
∗∗**Problem 4.9** A particle of mass m is placed in a *finite* spherical well:

$$V(r) = \begin{cases} -V_0, & \text{if } r \leq a; \\ 0, & \text{if } r > a. \end{cases}$$

Find the ground state, by solving the radial equation with $l = 0$. Show that there is no bound state if $V_0 a^2 < \pi^2 \hbar^2 / 8m$.

4.2 THE HYDROGEN ATOM

The hydrogen atom consists of a heavy, essentially motionless proton (we may as well put it at the origin), of charge e, together with a much lighter electron (charge $-e$) that orbits around it, bound by the mutual attraction of opposite charges (see Figure 4.3). From Coulomb's law, the potential energy (in SI units) is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}, \tag{4.52}$$

and the radial equation (Equation 4.37) says

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu. \tag{4.53}$$

Our problem is to solve this equation for $u(r)$, and determine the allowed energies, E. The hydrogen atom is such an important case that I'm not going to hand you the solutions this time—we'll work them out in detail, by the method we used in the analytical solution to the harmonic oscillator. (If any step in this process is unclear, you may wish to refer back to Section 2.3.2 for a more complete explanation.)

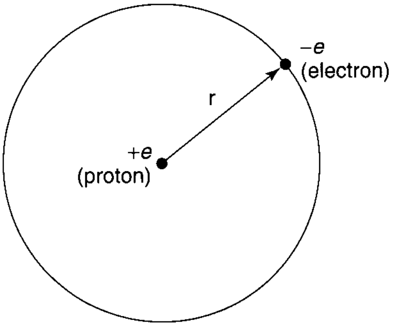

FIGURE 4.3: The hydrogen atom.

Incidentally, the Coulomb potential (Equation 4.52) admits *continuum* states (with $E > 0$), describing electron-proton scattering, as well as discrete *bound* states, representing the hydrogen atom, but we shall confine our attention to the latter.

4.2.1 The Radial Wave Function

Our first task is to tidy up the notation. Let

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}. \tag{4.54}$$

(For bound states, E is negative, so κ is *real*.) Dividing Equation 4.53 by E, we have

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{(\kappa r)} + \frac{l(l+1)}{(\kappa r)^2}\right]u.$$

This suggests that we introduce

$$\rho \equiv \kappa r, \quad \text{and} \quad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}, \tag{4.55}$$

so that

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u. \tag{4.56}$$

Next we examine the asymptotic form of the solutions. As $\rho \to \infty$, the constant term in the brackets dominates, so (approximately)

$$\frac{d^2u}{d\rho^2} = u.$$

The general solution is

$$u(\rho) = Ae^{-\rho} + Be^{\rho}, \tag{4.57}$$

but e^{ρ} blows up (as $\rho \to \infty$), so $B = 0$. Evidently,

$$u(\rho) \sim Ae^{-\rho}, \tag{4.58}$$

for large ρ. On the other hand, as $\rho \to 0$ the centrifugal term dominates;[12] approximately, then:

$$\frac{d^2u}{d\rho^2} = \frac{l(l+1)}{\rho^2}u.$$

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[12] This argument does not apply when $l = 0$ (although the conclusion, Equation 4.59, is in fact valid for that case too). But never mind: All I am trying to do is provide some *motivation* for a change of variables (Equation 4.60).

The general solution (check it!) is

$$u(\rho) = C\rho^{l+1} + D\rho^{-l},$$

but ρ^{-l} blows up (as $\rho \to 0$), so $D = 0$. Thus

$$u(\rho) \sim C\rho^{l+1}, \tag{4.59}$$

for small ρ.

