of a quantum mechanical system must satisfy all nine conditions. A classical vector sum need not satisfy the six conditions of quantization in length and quantization in the z–components.

Figure 14–1 is a semi–classical illustration of equation (14–1). The length of $\vec{\mathcal{J}}$, $\vec{\mathcal{L}}$, and $\vec{\mathcal{S}}$ are fixed, as are the z–components of each. The other two components are not fixed, however,

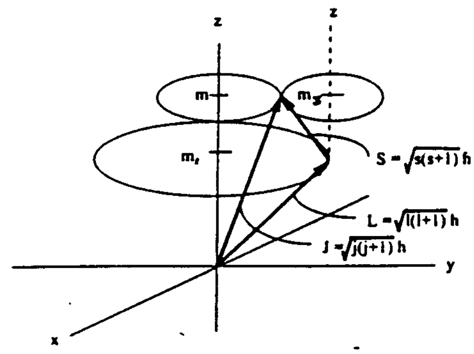

Figure 14 – 1. Semi – Classical Picture of Orbital and Spin Angular Momenta Composing Total Angular Momentum

and each can be pictured as precessing such that total length and z–component are constant. In this picture, orbital, spin, and total angular momenta vectors all precess about the z–axis. Further, it would appear from the figure that spin and orbital magnetic moments will sum to the total magnetic moment. We will soon show this is a fact.

Total Angular Momentum Commutators

In addition to the commutation relations (14–4), it is necessary to consider the commutation relations between orbital and spin angular momentum. Orbital and spin angular momentum operators commute. They exist in different spaces, cannot interact with an object in another space, therefore they must commute, *i.e.*,

$$\left[\,\mathcal{L}_i,\, \mathcal{S}_j\,\right] = 0, \qquad (14-6)$$

where i and j both represent all three components. This fact means that total angular momentum operators commute with orbital and spin angular momentum operators, or

$$\left[\,\mathcal{J}_i,\, \mathcal{L}_j\,\right] = \left[\,\mathcal{J}_i,\, \mathcal{S}_j\,\right] = 0, \qquad (14-7)$$

for any set of i and j; x, y, or z.

Example 14–1: Show $\left[\,\mathcal{J}_x,\, \mathcal{L}_x\,\right] = 0$.

$$\begin{aligned}\left[\,\mathcal{J}_x,\, \mathcal{L}_x\,\right] &= \mathcal{J}_x\mathcal{L}_x - \mathcal{L}_x\mathcal{J}_x \\ &= \left(\mathcal{L}_x + \mathcal{S}_x\right)\mathcal{L}_x - \mathcal{L}_x\left(\mathcal{L}_x + \mathcal{S}_x\right) \\ &= \mathcal{L}_x^2 + \mathcal{S}_x\mathcal{L}_x - \mathcal{L}_x^2 - \mathcal{L}_x\mathcal{S}_x \\ &= \left(\mathcal{L}_x^2 - \mathcal{L}_x^2\right) + \left(\mathcal{L}_x\mathcal{S}_x - \mathcal{L}_x\mathcal{S}_x\right) = 0,\end{aligned}$$

where we have used the fact $\mathcal{L}_x$ and $\mathcal{S}_x$ commute to reverse the order of the operations in the last line. This procedure can be used to show any set of operators $\mathcal{J}_i$ and $\mathcal{L}_j$ or $\mathcal{S}_j$ commute.

The $\mathcal{J}^2$ and $\mathcal{J}_z$ operators will have different eigenvalues when they operate on the same basis vector, so there are two indices for each basis vector. The first index is the eigenvalue for $\mathcal{J}^2$, denoted α, and the second index is the eigenvalue for $\mathcal{J}_z$, denoted β. The form of the eigenvalue equations must be

$$\mathcal{J}^2|\alpha,\beta> = \alpha|\alpha,\beta>, \qquad (14-11)$$

$$\mathcal{J}_z|\alpha,\beta> = \beta|\alpha,\beta>\,. \qquad (14-12)$$

Equations (14–11) and (14–12) are in total angular momentum state space which is the composition of orbital state space and spin state space. Using arguments similar to those of chapter 11 and chapter 13, $\mathcal{J}_\pm|\alpha,\beta>$ is an eigenvector of both $\mathcal{J}^2$ and $\mathcal{J}_z$, meaning

$$\mathcal{J}_z\left(\mathcal{J}_\pm|\alpha,\beta>\right) = (\beta\pm\hbar)\left(\mathcal{J}_\pm|\alpha,\beta>\right), \qquad (14-13)$$

and

$$\mathcal{J}^2\left(\mathcal{J}_\pm|\alpha,\beta>\right) = \alpha\left(\mathcal{J}_\pm|\alpha,\beta>\right). \qquad (14-14)$$

