Spin: Solved Problems

1. Consider an electron in the spin state

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

(a) Determine the normalization constant A.

(b) Find the expectation values of S_x, S_y, and S_z.

(c) Find the stasndard deviations ΔS_x, ΔS_y, and ΔS_z.

(d) Confirm that your results are consistent with all three uncertainty principles,

$$\Delta S_i \Delta S_j \geq \frac{\hbar}{2} \left| < S_k > \right|,$$

where the indices i, j, and k represent the cyclic permutations of x, y, and z.

Remember all that vector and matrix algebra you studied last quarter—before we started studying differential equations? Well, this problem is just plain vanilla matrix and vector algebra for spin 1/2 operators. This problem is designed to remind you of all the things you learned last quarter, and to increase your familiarity and dexterity with spin operators. In the S_z basis, the spin 1/2 operators are given by $\hbar/2$ times the 2×2 Pauli matrices:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and} \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Please remember to conjugate your bras!

1(a) Calculate the normalization constant

$$\langle \chi | \chi \rangle = (-3i, 4) A^* A \begin{pmatrix} 3i \\ 4 \end{pmatrix} = |A|^2 (9 + 16) = |A|^2\, 25$$

$$\boxed{\Rightarrow \quad A = \frac{1}{5}\,.}$$

(b) Calculate the expectation values

$$\begin{aligned} \langle S_x \rangle = \langle \chi | S_x | \chi \rangle &= (-3i, 4) \frac{1}{5} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\ &= \frac{\hbar}{50} (-3i, 4) \begin{pmatrix} 4 \\ 3i \end{pmatrix} = \frac{\hbar}{50} (-12i + 12i) \end{aligned}$$

$$\boxed{\Rightarrow \quad \langle S_x \rangle = 0.}$$

$$\langle S_y \rangle = \langle \chi | S_y | \chi \rangle = (-3i, 4) \frac{1}{5} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$
$$= \frac{\hbar}{50} (-3i, 4) \begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{\hbar}{50} (-12 - 12)$$

$$\boxed{\Rightarrow \quad \langle S_y \rangle = -\frac{12}{25} \hbar.}$$

$$\langle S_z \rangle = \langle \chi | S_z | \chi \rangle = (-3i, 4) \frac{1}{5} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$
$$= \frac{\hbar}{50} (-3i, 4) \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \frac{\hbar}{50} (9 - 16)$$

$$\boxed{\Rightarrow \quad \langle S_z \rangle = -\frac{7}{50} \hbar.}$$

(c) To calculate the standard deviations, we will use $\Delta \Omega = \left[\langle \Omega^2 \rangle - \langle \Omega \rangle^2 \right]^{1/2}$. We need the S_i^2's, which are given by

$$S_x^2 = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} I$$

$$S_y^2 = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} I$$

$$S_z^2 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} I.$$

So the corresponding expectation values are given by

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = (-3i, 4) \frac{1}{5} \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$
$$= \frac{\hbar^2}{100} (-3i, 4) \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \frac{\hbar^2}{100} (9 + 16) = \frac{\hbar^2}{4}.$$

Combining these results with the results from part b,

$$\langle S_x \rangle = 0, \qquad \langle S_y \rangle = -\frac{12}{25} \hbar, \qquad \text{and} \qquad \langle S_z \rangle = -\frac{7}{50} \hbar,$$

we find

$$\Delta S_x = \left[\langle S_x^2\rangle - \langle S_x\rangle^2\right]^{1/2} = \left[\frac{\hbar^2}{4} - 0\right]^{1/2} = \frac{\hbar}{2},$$

$$\Delta S_y = \left[\langle S_y^2\rangle - \langle S_y\rangle^2\right]^{1/2} = \left[\frac{\hbar^2}{4} - \left(-\frac{12}{25}\hbar\right)^2\right]^{1/2} = \left[\frac{625}{2500} - \frac{576}{2500}\right]^{1/2}\hbar = \left[\frac{49}{2500}\right]^{1/2}\hbar = \frac{7}{50}\hbar,$$

$$\Delta S_z = \left[\langle S_z^2\rangle - \langle S_z\rangle^2\right]^{1/2} = \left[\frac{\hbar^2}{4} - \left(-\frac{7}{50}\hbar\right)^2\right]^{1/2} = \left[\frac{625}{2500} - \frac{49}{2500}\right]^{1/2}\hbar = \left[\frac{576}{2500}\right]^{1/2}\hbar = \frac{12}{25}\hbar.$$

(d) To confirm that these results are consistent with all three uncertainty principles, we calculate

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2}\left|\langle S_z\rangle\right| \quad \Rightarrow \quad \left(\frac{\hbar}{2}\right)\left(\frac{7}{50}\hbar\right) \geq \frac{\hbar}{2}\left|\left(-\frac{7}{50}\hbar\right)\hbar\right| \quad \Rightarrow \quad \frac{7\hbar^2}{100} \geq \frac{7\hbar^2}{100} \quad \text{which checks,}$$

$$\Delta S_y \Delta S_z \geq \frac{\hbar}{2}\left|\langle S_x\rangle\right| \quad \Rightarrow \quad \left(\frac{7}{50}\hbar\right)\left(\frac{12}{25}\hbar\right) \geq \frac{\hbar}{2}\left|0\right| \quad \Rightarrow \quad \frac{84\hbar^2}{1250} \geq 0 \quad \text{which checks, and}$$

$$\Delta S_z \Delta S_x \geq \frac{\hbar}{2}\left|\langle S_y\hbar\rangle\right| \quad \Rightarrow \quad \left(\frac{12}{25}\hbar\right)\left(\frac{\hbar}{2}\right) \geq \frac{\hbar}{2}\left|\left(-\frac{12}{25}\hbar\right)\hbar\right| \quad \Rightarrow \quad \frac{12}{50}\hbar^2 \geq \frac{12}{50}\hbar^2 \quad \text{which also checks.}$$

Note that the first and third cases saturate the inequality.