The next step is to peel off the asymptotic behavior, introducing the new function $v(\rho)$:

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \tag{4.60}$$

in the hope that $v(\rho)$ will turn out to be simpler than $u(\rho)$. The first indications are not auspicious:

$$\frac{du}{d\rho} = \rho^l e^{-\rho}\left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right],$$

and

$$\frac{d^2u}{d\rho^2} = \rho^l e^{-\rho}\left\{\left[-2l-2+\rho+\frac{l(l+1)}{\rho}\right]v + 2(l+1-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right\}.$$

In terms of $v(\rho)$, then, the radial equation (Equation 4.56) reads

$$\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(l+1)]v = 0. \tag{4.61}$$

Finally, we assume the solution, $v(\rho)$, can be expressed as a power series in ρ:

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j. \tag{4.62}$$

Our problem is to determine the coefficients $(c_0, c_1, c_2, \dots)$. Differentiating term by term:

$$\frac{dv}{d\rho} = \sum_{j=0}^{\infty} j c_j \rho^{j-1} = \sum_{j=0}^{\infty} (j+1)c_{j+1}\rho^j.$$

[In the second summation I have renamed the "dummy index": $j \to j+1$. If this troubles you, write out the first few terms explicitly, and *check* it. You may object

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$$\frac{dv}{d\rho} = \sum_{j=0}^{\infty} j c_j \rho^{j-1} = \sum_{j=0}^{\infty}(j+1)c_{j+1}\rho^j.$$

[In the second summation I have renamed the "dummy index": $j \to j+1$. If this troubles you, write out the first few terms explicitly, and *check* it. You may object that the sum should now begin at $j = -1$, but the factor $(j + 1)$ kills that term anyway, so we might as well start at zero.] Differentiating again,

$$\frac{d^2v}{d\rho^2} = \sum_{j=0}^{\infty} j(j+1)c_{j+1}\rho^{j-1}.$$

Inserting these into Equation 4.61, we have

$$\sum_{j=0}^{\infty} j(j+1)c_{j+1}\rho^j + 2(l+1)\sum_{j=0}^{\infty}(j+1)c_{j+1}\rho^j$$

$$-2\sum_{j=0}^{\infty} jc_j\rho^j + [\rho_0 - 2(l+1)]\sum_{j=0}^{\infty} c_j\rho^j = 0.$$

Equating the coefficients of like powers yields

$$j(j+1)c_{j+1} + 2(l+1)(j+1)c_{j+1} - 2jc_j + [\rho_0 - 2(l+1)]c_j = 0,$$

or:

$$c_{j+1} = \left\{\frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)}\right\} c_j. \qquad [4.63]$$

This recursion formula determines the coefficients, and hence the function $v(\rho)$: We start with c_0 (this becomes an overall constant, to be fixed eventually by normalization), and Equation 4.63 gives us c_1; putting this back in, we obtain c_2, and so on.[13]

Now let's see what the coefficients look like for large j (this corresponds to large ρ, where the higher powers dominate). In this regime the recursion formula says[14]

$$c_{j+1} \cong \frac{2j}{j(j+1)}c_j = \frac{2}{j+1}c_j.$$

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[13] You might wonder why I didn't use the series method directly on $u(\rho)$—why factor out the asymptotic behavior before applying this procedure? Well, the reason for peeling off ρ^{l+1} is largely aesthetic: Without this, the sequence would begin with a long string of zeros (the first nonzero coefficient being c_{l+1}); by factoring out ρ^{l+1} we obtain a series that starts out with ρ^0. The $e^{-\rho}$ factor is more critical—if you *don't* pull that out, you get a three-term recursion formula, involving c_{j+2}, c_{j+1}, and c_j (*try* it!) and that is enormously more difficult to work with.

[14] You might ask why I don't drop the 1 in $j + 1$—after all, I am ignoring $2(l + 1) - \rho_0$ in the numerator, and $2l + 2$ in the denominator. In this approximation it would be fine to drop the 1 as well, but keeping it makes the argument a little cleaner. Try doing it without the 1, and you'll see what I mean.

Suppose for a moment that this were *exact*. Then

$$c_j = \frac{2^j}{j!}c_0, \qquad [4.64]$$

so

$$v(\rho) = c_0 \sum_{j=0}^{\infty} \frac{2^j}{j!}\rho^j = c_0 e^{2\rho},$$

and hence

$$u(\rho) = c_0 \rho^{l+1} e^{\rho}, \qquad [4.65]$$

which blows up at large ρ. The positive exponential is precisely the asymptotic behavior we *didn't* want, in Equation 4.57. (It's no accident that it reappears here; after all, it *does* represent the asymptotic form of *some* solutions to the radial equation—they just don't happen to be the ones we're interested in, because they aren't normalizable.) There is only one way out of this dilemma: *The series must terminate*. There must occur some maximal integer, $j_{\max}$, such that