The braket of eigenvectors and the operator $\mathcal{J}^2 - \mathcal{J}_z$ tells us

$$<\alpha,\beta|\,\mathcal{J}^2 - \mathcal{J}_z\,|\alpha,\beta> = \alpha - \beta^2 \geq 0 \quad \Rightarrow \quad \alpha \geq \beta^2,$$

so β is bounded by α. This means there is a maximum and minimum value of β for a given value of α, so the "ladder" has a top and a bottom. Operating with the raising operator on the eigenvector with a maximum value of β yields the zero vector, or more simply zero, so

$$\mathcal{J}_-\mathcal{J}_+|\alpha,\beta_{\text{max}}> = 0 \quad \Rightarrow \quad \alpha = \beta_{\text{max}}^2 + \hbar\beta_{\text{max}}.$$

Similarly, operating with the lowering operator on the eigenvector with a minimum value of β yields the zero vector so

$$\mathcal{J}_+\mathcal{J}_-|\alpha,\beta_{\text{min}}> = 0 \quad \Rightarrow \quad \alpha = \beta_{\text{min}}^2 - \hbar\beta_{\text{min}}.$$

This is true for any eigenvalue α, however, so we can equate these two equations and attain

$$\beta_{\text{max}} = -\beta_{\text{min}},$$

Since the eigenvector of $\mathcal{J}_z$, $\mathcal{J}_\pm|\alpha,\beta>$, is raised or lowered by $\hbar$, we assume the rungs of the ladder are separated by $\hbar$. If there are n steps between the bottom and top rungs of the ladder, there is a total separation of $n\hbar$ between the bottom and the top. From figure 14–2 we expect

$$\begin{aligned} 2\beta_{\text{max}} = n\hbar \quad &\Rightarrow \quad \beta_{\text{max}} = \frac{n\hbar}{2} \\ &\Rightarrow \quad \alpha = \beta_{\text{max}}(\beta_{\text{max}}+\hbar) \\ &\qquad = \frac{n\hbar}{2}\left(\frac{n\hbar}{2}+\hbar\right) \\ &\qquad = \hbar^2\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right). \qquad (14-15) \end{aligned}$$

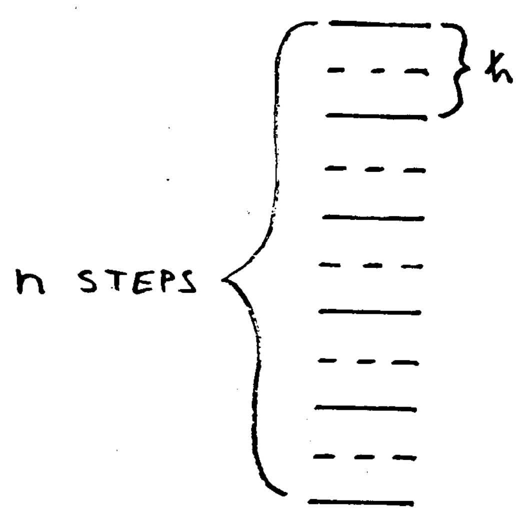

FIG 14-2. LADDER OF n STEPS.

The Problem of Combining Two Angular Momentum States

First, we want to clarify the problem and describe the form of the answer. The title to this section could be "The Problem of Combining Two Total Angular Momentum States," but the adjective total is understood at this point. If orbital or spin angular momentum is being addressed, it should be specified as a subset of total angular momentum.

The substance of this and the following sections is commonly called the <u>addition of angular momentum</u>. A better description is the one used; combination of angular momentum states of two or more particles. We have two individual eigenstates and want an eigenstate for the combination. In classical vector addition, the resultant must be the vector sum of the two component angular momenta. Also, the z–components must sum to that of the resultant. These constraints must be satisfied in a quantum mechanical system. A quantum mechanical system has additional constraints, the "length" and "length" of the z–component of an angular momentum "vector" is quantized. A semi–classical description is seen in figure 14–3. This could be a classical figure, other than the quantum mechanical length must be $\mathcal{J} = \sqrt{j(j+1)}\,\hbar$, in all three cases. As indicated in the figure, a resultant can be composed of an infinite number of classical vectors. Quantum mechanically, there are a limited number of possible combinations because of quantization of "length."

