$

(b) $S_- \mid 0, 0> = 0$

(c) $S_+ \mid 0, 0> = 0$

(d) $S^2 \mid 1, 1> = 2\hbar^2 \mid 1, 1>$

(e) $S^2 \mid 1, -1> = 2\hbar^2 \mid 1, 1>$

where the notation above corresponds to the compact arrow notation as follows

$$\begin{aligned} \mid 1,1> &= \uparrow\uparrow \\ \mid 1,0> &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ \mid 1,-1> &= \downarrow\downarrow \\ \mid 0,0> &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\ . \end{aligned}$$

Remember that you should operate on each particle spin state individually.

This problem is designed to remind you how single particls S^2, S_z, S_+ and S_- operators act on single particle spin states, to teach you how multiparticles S^2, S_z, S_+ and S_- operators act on multiparticle spin states, and to force you to study Griffith's *** presentation of the addition of spin 1/2 with spin 1/2 to get the three triplet states and the one singlet state. For parts a, b and c, you need to know how the two particles S_+ and S_- operators act. As we wrote in class, these operators are given by

$$S_+ = S_{1+}I_2 + I_1 S_{2+}$$
$$S_- = S_{1-}I_2 + I_1 S_{2-}.$$

So if we apply S_- to the $|1,1>$ state, we will obtain

$$\begin{aligned} S_- \,|\, 1,1> &= \left(S_-^{(1)} + S_-^{(2)}\right)(\uparrow\uparrow) \\ &= (S_- \uparrow)(I_2 \uparrow) + (I_1 \uparrow)(S_- \uparrow) \\ &= (\hbar \downarrow) \uparrow + \uparrow (\hbar \downarrow) \\ &= \sqrt{2}\hbar \frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow) \\ &= \sqrt{2}\hbar \,|\, 1, 0>\ . \end{aligned}$$

Parts a, b and c are similar. For parts d and e, you must learn how to apply the two particle S^2 operator. In class, we showed

$$\begin{aligned} S^2 &= \left(\vec{S}_1 + \vec{S}_2\right) \cdot \left(\vec{S}_1 + \vec{S}_2\right) \\ &= S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2 \\ &= S_1^2 + S_2^2 + 2S_{1x}S_{2x} + S_{1y}S_{2y} + 2S_{1z}S_{2z} \\ &= S_1^2 I_2 + I_1 S_2^2 + S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}. \end{aligned}$$

Since you know what all the single particl operators in last line of the above equation do, you can apply this expression to find what the composite S^2 operator does to the state $\mid 1,1>$ and $\mid 1,-1>$.

1(a) Apply the total S_- operator to the state $|1,0>$

$$\begin{aligned}
S_-|1,0> &= (S_{1-}I_2 + I_1 S_{2-})\,\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\
&= \frac{1}{\sqrt{2}}\,[(S_{1-}\uparrow)(I_2\downarrow) + (S_{1-}\downarrow)(I_2\uparrow) + (I_1\uparrow)(S_{2-}\downarrow) + (I_1\downarrow)(S_{2-}\uparrow)] \\
&= \frac{1}{\sqrt{2}}\,[\hbar(\downarrow\downarrow) + 0 + 0 + \hbar(\downarrow\downarrow)] \\
&= \frac{2\hbar}{\sqrt{2}}(\downarrow\downarrow)
\end{aligned}$$

$$\Rightarrow \quad S_-|1,0> = \sqrt{2}\hbar\,|1,-1>\,.$$

(b) Apply the total S_- operator to the state $|0,0>$

$$\begin{aligned}
S_-|0,0> &= (S_{1-}I_2 + I_1 S_{2-})\,\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \\
&= \frac{1}{\sqrt{2}}\,[(S_{1-}\uparrow)(I_2\downarrow) - (S_{1-}\downarrow)(I_2\uparrow) + (I_1\uparrow)(S_{2-}\downarrow) - (I_1\downarrow)(S_{2-}\uparrow)] \\
&= \frac{1}{\sqrt{2}}\,[\hbar(\downarrow\downarrow) - 0 + 0 - \hbar(\downarrow\downarrow)] \\
&= \frac{\hbar}{\sqrt{2}}\,[(\downarrow\downarrow) - (\downarrow\downarrow)]
\end{aligned}$$

$$\Rightarrow \quad S_-|0,0> = 0.$$

(c) Apply the total S_+ operator to the state $|0,0>$

$$\begin{aligned}
S_+|0,0> &= (S_{1+}I_2 + I_1 S_{2+})\,\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \\
&= \frac{1}{\sqrt{2}}\,[(S_{1+}\uparrow)(I_2\downarrow) - (S_{1+}\downarrow)(I_2\uparrow) + (I_1\uparrow)(S_{2+}\downarrow) - (I_1\downarrow)(S_{2+}\uparrow)] \\
&= \frac{1}{\sqrt{2}}\,[0 - \hbar(\uparrow\uparrow) + \hbar(\uparrow\uparrow) - 0] \\
&= \frac{\hbar}{\sqrt{2}}\,[-(\uparrow\uparrow) + (\uparrow\uparrow)]
\end{aligned}$$

$$\Rightarrow \quad S_+|0,0> = 0.$$

For parts d and e, it is very helpfull to know that

$$\begin{aligned} S^2 &= \left(\vec{S}_1 I_2 + I_1 \vec{S}_2\right) \cdot \left(\vec{S}_1 I_2 + I_1 \vec{S}_2\right) \\ &= S_1^2 I_2 + I_1 S_2^2 + S_{1+} S_{2-} + S_{1-} S_{2+} + 2 S_{1z} S2z \end{aligned}$$