$$c_{(j_{\max}+1)} = 0, \qquad [4.66]$$

(and beyond which all coefficients vanish automatically). Evidently (Equation 4.63)

$$2(j_{\max} + l + 1) - \rho_0 = 0.$$

Defining

$$n \equiv j_{\max} + l + 1 \qquad [4.67]$$

(the so-called **principal quantum number**), we have

$$\rho_0 = 2n. \qquad [4.68]$$

But ρ_0 determines E (Equations 4.54 and 4.55):

$$E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{me^4}{8\pi^2\epsilon_0^2\hbar^2\rho_0^2}, \qquad [4.69]$$

so the allowed energies are

$$\boxed{E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}, \quad n = 1, 2, 3, \ldots} \qquad [4.70]$$

This is the famous **Bohr formula**—by any measure the most important result in all of quantum mechanics. Bohr obtained it in 1913 by a serendipitous mixture of inapplicable classical physics and premature quantum theory (the Schrödinger equation did not come until 1924).

Combining Equations 4.55 and 4.68, we find that

$$\kappa = \left(\frac{me^2}{4\pi\epsilon_0\hbar^2}\right)\frac{1}{n} = \frac{1}{an}, \tag{4.71}$$

where

$$\boxed{a \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529 \times 10^{-10}\ \text{m}} \tag{4.72}$$

is the so-called **Bohr radius.**[15] It follows (again, from Equation 4.55) that

$$\rho = \frac{r}{an}. \tag{4.73}$$

The spatial wave functions for hydrogen are labeled by three quantum numbers (n, l, and m):

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)\, Y_l^m(\theta, \phi), \tag{4.74}$$

where (referring back to Equations 4.36 and 4.60)

$$R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho), \tag{4.75}$$

and $v(\rho)$ is a polynomial of degree $j_{\max} = n - l - 1$ in ρ, whose coefficients are determined (up to an overall normalization factor) by the recursion formula

$$c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)}c_j. \tag{4.76}$$

The **ground state** (that is, the state of lowest energy) is the case $n = 1$; putting in the accepted values for the physical constants, we get:

$$\boxed{E_1 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] = -13.6\ \text{eV}.} \tag{4.77}$$

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[15] It is traditional to write the Bohr radius with a subscript: a_0. But this is cumbersome and unnecessary, so I prefer to leave the subscript off.

Evidently the **binding energy** of hydrogen (the amount of energy you would have to impart to the electron in the ground state in order to ionize the atom) is 13.6 eV. Equation 4.67 forces $l = 0$, whence also $m = 0$ (see Equation 4.29), so

$$\psi_{100}(r, \theta, \phi) = R_{10}(r) Y_0^0(\theta, \phi). \tag{4.78}$$

The recursion formula truncates after the first term (Equation 4.76 with $j = 0$ yields $c_1 = 0$), so $v(\rho)$ is a constant (c_0), and

$$R_{10}(r) = \frac{c_0}{a} e^{-r/a}. \tag{4.79}$$

Normalizing it, in accordance with Equation 4.31:

$$\int_0^\infty |R_{10}|^2 r^2 \, dr = \frac{|c_0|^2}{a^2} \int_0^\infty e^{-2r/a} r^2 \, dr = |c_0|^2 \frac{a}{4} = 1,$$

so $c_0 = 2/\sqrt{a}$. Meanwhile, $Y_0^0 = 1/\sqrt{4\pi}$, and hence the ground state of hydrogen is

$$\boxed{\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}.} \tag{4.80}$$

If $n = 2$ the energy is

$$E_2 = \frac{-13.6 \text{ eV}}{4} = -3.4 \text{ eV}; \tag{4.81}$$

this is the first excited state—or rather, *states*, since we can have either $l = 0$ (in which case $m = 0$) or $l = 1$ (with $m = -1$, 0, or $+1$); evidently four different states share this same energy. If $l = 0$, the recursion relation (Equation 4.76) gives

$$c_1 = -c_0 \text{ (using } j = 0\text{)}, \quad \text{and} \quad c_2 = 0 \text{ (using } j = 1\text{)},$$

so $v(\rho) = c_0(1 - \rho)$, and therefore

$$R_{20}(r) = \frac{c_0}{2a}\left(1 - \frac{r}{2a}\right) e^{-r/2a}. \tag{4.82}$$