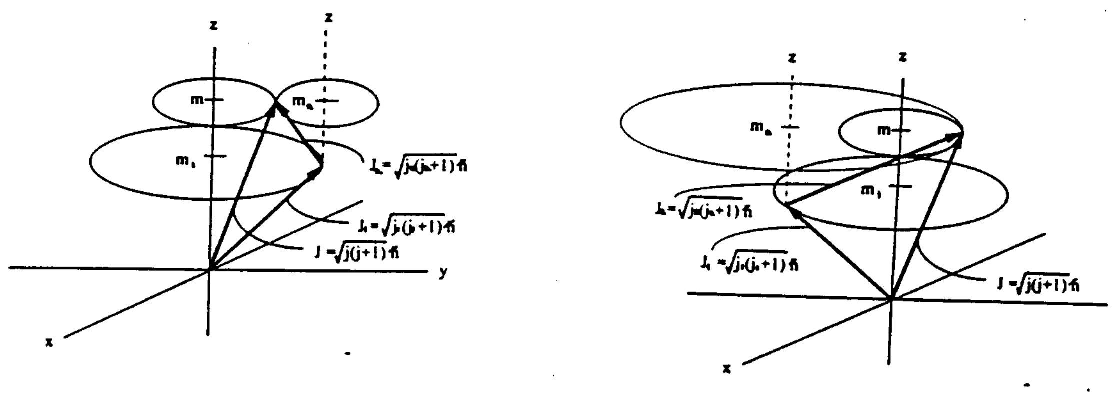

Figure 14 – 2. A Semi – Classical Picture of Combination of Angular Momentum

For simplicity, we will address only two particles, though three or more particles can be addressed by a similar conceptual development.

"Addition of angular momentum" means we want to find the angular momentum eigenstates of the system or possible angular momentum eigenstates of the system given the angular momentum eigenstates or possible angular momentum of the two particles composing the system. We are looking for eigenstates where

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}_1 + \vec{\mathcal{J}}_2.$$

The first step in solving this problem is to realize each of the three angular momenta have components which are canonical, *i.e.*,

$$\left[\,\mathcal{J}_{1_i},\, \mathcal{J}_{1_j}\,\right] = i\hbar\mathcal{J}_{1_k}, \qquad \left[\,\mathcal{J}_{2_i},\, \mathcal{J}_{2_j}\,\right] = i\hbar\mathcal{J}_{2_k}, \qquad \text{and} \qquad \left[\,\mathcal{J}_i,\, \mathcal{J}_j\,\right] = i\hbar\mathcal{J}_k.$$

TABLE 14-1: *Clebsch–Gordan Coefficients*

NOTE: A $\sqrt{\ }$ is to be understood over *every* coefficient: e.g., −8/15 read = $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

1/2 × 1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1 × 1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2 × 1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1 × 1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

2 × 1/2

m_1	m_2	5/2, +5/2
+2	1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2 × 1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

3/2 × 1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

2 × 3/2

m_1	m_2	7/2, +7/2
+2	+3/2	1

m_1	m_2	7/2, +5/2	5/2, +5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/7	16/35	2/5
+1	1/2	4/7	1/35	−2/5
0	3/2	2/7	−18/35	1/5

m_1	m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	1/2	18/35	−3/35	−1/5	1/5
−1	3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	1/2	12/35	−5/14	0	3/10
−2	3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	1/2	1/7	−16/35	2/5

m_1	m_2	7/2, −5/2	5/2, −5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2, −7/2
−2	−3/2	1

3/2 × 3/2

m_1	m_2	3, +3
+3/2	+3/2	1

m_1	m_2	3, +2	2, +2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3, +1	2, +1	1, +1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3, 0	2, 0	1, 0	0, 0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3, −1	2, −1	1, −1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3, −2	2, −2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3, −3
−3/2	−3/2	1

2 × 2

m_1	m_2	4, +4
+2	+2	1

m_1	m_2	4, +3	3, +3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4, +2	3, +2	2, +2
+2	0	3/14	1/2	2/7
+1	1	4/7	0	−3/7
0	2	3/14	−1/2	2/7

m_1	m_2	4, +1	3, +1	2, +1	1, +1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	1	3/7	−1/5	−1/14	3/10
−1	2	1/14	−3/10	3/7	−1/5

m_1	m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	1	8/35	−2/5	1/14	1/10	−1/5
−2	2	1/70	−1/10	2/7	−2/5	1/5

4, 3, 2, 1 / −1, −1, −1, −1