2. Given the most general normalized spin 1/2 spinor

$$\chi = A \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

where

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and where $|\,a\,|^2 + |\,b\,|^2 = 1$, compute:

(a) $< S_x >$

(b) $< S_y >$

(c) $< S_z >$

(d) $< S_x^2 >$

(e) $< S_y^2 >$

(f) $< S_z^2 >$

(g) Check that $< S_x^2 > + < S_y^2 > + < S_z^2 > = < S^2 >$.

This problem is very similar to Problem 1, but it treats the most general normalized spinor instead of a specific numerical example. You should find similar expectation values in parts e, f, and g. Since χ is normalized, you can do your calculations very simply. For example, the calculation for part c is just

$$< S_z > = < \chi\,|\,S_z\,|\,\chi > = (a^*, b^*)\,\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

2(a) Calculate the expectation value of S_x

$$\langle S_x \rangle = \langle \chi | S_x | \chi \rangle = (a^*, b^*)\,\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\,(a^*, b^*)\begin{pmatrix} b \\ a \end{pmatrix}$$

$$\Rightarrow \quad \langle S_x \rangle = \frac{\hbar}{2}(a^* b + b^* a) = \hbar\,\mathrm{Re}\,(ab^*)\,.$$

Note that the last step in the box is a simple exercise with complex numbers. Let $a = c + di$ and $b = e + fi$, so $a^*b + b^*a = (c - di)\,(e + fi) + (e - fi)\,(c + di) = ce + df + (cf - ed)\,i + ce + df - (cf - ed)\,i = 2\,(ce + df) = 2\,\mathrm{Re}\,(ab^*)$.

(b) Calculate the expectation value of S_y

$$\langle S_y \rangle = \langle \chi | S_y | \chi \rangle = (a^*, b^*) \, \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} (a^*, b^*) \begin{pmatrix} -bi \\ ai \end{pmatrix}$$

$$\boxed{\Rightarrow \quad \langle S_y \rangle = \frac{\hbar}{2} i(-a^* b + b^* a) = -\hbar \, \mathrm{Im}\,(ab^*)\,.}$$

Note that the argument for $\mathrm{Im}(ab^*)$ is similar to the argument for $\mathrm{Re}(ab^*)$.

(c) Calculate the expectation value of S_z

$$\langle S_z \rangle = \langle \chi | S_z | \chi \rangle = (a^*, b^*) \, \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} (a^*, b^*) \begin{pmatrix} a \\ -b \end{pmatrix}$$

$$\boxed{\Rightarrow \quad \langle S_z \rangle = \frac{\hbar}{2}(a^* a - b^* b) = \frac{\hbar}{2} \left(|a|^2 - |b|^2 \right).}$$

(d, e, f) The squares of the spin matrices were calculated in Problem 7.1. In all three cases they were found to be

$$S_i^2 = \frac{\hbar^2}{4} I.$$

Since the square of each matrix is identical, the expectation value calculations are identical, *i.e.*, we can find the expectation value of all three operators with one calculation:

$$\langle S_i^2 \rangle = \langle \chi | S_i^2 | \chi \rangle = (a^*, b^*) \, \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar^2}{4} (a^*, b^*) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar^2}{4}(a^* a + b^* b) = \frac{\hbar^2}{4} \left(|a|^2 + |b|^2 \right).$$

Remember that the spinor is normalized, so $|a|^2 + |b|^2 = 1$, and we conclude

$$\boxed{\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}.}$$

(g) To find $\langle S^2 \rangle$, we need the S^2 operator which is given by

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4} I + \frac{\hbar^2}{4} I + \frac{\hbar^2}{4} I = \frac{3\hbar^2}{4} I.$$

So the expectation value is given by

$$\begin{aligned} \langle S^2 \rangle = \langle \chi | S^2 | \chi \rangle &= (a^*, b^*) \, \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{3\hbar^2}{4} (a^*, b^*) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{3\hbar^2}{4}(a^* a + b^* b) \\ &= \frac{3\hbar^2}{4} \left(|a|^2 + |b|^2 \right) = \frac{3\hbar^2}{4}. \end{aligned}$$

In parts d, e and f, you found

$$\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \frac{\hbar^2}{4} + \frac{\hbar^2}{4} + \frac{\hbar^2}{4} = \frac{3\hbar^2}{4}.$$

So, both calculations produce the same result, namely

$$\boxed{\langle S^2 \rangle = \frac{3\hbar^2}{4}.}$$

3. Now consider the S_y operator for spin 1/2

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(a) Find the eigenvalues and the eigenspinors of the S_y operator.