(d) Apply the total S^2 operator to the state $|1,1>$

$$\begin{aligned} S^2|1,1> &= S^2(\uparrow\uparrow) \\ &= \left(S_1^2 \uparrow\right)(I_2 \uparrow) + (I_1 \uparrow)\left(S_2^2 \uparrow\right) + (S_{1+} \uparrow)(S_{2-} \uparrow) + (S_{1-} \uparrow)(S_{2+} \uparrow) + 2\,(S_{1z} \uparrow)(S_{2z} \uparrow) \\ &= \frac{3\hbar^2}{4} \uparrow\uparrow + \uparrow \frac{3\hbar^2}{4} \uparrow + (0)\,(\hbar \downarrow) + (\hbar \downarrow)\,(0) + 2\left(\frac{\hbar}{2} \uparrow\right)\left(\frac{\hbar}{2} \uparrow\right) \\ &= \frac{3\hbar^2}{2} \uparrow\uparrow + \frac{\hbar^2}{2} \uparrow\uparrow = 2\hbar^2 \uparrow\uparrow \end{aligned}$$

$$\Rightarrow \quad S^2|1,1> = 2\hbar^2\,|1,1>$$

(e) Apply the total S^2 operator to the state $|1,-1>$

$$\begin{aligned} S^2|1,-1> &= S^2(\downarrow\downarrow) \\ &= \left(S_1^2 \downarrow\right)(I_2 \downarrow) + (I_1 \downarrow)\left(S_2^2 \downarrow\right) + (S_{1+} \downarrow)(S_{2-} \downarrow) + (S_{1-} \downarrow)(S_{2+} \downarrow) + 2\,(S_{1z} \downarrow)(S_{2z} \downarrow) \\ &= \frac{3\hbar^2}{4} \downarrow\downarrow + \frac{3\hbar^2}{4} \downarrow\downarrow + (0)(0) + (0)(0) + 2\left(-\frac{\hbar}{2} \downarrow\right)\left(-\frac{\hbar}{2} \downarrow\right) \\ &= \frac{3\hbar^2}{2} \downarrow\downarrow + \frac{\hbar^2}{2} \downarrow\downarrow = 2\hbar^2 \downarrow\downarrow \end{aligned}$$

$$\Rightarrow \quad S^2|1,-1> = 2\hbar^2\,|1,-1>$$

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2. Quarks carry spin 1/2. Three quarks bind together to make a baryon, such as a proton or neutron, and two quarks bind together to make a meson, such as a pion or a kaon.

(a) What spins are possible for baryons?

(b) What spins are possible for mesons?

You should assume that the quarks are in their orbital ground state (or at rest), so that their orbital angular momentum is zero—*i.e.*, just add the spins not the j's.

For this problem, you must add three spins to calculate the possible spins of the baryons, and you must add two spins to calculate the possible spins of the mesons.

(a) First add two of the spins to get the two possible values when you add spin 1/2 (namely the singlet and triplet states which have $S = ?$ and ??). Then add ? and ?? to the third spin 1/2 to get the two possible spin states of the baryons.

(b) For the mesons, you only need to add the spins of two spin 1/2 quarks, so you'll get a singlet and a triplet with spin? and spin ??.

2(a) To calculate the spins of the baryons, with three spin 1/2 quarks, we have to add three spin 1/2's. When we add the first two, we get spin 1 and spin 0, since we get all spins in integer steps from $|1/2| + |1/2|$ to $|1/2 - 1/2|$. Then we must add these two spin values (spin 1 and spin 0) to spin 1/2. When we add spin 1 to spin 1/2, and we get spin 3/2 and spin 1/2 since we get all spins from $|1| + |1/2|$ down to $|1 - 1/2|$ in integer steps. Finally, we must add spin 0 to spin 1/2, and we get spin 1/2 since we get all spins from $|1/2| + |0|$ down to $|1/2 - 0|$ in integer steps. So the possible spin states for a ground state baryon are

$$\boxed{\frac{3}{2} \quad \text{or} \quad \frac{1}{2}.}$$

(b) To calculate the spins of the mesons, with two spin 1/2 quarks, we must add two spin 1/2's. Then we get all spins from $|1/2| + |1/2|$ down to $|1/2 - 1/2|$ in integer steps. So the possible spin states for a ground state meson are

$$\boxed{1 \quad \text{or} \quad 0.}$$

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3. A particle of spin 1 and a particle of spin 2 are at rest in a configuration such that the total spin is 3, and the z component of the system is 1. If you measure the z component of the angular momentum of the spin 2 particle, what are the possible values that you chould obtain, and with what probabilities would you obtain them?

You need to expand the composite spin $\mid 3,1>$ in terms of the individual spins. For spin 2, the possible states are $\mid 2,2>$, $\mid 2,1>$, $\mid 2,0>$, $\mid 2,-1>$ and $\mid 2,-2>$, and for spin 1, the possible states are $\mid 1,1>$, $\mid 1,0>$ and $\mid 1,-1>$. But we only want the combinations that have a z-projection equal to one, so the expansion we seek is

$$\mid 3,1> = \alpha \mid 2,2> \mid 1,-1> + \beta \mid 2,1> \mid 1,0> + \gamma \mid 2,0> \mid 1,1> .$$

So you need to find the three expansion coefficients α, β and γ in the Clebsch-Gordon tables. Then the probabilities you seek are given by $\mid \alpha \mid^2$, $\mid \beta \mid^2$ and $\mid \gamma \mid^2$.

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3. From the 2 x 1 Clebsch-Gordon table, we obtain

$$|3,1> = \sqrt{\frac{1}{15}}\,|2,2>|1,-1> + \sqrt{\frac{8}{15}}\,|2,1>|1,0> + \sqrt{\frac{6}{15}}\,|2,0>|1,1> .$$

The corresponding probabilities we seek are the squares of the coefficients of the respective components. So, we obtain

$$\boxed{P\left(S_{2z} = 2\hbar\right) = \frac{1}{15}, \quad P\left(S_{2z} = \hbar\right) = \frac{8}{15}, \quad \text{and} \quad P\left(S_{2z} = 0\right) = \frac{6}{15}.}$$

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4. Consider an electron with spin down in the ψ_{510} state in a hydrogen atom. If you measure the total angular momentum squared of the electron alone, what are the possible values that you chould obtain, and with what probabilities would you obtain them?