[Notice that the expansion coefficients $\{c_j\}$ are completely different for different quantum numbers n and l.] If $l = 1$ the recursion formula terminates the series after a single term; $v(\rho)$ is a constant, and we find

$$R_{21}(r) = \frac{c_0}{4a^2} r e^{-r/2a}. \tag{4.83}$$

(In each case the constant c_0 is to be determined by normalization—see Problem 4.11.)

For arbitrary n, the possible values of l (consistent with Equation 4.67) are

$$l = 0, 1, 2, \ldots, n-1, \tag{4.84}$$

and for each l there are $(2l+1)$ possible values of m (Equation 4.29), so the total degeneracy of the energy level E_n is

$$d(n) = \sum_{l=0}^{n-1} (2l+1) = n^2. \tag{4.85}$$

The polynomial $v(\rho)$ (defined by the recursion formula, Equation 4.76) is a function well known to applied mathematicians; apart from normalization, it can be written as

$$v(\rho) = L_{n-l-1}^{2l+1}(2\rho), \tag{4.86}$$

where

$$L_{q-p}^{p}(x) \equiv (-1)^p \left(\frac{d}{dx}\right)^p L_q(x) \tag{4.87}$$

is an **associated Laguerre polynomial**, and

$$L_q(x) \equiv e^x \left(\frac{d}{dx}\right)^q \left(e^{-x} x^q\right) \tag{4.88}$$

is the qth **Laguerre polynomial.**[16] (The first few Laguerre polynomials are listed in Table 4.5; some associated Laguerre polynomials are given in Table 4.6. The first few radial wave functions are listed in Table 4.7, and plotted in Figure 4.4.) The normalized hydrogen wave functions are[17]

$$\boxed{\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}(2r/na)\right] Y_l^m(\theta, \phi).} \tag{4.89}$$

They are not pretty, but don't complain—this is one of the very few realistic systems that can be solved at all, in exact closed form. Notice that whereas the wave functions depend on all three quantum numbers, the *energies* (Equation 4.70) are determined by n alone. This is a peculiarity of the Coulomb potential; in the

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[16] As usual, there are rival normalization conventions in the literature; I have adopted the most nearly standard one.

[17] If you want to see how the normalization factor is calculated, study (for example), L. Schiff, *Quantum Mechanics*, 2nd ed., (McGraw-Hill, New York, 1968), page 93.

TABLE 4.5: The first few Laguerre polynomials, $L_q(x)$.

$L_0 = 1$

$L_1 = -x + 1$

$L_2 = x^2 - 4x + 2$

$L_3 = -x^3 + 9x^2 - 18x + 6$

$L_4 = x^4 - 16x^3 + 72x^2 - 96x + 24$

$L_5 = -x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120$

$L_6 = x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$

TABLE 4.6: Some associated Laguerre polynomials, $L_{q-p}^p(x)$.

$L_0^0 = 1$

$L_1^0 = -x + 1$

$L_2^0 = x^2 - 4x + 2$

$L_0^1 = 1$

$L_1^1 = -2x + 4$

$L_2^1 = 3x^2 - 18x + 18$

$L_0^2 = 2$

$L_1^2 = -6x + 18$

$L_2^2 = 12x^2 - 96x + 144$

$L_0^3 = 6$

$L_1^3 = -24x + 96$

$L_2^3 = 60x^2 - 600x + 1200$

case of the spherical well, you may recall, the energies depend also on l (Equation 4.50). The wave functions are mutually orthogonal:

$$\int \psi_{nlm}^* \, \psi_{n'l'm'} \, r^2 \sin\theta \, dr \, d\theta \, d\phi = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \qquad [4.90]$$

This follows from the orthogonality of the spherical harmonics (Equation 4.33) and (for $n \neq n'$) from the fact that they are eigenfunctions of H with distinct eigenvalues.