(b) If you measure S_y on a particle in the general state χ given in Problem 2, what values could you obtain, and with what probabilities would you obtain them? Check that the probabilities add up to 1.

(c) If you measure S_y^2, what values could you obtain, and with what probabilities would you obtain them?

Spinors are nick names for the eigenvectors of a spin operator. For part b, remember that a general state can always be written as a superposition of eigenvectors, or eigenspinors. In this case,

$$\chi = C_+\chi_+^{(y)} + C_-\chi_-^{(y)}$$

where $\chi_i^{(y)}$ are the eigenspinors corresponding to the S_y matrix. The tricky part here arises because the coefficients can be complex. Remember to use $|\,a\,|^2 + |\,b\,|^2 = 1$.

3(a) First, find the eigenvalues and eigenvectors of S_y in the usual manner. Given

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & 0 \end{pmatrix},$$

calculate

$$\det\begin{pmatrix} -\omega & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\omega \end{pmatrix} = \omega^2 - \frac{\hbar^2}{4} = 0.$$

Then, solving the characteristic equation gives us the eigenvalues

$$\boxed{\Rightarrow \quad \omega = \pm\frac{\hbar}{2}.}$$

Next, find the eigenvectors. First

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -ib & = & a \\ ia & = & b \end{matrix} \quad \Rightarrow \quad |\,S_y = \frac{\hbar}{2}\rangle = A\begin{pmatrix} 1 \\ i \end{pmatrix}$$

and normalizing

$$\Rightarrow \quad (1, -i)A^*A\begin{pmatrix} 1 \\ i \end{pmatrix} = |A|^2(1+1) \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}}.$$

And second,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -ib & = & -a \\ ia & = & -b \end{matrix} \quad \Rightarrow \quad | S_y = -\frac{\hbar}{2}\rangle = A\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\Rightarrow \quad (1, i)A^*A\begin{pmatrix} 1 \\ -i \end{pmatrix} = |A|^2(1+1) \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}}.$$

And we conclude

$$\boxed{\Rightarrow \quad \chi_+^{(y)} = S_y = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \text{ and } \chi_-^{(y)} = S_y = -\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.}$$

(b) The possible results for a measurement of S_y are the eigenvalues of S_y

$$\boxed{+\frac{\hbar}{2} \text{ and } -\frac{\hbar}{2}.}$$

For any general state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, we can express χ as a superposition of the eigenvectors of S_y,

$$\chi = C_+\chi_+^{(y)} + C_-\chi_-^{(y)} \qquad \text{where} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \text{and} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

We can find C_+ and C_- by calculating the inner products, that is

$$C_+ = \langle\chi_+^{(y)}|\chi\rangle = \frac{1}{\sqrt{2}}(1, -i)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a - ib),$$

$$C_- = \langle\chi_-^{(y)}|\chi\rangle = \frac{1}{\sqrt{2}}(1, i)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a + ib).$$

If the general state χ is normalized, the probabilities are given by

$$\boxed{\begin{aligned} |C_+|^2 = |\frac{1}{\sqrt{2}}(a - ib)|^2 = \quad &\text{for spin up, a.k.a. } S_y = +\frac{\hbar}{2} \\ |C_-|^2 = |\frac{1}{\sqrt{2}}(a + ib)|^2 \quad &\text{for spin down, a.k.a. } S_y = -\frac{\hbar}{2}. \end{aligned}}$$

Note that a and b can be complex! The sum of these two probabilities is given by

$$\begin{aligned} |C_+|^2 + |C_-|^2 &= \left|\frac{1}{\sqrt{2}}(a - ib)\right|^2 + \left|\frac{1}{\sqrt{2}}(a + ib)\right|^2 \\ &= \frac{1}{2}\left((a^* + ib^*)(a - ib) + (a^* - ib^*)(a + ib)\right) \\ &= \frac{1}{2}\left(|a|^2 + iab^* - ia^*b + |b|^2 + |a|^2 + ia^*b - iab^* + |b|^2\right) \\ &= \frac{1}{2}\left(2|a|^2 + 2|b|^2\right) = |a|^2 + |b|^2 = 1. \end{aligned}$$

And we can conclude that

$$\boxed{\sum_i P_i = 1.}$$

(c) In problem 1, we found $S_y^2 = \frac{\hbar^2}{4} I$, so

$$\boxed{\text{the only possible result of a measurement of } S_y^2 \text{ is } \frac{\hbar^2}{4} \text{ with probability } P\left(\frac{\hbar^2}{4}\right) = 1.}$$

Note that an eigenvalue/eigenvector calculation will yield the same result, since the eigenvalues are degenerate.

4. Consider an electron at rest at rest ($i.e., \vec{v}=0$) in a uniform magnetic field $\vec{B}_0 = B_0\hat{z}$. At $t=0$, the spin is pointing in the $+\hat{x}$ direction, $i.e.$, $< S_x(t=0) > = +\hbar/2$. Calculate the expectation value $< \vec{S}(t) >$ for all times t.