The orbital angular momentum of the electron is $l = 1$ (remember it is ψ_{nlm}!). And the spin of the electron is 1/2. So we seek all the possible results of adding $l = 1$ to $s = 1/2$. Since the elctron is spin down, $s_z = -1/2$ and we want the expansion coefficients for

$$|\, 1,0> \;|\, 1/2,-1/2> = \alpha \,|\, 3/2,-1/2> + \beta \,|\, 1/2,-1/2> .$$

Look these up in your favorite Clebsch-Gordon coefficient table, and then calculate $|\, \alpha \,|^2$ and $|\, \beta \,|^2$ to obtain the probabilities.

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4. The electron has intrinsic spin angular momentum 1/2, and z component $m_s = 1/2$. The system has total angular momentum $l = 1$ and z component $m_l = 0$ according to the indices on the wave function ψ_{510}. In the direct product spin space, we have the state vector $|1,0>|\frac{1}{2},-\frac{1}{2}>$. The possible values of the total angular momentum are all j's from $|l| + |s| = |1| + |1/2|$ down to $|l - s| = |1 - 1/2|$ in integer steps, so $j = 3/2$ and $1/2$. From the $1 \times 1/2$ Clebsch-Gordon table, we find

$$|1,0> \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Since the total angular momentum must be either 3/2 or 1/2, and the eigenvalues of J^2 are $j(j+1)\hbar^2$, the possible values of J^2 measurements are

$$J^2 = \frac{15}{4}\hbar^2 \quad \text{or} \quad J^2 = \frac{3}{4}\hbar^2.$$

The corresponding probabilities are the squares of the respective coefficients, so we obtain

$$\boxed{P(j = \frac{3}{2}) = P\left(J^2 = \frac{15}{4}\hbar^2\right) = \frac{2}{3}, \quad P(j = \frac{1}{2}) = P\left(J^2 = \frac{3}{4}\hbar^2\right) = \frac{1}{3}.}$$

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5. Consider adding spin 1/2 to any arbitrary spin s_2. Show that the Clebsch-Gordon coefficients for the case $s_1 = 1/2,\ s_2 =$ anything, are given by

$$\sqrt{\frac{s_2 \pm m + 1/2}{2s_2+1}}, \qquad \pm\sqrt{\frac{s_2 \mp m + 1/2}{2s_2+1}}.$$

To find the Clebsch-Gordon coefficients to add $s_1 = 1/2$ to $s_2 =$ anything, means we must find the expansion coefficient A and B in the expression

$$|\,s,m\!> = A\left|\,\frac{1}{2},\frac{1}{2}\right\rangle\left|\,s_2,\left(m-\frac{1}{2}\right)\right\rangle + B\left|\,\frac{1}{2},-\frac{1}{2}\right\rangle\left|\,s_2,\left(m+\frac{1}{2}\right)\right\rangle.$$

Now since $|\,s,m\!>$ is an eigenstate of S^2, we are seeking the eigenstates of S^2. So the solution to this problem is to apply the S^2 operator in the form

$$S^2 = S_1^2 I_2 + I_1 S_2^2 + S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}$$

to the equation above, and to carefully collect all of the therms... the algebra is fierce, but the idea is simple!
When you are all done, you should obtain

$$S^2\,|\,s,m\!> = \hbar^2\left\{A\left[\frac{3}{4} + s_2(s_2+1) + m - \frac{1}{2}\right] + B\sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}}\right\}|\,\frac{1}{2},\frac{1}{2}\rangle|\,s_2,m-\frac{1}{2}\rangle$$

$$+\hbar^2\left\{B\left[\frac{3}{4} + s_2(s_2+1) - m - \frac{1}{2}\right] + A\sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}}\right\}|\,\frac{1}{2},-\frac{1}{2}\rangle|\,s_2,m+\frac{1}{2}\rangle.$$

The next step is to use the fact that

$$S^2\,|\,s,m\!> = \hbar^2 s(s+1)\,|\,s,m\!>$$

to obtain the second equation

$$s^2\,|\,s,m\!> == \hbar^2 s(s+1)A\left|\,\frac{1}{2},\frac{1}{2}\right\rangle\left|\,s_2,\left(m-\frac{1}{2}\right)\right\rangle + B\left|\,\frac{1}{2},-\frac{1}{2}\right\rangle\left|\,s_2,\left(m+\frac{1}{2}\right)\right\rangle$$

Since the coefficients of each term must be the same, you can conclude that

$$A\left[s_2(s_2+1) - s(s+1) + m + \frac{1}{4}\right] + B\sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}} = 0$$

and

$$B\left[s_2(s_2+1) - s(s+1) - m + \frac{1}{4}\right] + A\sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}} = 0.$$

Now you need to solve for A and B in terms of s_2 and m. This is still a little bit tricky. However, once you simplify the problem by letting

$$a = s_2(s_2+1) - s(s+1) + \frac{1}{4}$$

and

$$b = \sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}}$$

to simplify the notation, you will be able to eliminate B. You should find that the solution requires that A is non-zero and that this occures when $a^2 - b^2 = m^2$. Express this in terms of the quantum numbers s_2 and m. You have to analyze the four possibilities carefully. Since $s \geq 0$, you should find that the possibilities are

$$s = s_2 \pm \frac{1}{2}.$$

Use this to solve first for a and b, then for A and B, (or just solve for A and B directly). You should obtain

$$A\sqrt{s_2 + \frac{1}{2} \mp m} = \pm B\sqrt{\left(s_2 + \frac{1}{2} \pm m\right)}.$$

Now you are almost done. All that remains is to normalize. Use the normalization condition

$$|, A\,|^2 + |\,B\,|^2 = 1$$

to obtain the expansion coefficients given in the problem.

