(The subscript x denotes differentiation.) Comparison of this equation with (7.101) permits the identification

$$\mathbb{J}_x = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) \tag{7.106}$$

Note that the dimensions of $\mathbb{J}_x$ are number per second. In three dimensions the current density is written

$$\mathbf{J} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*) \tag{7.107}$$

and has dimensions $cm^{-2}\,s^{-1}$.

Transmission and Reflection Coefficients

For one-dimensional scattering problems, the particles in the beam are in plane-wave states with definite momentum. Given the wavefunctions relevant to incident, reflected, and transmitted beams, one may calculate the corresponding current densities according to (7.106). The *transmission coefficient* T and *reflection coefficient* R are defined as

$$T \equiv \left|\frac{\mathbb{J}_{\text{trans}}}{\mathbb{J}_{\text{inc}}}\right|, \qquad R \equiv \left|\frac{\mathbb{J}_{\text{refl}}}{\mathbb{J}_{\text{inc}}}\right| \tag{7.108}$$

These one-dimensional barrier problems are closely akin to problems on the transmission and reflection of electromagnetic plane waves through media of varying index of refraction (see Fig. 7.16). In the quantum mechanical case, the scattering is also of waves.

For one-dimensional barrier problems there are three pertinent beams. Particles in the incident beam have momentum

$$p_{\text{inc}} = \hbar k_1 \tag{7.109}$$

Particles in the reflected beam have the opposite momentum

$$p_{\text{refl}} = -\hbar k_1 \tag{7.110}$$

In the event that the environment (i.e., the potential) in the domain of the transmitted beam ($x = +\infty$) is different from that of the incident beam ($x = -\infty$), the momenta in these two domains will differ. Particles in the transmitted beam will have momentum $\hbar k_2 \neq \hbar k_1$,

$$p_{\text{trans}} = \hbar k_2 \tag{7.111}$$

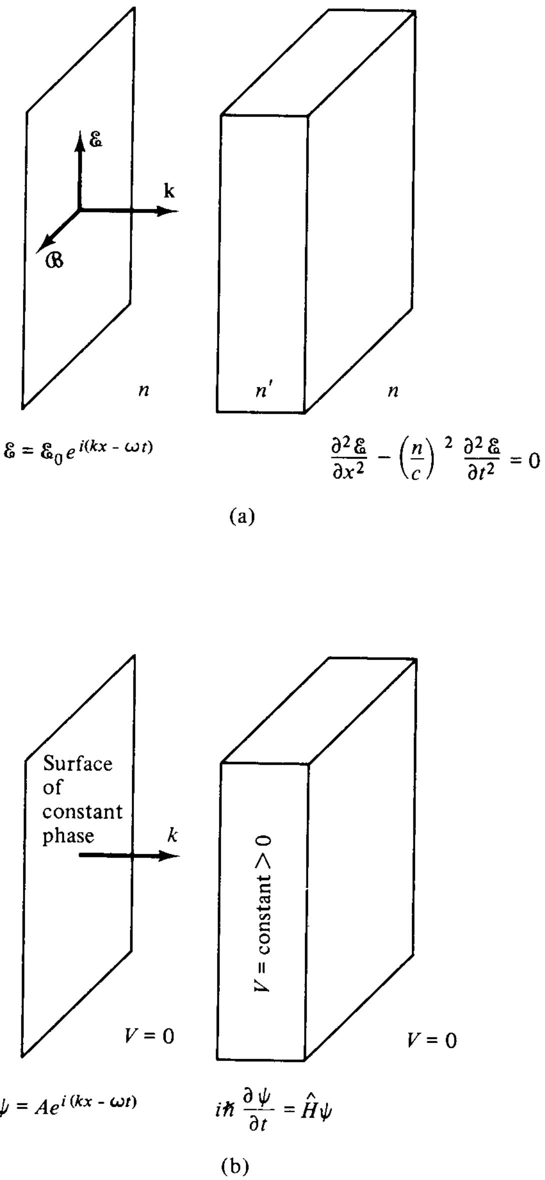


FIGURE 7.16 (a) Scattering of plane electromagnetic waves through domains of different index of refraction n. (b) Scattering of plane, free-particle wavefunctions through domains of different potential.

