Homework Set 5 Solutions

(a) The four matrices are obtained by straightfoward calculation

$$L_x^2 = L_x L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{\Rightarrow \quad L_x^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

$$L_y^2 = L_y L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\boxed{\Rightarrow \quad L_y^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}}$$

$$L_z^2 = L_z L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boxed{\Rightarrow \quad L_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = \hbar^2 \left[\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$= \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\boxed{\Rightarrow \quad L^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 I}$$

(b) The ladder operator matrices are also obtained by straightforward calculation

$$L_+ = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + i \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\Rightarrow \quad L_+ = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

and

$$L_- = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - i \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\boxed{\Rightarrow \quad L_- = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

(c) First, check the effect of the L_- operator on the three $l = 1$ eigenvectors:

$$L_- \,|\, 1, 1 > = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{2}\,\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$L_- \,|\, 1, 0 > = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\,\hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L_- \,|\, 1, -1 > = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2}\,\hbar \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So

$$L_- \,|\, l,m\!> \; = \sqrt{l(l+1) - m(m-1)}\; \hbar \,|\, l, m-1\!> .$$

Next, check the effect of the L_+ operator on the three $l = 1$ eigenvectors:

$$L_+ \,|\, 1,1\!> \; = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{2}\,\hbar \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$L_+ \,|\, 1,0\!> \; = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\,\hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$L_+ \,|\, 1,-1\!> \; = \sqrt{2}\,\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2}\,\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So $L_+ \,|\, l,m\!> \; = \sqrt{l(l+1) - m(m+1)}\; \hbar^2 \,|\, l, m+1\!>$.

Now, check the effect of the L^2 operator on the three $l = 1$ eigenvectors:

$$L^2 \,|\, 1,1\!> \; = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$L^2 \,|\, 1,0\!> \; = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$L^2 \,|\, 1,-1\!> \; = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So $L^2 \,|\, l,m\!> \; = l(l+1)\hbar^2 \,|\, l,m\!>$.

Finally, check the effect of the L_z operator on the three $l = 1$ eigenvectors:

$$L_z \,|\, 1,1\!> \; = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$L_z \,|\, 1,0\!> \; = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$L_z \,|\, 1,-1\!> \; = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So $L_z \,|\, l, m\!> \,=\, m\hbar \,|\, l, m\!>$.

(d) To calculate the normalization constant, we want to choose A so that $<\psi(0)\,|\,\psi(0)> \,=\, 1$. Then, since we have

$$|\,\psi(t=0)\!> \,=\, A \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

$$\Rightarrow \quad (3,\ 2,\ 4)^*\, A^*\, A \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 1$$

$$\Rightarrow \quad |\,A\,|^2\,(9+4+16) = \,|\,A\,|^2\,(29) = 1$$

$$\boxed{\Rightarrow \quad A = \frac{1}{\sqrt{29}}.}$$

(e) To calculate the possibilities and the probabilities of L^2 measurements, we need to know the eigenvalues and the eigenvectors of L^2. In part a, we found

$$L^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix}.$$

So there is a triply degenerate eigenvalue equal to $2\hbar^2$. Consequently, that is the only possible result of a measurement. Since there is only one possible result of an L^2 measurement, the probability is 1. We conclude

$$\boxed{2\hbar^2 \quad \text{with} \quad P(L^2 = 2\hbar^2) = 1}$$

(f) We need to find the eigenvalues and eigenvectors of L_z, where

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}.$$

So the eigenvalues are $\hbar,\ 0,$ and $-\hbar$, and these are the only possible results of an L_z measurement.

The corresponding probabilities are given by

$$P(L_z = \hbar) = \left|(1,\ 0,\ 0)\frac{1}{\sqrt{29}}\begin{pmatrix}3\\2\\4\end{pmatrix}\right|^2 = \left|\frac{3}{\sqrt{29}}\right|^2 = \frac{9}{29},$$

$$P(L_z = 0) = \left|(0,\ 1,\ 0)\frac{1}{\sqrt{29}}\begin{pmatrix}3\\2\\4\end{pmatrix}\right|^2 = \left|\frac{2}{\sqrt{29}}\right|^2 = \frac{4}{29},$$

$$P(L_z = -\hbar) = \left|(0,\ 0,\ 1)\frac{1}{\sqrt{29}}\begin{pmatrix}3\\2\\4\end{pmatrix}\right|^2 = \left|\frac{4}{\sqrt{29}}\right|^2 = \frac{16}{29}.$$