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5. Finding the Clebsch-Gordan coefficients for the sum of $s_1 = 1/2$ with $s_2 =$ anything means that we are looking for the coefficients A and B in the expansion

$$|s, m\!> = A\left|\frac{1}{2}, \frac{1}{2}\right\rangle\left|s_2, m - \frac{1}{2}\right\rangle + B\left|\frac{1}{2}, -\frac{1}{2}\right\rangle\left|s_2, m + \frac{1}{2}\right\rangle.$$

Here $|s, m\!>$ is an eigenstate of total S^2, and we are looking for the eigenstates of S^2, so we know that

$$S^2|s, m\!> = s(s+1)\,\hbar^2\,|s, m\!> .$$

Our strategy is to expand the total S^2 operator in the usual way, namely

$$S^2 = S_1^2 I_2 + I_1 S_2^2 + S_{1+}S_{1+} + 2S_{1-}S_{2+} + 2S_{1z}S_{2z},$$

and then to apply this expansion of S^2 to the expanded state vector, to collect all of the terms, to set the two expressions equal to each other, and then to solve for A and B. So first, we apply the total S^2 operator to obtain

$$
\begin{aligned}
S^2|S_1,m\!> = A\Big\{ & S_1^2\Big|\frac{1}{2},\frac{1}{2}\Big\rangle\ I_2\Big|s_2,m-\frac{1}{2}\Big\rangle \\
& + I_2\Big|\frac{1}{2},\frac{1}{2}\Big\rangle\ S_2^2\Big|s_2,m-\frac{1}{2}\Big\rangle \\
& + S_{1+}\Big|\frac{1}{2},\frac{1}{2}\Big\rangle\ S_{2-}\Big|s_2,m-\frac{1}{2}\Big\rangle \\
& + S_{1-}\Big|\frac{1}{2},\frac{1}{2}\Big\rangle\ S_{2+}\Big|s_2,m-\frac{1}{2}\Big\rangle \\
& + 2S_{1z}\Big|\frac{1}{2},\frac{1}{2}\Big\rangle\ S_{2z}\Big|s_2,m-\frac{1}{2}\Big\rangle\Big\} \\
+B\Big\{ & S_1^2\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle\ I_2\Big|s_2,m+\frac{1}{2}\Big\rangle \\
& + I_2\frac{1}{2},-\frac{1}{2}\Big\rangle\ S_2^2\Big|s_2,m+\frac{1}{2}\Big\rangle \\
& + S_{1+}\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle\ S_{2-}\Big|s_2,m+\frac{1}{2}\Big\rangle \\
& + S_{1-}\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle\ S_{2-}\Big|s_2,m+\frac{1}{2}\Big\rangle \\
& + 2S_{1z}\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle\ S_{2z}\Big|s_2,m+\frac{1}{2}\Big\rangle\Big\} \\
= {} & s(s+1)\hbar^2|s,m\!>\ .
\end{aligned}
$$

Next, we must evaluate the action of each of the single-spin operators, using the usual relations:

$$S_+\,|s,m\!> = \sqrt{s(s+1)-m(m+1)}\,\hbar\,|s,m+1\!>$$

$$S_-\,|s,m\!> = \sqrt{s(s+1)-m(m-1)}\,\hbar\,|s,m-1\!>$$

$$S^2\,|s,m\!> = s(s+1)\,\hbar^2\,|s,m\!>$$

$$S_z\,|s,m\!> = m\,\hbar\,|s,m\!>,$$

and then we must collect all of the terms:

$$\begin{aligned}
S^2|s,m> = A\Bigg\{&\frac{3}{4}\hbar^2\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|s_2,m-\frac{1}{2}\right\rangle \\
&+\left|\frac{1}{2},\frac{1}{2}\right\rangle s_2(s_2+1)\hbar^2\left|s_2,m-\frac{1}{2}\right\rangle \\
&+0 \\
&+\hbar\left|\frac{1}{2},\frac{1}{2}\right\rangle\sqrt{s_2(s_2+1)-(m-\frac{1}{2})(m+\frac{1}{2})}\,\hbar\left|s_2,m+\frac{1}{2}\right\rangle \\
&+2\,\frac{\hbar}{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle(m-\frac{1}{2})\hbar\left|s_2,m-\frac{1}{2}\right\rangle\Bigg\} \\
+B\Bigg\{&\frac{3}{4}\hbar^2\left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|s_2,m+\frac{1}{2}\right\rangle \\
&+\left|\frac{1}{2},-\frac{1}{2}\right\rangle s_2(s_2+1)\hbar^2\left|s_2,m+\frac{1}{2}\right\rangle \\
&+\hbar\left|\frac{1}{2},\frac{1}{2}\right\rangle\sqrt{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})}\,\hbar\left|s_2,m-\frac{1}{2}\right\rangle \\
&+0 \\
&+2\left(-\frac{\hbar}{2}\right)\left|\frac{1}{2},-\frac{1}{2}\right\rangle(m+\frac{1}{2})\hbar\left|s_2,m+\frac{1}{2}\right\rangle\Bigg\} \\
= s(s+1)\hbar^2&\left\{A\left\{\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|s_2,m-\frac{1}{2}\right\rangle\right\}+B\left\{\left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|s_2,m+\frac{1}{2}\right\rangle\right\}\right\}
\end{aligned}$$

Collecting the terms, we have

$$\begin{aligned}
&A\left\{\frac{3}{4}\hbar^2+s_2(s_2+1)\hbar^2+(m-\frac{1}{2})\hbar^2\right\}\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|s_2,m-\frac{1}{2}\right\rangle \\
&\quad+A\sqrt{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})}\,\hbar^2\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|s_2,m+\frac{1}{2}\right\rangle \\
&+B\left\{\frac{3}{4}\hbar^2+s_2(s_2+1)\hbar^2-(m+\frac{1}{2})\hbar^2\right\}\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|s_2,m+\frac{1}{2}\right\rangle \\
&\quad+B\sqrt{s_2(s_2+1)-(m-\frac{1}{2})(m+\frac{1}{2})}\,\hbar^2\left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|s_2,m-\frac{1}{2}\right\rangle \\
&=A\left\{s(s+1)\hbar^2\right\}\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|s_2,m-\frac{1}{2}\right\rangle+B\left\{s(s+1)\hbar^2\right\}\left|\frac{1}{2},\frac{-1}{2}\right\rangle\left|s_2,m+\frac{1}{2}\right\rangle
\end{aligned}$$