Visualizing the hydrogen wave functions is not easy. Chemists like to draw "density plots," in which the brightness of the cloud is proportional to $|\psi|^2$ (Figure 4.5). More quantitative (but perhaps harder to read) are surfaces of constant probability density (Figure 4.6).

$*$**Problem 4.10** Work out the radial wave functions R_{30}, R_{31}, and R_{32}, using the recursion formula (Equation 4.76). Don't bother to normalize them.

TABLE 4.7: The first few radial wave functions for hydrogen, $R_{nl}(r)$.

$$R_{10} = 2a^{-3/2}\exp(-r/a)$$

$$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)\exp(-r/2a)$$

$$R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\frac{r}{a}\exp(-r/2\mathrm{a})$$

$$R_{30} = \frac{2}{\sqrt{27}}a^{-3/2}\left(1 - \frac{2}{3}\frac{r}{a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right)\exp(-r/3a)$$

$$R_{31} = \frac{8}{27\sqrt{6}}a^{-3/2}\left(1 - \frac{1}{6}\frac{r}{a}\right)\left(\frac{r}{a}\right)\exp(-r/3a)$$

$$R_{32} = \frac{4}{81\sqrt{30}}a^{-3/2}\left(\frac{r}{a}\right)^2\exp(-r/3a)$$

$$R_{40} = \frac{1}{4}a^{-3/2}\left(1 - \frac{3}{4}\frac{r}{a} + \frac{1}{8}\left(\frac{r}{a}\right)^2 - \frac{1}{192}\left(\frac{r}{a}\right)^3\right)\exp(-r/4a)$$

$$R_{41} = \frac{\sqrt{5}}{16\sqrt{3}}a^{-3/2}\left(1 - \frac{1}{4}\frac{r}{a} + \frac{1}{80}\left(\frac{r}{a}\right)^2\right)\frac{r}{a}\exp(-r/4a)$$

$$R_{42} = \frac{1}{64\sqrt{5}}a^{-3/2}\left(1 - \frac{1}{12}\frac{r}{a}\right)\left(\frac{r}{a}\right)^2\exp(-r/4a)$$

$$R_{43} = \frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3\exp(-r/4a)$$

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*Problem 4.11

(a) Normalize R_{20} (Equation 4.82), and construct the function ψ_{200}.

(b) Normalize R_{21} (Equation 4.83), and construct ψ_{211}, ψ_{210}, and ψ_{21-1}.

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*Problem 4.12

(a) Using Equation 4.88, work out the first four Laguerre polynomials.

(b) Using Equations 4.86, 4.87, and 4.88, find $v(\rho)$, for the case $n = 5$, $l = 2$.

(c) Find $v(\rho)$ again (for the case $n = 5$, $l = 2$), but this time get it from the recursion formula (Equation 4.76).