In all cases the potential is constant in the domains of the incident and transmitted beams (see Fig. 7.14), so the wavefunctions in these domains describe free particles, and we may write

$$\begin{aligned}
\psi_{\text{inc}} &= Ae^{i(k_1x-\omega_1t)}, & \hbar\omega_1 &= E_{\text{inc}} = \frac{\hbar^2{k_1}^2}{2m} \\
\psi_{\text{refl}} &= Be^{-i(k_1x+\omega_1t)}, & \hbar\omega_1 &= E_{\text{refl}} = E_{\text{inc}} \\
\psi_{\text{trans}} &= Ce^{i(k_2x-\omega_2t)}, & \hbar\omega_2 &= E_{\text{trans}} = \frac{\hbar^2{k_2}^2}{2m} + V \\
& & &= E_{\text{inc}} = \hbar\omega_1
\end{aligned} \tag{7.112}$$

Energy is conserved across the potential hill so that frequency remains constant ($\omega_1 = \omega_2$). The change in wavenumber k corresponds to changes in momentum and kinetic energy. Using (7.106) permits calculation of the currents

$$\begin{aligned}
\mathbb{J}_{\text{inc}} &= \frac{\hbar}{2mi}2ik_1|A|^2 \\
\mathbb{J}_{\text{trans}} &= \frac{\hbar}{2mi}2ik_2|C|^2 \\
\mathbb{J}_{\text{refl}} &= -\frac{\hbar}{2mi}2ik_1|B|^2
\end{aligned} \tag{7.113}$$

It should be noted that these relations are equivalent to the classical prescription for particle current, $\mathbb{J} = \rho v$, with $\rho = |\psi|^2$ and $v = \hbar k/m$. These formulas, together with (7.108), give the T and R coefficients

$$T = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_1}, \qquad R = \left|\frac{B}{A}\right|^2 \tag{7.114}$$

In the event that the potentials in domains of incident and transmitted beams are equal, $k_1 = k_2$ and $T = |C/A|^2$. More generally, to calculate C/A and B/A as functions of the parameters of the scattering experiment (namely, incident energy, structure of potential barrier), one must solve the Schrödinger equation across the domain of the potential barrier.

PROBLEMS

7.34 Show that the current density $\mathbf{J}$ may be written

$$\mathbf{J} = \frac{1}{2m}[\psi^*\hat{\mathbf{p}}\psi + (\psi^*\hat{\mathbf{p}}\psi)^*]$$

where $\hat{\mathbf{p}}$ is the momentum operator.

7.35 Show that for a one-dimensional wavefunction of the form

$$\psi(x, t) = A \exp [i\phi(x, t)]$$

$$\mathbb{J} = \frac{\hbar}{m} |A|^2 \frac{\partial \phi}{\partial x}$$

7.36 Show that for a *wave packet* $\psi(x, t)$, one may write

$$\int_{-\infty}^{\infty} \mathbb{J}\, dx = \frac{1}{2m} (\langle p \rangle + \langle p \rangle^*) = \frac{\langle p \rangle}{m}$$

7.37 Show that a complex potential function, $V^*(x) \neq V(x)$, contradicts the continuity equation (7.97).

7.38 (a) Show that if $\psi(x, t)$ is real, then

$$\mathbb{J} = 0$$

for all x.

(b) What type of wave structure does a real state function correspond to?

7.6 ONE-DIMENSIONAL BARRIER PROBLEMS

In a one-dimensional scattering experiment, the intensity and energy of the particles in the incident beam are known in addition to the structure of the potential barrier $V(x)$. Three fundamental scattering configurations are depicted in Fig. 7.17. The energy of the particles in the beam is denoted by E.