Since the coefficient of the $|s_2, m \pm 1/2>$ terms on the left hand side must be equal to the corresponding coefficient of the $|s_2, m \pm 1/2>$ term on the right hand side, we have two equations

$$A\left[s_2(s_2+1)+m+\frac{1}{4}\right]+B\sqrt{s_2(s_2+1)-m^2+\frac{1}{4}}=s(s+1)A$$

and

$$B\left[s_2(s_2+1)-m+\frac{1}{4}\right]+A\sqrt{s_2(s_2+1)-m^2+\frac{1}{4}}=s(s+1)B.$$

Rearranging the terms in these two equations, we find

$$A\left[s_2(s_2+1) - s(s+1) + m + \frac{1}{4}\right] + B\sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}} = 0$$

and

$$B\left[s_2(s_2+1) - s(s+1) - m + \frac{1}{4}\right] + A\sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}} = 0.$$

To simplify the notation, let

$$a = s_2(s_2+1) - s(s+1) + \frac{1}{4}, \quad \text{and} \quad b = \sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}}$$

and solve for A. The two equations become

$$A(a+m) + Bb = 0 \quad \text{and} \quad B(a-m) + Ab = 0$$

$$\Rightarrow \quad B = -\frac{Ab}{a-m} \quad \Rightarrow \quad A(a+m) - \frac{Ab^2}{a-m} = 0$$

$$\Rightarrow \quad A(a^2 - m^2) - Ab^2 = 0 \quad \Rightarrow \quad A(a^2 - m^2 - b^2) = 0.$$

This can occur for non-zero A if $a^2 - m^2 - b^2 = 0 \quad \Rightarrow \quad a^2 - b^2 = m^2$. Re-expressing this in terms of the quantum numbers yields

$$\left[s_2(s_2+1) - s(s+1) + \frac{1}{4}\right]^2 - s_2(s_2+1) + m^2 - \frac{1}{4} = m^2$$

$$\Rightarrow \quad \left[s_2(s_2+1) - s(s+1) + \frac{1}{4}\right]^2 = s_2^2 + s_2 + \frac{1}{4} = \left(s_2 + \frac{1}{2}\right)^2$$

$$\Rightarrow \quad s_2(s_2+1) - s(s+1) + \frac{1}{4} = \pm\left(s_2 + \frac{1}{2}\right)$$

$$\Rightarrow \quad s(s+1) = s_2(s_2+1) \mp \left(s_2 + \frac{1}{2}\right) + \frac{1}{4}$$

$$\Rightarrow \quad s^2 + s + \frac{1}{4} = s_2(s_2+1) \mp \left(s_2 + \frac{1}{2}\right) + \frac{1}{2}$$

$$\Rightarrow \quad \left(s + \frac{1}{2}\right)^2 = s_2(s_2+1) \mp \left(s_2 + \frac{1}{2}\right) + \frac{1}{2}.$$

If you carefully solve this equation, subject to the condition that $s \geq 0$, you will find

$$\boxed{s = s_2 \pm \frac{1}{2}.}$$

The solution is tricky because of all the signs you must keep track of, but this is the formal way to prove that the sum of spin 1/2 with spin s_2 yields total spin $s_2 + 1/2$ and $s_2 - 1/2$. It's okay if you just assumed this. The next step is to find A and B. In terms of the earlier defined a and b, we have

$$\begin{aligned}
a &= s_2(s_2+1) - s(s+1) + \frac{1}{4} = s_2^2 + s_2 - s^2 - s + \frac{1}{4} \\
&= s_2^2 + s_2 - \left(s_2 \pm \frac{1}{2}\right)^2 - \left(s_2 \pm \frac{1}{2}\right) + \frac{1}{4} \\
&= s_2^2 + s_2 - \left(s_2^2 \pm s_2 + \frac{1}{4}\right) - s_2 \mp \frac{1}{2} + \frac{1}{4} \\
&= s_2^2 + s_2 - s_2^2 \mp s_2 - \frac{1}{4} - s_2 \mp \frac{1}{2} + \frac{1}{4} \\
&= \mp\left(s_2 + \frac{1}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
b &= \sqrt{s_2(s_2+1) - m^2 + \frac{1}{4}} = \sqrt{s_2^2 + s_2 + \frac{1}{4} - m^2} \\
&= \sqrt{\left(s_2 + \frac{1}{2}\right)^2 - m^2} = \sqrt{\left(s_2 + \frac{1}{2} + m\right)\left(s_2 + \frac{1}{2} - m\right)}.
\end{aligned}$$

Remembering that $A(a+m) = -Bb$, we find

$$\begin{aligned}
& A\left[\mp\left(s_2 + \frac{1}{2}\right) + m\right] = -B\sqrt{\left(s_2 + \frac{1}{2} + m\right)\left(s_2 + \frac{1}{2} - m\right)} \\
\Rightarrow\quad & \mp A\left[\left(s_2 + \frac{1}{2}\right) \mp m\right] = -B\sqrt{\left(s_2 + \frac{1}{2} + m\right)\left(s_2 + \frac{1}{2} - m\right)} \\
\Rightarrow\quad & A\sqrt{s_2 + \frac{1}{2} \mp m} = \pm B\sqrt{\left(s_2 + \frac{1}{2} \pm m\right)}\;.
\end{aligned}$$

We are finally almost done. We can use the normalization condition

$$|A|^2 + |B|^2 = 1,$$

to find A and B separately, instead of their ratio.

Eliminating $|B|^2$, we find

$$\begin{aligned}
& |A|^2 + |A|^2\left(\frac{s_2 + \frac{1}{2} \mp m}{s_2 + \frac{1}{2} \pm m}\right) = 1 \\
\Rightarrow\quad & |A|^2 \frac{s_2 + \frac{1}{2} \pm m + s_2 + \frac{1}{2} \mp m}{s_2 + \frac{1}{2} \pm m} = 1 \\
\Rightarrow\quad & |A|^2 \frac{2s_2 + 1}{s_2 + \frac{1}{2} \pm m} = 1
\end{aligned}$$

$$\boxed{\Rightarrow \quad A = \sqrt{\frac{s_2 + \frac{1}{2} \pm m}{2s_2 + 1}}.}$$

And, finally, solving for B, we find

$$\begin{aligned} B &= A \frac{\sqrt{s_2 + \frac{1}{2} \mp m}}{\sqrt{s_2 + \frac{1}{2} \pm m}} \\ &= \sqrt{\frac{s_2 + \frac{1}{2} \pm m}{2s_2 + 1}} \frac{\sqrt{s_2 + \frac{1}{2} \mp m}}{\sqrt{s_2 + \frac{1}{2} \pm m}} \end{aligned}$$

$$\boxed{\Rightarrow \quad B = \sqrt{\frac{s_2 + \frac{1}{2} \mp m}{2s_2 + 1}}.}$$

Whew!!!