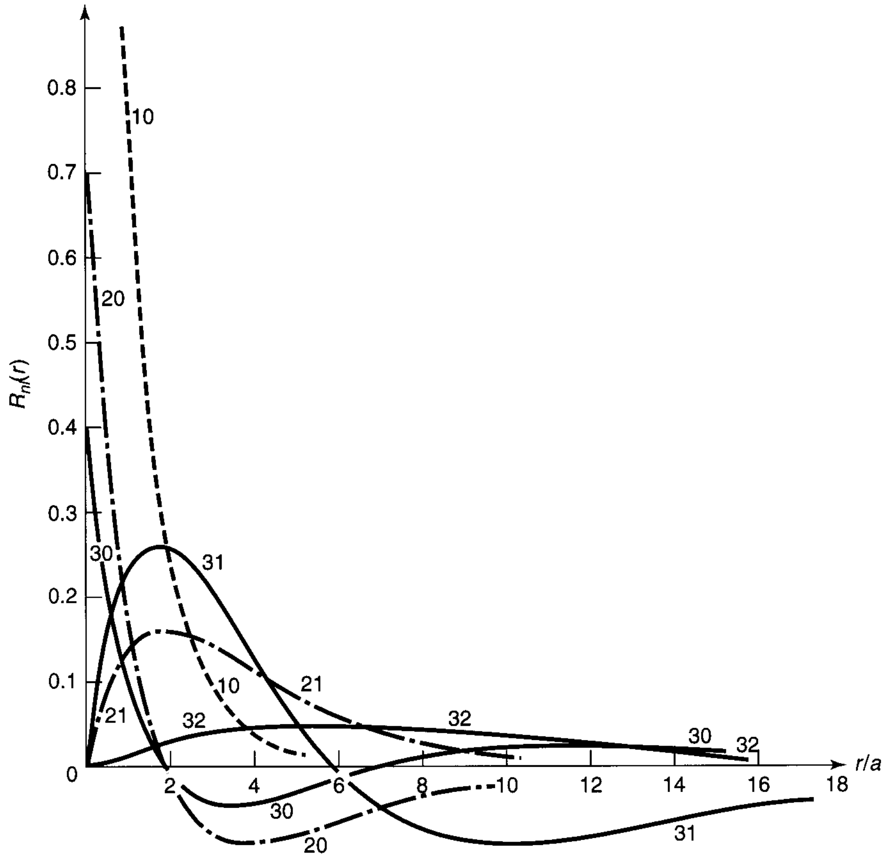


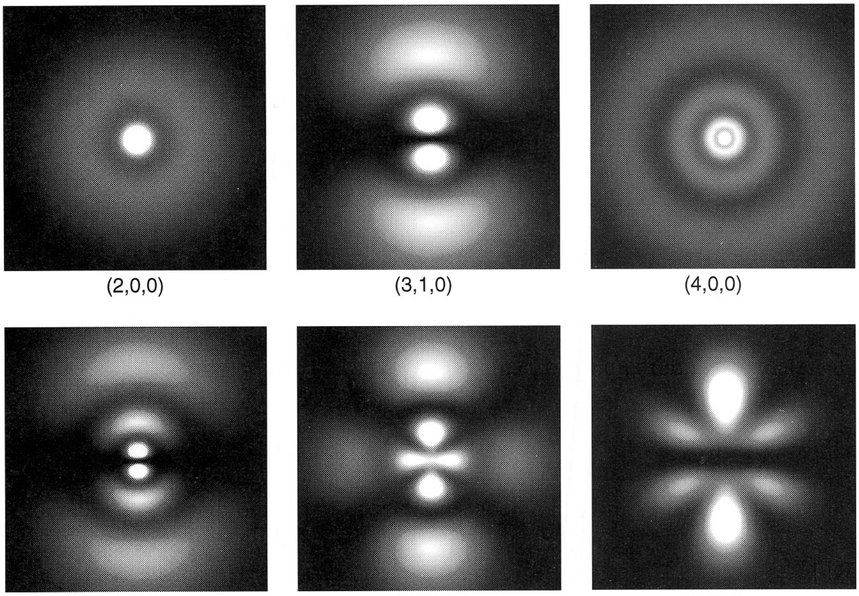
FIGURE 4.4: **Graphs of the first few hydrogen radial wave functions, $R_{nl}(r)$.**

∗Problem 4.13

(a) Find $\langle r\rangle$ and $\langle r^2\rangle$ for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.

(b) Find $\langle x\rangle$ and $\langle x^2\rangle$ for an electron in the ground state of hydrogen. *Hint:* This requires no new integration—note that $r^2 = x^2 + y^2 + z^2$, and exploit the symmetry of the ground state.

(c) Find $\langle x^2\rangle$ in the state $n = 2$, $l = 1$, $m = 1$. *Warning:* This state is *not* symmetrical in x, y, z. Use $x = r\sin\theta\cos\phi$.



(2,0,0) (3,1,0) (4,0,0)

(4,1,0) (4,2,0) (4,3,0)

FIGURE 4.5: Density plots for the hydrogen wave functions (n, l, m). Imagine each plot to be rotated about the (vertical) z axis. Printed by permission using "Atom in a Box," v1.0.8, by Dauger Research. You can make your own plots by going to the Web site http://dauger.com.

Problem 4.14 What is the *most probable* value of r, in the ground state of hydrogen? (The answer is *not* zero!) *Hint:* First you must figure out the probability that the electron would be found between r and $r + dr$.

