A Semi–Classical Picture of Spin

Since there is no classical analogy to quantum mechanical spin angular momentum, it is difficult to describe. A picture is often helpful, and though it is a bad analogy, the initial picture in the Bohr hydrogen atom was an electron orbiting a proton as in figure 13–1. Both the electron and proton are charged particles. The "orbit" of the electron around the proton is a current loop, which means there is an orbital magnetic moment. Further, since each of the charged particles has intrinsic spin, each particle constitutes a smaller current loop so each has a spin magnentic moment. The the potential energy of interaction of a magnetic moment and a magnetic field is $V = -\vec{\mu} \cdot \vec{B}$. We will find though quantized, instrinsic spin can have different orientations, so coupling of various magnetic moments with a magnetic field will result in different energies. In an external magnetic field, the coupling due to orbital and spin magnetic moments is known as <u>Zeeman effect</u>. The dominant effect is between the external magnetic field and the orbital magnetic moment because the orbital magnetic moment is much larger than the spin magnetic moments.

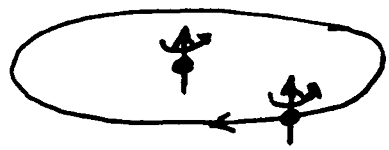

FIG 13-1. THE PROTON IN A HYDROGEN ATOM HAS A SPIN MAGNETIC MOMENT AND THE ELECTRON HAS A SPIN AND ORBITAL MAGNETI MOMENT.

Of more importance at the moment, the relative motions of the proton and electron create an internal magnetic field. Coupling between the different possibilities of the orbital magnetic moment and the internal magnetic field also results in different energies. When discovered, the splitting of energies was considered small or very fine, so this name given to this effect is <u>fine structure</u>. The differences in the energy levels for fine structure splitting is on the order of $0.01\,eV$. Finally, there is interaction between the magnetic moments of the particles and the magnetic fields created by spin alone. This results in further splitting of energy levels, finer than that of the fine structure, and is known as <u>hyperfine structure</u>. The differences in the energy levels for hyperfine structure splitting is on the order of $10^{-7}\,eV$.

The differences in energy levels due to these effects is observable in the spectra of a sample of hydrogen, for instance. Figure 13–2 illustrates the energy level differences due to spin–orbit coupling, fine structure, and spin–spin coupling, hyperfine structure. Significantly, the degeneracy of the energy levels of hydrogen is removed when these smaller effects are considered.

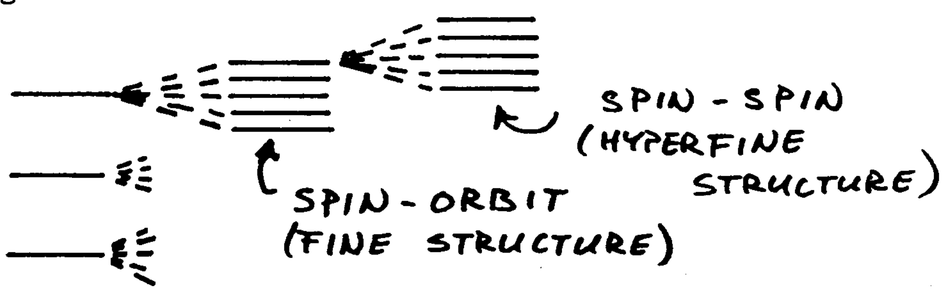

Figure 13 – 2. A principal energy level, and spectral splitting due to spin – orbit and spin – spin coupling

which seems reasonable for atoms with radii on the order of an Angstrom. How fast does this electron spin? Equivalently, what is the speed of a point on the "equator" of this electron? Setting the angular momentum of a solid sphere equal to $\hbar/2$,

$$L = I\omega = \frac{2}{5} m_e r_e^2 \left(\frac{v}{r_e} \right) = \frac{2}{5} m_e r_e^2 v = \frac{\hbar}{2}$$

$$\Rightarrow \quad v = \frac{5}{4} \frac{\hbar}{m_e r_e} = \frac{5}{4} \frac{\hbar}{m_e \left(e^2 / m_e c^2 \right)} = \frac{5}{4} \left(\frac{\hbar c}{e^2} \right) c = \frac{5}{4} (137) c \approx 171 c,$$

which is definitely superluminal, so is dubious....

Example 13–3: What would be the speed of a point on the "equator" of a billiard ball model electron if the Compton radius is used?

$$v = \frac{5}{4} \frac{\hbar}{m_e r_c} = \frac{5}{4} \frac{\hbar}{m_e (2\pi r_e)} = \frac{5}{8\pi} (137) c \approx 27c,$$

which is not much better, and the model has the electron billiard ball radius approach an atomic radius, which is not good for many electron atoms, in particular. The conclusion is these are not good estimates for a billiard ball electron radius. And this is consistent with the current picture that the billiard ball model does not apply to an electron, and the electron has no radius at all....