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6. Consider an electron in a hydrogen atom which is in the combined spin and position state

$$R_{21}\left(\sqrt{\frac{1}{3}}Y_{1,0}\,\chi_+ + \sqrt{\frac{2}{3}}Y_{1,1}\,\chi_-\right).$$

If you measure the following quantitites, what are the possible values that you chould obtain, and with what probabilities would you obtain them?

(a) L^2

(b) L_z

(c) S^2

(d) S_z

(e) J^2

(f) J_z

Now consider two of the continuous probability distributions:

(g) If you measure the position of the electron without determining its spin, what is the probability density as a function of $r,\ \theta,$ and ϕ?

(h) If you measure both the z component of the spin and the distance from the origin, what is the probability density for finding the electron with spin up at radius r?

Since the wave function $\Psi(r,\theta,\phi)$,

$$R_{21}\left(\sqrt{\frac{1}{3}}Y_{1,0}\chi_+ + \sqrt{\frac{2}{3}}Y_{1,1}\chi_-\right) = R_{21}\left[\sqrt{\frac{1}{3}}Y_{1,0}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{2}{3}}Y_{1,1}\begin{pmatrix}0\\1\end{pmatrix}\right] = R_{21}\begin{pmatrix}\sqrt{\frac{1}{3}}Y_{1,0}\\ \sqrt{\frac{2}{3}}Y_{1,1}\end{pmatrix}$$

is normalized, the probabilities we seek are given by

$$P(\Lambda=\alpha) = \left|\ <\Lambda=\alpha\right|\ \Psi(r,\theta,\phi)>\left|^2 = \left|\int_{-\infty}^{\infty}\Lambda_\alpha^\dagger(r,\theta,\phi)\Psi(r,\theta,\phi)d^3r\right|^2.$$

So we must calculate the integral

$$= \left|\int_{-\infty}^{\infty}(\Lambda_{\alpha 1}^*,\Lambda_{\alpha 2}^*)\begin{pmatrix}\sqrt{\frac{1}{3}}Y_{1,0}\\ \sqrt{\frac{2}{3}}Y_{1,1}\end{pmatrix}d^3r\right|^2$$

$$= \int_{-\infty}^{\infty}|\,R_{21}\,|^2 r^2 dr\left|\int_{-\infty}^{\infty}\left(\sqrt{\frac{1}{3}}\Lambda_{\alpha 1}^* Y_{1,0} + \sqrt{\frac{2}{3}}\Lambda_{\alpha 2}^* Y_{1,1}\right)d\Omega\right|^2$$

There is no radial dependence in Λ because we are only seeking the results of spin and orbital angular momentum measurements. The spin dependence is contained in the spinors??, and the orbital angular momentum information is contained in the spherical harmonics. Remember that the spherical harmonics are orthonormal, so we have

$$\int Y_{l,m^*} Y_{l',m'} d\Omega = \delta_{ll'}\delta_{mm'},$$

and spherical harmonics which are not identical will not contribute. Let's verify that the initial wavefunction is normalized

$$< \Psi(r,\theta,\phi) \Big| \Psi(r,\theta,\phi) > = \int_{-\infty}^{\infty} |R_{2,1}|^2 r^2 dr \int_{-\infty}^{\infty} \left(\sqrt{\frac{1}{3}} Y_{1,0}^*, \sqrt{\frac{2}{3}} Y_{1,1}^* \right) \begin{pmatrix} \sqrt{\frac{1}{3}} Y_{1,0} \\ \sqrt{\frac{2}{3}} Y_{1,1} \end{pmatrix} d\Omega$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{3} Y_{1,0}^* Y_{1,0} + \frac{2}{3} Y_{1,1}^* Y_{1,1} \right) d\Omega = \frac{1}{3} \int_{-\infty}^{\infty} Y_{1,0}^* Y_{1,0} d\Omega + \frac{2}{3} \int_{-\infty}^{\infty} Y_{1,1}^* Y_{1,1} d\Omega = \frac{1}{3} + \frac{2}{3} = 1.$$

The integral over all solid angles of any spherical harmonic times its complex conjugate is 1, and the integral over all solid angles of any spherical harmonic times any other spherical harmonic is 0.

(a) The only $Y_{l,m}$'s that appear in the wavefunction have $l = 1$. So the probability that you must calculate is $P(l = 1)$.

(b) The only $Y_{l,m}$'s that appear in the wavefunction have $m = 0$ and 1. So the probabilities you must calculate are $P(m_l = 0)$ and $P(m_l = 1)$. You should obtain 1/3 and 2/3 respectively.

(c) The only spin that appears in the wavefunction has $s = 1/2$. So the probability that you must calculate is $P(s = 1/2)$.

(d) The only m_s values that appear in the wavefunction are $m_s = +1/2$ and $m_s = -1/2$. So the probabilitis that you must calculate are $P(m_s = +1/2)$ and $P(m_s = -1/2)$. You should obtain 1/3 and 2/3, respectively.