Problem 4.15 A hydrogen atom starts out in the following linear combination of the stationary states $n = 2$, $l = 1$, $m = 1$ and $n = 2$, $l = 1$, $m = -1$:

$$\Psi(\mathbf{r}, 0) = \frac{1}{\sqrt{2}}(\psi_{211} + \psi_{21-1}).$$

(a) Construct $\Psi(\mathbf{r}, t)$. Simplify it as much as you can.

(b) Find the expectation value of the potential energy, $\langle V \rangle$. (Does it depend on t?) Give both the formula and the actual number, in electron volts.

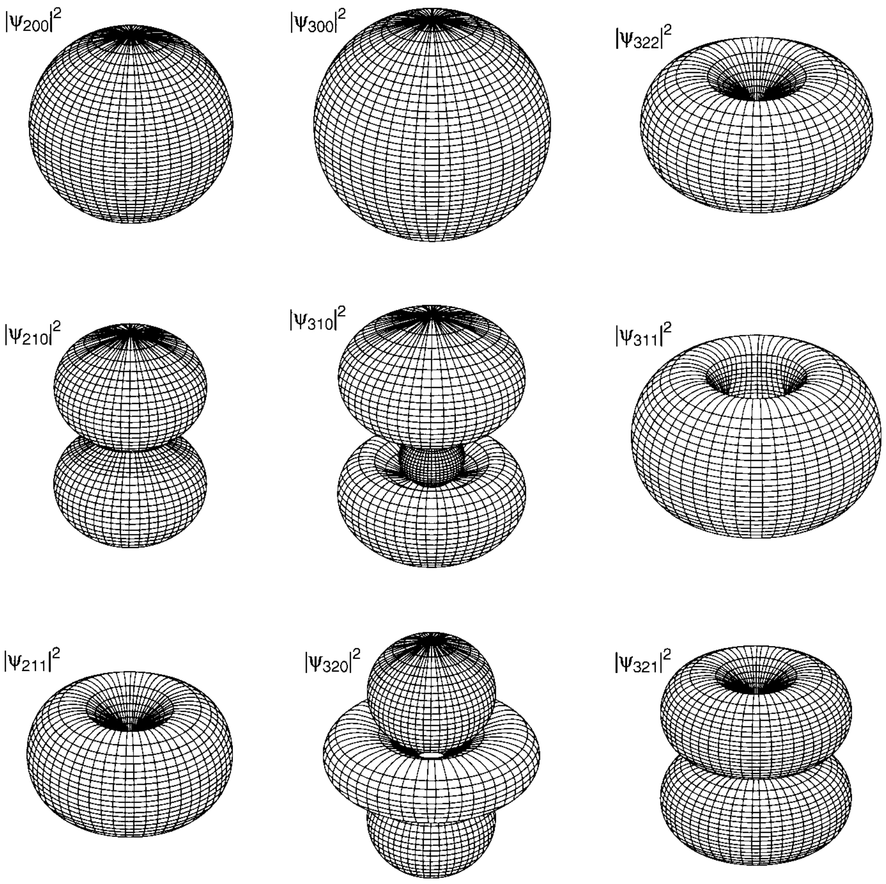


FIGURE 4.6: Surfaces of constant $|\psi|^2$ for the first few hydrogen wave functions. Reprinted by permission from Siegmund Brandt and Hans Dieter Dahmen, *The Picture Book of Quantum Mechanics*, 3rd ed., Springer, New York (2001).

4.2.2 The Spectrum of Hydrogen

In principle, if you put a hydrogen atom into some stationary state Ψ_{nlm}, it should stay there forever. However, if you *tickle* it slightly (by collision with another atom, say, or by shining light on it), the electron may undergo a **transition** to some other stationary state—either by *absorbing* energy, and moving up to a higher-energy state, or by *giving off* energy (typically in the form of electromagnetic radiation), and moving down.[18] In practice such perturbations are *always* present; transitions (or, as they are sometimes called, "quantum jumps") are constantly occurring, and the result is that a container of hydrogen gives off light (photons), whose energy corresponds to the *difference* in energy between the initial and final states:

$$E_\gamma = E_i - E_f = -13.6\text{ eV}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right). \qquad [4.91]$$