The Simple Step

Let us first consider the simple step (Fig. 7.17a) for the case $E > V$. We wish to obtain the space-dependent wavefunction φ for all x. The potential function is zero for $x < 0$ and is the constant V, for $x \geq 0$. The incident beam comes from $x = -\infty$. To construct φ we divide the x axis into two domains: region I and region II, depicted in Fig. 7.18. In region I, $V = 0$, and the time-independent Schrödinger equation appears as

$$-\frac{\hbar^2}{2m} \varphi_{xx} = E\varphi \tag{7.115}$$

In this domain the energy is entirely kinetic. If we set

$$\frac{\hbar^2 k_1{}^2}{2m} = E \tag{7.116}$$

then the latter equation becomes

$$\varphi_{xx} = -k_1{}^2\varphi \qquad \text{in region I} \tag{7.117}$$

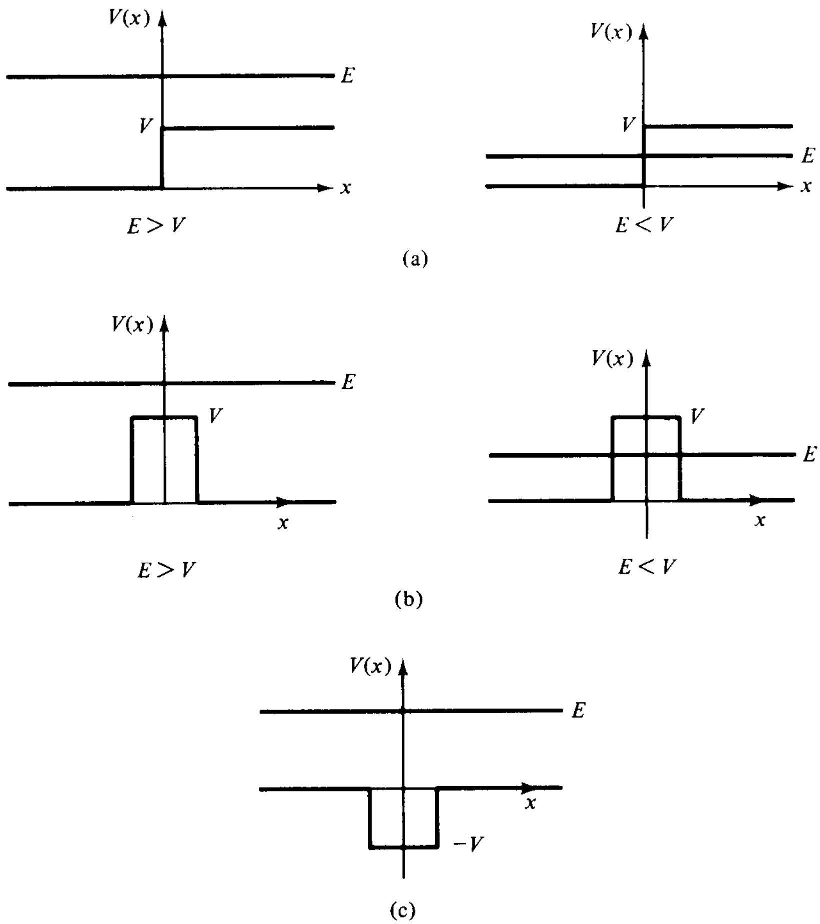


FIGURE 7.17 (a) The simple step. (b) The rectangular barrier. (c) The rectangular well.

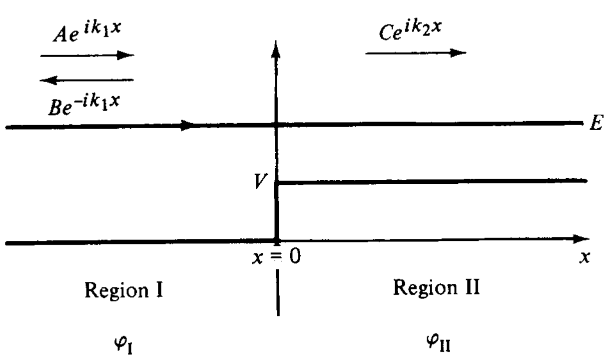


FIGURE 7.18 Domains relevant to the simple-step scattering problem for the case $E \geq V$.