To work parts e and f, the probabilities of measuring J^2 and J_z, we must add the orbital angular momentum and the spin angular momentum to obtain the total angular momentum. Translating the $l,\ m$ and $s,\ m_s$ information in initial wavefunction into Dirac notation we find

$$\Psi = \sqrt{\frac{1}{3}} \big| 1,0 \big\rangle_l \Big| \frac{1}{2}, \frac{1}{2} \Big\rangle_s + \sqrt{\frac{2}{3}} \big| 1,1 \big\rangle_l \Big| \frac{1}{2}, -\frac{1}{2} \Big\rangle_s.$$

Use the Clebsch-Gordon tables to rewrite the wavefunction in terms of the $|\, j, m_j >$ states. You should obtain

$$\Psi = 2\frac{\sqrt{2}}{3} \Big| \frac{3}{2}, \frac{1}{2} \Big\rangle + \frac{1}{3} \Big| \frac{1}{2}, \frac{1}{2} \Big\rangle.$$

(e) The only total angular momentum that apper are $j = 3/2$ and 1/2. So the probabilities that you must calculate are $P(j = 3/2)$ and $P(j = 1/2)$. You should obtain 8/9 and 1/9, respectively.

(f) The only z-component of the total angular momentum that appears in the wavefunction is $m_j = 1/2$. So the probability that you must calculate is $P(m_j = 1/2)$.

Finally, calculate the probabilities that you will obtain j and the probabilities that you will obtain m_j by using thie expansion of the wavefunction in thje $|\, j, m_j >$ basis.

(g) The probability density in spherical coordinate is given by $|\,\Psi(r,\theta,\phi)\,|^2$, where

$$\Big| \Psi(r,\theta,\phi) \Big|^2 = < \Psi(r,\theta,\phi) \Big| \Psi(r,\theta,\phi) > .$$

You will need the radial wave function, which is given in Table 4.6 on page 141 of Griffiths. You should obtain

$$\Big| \Psi(r,\theta,\phi) \Big|^2 = \frac{1}{96\pi} \frac{r^2}{a^5} e^{-r/a}.$$

(h) The possibility of measuring spin up corresponds to the portion with χ_+, so the probability density for measuring spin up at a radius r is given by

$$\Big| \Psi(r,\theta,\phi) \Big|^2_{\text{spin up}} = R_{21}^* R_{21} \sqrt{\frac{1}{3}} Y_{1,0}^* \sqrt{\frac{1}{3}} Y_{1,0}.$$

It is easiest to use the radial result from part g and the orthonormality property of the spherical harmonics. You should obtain

$$\Big| \Psi(r,\theta,\phi) \Big|^2 = \frac{1}{72} \frac{r^2}{a^5} e^{-r/a}.$$

6(a) Since both the spherical harmonics comprising our wave function have $l = 1$, the only possibility for a measurement of L^2 is $l(l+1)\hbar^2 = 2\hbar^2$.

$$\Rightarrow \quad P(l = 1) = P(L^2 = 2\hbar^2) = 1.$$

(b) The possibilities for measurements of L_z are the eigenvalues of L_z which are $m\hbar$. The m's in our wave function are 0 and 1, so possibilities are $0\hbar$ and $\hbar$. The associated probabilities are

$$P(L_z = 0) = \int_{-\infty}^{\infty} d\Omega \left| (1,\ 0) \begin{pmatrix} \sqrt{\frac{1}{3}}\, Y_{1,0} \\ \sqrt{\frac{2}{3}}\, Y_{1,1} \end{pmatrix} \right|^2 = \int_{-\infty}^{\infty} d\Omega \left| \sqrt{\frac{1}{3}}\, Y_{1,0} \right|^2 = \frac{1}{3} \int_{-\infty}^{\infty} d\Omega \, |Y_{1,0}|^2 = \frac{1}{3}$$

$$P(L_z = \hbar) = \int_{-\infty}^{\infty} d\Omega \left| (0,\ 1) \begin{pmatrix} \sqrt{\frac{1}{3}}\, Y_{1,0} \\ \sqrt{\frac{2}{3}}\, Y_{1,1} \end{pmatrix} \right|^2 = \int_{-\infty}^{\infty} d\Omega \left| \sqrt{\frac{2}{3}}\, Y_{1,1} \right|^2 = \frac{2}{3} \int_{-\infty}^{\infty} d\Omega \, |Y_{1,1}|^2 = \frac{2}{3}.$$

$$\Rightarrow \quad P(L_z = 0\,\hbar) = \frac{1}{3}, \qquad P(L_z = \hbar) = \frac{2}{3}.$$

(c) The possibilities for measurements of S^2 are the eigenvalues of S^2 which are $s(s+1)\hbar^2$. For the electron, the intrinsic spin angular momentum $s = 1/2$, so the only possibility is $S^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = 3\hbar^2/4$.

$$\Rightarrow \quad P\left(S = \frac{1}{2}\right) = P\left(S^2 = \frac{3}{4}\hbar^2\right) = 1.$$

(d) The possibilities for measurements of S_z are the eigenvalues of S_z which are $m_s\hbar$. Note that spin up corresponds to the $Y_{1,0}$ portion of the wave function, and spin down corresponds to the $Y_{1,1}$ portion of the wave function. The calculations, therefore, look identical to those of part b, and we obtain

$$\boxed{P\left(S_z = \frac{\hbar}{2}\right) = \frac{1}{3}, \quad P\left(S_z = -\frac{\hbar}{2}\right) = \frac{2}{3}.}$$

(e) Here, $\vec{J} = \vec{L} + \vec{S}$, so this is a problem in addition of angular momentum. We first write down the information included in the original wave function in Dirac notation to explicitly display the form in the direct product space:

$$|\,\Psi> = \sqrt{\frac{1}{3}}\,|1,0\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,|1,1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

This is now very similar to Problem 4. The linear combinations we seek are

$$|1,0\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = A_1\left|\frac{3}{2},\frac{1}{2}\right\rangle + B_1\left|\frac{1}{2},\frac{1}{2}\right\rangle,$$

and

$$|1,1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle = A_2\left|\frac{3}{2},\frac{1}{2}\right\rangle + B_2\left|\frac{1}{2},\frac{1}{2}\right\rangle.$$

Using the 1 x 1/2 Clebsch-Gordan table, we find

$$A_1 = \sqrt{\frac{2}{3}}, \qquad B_1 = -\sqrt{\frac{1}{3}}, \qquad A_2 = \sqrt{\frac{1}{3}}, \quad \text{and} \quad B_2 = \sqrt{\frac{2}{3}}.$$