Now, according to the **Planck formula**,[19] the energy of a photon is proportional to its frequency:

$$E_\gamma = h\nu. \qquad [4.92]$$

Meanwhile, the *wavelength* is given by $\lambda = c/\nu$, so

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right), \qquad [4.93]$$

where

$$R \equiv \frac{m}{4\pi c\hbar^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = 1.097 \times 10^7\text{ m}^{-1} \qquad [4.94]$$

is known as the **Rydberg constant**. Equation 4.93 is the **Rydberg formula** for the spectrum of hydrogen; it was discovered empirically in the nineteenth century, and the greatest triumph of Bohr's theory was its ability to account for this result—and to calculate R in terms of the fundamental constants of nature. Transitions to the ground state ($n_f = 1$) lie in the ultraviolet; they are known to spectroscopists as the **Lyman series**. Transitions to the first excited state ($n_f = 2$) fall in the visible region; they constitute the **Balmer series**. Transitions to $n_f = 3$ (the **Paschen series**) are in the infrared; and so on (see Figure 4.7). (At room temperature, most hydrogen atoms are in the ground state; to obtain the emission spectrum you must first populate the various excited states; typically this is done by passing an electric spark through the gas.)

∗Problem 4.16 A **hydrogenic atom** consists of a single electron orbiting a nucleus with Z protons ($Z = 1$ would be hydrogen itself, $Z = 2$ is ionized helium, $Z = 3$

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[18] By its nature, this involves a time-*de*pendent interaction, and the details will have to wait for Chapter 9; for our present purposes the actual mechanism involved is immaterial.

[19] The photon is a quantum of electromagnetic radiation; it's a relativistic object if there ever was one, and therefore outside the scope of nonrelativistic quantum mechanics. It will be useful in a few places to speak of photons, and to invoke the Planck formula for their energy, but please bear in mind that this is external to the theory we are developing.

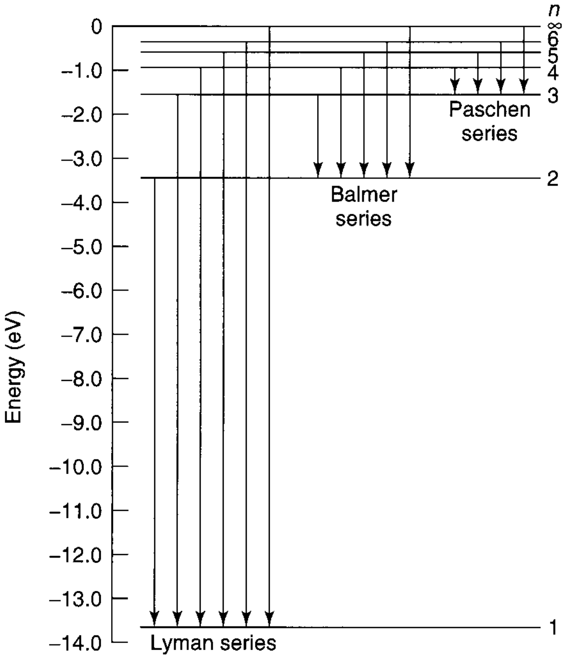

FIGURE 4.7: **Energy levels and transitions in the spectrum of hydrogen.**

is doubly ionized lithium, and so on). Determine the Bohr energies $E_n(Z)$, the binding energy $E_1(Z)$, the Bohr radius $a(Z)$, and the Rydberg constant $R(Z)$ for a hydrogenic atom. (Express your answers as appropriate multiples of the hydrogen values.) Where in the electromagnetic spectrum would the Lyman series fall, for $Z = 2$ and $Z = 3$? *Hint:* There's nothing much to *calculate* here—in the potential (Equation 4.52) $e^2 \rightarrow Ze^2$, so all you have to do is make the same substitution in all the final results.

Problem 4.17 Consider the earth-sun system as a gravitational analog to the hydrogen atom.

(a) What is the potential energy function (replacing Equation 4.52)? (Let m be the mass of the earth, and M the mass of the sun.)

(b) What is the "Bohr radius," a_g, for this system? Work out the actual number.

(c) Write down the gravitational "Bohr formula," and, by equating E_n to the classical energy of a planet in a circular orbit of radius r_o, show that $n = \sqrt{r_o/a_g}$. From this, estimate the quantum number n of the earth.