So the possible results, and the corresponding probabilities are

$$\boxed{\begin{aligned} L_z &= \hbar \quad \text{with} \quad P(L_z = \hbar) = \frac{9}{29}, \\ L_z &= 0 \quad \text{with} \quad P(L_z = 0) = \frac{4}{29}, \\ L_z &= -\hbar \quad \text{with} \quad P(L_z = -\hbar) = \frac{16}{29}. \end{aligned}}$$

(g) To calculate the expectation value of the L^2 operator, we can use the L^2 matrix which is given by

$$L^2 = 2\hbar^2 \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}.$$

$$\begin{aligned}\Rightarrow \quad <L^2> &= <\psi\,|\,L^2\,|\,\psi> = \frac{1}{\sqrt{29}}\big(3,\ 2,\ 4\big)2\hbar^2\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\frac{1}{\sqrt{29}}\begin{pmatrix}3\\2\\4\end{pmatrix} \\ &= \frac{2\hbar^2}{29}\big(3,\ 2,\ 4\big)\begin{pmatrix}3\\2\\4\end{pmatrix} \\ &= \frac{2\hbar^2}{29}\big(9+4+16\big) = \frac{2\hbar^2}{29}\big(29\big)\end{aligned}$$

$$\boxed{\Rightarrow \quad <L^2> = 2\hbar^2.}$$

Next, calculate the standard deviation of the L^2 operator,

$$\boxed{\Delta L^2 = <(L^2 - <L^2> I)^2>^{1/2} = <(2\hbar^2 I - 2\hbar^2 I)^2>^{1/2} = 0.}$$

Since there is only one possible value of L^2, and there can be no variation in it, we also could have immediately concluded that $\Delta L^2 = 0$ without a detailed calculation.

To calculate the expectation value of the L_z operator, we can use the L_z matrix which is given by

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$\begin{aligned} \Rightarrow \quad <L_z> &= <\psi \,|\, L_z \,|\, \psi> = \frac{1}{\sqrt{29}} \big(3,\ 2,\ 4\big) \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{29}} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \\ &= \frac{\hbar}{29} \big(3,\ 2,\ 4\big) \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \\ &= \frac{\hbar}{29} \big(9 + 0 - 16\big) = \frac{\hbar}{29} \big(-7\big) \end{aligned}$$

$$\boxed{\Rightarrow \quad <L_z> = -\frac{7}{29}\, \hbar.}$$

To calculate the standard deviation of the L_z operator, we'll use

$$\Delta L_z = \left[<L_z^2> - <L_z>^2 \right]^{1/2}.$$

So we need L_z^2, and in part a we found

$$L_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

First calculate $<L_z^2>$,

$$\begin{aligned} <L_z^2> &= <\psi \,|\, L_z^2 \,|\, \psi> = \frac{1}{\sqrt{29}} \big(3,\ 2,\ 4\big) \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{29}} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \\ &= \frac{\hbar^2}{29} \big(3,\ 2,\ 4\big) \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \\ &= \frac{\hbar^2}{29} \big(9 + 0 + 16\big) = \frac{\hbar^2}{29} \big(25\big) \end{aligned}$$

Then combine $<L_z^2>$ with $<L_z>^2$ to obtain ΔL_z,

$$\begin{aligned}\Rightarrow \quad \Delta L_z &= \left[\frac{25}{29}\hbar^2 - \left(-\frac{7}{29}\hbar\right)^2\right]^{1/2} \\ &= \left[\frac{25}{29}\hbar^2 - \frac{49}{29^2}\hbar^2\right]^{1/2} \\ &= \hbar\left[\frac{29\cdot 25 - 49}{29^2}\right]^{1/2} \\ &= \hbar\left[\frac{676}{29^2}\right]^{1/2}\end{aligned}$$

$$\boxed{\Rightarrow \quad \Delta L_z = \frac{26}{29}\,\hbar.}$$

(h) Combining our results from parts e, f and g, we will now evaluate

$$<\Omega> = \sum_i P(\omega_i)\,\omega_i \qquad \text{and} \qquad \Delta\Omega = \sqrt{\sum_i P(\omega_i)\,(\omega_i - <\Omega>)^2}$$

first for $\Omega = L^2$, and then for $\Omega = L_z$.