This problem combines many concepts and topics from last quarter in a spin problem. Take a look at Griffiths, pages 160 - 165 where he presents very useful comments and example calculations for this problem. You will probably want to develop your arguments by following the first part of his discussion on Larmor precession. You need to be able to express a Hamiltonian in terms of the gyromagnetic ratio. Problems 6 and 7, in addition to this problem, depend on this. Remember that if you measure, the state immediately after the measurement is the eigenvector which corresponds to the eigenvalue you obtained. So, you will want to find the eigenvalues and eigenvectors of S_x to determine $\mid \psi(0) >$. Then you will want to find the time-dependent state vector by expanding the $t=0$ state vector in eigenstates of S_z, and using

$$\mid \psi(t) > = \sum_i \mid i >< i \mid \psi(0) > e^{-iE_n t/\hbar}.$$

Here the $\mid i >$'s are the normalized eigenvectors of the Hamiltonian. You will obtain

$$\mid \psi(t) > = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma B_0 t/2} \\ e^{-i\gamma B_0 t/2} \end{pmatrix}$$

if you do everything correctly. Finally, find the expectation values of each component, and then calculate the vector sum

$$< \vec{S}(t) > = < S_x > \hat{x} + < S_y > \hat{y} + < S_z > \hat{z}$$

to complete the problem.

4(a) Given $\vec{B}_0 = B_0\hat{z}$, the corresponding Hamiltonian is given by

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma\, \vec{S}\cdot\vec{B} = -\gamma\, \vec{S}\cdot B_0\hat{z} = -\gamma\, B_0 S_z$$

where μ is the magnetic dipole moment and γ is the gyromagnetic ratio. In matrix form, we have

$$H = -\gamma\, B_0 \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\gamma\, B_0 \hbar}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

so we can find the eigenvalues by inspection

$$E = -\frac{\gamma\, B_0 \hbar}{2} \quad \text{and} \quad -\frac{\gamma\, B_0 \hbar}{2}.$$

Next, we find the eigenvectors in the usual way,

$$\frac{\gamma B_0 \hbar}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\gamma B_0 \hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -a & = & a \\ b & = & b \end{matrix} \quad \Rightarrow \quad |E = \frac{\gamma B_0 \hbar}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{\gamma B_0\hbar}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -\frac{\gamma B_0\hbar}{2}\begin{pmatrix}a\\ b\end{pmatrix} \quad\Rightarrow\quad \begin{matrix}-a & = & -a\\ b & = & -b\end{matrix} \quad\Rightarrow\quad |E = -\frac{\gamma B_0\hbar}{2}\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$$

At time $t = 0$, the spin is pointing in the $\hat{x}$ direction, *i.e.*, $\langle S_x(t=0)\rangle = \hbar/2$, which means the state vector is the same as the eigenvector associated with eigenvalue $\hbar/2$ of the S_x operator. So, we need to find the corresponding eigenvector of the S_x operator:

$$\det\begin{pmatrix}-\omega & \hbar/2\\ \hbar/2 & -om\end{pmatrix} = \omega^2 - \hbar^2/4 \quad\Rightarrow\quad \omega = \pm\hbar/2 \text{ are the eigenvalues and}$$

$$\hbar/2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \hbar/2\begin{pmatrix}a\\ b\end{pmatrix} \quad\Rightarrow\quad \begin{matrix}b & = & a\\ a & = & b\end{matrix} \quad\Rightarrow\quad |S_y = \hbar/2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

when normalized. This is the initial state vector $|\psi(0)\rangle$. The problem now is to calculate the time-dependent expectation value, so we must first calculate the time evolution of the state vector:

$$\begin{aligned}|\psi(t)\rangle &= \sum_i |i\rangle\langle i|\psi(0)\rangle e^{-iE_nt/\hbar}\\ &= \begin{pmatrix}1\\ 0\end{pmatrix}(1,0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}e^{-i(-\gamma B_0t/2)} + \begin{pmatrix}0\\ 1\end{pmatrix}(0,1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}e^{-i(\gamma B_0t/2)}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\end{pmatrix}e^{i\gamma B_0t/2} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\end{pmatrix}e^{-i\gamma B_0t/2}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{+i\gamma B_0t/2}\\ e^{-i\gamma B_0t/2}\end{pmatrix}.\end{aligned}$$

We want the expectation value of the vector spin $\langle\vec{S}(t)\rangle$, which is given by

$$\langle\vec{S}(t)\rangle = \langle S_x\rangle\hat{x} + \langle S_y\rangle\hat{y} + \langle S_z\rangle\hat{z}.$$

So, we need the expectation value of each components:

$$\begin{aligned}\langle S_x\rangle &= \frac{1}{\sqrt{2}}\left(e^{-i\gamma B_0t/2}, e^{i\gamma B_0t/2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma B_0t/2}\\ e^{-i\gamma B_0t/2}\end{pmatrix}\\ &= \frac{\hbar}{4}\left(e^{-i\gamma B_0t/2}, e^{i\gamma B_0t/2}\right)\begin{pmatrix}e^{-i\gamma B_0t/2}\\ e^{i\gamma B_0t/2}\end{pmatrix}\\ &= \frac{\hbar}{4}\left(e^{-i\gamma B_0t} + e^{i\gamma B_0t}\right)\\ &= \frac{\hbar}{2}\cos(\gamma B_0t)\end{aligned}$$