In region II the potential is the constant V and the time-independent Schrödinger equation appears as

$$-\frac{\hbar^2}{2m}\varphi_{xx} = (E - V)\varphi \tag{7.118}$$

The kinetic energy decreases by V and is given by

$$\frac{\hbar^2 k_2^{\,2}}{2m} = E - V \tag{7.119}$$

In terms of k_2, (7.118) appears as

$$\varphi_{xx} = -k_2^{\,2}\varphi \qquad \text{in region II} \tag{7.120}$$

Writing φ_{I} for the solution to (7.117) and φ_{II} for the solution to (7.120), one obtains

$$\begin{aligned} \varphi_{\mathrm{I}} &= Ae^{ik_1x} + Be^{-ik_1x} \\ \varphi_{\mathrm{II}} &= Ce^{ik_2x} + De^{-ik_2x} \end{aligned} \tag{7.121}$$

Since the term De^{-ik_2x} (together with the time-dependent factor $e^{-i\omega_2 t}$) represents a wave emanating from the right ($x = +\infty$ in Fig. 7.18), and there is no such wave, we may conclude that $D = 0$. The interpretation of the remaining A, B, and C terms is given in Eq. (7.112). To repeat, $A \exp(ik_1x)$ represents the incident wave; $B \exp(-ik_1x)$, the reflected wave; and $C \exp(ik_2x)$, the transmitted wave.

It is important at this time to realize that φ_{I} and φ_{II} (with $D \equiv 0$) represent a single solution to the Schrödinger equation for all x, for the potential curve depicted in Fig. 7.18. Since any wavefunction and its first derivative are continuous (see Section 3.3), at the point $x = 0$ where φ_{I} and φ_{II} join it is required that

$$\begin{aligned} \varphi_{\mathrm{I}}(0) &= \varphi_{\mathrm{II}}(0) \\ \frac{\partial}{\partial x}\varphi_{\mathrm{I}}(0) &= \frac{\partial}{\partial x}\varphi_{\mathrm{II}}(0) \end{aligned} \tag{7.122}$$

These equalities give the relations

$$\begin{aligned} A + B &= C \\ A - B &= \frac{k_2}{k_1}C \end{aligned} \tag{7.123}$$

Solving for C/A and B/A, one obtains

$$\frac{C}{A} = \frac{2}{1 + k_2/k_1}, \qquad \frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1} \tag{7.124}$$

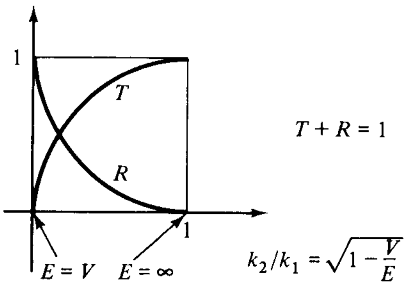


FIGURE 7.19 *T* **and** *R* **versus** k_2/k_1 **for the simple-step scattering problem for** $E \geq V$.

Substituting these values into (7.114) gives

(7.125) $$T = \frac{4k_2/k_1}{[1 + (k_2/k_1)]^2}, \qquad R = \left|\frac{1 - k_2/k_1}{1 + k_2/k_1}\right|^2$$

The ratio k_2/k_1 is obtained from (7.116) and (7.119).

(7.126) $$\left(\frac{k_2}{k_1}\right)^2 = 1 - \frac{V}{E}$$

In the present case $E \geq V$, so $0 \leq k_2/k_1 \leq 1$. For $E \gg V$, $k_2/k_1 \to 1$ and $T \to 1, R \to 0$. There is total transmission. For $E = V$, $k_2/k_1 = 0$ and $T = 0$, $R = 1$. There is total reflection and zero transmission. The T and R curves for the simple-step potential are sketched in Fig. 7.19. For all values of (k_2/k_1) we note that

(7.127) $$T + R = 1$$

The validitiy of this relation for all one-dimensional barrier problems is proved in Problem 7.39.