The Gyromagnetic Ratio

The intent of this section is to introduce some terminology, and more importantly, illustrate how angular momentum associated with a charged particle means a magnetic moment is present.

The classical definition of a <u>gyromagnetic ratio</u> is the ratio of the magnetic moment to the angular momentum. In other words

$$\mu = \gamma \vec{L},$$

where γ is the gyromagnetic ratio.

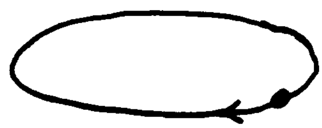

FIG 13-3. AN ELECTRON IN A "CIRCULAR ORBIT"

Example 13–4: Calculate the gyromagnetic ratio for a single electron in a circular orbit using classical arguments.

This example illustrates how angular momentum and magnetic moment are related. A circular current loop has a circumference of $2\pi r$ and an area of πr^2. For a current I in a circular loop of area A, the magnetic moment is $\vec{\mu} = I\vec{A}$. If the current is composed of one electron completing a circle during a period T,

$$\vec{\mu} = I\vec{A} \quad \Rightarrow \quad |\vec{\mu}| = \mu = IA = \frac{e}{T} \pi r^2,$$

where the direction is normal to the plane of the circle in accordance with the right hand rule. One circumference is traversed each period, or $2\pi r = vT$, so

$$\mu = \frac{ev}{2\pi r} \pi r^2 = \frac{1}{2} evr.$$

state space as ξ_s, and full state space as ξ, the sets of eigenvalues of appropriate operators span the respective spaces. Schematically

$$\left.\begin{array}{lcc} & & \{\mathcal{X},\mathcal{Y},\mathcal{Z}\} \\ \xi_{\mathrm{r}} & \longrightarrow & \{\mathcal{P}_x,\mathcal{P}_y,\mathcal{P}_z\} \\ & & \{\mathcal{H},\mathcal{L}^2,\mathcal{L}_z\} \\ & & \\ \xi_s & \longrightarrow & \{\mathcal{S}^2,\mathcal{S}_z\} \end{array}\right\} \qquad \xi \longrightarrow \begin{array}{c} \{\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{S}^2,\mathcal{S}_z\} \\ \{\mathcal{P}_x,\mathcal{P}_y,\mathcal{P}_z,\mathcal{S}^2,\mathcal{S}_z\} \\ \{\mathcal{H},\mathcal{L}^2,\mathcal{L}_z,\mathcal{S}^2,\mathcal{S}_z\} \end{array}$$

Figure 13 – 4. Combination of any representation of orbital state space and spin space is the set of eigenvalues of orbital and spin operators in full state space

In general, the combination of spaces is done by forming a <u>direct product</u> of the component spaces. In this case we would write $\xi = \xi_{\mathrm{r}} \bigotimes \xi_s$, to denote a direct product. A direct product is a generalization of the outer product of vectors to operators, or in this case, spaces. In general,

$$\mathcal{A} \bigotimes \mathcal{B} = \mathcal{C} \quad \Rightarrow \quad \mathcal{C}_{ikjl} = \mathcal{A}_{ij}\mathcal{B}_{kl}.$$

If both $\mathcal{A}$ and $\mathcal{B}$ are represented by $2\,\mathrm{X}\,2$ matrices,

$$\mathcal{A}\bigotimes\mathcal{B} = \begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

You can imagine this is an impractical calculation to do explicitly where one of the component spaces is an infinite dimensional Hilbert space. Instead, we will develop just the spin space and combine the results explicitly for a wavefunction of hydrogen to illustrate the meaning.

Though we will develop eigenvalues and eigenvectors in general, spin 1/2 particles will dominate our discussion so that we can address a specific spin space. Spin spaces are infinite dimensional Hilbert spaces, however, we can represent the operators of spin spaces in subspaces. In particular, spin 1/2 uses a two dimensional subspace. Therefore, our discussion of spin space will involve a number of $2\,\mathrm{X}\,2$ matrix operators.