$$\begin{aligned}\langle S_y\rangle &= \frac{1}{\sqrt{2}}\left(e^{-i\gamma B_0t/2}, e^{i\gamma B_0t/2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma B_0t/2}\\ e^{-i\gamma B_0t/2}\end{pmatrix}\\ &= \frac{\hbar}{4}\left(e^{-i\gamma B_0t/2}, e^{i\gamma B_0t/2}\right)\begin{pmatrix}-ie^{-i\gamma B_0t/2}\\ ie^{i\gamma B_0t/2}\end{pmatrix}\\ &= \frac{\hbar}{4}\left(-ie^{-i\gamma B_0t} + ie^{i\gamma B_0t}\right)\\ &= -\frac{\hbar}{2}\sin(\gamma B_0t)\end{aligned}$$

$$\begin{aligned}\langle S_z \rangle &= \frac{1}{\sqrt{2}} \left(e^{-i\gamma B_0 t/2}, e^{i\gamma B_0 t/2} \right) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma B_0 t/2} \\ e^{-i\gamma B_0 t/2} \end{pmatrix} \\ &= \frac{\hbar}{4} \left(e^{-i\gamma B_0 t/2}, e^{i\gamma B_0 t/2} \right) \begin{pmatrix} e^{i\gamma B_0 t/2} \\ -e^{-i\gamma B_0 t/2} \end{pmatrix} \\ &= \frac{\hbar}{4} \left(e^0 - e^0 \right) = 0.\end{aligned}$$

Then the vector sum is given by

$$\langle \vec{S}(t) \rangle = \langle S_x \rangle \hat{x} + \langle S_y \rangle \hat{y} + \langle S_z \rangle \hat{z} = \frac{\hbar}{2} \cos(\gamma B_0 t) \hat{x} - \frac{\hbar}{2} \sin(\gamma B_0 t) \hat{y} + 0 \hat{z}.$$

$$\boxed{\Rightarrow \quad \langle \vec{S}(t) \rangle = \frac{\hbar}{2} \left[\cos(\gamma B_0 t)\ \hat{x} - \sin(\gamma B_0 t)\ \hat{y} \right]}$$

5. Consider another spin 1/2 system.

(a) What are the eigenvalues and eigenvectors of the $S_x + S_y$ operator?

(b) Suppose a measurement of this operator is made, and the system is found to be in the state corresponding to the larger eigenvalue. What is the probability that an immediately following measurement of S_z yields $\hbar/2$?

Add the S_x and the S_y matrices to get the $S_x + S_y$ matrix. Then find the eigenvalues and the eigenvectors of $S_x + S_y$ in the usual way. The eigenvalues will look very familiar, and the eigenvectors will be complex valued. The state vector immediately after a measurement is the eigenvector which corresponds to the eigenvalue measured. So you will need the eigenvectors of S_z, since that is what is being measured. Your probability should be 1/2.

5(a) First, find the eigenvalues of the $S_x + S_y$ operator:

$$\begin{aligned} S_x + S_y &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\hbar}{2}(1-i) \\ \frac{\hbar}{2}(1+i) & 0 \end{pmatrix} \end{aligned}$$

so the characteristic equation is given by

$$\det\begin{pmatrix} -\omega & \frac{\hbar}{2}(1-i) \\ \frac{\hbar}{2}(1+i) & -\omega \end{pmatrix} = \omega^2 - \frac{\hbar^2}{4}(1-i)(1+i) = \omega^2 - \frac{\hbar^2}{4}(1+1) = \omega^2 - \frac{\hbar^2}{2} = 0.$$

And solving this equation yields the eigenvalues

$$\boxed{+\frac{\hbar}{\sqrt{2}} \text{ and } -\frac{\hbar}{\sqrt{2}}.}$$

Next, find the eigenvectors:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} (i-i)b &= \tfrac{2}{\sqrt{2}}a \\ (1+i)a &= \tfrac{2}{\sqrt{2}}b \end{aligned}$$

$$\Rightarrow \quad a = \frac{\sqrt{2}}{2}(1-i)b \;\text{ and }\; a = 1 \quad \Rightarrow \quad b = \frac{\sqrt{2}}{2}(1+i) \quad \Rightarrow \quad |\frac{\hbar}{\sqrt{2}}\rangle = A\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}(1+i) \end{pmatrix}.$$

Calculate the normalization constant:

$$\left(1, \frac{\sqrt{2}}{2}(1-i)\right) A^* A \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} = |A|^2\left(1 + \frac{2}{4}(2)\right) = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}}.$$

Find the second eigenvector

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{\sqrt{2}}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} (i-i)b & = & -\frac{2}{\sqrt{2}}a \\ (1+i)a & = & -\frac{2}{\sqrt{2}}b \end{matrix}$$

$$\Rightarrow \quad a = -\frac{\sqrt{2}}{2}(1-i)b \text{ and } a = 1 \quad \Rightarrow \quad b = -\frac{\sqrt{2}}{2}(1+i) \quad \Rightarrow \quad |-\frac{\hbar}{\sqrt{2}}\rangle = A\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix}.$$

And calculate the normalization constant:

$$\left(1, -\frac{\sqrt{2}}{2}(1-i)\right) A^* A \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix} = |A|^2\left(1 + \frac{2}{4}(2)\right) = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}}.$$

So, the eigenvectors are given by

$$|\frac{\hbar}{\sqrt{2}}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} \quad \text{and} \quad |-\frac{\hbar}{\sqrt{2}}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2}(1+i) \end{pmatrix}.$$