For $<L^2>$, we find

$$\boxed{<L^2> = P\,(2\hbar^2)\,2\hbar^2 = 1\cdot 2\hbar^2 = 2\hbar^2,}$$

and for ΔL^2, we find

$$\boxed{\Delta L^2 = \sqrt{P\,(2\hbar^2)\,\left(2\hbar^2 - 2\hbar^2\right)^2} = \sqrt{1\cdot 0} = 0.}$$

For $<L_z>$, we find

$$\boxed{<L_z> = \sum_i P(\omega_i)\,\omega_i = \frac{9}{29}(+\hbar) + \frac{4}{29}(0) + \frac{16}{29}(-\hbar) = \frac{9-16}{29}(\hbar) = -\frac{7}{29}\hbar,}$$

and for ΔL_z, we find

$$\Delta L_z = \sqrt{\sum_i P(\omega_i)\;(\omega_i - <L_z>)^2}$$

$$= \left[\frac{9}{29}\left(+\hbar - \left(\frac{-7}{29}\hbar\right)\right)^2 + \frac{4}{29}\left(0 - \left(\frac{-7}{29}\hbar\right)\right)^2 + \frac{16}{29}\left(-\hbar - \left(\frac{-7}{29}\hbar\right)\right)^2\right]^{1/2}$$

$$= \left[\frac{9}{29}\left(\frac{36}{29}\hbar\right)^2 + \frac{4}{29}\left(\frac{7}{29}\hbar\right)^2 + \frac{16}{29}\left(\frac{-22}{29}\hbar\right)^2\right]^{1/2}$$

$$= \left[\frac{9}{29}\left(\frac{1296}{29^2}\hbar^2\right) + \frac{4}{29}\left(\frac{49}{29^2}\hbar^2\right) + \frac{16}{29}\left(\frac{484}{29^2}\hbar^2\right)\right]^{1/2}$$

$$= \left[\frac{11664 + 196 + 7744}{29^3}\right]^{1/2}\hbar$$

$$= \left[\frac{19604}{29^3}\right]^{1/2}\hbar = \left[\frac{676}{29^2}\right]^{1/2}\hbar$$

$$\boxed{\Rightarrow \quad \Delta L_z = \frac{26}{29}\hbar.}$$

We conclude that the probability-based method produces exactly the same results as the direct matrix calculation.

(i) The plot of $P(L^2)$ versus L^2 looks like this and the plot of $P(L_z)$ versus L_z looks like this

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(j) First, calculate the expectation value of the L_x operator for the three eigenstates of L_z:

$$<1\,|\,L_x\,|\,1> = (1,\ 0,\ 0)\,\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\,(1,\ 0,\ 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$<2\,|\,L_x\,|\,2> = (0,\ 1,\ 0)\,\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\,(0,\ 1,\ 0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$<3\,|\,L_x\,|\,3> = (0,\ 0,\ 1)\,\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\,(0,\ 0,\ 1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\boxed{\Rightarrow \quad <L_x> = 0 \text{ for any eigenstate of } L_z.}$$

Then, calculate the expectation values of the L_y operator for the three eigenstates of L_z:

$$<1\,|\,L_y\,|\,1> = (1,\ 0,\ 0)\,\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\,(1,\ 0,\ 0)\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix} = 0$$

$$<2\,|\,L_y\,|\,2> = (0,\ 1,\ 0)\,\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\,(0,\ 1,\ 0)\begin{pmatrix} -i \\ 0 \\ i \end{pmatrix} = 0$$

$$<3\,|\,L_y\,|\,3> = (0,\ 0,\ 1)\,\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\,(0,\ 0,\ 1)\begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \quad <L_y> = 0 \text{ for any eigenstate of } L_z.$$

Now, to show that $<L^2 - L_z^2> = <L_x^2 + L_y^2>$, note that we can just show that the $L^2 - L_z^2$ matrix is equal to the $L_x^2 + L_y^2$ matrix. So, calculating we find

$$L^2 - L_z^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hbar^2 \begin{pmatrix} 2-1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2-1 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$L_x^2 + L_y^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{\hbar^2}{2} \begin{pmatrix} 1+1 & 0 & 1-1 \\ 0 & 2+2 & 0 \\ 1-1 & 0 & 1+1 \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrices are identical, so the corresponding operators must have the same expectation values.