In the second configuration for the simple-step barrier, $E < V$ (see Fig. 7.17a). Again the x domain is divided into two regions, as shown in Fig. 7.20. In region I the Schrödinger equation becomes

(7.128) $$\varphi_{xx} = -k_1{}^2\varphi \qquad \text{in region I}$$

where

(7.129) $$\frac{\hbar^2 k_1{}^2}{2m} = E$$

In region II the Schrödinger equation is

(7.130) $$\varphi_{xx} = \kappa^2\varphi \qquad \text{in region II}$$

where

(7.131) $$\frac{\hbar^2\kappa^2}{2m} = V - E > 0$$

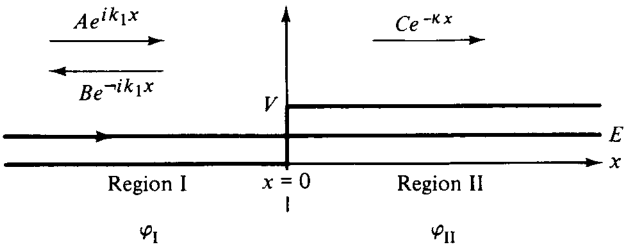


FIGURE 7.20 Domains relevant to the simple-step scattering problem for the case $E \leq V$.

The kinetic energy in this domain is negative $(-\hbar^2\kappa^2/2m)$. In classical physics region II is a "forbidden" domain. In quantum mechanics, however, it is possible for particles to penetrate the barrier.

Again calling the solution to (7.128) φ_I and the solution to (7.130) φ_{II}, we obtain

$$\begin{aligned} \varphi_I &= Ae^{ik_1x} + Be^{-ik_1x} \\ \varphi_{II} &= Ce^{-\kappa x} \end{aligned} \tag{7.132}$$

Continuity of φ and φ_x at $x = 0$ gives

$$\begin{aligned} 1 + \frac{B}{A} &= \frac{C}{A} \\ 1 - \frac{B}{A} &= i\frac{\kappa}{k_1}\frac{C}{A} \end{aligned} \tag{7.133}$$

Solving for (C/A) and (B/A) one obtains

$$\begin{aligned} \frac{C}{A} &= \frac{2}{1 + i\kappa/k_1} \\ \frac{B}{A} &= \frac{1 - i\kappa/k_1}{1 + i\kappa/k_1} \end{aligned} \tag{7.134}$$

The coefficient B/A is of the form z^*/z, where z is a complex number. It follows that $|B/A| = 1$, so

$$R = \left|\frac{B}{A}\right|^2 = 1, \qquad T = 0 \tag{7.135}$$

There is total reflection, hence the transmission must be zero.

To obtain the latter result analytically from our equations above, we must calculate the transmitted current. The function φ_{II} is of the form of a complex amplitude times a real function of x (7.132). Such wavefunctions do not represent propagating waves. They are sometimes called *evanescent waves*. That they carry no current is most simply seen by constructing $\mathbb{J}_{\text{trans}}$ (7.106).

$$\mathbb{J}_{\text{trans}} = \frac{\hbar}{2mi}|C|^2\left(e^{-\kappa x}\frac{\partial}{\partial x}e^{-\kappa x} - e^{-\kappa x}\frac{\partial}{\partial x}e^{-\kappa x}\right) = 0 \tag{7.136}$$

We conclude that $T = 0$.

PROBLEMS

7.39 Show that

$$T + R = 1$$

for all one-dimensional barrier problems.

Answer

Since the scattering process is assumed to be steady-state, the continuity equation (7.101) becomes

$$\frac{\partial \mathbb{J}_x}{\partial x} = 0$$

Integrating this equation, one obtains

$$\int_{-\infty}^{\infty}\left(\frac{\partial \mathbb{J}_x}{\partial x}\right)dx = \mathbb{J}_{+\infty} - \mathbb{J}_{-\infty} = 0$$

But

$$\mathbb{J}_{-\infty} = \mathbb{J}_{\text{inc}} - \mathbb{J}_{\text{refl}}$$

$$\mathbb{J}_{+\infty} = \mathbb{J}_{\text{trans}}$$

so that the equation above becomes

$$\mathbb{J}_{\text{trans}} + \mathbb{J}_{\text{refl}} = \mathbb{J}_{\text{inc}}$$

Dividing through by $\mathbb{J}_{\text{inc}}$ gives the desired result.