(b) A measurement that yields the larger eigenvalue of $S_x + S_y$ indicates that the state vector is then equal to the eigenvector of $S_x + S_y$ that corresponds to that eigenvalue. That is to say:

$$\text{measuring} \quad \frac{\hbar}{\sqrt{2}} \quad \Rightarrow \quad |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}(1+i) \end{pmatrix}.$$

Now for the S_z operator, the eigenvalues and eigenvectors are given by

$$|S_z = \frac{\hbar}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |S_z = -\frac{\hbar}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A measurement of S_z which yields $\hbar/2$ corresponds to the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so the probability is given by

$$P(S_z = \frac{\hbar}{2}) = \left|(1,0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2}(1+i) \end{pmatrix}\right|^2 = \frac{1}{2}\left|1+0\right|^2 = \frac{1}{2}.$$

6. A spin 1/2 particle is in an eigenstate of S_x with eigenvalue $+\hbar/2$ at time $t = 0$. At time $t = 0$, it is placed in a magnetic field $\vec{B} = (0, 0, B)$, in which it is allowed to precess for a time T. At time $t = T$, the magnetic field is very rapidly rotated to the $\hat{y}$ direction, so that its components are $(0, B, 0)$. After another time interval T, a measurement of S_x is conducted. What is the probability that the value $\hbar/2$ will be found?

There's a lot in this problem. First you need to construct $|\,\psi(0)\!>$ from the measured eigenvalue. Then you will need to calculate the time-dependent $|\,\psi(t)\!>$ from the initial state $|\,\psi(0)\!>$. Remember that you have already calculated an analogous time-dependent $|\,\psi(t)\!>$ in Problem 4. However, here you will need to use the Hamiltonian appropriate to each direction of the field. For the times $t > T$, you will start with the new initial state vector $|\,\psi(0')\!> \;=\; |\,\psi(T)\!>$, and then you will evaluate its time dependence using the new Hamiltonian $H = -\gamma\, B_0\, S_y$. So for $t > T$, you should find

$$|\,\psi(t')\!> = \sum_{i'} |\,i'\!><\!i'\,|\,\psi(T)\!>\, e^{-iE_n' t'/\hbar}.$$

Here $t' = t - T$, the total time the system has evolved in the new field, the E_n's are the eigenvalues of the new Hamiltonian, and the $|\,i'\!>$'s are the eigenvectors of the new Hamiltonian. When you are all done, you should obtain

$$|\,\psi(t = 2T)\!> = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 - i + e^{i\gamma BT} + ie^{-i\gamma BT} \\ 1 - i + e^{-i\gamma BT} + ie^{i\gamma BT} \end{pmatrix}.$$

To calculate the probabilities, calculate the inner product with the appropriate eigenvector of the observable operator being measured, namely $|\,S_x = +1/2\!>$. You should obtain

$$P\left(S_x = \frac{\hbar}{2}\right) = \frac{1}{2}\left[1 + \cos^2(\frac{\gamma Bt}{2})\right].$$

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6. A spin 1/2 object with an eigenvalue of $+\hbar/2$ for the operator S_x at $t = 0$, has the initial state vector

$$|S_x = +\hbar/2\rangle = |\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We calculated this eigenvector in Problem 4. This system is placed in a magnetic field in the z direction. We need the time-dependent state vector in this magnetic field. We also calculated this in Problem 4, where we found

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\gamma B_0 t/2} \\ e^{-i\gamma B_0 t/2} \end{pmatrix}.$$

At time $t = T$, the direction of the magnetic field is switched $B\hat{z} \rightarrow B\hat{y}$. So the new Hamiltonian is given by

$$H' = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = -\gamma\vec{S}\cdot B\hat{y} = -\gamma B S_y = -\frac{\gamma B\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The new eigenvalues and eigenvectors of the Hamiltonian follow directly from the results of Problem 3, and are given by

$$|E = \frac{\gamma B\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \qquad |E = -\frac{\gamma B\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}.$$

For times $t > T$, $|\psi(t)\rangle$ evolves as if $|\psi(t=T)\rangle$ was the initial state $|\psi(t=0)\rangle$ in the new field. We can use this fact to simply calculate $|\psi(t)\rangle$ and we find

$$|\psi(t > T)\rangle = \sum_{i'} |i'\rangle\langle i'|\psi(t=T)\rangle e^{-iE_{i'}t'/\hbar}$$

where $t' = t - T$ the time the system has evolved in the new field, the $E_{i'}$'s are the eigenvalues of H', and the $|i'\rangle$'s are the eigenvectors of H'.

To measure after a second time interval of T, means that $t' = 2T - T = T$, so we have

$$\begin{aligned}
|\psi(2T)\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\frac{1}{\sqrt{2}}(1,+i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma BT/2}\\e^{-i\gamma BT/2}\end{pmatrix}e^{-i\gamma BT/2} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\frac{1}{\sqrt{2}}(1,-i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma BT/2}\\e^{-i\gamma BT/2}\end{pmatrix}e^{i\gamma BT/2}\\
&= \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\-i\end{pmatrix}(1,+i)\begin{pmatrix}1\\e^{-i\gamma BT}\end{pmatrix} + \begin{pmatrix}1\\i\end{pmatrix}(1,-i)\begin{pmatrix}e^{i\gamma BT}\\1\end{pmatrix}\right]\\
&= \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\-i\end{pmatrix}\left(1 + ie^{-i\gamma BT}\right) + \begin{pmatrix}1\\i\end{pmatrix}\left(e^{i\gamma BT} - i\right)\right]\\
&= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 - i + e^{i\gamma BT} + ie^{-i\gamma BT}\\1 - i + e^{-i\gamma BT} + ie^{i\gamma BT}\end{pmatrix}.
\end{aligned}$$