We also could have used the results of Problem 5 to obtain these coefficients. So, now we have

$$\begin{aligned}
|\,\Psi> &= \sqrt{\frac{1}{3}}\left(\sqrt{\frac{2}{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\right) + \sqrt{\frac{2}{3}}\left(\sqrt{\frac{1}{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\right) \\
&= \frac{\sqrt{2}}{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle - \frac{1}{3}\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{\sqrt{2}}{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle + \frac{2}{3}\left|\frac{1}{2},\frac{1}{2}\right\rangle \\
&= 2\frac{\sqrt{2}}{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle + \frac{1}{3}\left|\frac{1}{2},\frac{1}{2}\right\rangle.
\end{aligned}$$

This tells us that the only possible values of j are 3/2 and 1/2. The corresponding eigenvalues of J^2 are $j(j+1)\hbar^2$. The associated probabilities are the squares of the coefficients for the respective values of j, and we obtain

$$\boxed{P\left(J^2 = \frac{15}{4}\hbar^2\right) = \left(2\frac{\sqrt{2}}{3}\right)^2 = \frac{8}{9}, \quad P\left(J^2 = \frac{3}{4}\hbar^2\right) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.}$$

(f) Note that the z-component in both kets is 1/2, so that is the only possible value of J_z. The corresponding eigenvalues of J_z are $m_j\hbar$, so we conclude that

$$\boxed{P\left(J_z = \frac{\hbar}{2}\right) = 1.}$$

(g) The probability density as a function of $r,\ \theta,$ and ϕ is $|\Psi(r,\theta,\phi)|^2$ which is equal to $|\,R_{21}\,|^2$ times

$$\left(\sqrt{\frac{1}{3}}\,Y_{1,0}^*\sqrt{\frac{1}{3}}\,Y_{1,0}\chi_+^*\chi_+ + \sqrt{\frac{1}{3}}\,Y_{1,0}^*\sqrt{\frac{2}{3}}\,Y_{1,1}\chi_+^*\chi_- + \sqrt{\frac{2}{3}}\,Y_{1,1}^*\sqrt{\frac{1}{3}}\,Y_{1,0}\chi_-^*\chi_+ + \sqrt{\frac{2}{3}}\,Y_{1,1}^*\sqrt{\frac{2}{3}}\,Y_{1,1}\chi_-^*\chi_-\right).$$

The second and third terms inside the parentheses are zero because

$$\chi_+^*\chi_- = (1,\ 0)\begin{pmatrix}0\\1\end{pmatrix} = 0 \quad \text{and} \quad \chi_-^*\chi_+ = (0,\ 1)\begin{pmatrix}1\\0\end{pmatrix} = 0.$$

Similar calculations for the first and fourth terms shows us that the inner product of these spinors are both 1, so the probability density above reduces to

$$\Big|\Psi(r,\theta,\phi)\Big|^2 = |\,R_{21}\,|^2\left(\sqrt{\frac{1}{3}}\,Y_{1,0}^*\sqrt{\frac{1}{3}}\,Y_{1,0} + \sqrt{\frac{2}{3}}\,Y_{1,1}^*\sqrt{\frac{2}{3}}\,Y_{1,1}\right). \tag{1}$$

The radial wave functions are given in table 4.6 on page 141 of Griffiths as

$$R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\frac{r}{a}e^{-r/2a} \quad \Rightarrow \quad |\,R_{21}\,|^2 = \frac{1}{24}\frac{r^2}{a^5}e^{-r/a}.$$

The spherical harmonics are given in table 4.2 on page 128 of Griffiths as

$$\sqrt{\frac{1}{3}}Y_{1,0}^*\sqrt{\frac{1}{3}}Y_{1,0} = \frac{1}{3}\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta = \frac{1}{4\pi}\cos^2\theta,$$

and

$$\sqrt{\frac{2}{3}}\,Y_{1,1}^*\sqrt{\frac{2}{3}}\,Y_{1,1} = \frac{2}{3}\left[-\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{-i\phi}\right]\left[-\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{+i\phi}\right] = \frac{2}{3}\frac{3}{8\pi}\sin^2\theta = \frac{1}{4\pi}\sin^2\theta.$$

Substituting these results in equation (1), we obtain

$$\Big|\Psi(r,\theta,\phi)\Big|^2 = \frac{1}{24}\frac{r^2}{a^5}e^{-r/a}\left(\frac{1}{4\pi}\cos^2\theta + \frac{1}{4\pi}\sin^2\theta\right) = \frac{1}{24}\frac{1}{4\pi}\frac{r^2}{a^5}e^{-r/a}\left(\cos^2\theta + \sin^2\theta\right).$$

$$\boxed{\Rightarrow \quad \Big|\Psi(r,\theta,\phi)\Big|^2 = \frac{1}{96\pi}\frac{r^2}{a^5}e^{-r/a}}$$

(h) The possibility of measuring spin up corresponds to the portion with χ_+, so the probability density for measuring spin up at a radius r is given by

$$\Big|\Psi(r,\theta,\phi)\Big|^2_{\text{spin up}} = |\,R_{21}\,|^2\sqrt{\frac{1}{3}}\;Y^*_{1,0}\sqrt{\frac{1}{3}}\;Y_{1,0}.$$

Since we want the probability density at a fixed r independent of θ and ϕ, we must integrate over the angles to arrive at the probability density as a function of r alone. The integral is simple if we use the orthonormality of the spherical harmonics.

$$\int\sqrt{\frac{1}{3}}\;Y^*_{1,0}\sqrt{\frac{1}{3}}\;Y_{1,0}\,d\Omega = \frac{1}{3}\int Y^*_{1,0}\,Y_{1,0}\,d\Omega = \frac{1}{3}.$$

Putting this together with the radial dependence, we obtain

$$\boxed{\Big|\Psi(r)\Big|^2_{\text{spin up}} = \frac{1}{72}\,\frac{r^2}{a^5}\,e^{-r/a}.}$$