7.40 Electrons in a beam of density $\rho = 10^{15}$ electrons/m are accelerated through a potential of 100 V. The resulting current then impinges on a potential step of height 50 V.

(a) What are the incident, reflected, and transmitted currents?

(b) Design an electrostatic configuration that gives a simple-step potential.

7.41 Show that the reflection coefficients for the two cases depicted in Fig. 7.21 are equal.

7.42 For the scattering configuration depicted in Fig. 7.20, given that $V = 2E$, at what value of x is the density in region II half the density of particles in the incident beam?

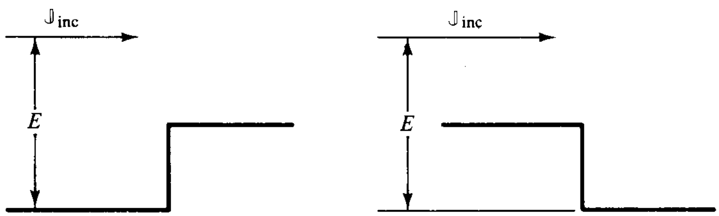


FIGURE 7.21 Reflection coefficients for these two configurations are equal. (See Problem 7.41.)

7.43 Equation (7.123) may be written in the matrix form

$$\begin{pmatrix} -1 & 1 \\ 1 & k_2/k_1 \end{pmatrix} \begin{pmatrix} B/A \\ C/A \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Calling the 2×2 matrix $\mathscr{D}$, the left column vector $\mathscr{V}$, and the right column vector $\mathscr{U}$ permits this equation to be more simply written

$$\mathscr{D}\mathscr{V} = \mathscr{U}$$

This inhomogeneous matrix equation has the solution

$$\mathscr{V} = \mathscr{D}^{-1}\mathscr{U}$$

where $\mathscr{D}^{-1}$ is the inverse of $\mathscr{D}$, that is,

$$\mathscr{D}^{-1}\mathscr{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) Find $\mathscr{D}^{-1}$ and then construct $\mathscr{V}$ using the technique above. Check your answer with (7.124).

(b) Do the same for (7.133) and (7.134).

7.7 THE RECTANGULAR BARRIER. TUNNELING

The scattering configuration we now wish to examine is depicted in Fig. 7.17b. The energy of the particles in the beam is greater than the height of the potential barrier, $E > V$. For the case at hand there are three relevant domains (see Fig. 7.22):

$$\begin{aligned} &\text{Region I:} && x < -a, \quad V = 0. \\ (7.137)\qquad &\text{Region II:} && -a \leq x \leq +a, \quad V > 0, \quad \text{and constant.} \\ &\text{Region III:} && a < x, \quad V = 0. \end{aligned}$$

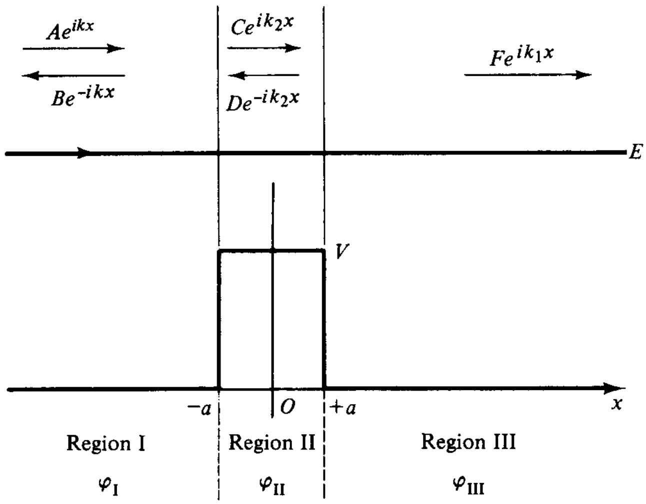


FIGURE 7.22 Domains relevant to the rectangular barrier scattering problem for the case $E \geq V$.