The probability of measuring $\hbar/2$ for S_x at time $2T$ can be calculated as the inner product of the state vector and the eigenvector of S_x associated with the eigenvalue $\hbar/2$, or

$$\begin{aligned}
P\left(S_x = \frac{\hbar}{2}\right) &= \left|\frac{1}{\sqrt{2}}(1,1)\frac{1}{2\sqrt{2}}\begin{pmatrix}1 - i + e^{i\gamma BT} + ie^{-i\gamma BT}\\1 - i + e^{-i\gamma BT} + ie^{i\gamma BT}\end{pmatrix}\right|^2\\
&= \frac{1}{16}\left|1 - i + e^{i\gamma BT} + ie^{-i\gamma BT} + 1 - i + e^{-i\gamma BT} + ie^{i\gamma BT}\right|^2\\
&= \frac{1}{16}\left|2 - 2i + \left(e^{i\gamma BT} + e^{-i\gamma BT}\right) + i\left(e^{i\gamma BT} + e^{-i\gamma BT}\right)\right|^2\\
&= \frac{1}{4}\left|1 - i + \cos(\gamma BT) + i\cos(\gamma BT)\right|^2\\
&= \frac{1}{4}\left|(1 + \cos(\gamma BT)) - i\,(1 - \cos(\gamma BT))\right|^2 \qquad (\ddagger \text{ see note below})\\
&= \frac{1}{4}\left[(1 + \cos(\gamma BT))^2 + (1 - \cos(\gamma BT))^2\right]\\
&= \frac{1}{4}\left[1 + 2\cos(\gamma BT) + \cos^2(\gamma BT) + 1 - 2\cos(\gamma BT) + \cos^2(\gamma BT)\right]\\
&= \frac{1}{4}\left[2 + 2\cos^2(\gamma BT)\right] = \frac{1}{2}\left[1 + \cos^2(\gamma BT)\right].
\end{aligned}$$

In terms of the original time parameter, $t = 2T \;\Rightarrow\; T = t/2$, so we have

$$\boxed{P\left(S_x = \frac{\hbar}{2}\right) = \frac{1}{2}\left[1 + \cos^2(\frac{\gamma Bt}{2})\right].}$$

‡ There was some confusion about this in class. Remember that $|A|^2 = A^*A$. In this case, we put the expression in the form where a complex conjugate is easily identified, *i.e.*, $|(1+\alpha)+i(1-\alpha)|^2 = [(1+\alpha)+i(1-\alpha)][(1+\alpha)-i(1-\alpha)]$. The product of the complex conjugates will be the sum of the squares of the real and imaginary terms.

7. Consider an electron at rest in the oscillating magnetic field

$$\vec{B} = B_0 \cos(\omega t)\,\hat{z},$$

where B_0 and ω are constants.

(a) Construct the Hamiltonian matrix for this system.

(b) The electron starts out at $t = 0$ in the spin up state with respect to the x-axis. Determine $\chi(t)$ at all subsequent times by solving the TDSE.

(c) Show that the probability of getting $-\hbar/2$ for a measurement of S_x at time t is

$$P(S_x = -\frac{\hbar}{2}) = \sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right).$$

(d) Calculate the minimum value of B_0 required to force a complete "spin-flip" in S_x.

As before, the Hamiltonian is just given by

$$H = -\gamma\, B_0\, \cos(\omega t)\, S_z.$$

However, note that now the Hamiltonian is time dependent because the field is time dependent! Because we have a time-dependent Hamiltonian, we must use the TDSE—we cannot use the TISE. This problem is one of the very few time-dependent Hamiltonian problems that we can solve!!! To solve the TDSE, assume the most general form of the state vector

$$|\,\psi(t)\!> = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}.$$

Put this into the TDSE, and take the time derivatives to obtain two first-order differential equations, one for α and another for β. We can solve this problem because the α and β dependence is completely decoupled! Solve these two completely decoupled differential equations to obtain $\alpha(t)$ and $\beta(t)$. You should obtain

$$|\,\psi(t)\!> = A \begin{pmatrix} \exp\left(\frac{i\gamma B_0}{2\omega}\sin(\omega t)\right) \\ \exp\left(-\frac{i\gamma B_0}{2\omega}\sin(\omega t)\right) \end{pmatrix}.$$

Now the only things left to do, are (1) to normalize your time-dependent state vector, and (2) to calculate the probabilities of the measurement. Note that these probabilities are time-dependent because α and β are. Also note that you have calculated the probability of measuring spin down. In part b, the electron was in the spin up state. So if your probability from part c is equal to one, you know that a "spin-flip" has occurred. To conclude this problem, set the probability function of part c equal to one, and solve for the minimum value of B_0.