Ladder Operators for Spin Angular Momentum

We are going to address spin angular momentum in a development similar to orbital angular momentum. The parallels are striking. In fact, spin angular momentum arguments are the same arguments made for orbital angular momentum in chapter 11. The reason is the commutators of the components of all types of angular momentum are canonical, and specifically for spin are

$$\left[\mathcal{S}_x,\,\mathcal{S}_y\right] = i\hbar\,\mathcal{S}_z, \qquad \left[\mathcal{S}_y,\,\mathcal{S}_z\right] = i\hbar\,\mathcal{S}_x, \qquad \text{and} \qquad \left[\mathcal{S}_z,\,\mathcal{S}_x\right] = i\hbar\,\mathcal{S}_y. \tag{13 – 5}$$

Like orbital angular momentum, the square of the spin angular momentum operator commutes with all its components,

$$\left[\mathcal{S}^2,\,\mathcal{S}_i\right] = 0, \tag{13 – 6}$$

$$\Rightarrow \quad \mathcal{S}_z \mathcal{S}_+ |\alpha, \beta> = (\mathcal{S}_+ \mathcal{S}_z + \hbar \mathcal{S}_+) |\alpha, \beta>$$
$$= \mathcal{S}_+ \mathcal{S}_z |\alpha, \beta> + \hbar \mathcal{S}_+ |\alpha, \beta>$$
$$= \mathcal{S}_+ \beta |\alpha, \beta> + \hbar \mathcal{S}_+ |\alpha, \beta>$$
$$= (\beta + \hbar) \mathcal{S}_+ |\alpha, \beta>,$$

$$\Rightarrow \quad \mathcal{S}_z \left(\mathcal{S}_+ |\alpha, \beta> \right) = (\beta + \hbar) \left(\mathcal{S}_+ |\alpha, \beta> \right).$$

Example 13–8: Show $\mathcal{S}_+ |\alpha, \beta>$ is an eigenvector of $\mathcal{S}^2$.

This is a carbon copy of example 11–12. Here

$$\left[\mathcal{S}^2, \mathcal{S}_+ \right] = \mathcal{S}^2 \mathcal{S}_+ - \mathcal{S}_+ \mathcal{S}^2 = 0$$

$$\Rightarrow \quad \mathcal{S}^2 \mathcal{S}_+ = \mathcal{S}_+ \mathcal{S}^2.$$

$$\Rightarrow \quad \mathcal{S}^2 \mathcal{S}_+ |\alpha, \beta> = \mathcal{S}_+ \mathcal{S}^2 |\alpha, \beta> = \mathcal{S}_+ \alpha |\alpha, \beta> = \alpha \mathcal{S}_+ |\alpha, \beta>,$$

$$\Rightarrow \quad \mathcal{S}^2 \left(\mathcal{S}_+ |\alpha, \beta> \right) = \alpha \left(\mathcal{S}_+ |\alpha, \beta> \right).$$

Eigenvalues of S^2

The linear algebra arguments for spin angular momentum are the same as for orbital angular momentum with the exception the operator $\mathcal{S}$ takes the place of the operator L. This is because the commutation relations are the same. Were we to follow the calculations of chapter 11, explicitly substituting $\mathcal{S}$ for $\mathcal{L}$, we would arrive at equations (11–22) and (11–23),

$$\alpha = \beta_{\text{max}}^2 + \hbar \beta_{\text{max}},$$

and

$$\alpha = \beta_{\text{min}}^2 - \hbar \beta_{\text{min}}.$$

Equating these two equations and solving for β_{max} results in

$$\beta_{\text{max}} = -\beta_{\text{min}},$$

which is the maximum ladder separation. It gives us the top and bottom of the ladder. We assume the rungs of the ladder are separated by $\hbar$, because that is the amount of change indicated by the raising and lowering operators. If there is other than minimum separation, say there are n steps between the bottom and top rungs of the ladder, there is a total separation of $n\hbar$ between the bottom and the top. From figure 13–5 we expect

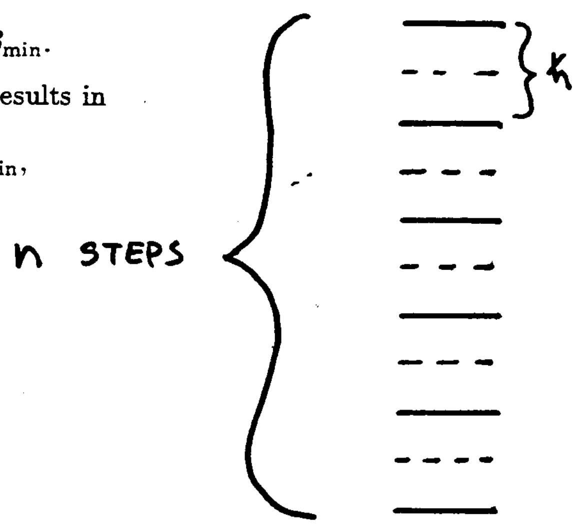

FIG 13-5. LADDER OF n STEPS.

$$2\beta_{\text{max}} = n\hbar \quad \Rightarrow \quad \beta_{\text{max}} = \frac{n\hbar}{2}.$$

$$\Rightarrow \quad \alpha = \beta_{\text{max}} (\beta_{\text{max}} + \hbar)$$