(k) The standard sketch of the semiclassical vector model for $l = 1$ angular momentum states looks like this

Note that this picture shows all three eigenvectors of L_z to be vectors of fixed length $\sqrt{l(l+1)\,\hbar^2} = 2\hbar$ precessing around the z-axis. However, even though they are precessing, the three eigenvectors of L_z have fixed projections of L_z: $\hbar$, 0, and $-\hbar$.

Because the three eigenvectors of L_z are precessing around the z-axis, the time-averaged expectation values of L_x and of L_y for the eigenstates of L_z are zero.

The geometry also shows the physical significance of the equation $< L^2 - L_z^2 > \, = \, < L_x^2 + L_y^2 >$: the in-plane projection of the precessing vector is fixed and has length squared $L_x^2 + L_y^2$.

(l) The Hamiltonian is given by

$$H = \frac{L_z^2}{2I}.$$

And we know that

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad L_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So we find

$$\boxed{H = \frac{\hbar^2}{2I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \hbar^2/2I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar^2/2I \end{pmatrix}.}$$

This is diagonal, so we can read off the eigenvalues from the diagonal entries. They are

$$\boxed{\frac{\hbar^2}{2I}, \; 0, \; \text{and} \; \frac{\hbar^2}{2I,}}$$

and the corresponding eigenvectors are

$$\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.}$$

(m) The initial state vector is given by

$$|\,\psi(t=0)> \, = \frac{1}{\sqrt{29}} \left[3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right].$$

So, the time-dependent state vector is given by

$$|\,\psi(t)\,> = \frac{1}{\sqrt{29}}\left[3\begin{pmatrix}1\\0\\0\end{pmatrix} e^{-i\hbar t/2I} + \; 2\begin{pmatrix}0\\1\\0\end{pmatrix} + \; 4\begin{pmatrix}0\\0\\1\end{pmatrix} e^{-i\hbar t/2I}\right].$$

(n) The time-dependent state vector can be rewritten as

$$|\,\psi(t)\,> = \frac{1}{\sqrt{29}}\begin{pmatrix}3e^{-i\alpha}\\2\\4e^{-i\alpha}\end{pmatrix},$$

where we have introduced $\alpha = \hbar t/2I$ for compactness.

The time-dependent expectation value of the energy is given by

$$\begin{aligned}<\psi\,|\,H\,|\,\psi> &= \frac{1}{\sqrt{29}}\left(3e^{+i\alpha},\; 2,\; 4e^{+i\alpha}\right)\frac{\hbar^2}{2I}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\frac{1}{\sqrt{29}}\begin{pmatrix}3e^{-i\alpha}\\2\\4e^{-i\alpha}\end{pmatrix}\\ &= \frac{\hbar^2}{58I}\left(3e^{+i\alpha},\; 2,\; 4e^{+i\alpha}\right)\begin{pmatrix}3e^{-i\alpha}\\0\\4e^{-i\alpha}\end{pmatrix}\\ &= \frac{\hbar^2}{58I}\,(9+16)\end{aligned}$$

$$\Rightarrow \quad <\psi\,|\,H\,|\,\psi>= \frac{25}{29}\,\hbar^2.$$

The time-dependent expectation value of the L^2 operator is given by

$$\begin{aligned}<\psi\,|\,L^2\,|\,\psi> &= \frac{1}{\sqrt{29}}\left(3e^{+i\alpha},\; 2,\; 4e^{+i\alpha}\right)2\hbar^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\frac{1}{\sqrt{29}}\begin{pmatrix}3e^{-i\alpha}\\2\\4e^{-i\alpha}\end{pmatrix}\\ &= \frac{2\hbar^2}{29}\left(3e^{+i\alpha},\; 2,\; 4e^{+i\alpha}\right)\begin{pmatrix}3e^{-i\alpha}\\2\\4e^{-i\alpha}\end{pmatrix}\\ &= \frac{2\hbar^2}{29}\,(9+4+16)\end{aligned}$$