The solutions to the time-independent Schrödinger equation in each of the three domains are:

$$\varphi_I = Ae^{ik_1x} + Be^{-ik_1x}, \qquad \frac{\hbar^2 k_1^2}{2m} = E$$

$$\varphi_{II} = Ce^{ik_2x} + De^{-ik_2x}, \qquad \frac{\hbar^2 k_2^2}{2m} = E - V$$

(7.138)

$$\varphi_{III} = Fe^{ik_1x}, \qquad \frac{\hbar^2 k_1^2}{2m} = E$$

$$(ak_1)^2 - (ak_2)^2 = \frac{2ma^2V}{\hbar^2} \equiv \frac{g^2}{4}$$

The parameter g contains all the barrier (or well) characteristics. The latter equation (conservation of energy) reveals the simple manner in which ak_1 and ak_2 are related. In Cartesian ak_1, ak_2 space they lie on a hyperbola (Fig. 7.23). The permitted values of k_1 (and therefore E) comprise a positive unbounded continuum. For each such eigen-k_1-value, there is a corresponding eigenstate (φ_I, φ_{II}, φ_{III}) which is determined in terms of the coefficients, $(B/A, C/A, D/A, F/A)$. Knowledge of these coefficients gives the scattering parameters

$$T = \left|\frac{F}{A}\right|^2; \qquad R = \left|\frac{B}{A}\right|^2$$

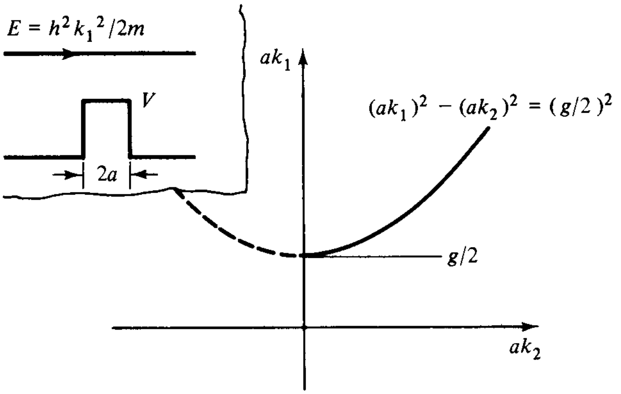


FIGURE 7.23 **For rectangular-barrier scattering with $E \geq V$, ak_1 and ak_2 lie on a hyperbola.**

$$ak_1 \geq ak_2 \geq 0$$

The energy spectrum $\hbar^2 k_1{}^2/2m$ comprises an unbounded continuum.

The coefficients are determined from the boundary conditions at $x = a$ and $x = -a$,

$$\begin{aligned} e^{-ik_1 a} + \left(\frac{B}{A}\right)e^{ik_1 a} &= \left(\frac{C}{A}\right)e^{-ik_2 a} + \left(\frac{D}{A}\right)e^{ik_2 a} \\ k_1\left[e^{-ik_1 a} - \left(\frac{B}{A}\right)e^{ik_1 a}\right] &= k_2\left[\left(\frac{C}{A}\right)e^{-ik_2 a} - \left(\frac{D}{A}\right)e^{ik_2 a}\right] \\ \left(\frac{C}{A}\right)e^{ik_2 a} + \left(\frac{D}{A}\right)e^{-ik_2 a} &= \left(\frac{F}{A}\right)e^{ik_1 a} \\ k_2\left[\left(\frac{C}{A}\right)e^{ik_2 a} - \left(\frac{D}{A}\right)e^{-ik_2 a}\right] &= k_1\left(\frac{F}{A}\right)e^{ik_1 a} \end{aligned} \tag{7.139}$$

These are four linear, algebraic, inhomogeneous equations for the four unknowns: (B/A), (C/A), (D/A), and (F/A). Solving the last two for (D/A) and (C/A) as functions of (F/A) and substituting into the first two permits one to solve for (B/A) and (F/A). These appear as

$$\begin{aligned} \frac{F}{A} &= e^{2ik_1 a}\left[\cos(2k_2 a) - \frac{i}{2}\left(\frac{k_1{}^2 + k_2{}^2}{k_1 k_2}\right)\sin(2k_2 a)\right]^{-1} \\ 2\left(\frac{B}{A}\right) &= i\left(\frac{F}{A}\right)\frac{k_2{}^2 - k_1{}^2}{k_1 k_2}\sin(2k_2 a) \end{aligned} \tag{7.140}$$