7(a) First, we need the Hamiltonian. Since $\vec{B} = B_0 \cos(\omega t)\hat{z}$, the Hamiltonian is given by

$$\begin{aligned} H &= -\vec{\mu} \cdot \vec{B} = -\gamma \vec{S} \cdot B_0 \cos(\omega t)\hat{z} \\ &= -\gamma S_z B_0 \cos(\omega t) = -\gamma B_0 \cos(\omega t) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$$\boxed{\Rightarrow \quad H = \frac{\gamma B_0 \hbar}{2} \cos(\omega t) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.}$$

(b) Since the electron starts out at $t = 0$ in the spin up state with respect to the x-axis, we have

$$|\psi(0)\rangle = |S_x = \hbar/2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The TDSE is given by

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle.$$

Now we have $|\psi(0)\rangle$, but we need $|\psi(t)\rangle$, which is unknown to us at the moment. Let

$$|\psi(t)\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix},$$

and plug this into the TDSE to find

$$\begin{aligned} i\hbar \frac{d}{dt} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} &= \frac{\gamma B_0 \hbar}{2} \cos(\omega t) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \\ \Rightarrow \quad i\hbar \begin{pmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \end{pmatrix} &= \frac{\gamma B_0 \hbar}{2} \cos(\omega t) \begin{pmatrix} -\alpha(t) \\ \beta(t) \end{pmatrix}. \end{aligned}$$

This gives us two equations

$$i\hbar \dot{\alpha}(t) = -\frac{\gamma B_0 \hbar}{2} \cos(\omega t) \alpha(t) \qquad \text{and} \qquad i\hbar \dot{\beta}(t) = \frac{\gamma B_0 \hbar}{2} \cos(\omega t) \beta(t).$$

Solving the equation with α, we find

$$\begin{aligned} \dot{\alpha} &= \frac{i\gamma B_0}{2} \cos(\omega t) \alpha \\ \Rightarrow \quad \frac{d\alpha}{dt} &= \frac{i\gamma B_0}{2} \cos(\omega t) \alpha \\ \Rightarrow \quad \frac{d\alpha}{\alpha} &= \frac{i\gamma B_0}{2} \cos(\omega t) dt \\ \Rightarrow \quad \ln \alpha &= \frac{i\gamma B_0}{2\omega} \sin(\omega t) \\ \Rightarrow \quad \alpha(t) &= \exp\left(\frac{i\gamma B_0}{2\omega} \sin(\omega t) \right) \end{aligned}$$

Similarly, solving the equation with β yields

$$\beta(t) = \exp\left(-\frac{i\gamma B_0}{2\omega}\sin(\omega t)\right).$$

So, the time dependent state vector is given by

$$|\psi(t)\rangle = A\begin{pmatrix} \exp\left((i\gamma B_0/2\omega)\sin(\omega t)\right) \\ \exp\left((-i\gamma B_0/2\omega)\sin(\omega t)\right) \end{pmatrix}.$$

To find the normalization constant, we calculate

$$\left(\exp\left(-\frac{i\gamma B_0}{2\omega}\sin(\omega t)\right),\ \exp\left(\frac{i\gamma B_0}{2\omega}\sin(\omega t)\right)\right) A^* A \begin{pmatrix} \exp\left((i\gamma B_0/2\omega)\sin(\omega t)\right) \\ \exp\left((-i\gamma B_0/2\omega)\sin(\omega t)\right) \end{pmatrix}$$

$$= |A|^2\left(e^0 + e^0\right) = 2|A|^2 = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}}.$$

So, the normalized, time-dependent state vector is given by

$$\boxed{|\psi(t)\rangle = \chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left((i\gamma B_0/2\omega)\sin(\omega t)\right) \\ \exp\left((-i\gamma B_0/2\omega)\sin(\omega t)\right) \end{pmatrix}.}$$

(c) If a measurement of S_x yields $-\hbar/2$, we know the state vector is identical to the eigenvector corresponding to that eigenvalue, which is

$$|S_x = -\hbar/2\rangle = |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So, the probability is given by

$$\begin{aligned} P\left(S_x = -\frac{\hbar}{2}\right) &= \left|\frac{1}{\sqrt{2}}(1,\ -1)\frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left((i\gamma B_0/2\omega)\sin(\omega t)\right) \\ \exp\left((-i\gamma B_0/2\omega)\sin(\omega t)\right) \end{pmatrix}\right|^2 \\ &= \frac{1}{4}\left|\exp\left(\frac{i\gamma B_0}{2\omega}\sin(\omega t)\right) - \exp\left(-\frac{i\gamma B_0}{2\omega}\sin(\omega t)\right)\right|^2 \\ &= \frac{1}{4}\left|i2\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)\right|^2 \\ &= \left(i\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)\right)\left(-i\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)\right) \end{aligned}$$

$$\boxed{\Rightarrow \quad P\left(S_x = -\frac{\hbar}{2}\right) = \sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right).}$$

(d) The electron was spin up in part b, and in part c we calculated the probability that it will be measured spin down. If the probability it is measured spin down is equal to 1, we know that a complete spin flip has occurred. So we have

$$\sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right) = 1 \quad \Rightarrow \quad \frac{\gamma B_0}{2\omega}\sin(\omega t) = \frac{\pi}{2} + n\pi.$$

The minimum value of the field B_0 which makes this true occurs when $\sin(\omega t) = 1$. So, the minimum field occurs when

$$\frac{\gamma B_0}{2\omega} = \frac{\pi}{2}$$

$$\boxed{\Rightarrow \quad B_0 = \frac{\pi\omega}{\gamma}.}$$