$$\Rightarrow \quad <\psi\,|\,L^2\,|\,\psi>= 2\hbar^2.$$

The time-dependent expectation value of the L_z operator is given by

$$
\begin{aligned}
<\psi\,|\,L_z\,|\,\psi> &= \frac{1}{\sqrt{29}}\left(3e^{+i\alpha},\ 2,\ 4e^{+i\alpha}\right)\hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\frac{1}{\sqrt{29}}\begin{pmatrix}3e^{-i\alpha}\\ 2\\ 4e^{-i\alpha}\end{pmatrix}\\
&= \frac{\hbar}{29}\left(3e^{+i\alpha},\ 2,\ 4e^{+i\alpha}\right)\begin{pmatrix}3e^{-i\alpha}\\ 0\\ -4e^{-i\alpha}\end{pmatrix}\\
&= \frac{\hbar}{29}\left(9+0-16\right)
\end{aligned}
$$

$$
\boxed{\Rightarrow \quad <\psi\,|\,L_z\,|\,\psi>= -\frac{7}{29}\hbar.}
$$

The time-dependent expectation value of the L_x operator is given by

$$
\begin{aligned}
<\psi\,|\,L_x\,|\,\psi> &= \frac{1}{\sqrt{29}}\left(3e^{+i\alpha},\ 2,\ 4e^{+i\alpha}\right)\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\frac{1}{\sqrt{29}}\begin{pmatrix}3e^{-i\alpha}\\ 2\\ 4e^{-i\alpha}\end{pmatrix}\\
&= \frac{\hbar}{29\sqrt{2}}\left(3e^{+i\alpha},\ 2,\ 4e^{+i\alpha}\right)\begin{pmatrix}2\\ 7e^{-i\alpha}\\ 2\end{pmatrix}\\
&= \frac{\hbar}{29\sqrt{2}}\left(6e^{+i\alpha}+14e^{-i\alpha}+8e^{+i\alpha}\right)\\
&= \frac{14\hbar}{29\sqrt{2}}\left(e^{+i\alpha}+e^{-i\alpha}\right) = \frac{14\sqrt{2}\hbar}{29}\left(\frac{e^{+i\alpha}+e^{-i\alpha}}{2}\right)\\
&= \frac{14\sqrt{2}\hbar}{29}\cos\alpha
\end{aligned}
$$

$$
\boxed{\Rightarrow \quad <\psi\,|\,L_x\,|\,\psi>= \frac{14\sqrt{2}\hbar}{29}\ \cos\left(\frac{\hbar t}{2I}\right).}
$$

And, the time-dependent expectation value of the L_y operator is given by

$$\begin{aligned} <\psi\,|\,L_y\,|\,\psi> &= \frac{1}{\sqrt{29}}\left(3e^{+i\alpha},\ 2,\ 4e^{+i\alpha}\right)\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\frac{1}{\sqrt{29}}\begin{pmatrix} 3e^{-i\alpha} \\ 2 \\ 4e^{-i\alpha} \end{pmatrix} \\ &= \frac{\hbar}{29\sqrt{2}}\left(3e^{+i\alpha},\ 2,\ 4e^{+i\alpha}\right)\begin{pmatrix} -2i \\ -ie^{-i\alpha} \\ 2i \end{pmatrix} \\ &= \frac{i\hbar}{29\sqrt{2}}\left(-6e^{+i\alpha} - 2e^{-i\alpha} + 8e^{+i\alpha}\right) \\ &= \frac{i\hbar}{29\sqrt{2}}\left(2e^{+i\alpha} - 2e^{-i\alpha}\right) = -\frac{2\sqrt{2}\hbar}{29}\left(\frac{e^{+i\alpha} - e^{-i\alpha}}{2i}\right) \\ &= -\frac{2\sqrt{2}\hbar}{29}\sin\alpha \end{aligned}$$

$$\boxed{\Rightarrow\quad <\psi\,|\,L_y\,|\,\psi> = -\frac{2\sqrt{2}\,\hbar}{29}\sin\left(\frac{\hbar t}{2I}\right).}$$

Now, why are L^2 and L_z time-independent, in contrast to L_x and L_y which are time-dependent? Time dependence is related to whether the operator commutes with the Hamiltonian, or not.

Let's check which operators commute with the Hamiltonian: H, L^2, and L_z are all diagonal, so they all commute with one another. Remember that you demonstrated that any two diagonal matrices commute in Problem Set 3. Now, let's check whether L_x and L_y commute with the Hamiltonian:

$$[H, L_x] = \frac{\hbar^2}{2I}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{\hbar^2}{2I}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{\hbar^3}{2\sqrt{2}I}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \frac{\hbar^3}{2\sqrt{2}I}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\hbar^3}{2\sqrt{2}I}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \neq 0$$

so H and L_x do not commute. Next check

$$[H, L_y] = \frac{\hbar^2}{2I}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\frac{\hbar^2}{2I}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{\hbar^3}{2\sqrt{2}I}\begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix} - \frac{\hbar^3}{2\sqrt{2}I}\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & -i \\ 0 & 0 & 0 \end{pmatrix} = \frac{\hbar^3}{2\sqrt{2}I}\begin{pmatrix} 0 & -i & 0 \\ -i & 0 & i \\ 0 & i & 0 \end{pmatrix} \neq 0$$

so H and L_y do not commute.

Since L^2 and L_z commute with H, they share a common eigenbasis with H, and therefore their commom eigenvectors all have the same time-dependence. Consequently, $<L^2>$ and $<L_z>$ are time-independent.

Since L_x and L_y do not commute with H, they do not share a common eigenbasis with H. Consequently, L_x and L_y will necessarily evolve in time differently than H, because the eigenvectors of L_x and L_y are both combinations of several eigenvectors of H with different time-dependent exponential phase factors.

(o) The three relevant spherical harmonics are

$$Y_{1,1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad \text{and} \quad Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}.$$

These are the energy eigenfunctions pertinent to this problem. If our initial wave function is rewritten as a function of polar and azimuthal angles, it becomes

$$\boxed{\psi(\theta,\phi,t=0) = \frac{1}{\sqrt{29}}\left[3\sqrt{\frac{3}{8\pi}}\sin\theta\; e^{i\phi} + \;2\sqrt{\frac{3}{4\pi}}\cos\theta + \;4\sqrt{\frac{3}{8\pi}}\sin\theta\; e^{-i\phi}\right].}$$

Since the Hamiltonian is given by

$$H = \frac{L_z^2}{2I},$$

and the L_z operator is given by

$$L_z = -i\hbar\frac{\partial}{\partial\phi},$$

we find

$$H = \frac{1}{2I}\left(-i\hbar\frac{\partial}{\partial\phi}\right)\left(-i\hbar\frac{\partial}{\partial\phi}\right) = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}.$$

And, in position space, the TISE becomes the differential equation

$$\boxed{-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}\;\psi_{lm}(\theta,\phi) = E_n\;\psi_{lm}(\theta,\phi).}$$

Here the energy eigenfunctions are the three $l=1$ spherical harmonics (*i.e.*, the ψ_{lm}'s), and for $l=1$, the equation above represents three equations: one for $m=1$, one for $m=0$ and one for $m=-1$.

The general form of the TDSE is given by

$$H \,|\, \psi_n > = i\hbar \frac{d}{dt} \,|\, \psi_n > .$$

So, in position space, the TDSE becomes the differential equation

$$-\frac{\hbar^2}{2I} \frac{\partial^2}{\partial \phi^2}\, \psi(\theta, \phi, t) = i\, \hbar \frac{d}{dt}\, \psi(\theta, \phi, t),$$

where full time-dependent position space wave function is given by

$$\psi(\theta, \phi, t) = \frac{1}{\sqrt{29}} \left[3\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}\, e^{-i\hbar t/2I} + \, 2\sqrt{\frac{3}{4\pi}} \cos\theta + \, 4\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}\, e^{-i\hbar t/2I} \right].$$

(p) By combining the Hamiltonian matrix from part l, with the three $l = 1$ eigenvectors, the TISE becomes the following three matrix equations

$$\frac{\hbar^2}{2I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = E_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\frac{\hbar^2}{2I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = E_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\frac{\hbar^2}{2I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = E_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

And, the TDSE becomes the following matrix equation

$$\frac{\hbar^2}{2I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{29}} \begin{pmatrix} 3e^{-i\hbar t/2I} \\ 2 \\ 4e^{-i\hbar t/2I} \end{pmatrix} = \frac{i\hbar}{\sqrt{29}} \frac{d}{dt} \begin{pmatrix} 3e^{-i\hbar t/2I} \\ 2 \\ 4e^{-i\hbar t/2I} \end{pmatrix}.$$

Note that if you do the matrix multiplication on the left hand side, and the differentiation on the right hand side, it works—it's a bona fide equality!