The transmission coefficient is most simply obtained from the second of these, together with the relation

$$T + R = \left|\frac{F}{A}\right|^2 + \left|\frac{B}{A}\right|^2 = 1 \tag{7.141}$$

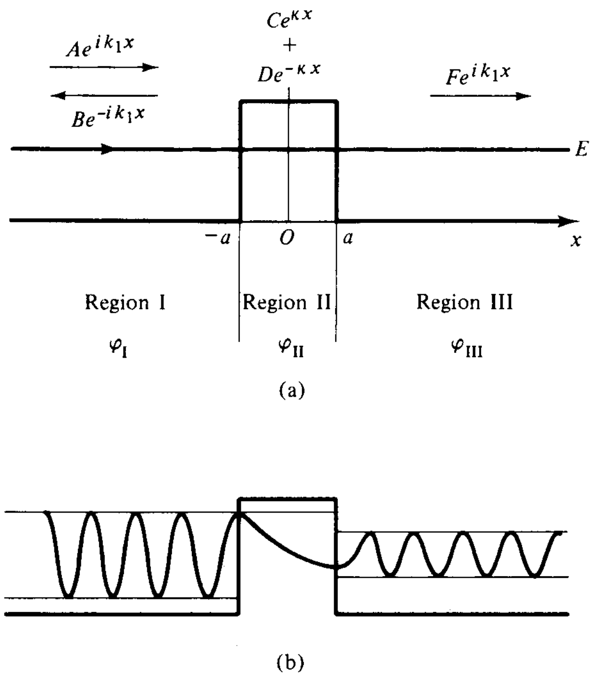


FIGURE 7.24 **(a) Domains relevant to the rectangular barrier scattering problem, for the case $E \leq V$. (b) Real part of φ for the case above, showing the hyperbolic decay in the barrier domain and decrease in amplitude of the transmitted wave.**

There results

$$\frac{1}{T} = \left|\frac{A}{F}\right|^2 = 1 + \frac{1}{4}\left(\frac{k_1{}^2 - k_2{}^2}{k_1 k_2}\right)^2 \sin^2(2k_2 a) \tag{7.142}$$

Rewriting k_1 and k_2 in terms of E and V as given by (7.138), one obtains

$$\boxed{\frac{1}{T} = 1 + \frac{1}{4}\frac{V^2}{E(E-V)}\sin^2(2k_2 a) \qquad E > V} \tag{7.143}$$

The reflection coefficient is $1 - T$.

For the case $E < V$, as depicted in Fig. 7.24a, we find that the structure of the solutions (7.138) are still appropriate, with the simple modification

$$\begin{gathered} ik_2 \to \kappa, \qquad \frac{\hbar^2\kappa^2}{2m} = V - E > 0 \\ (ak_1)^2 + (a\kappa)^2 = \frac{2ma^2V}{\hbar^2} \equiv \frac{g^2}{4} \end{gathered} \tag{7.144}$$

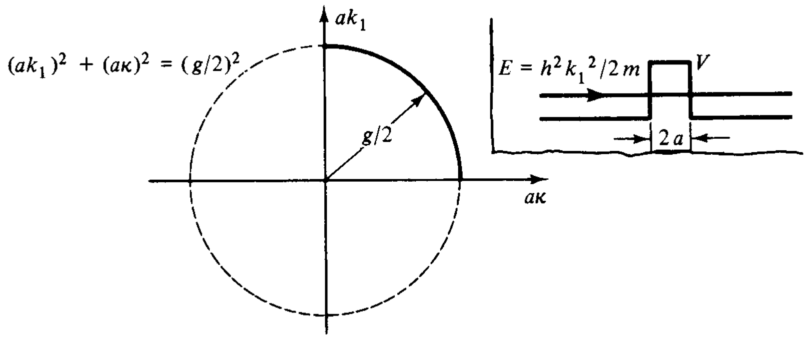


FIGURE 7.25 **For rectangular barrier scattering with $E \leq V$, ak_1 and $a\kappa$ lie on a circle $ak_1 \geq 0$, $a\kappa \geq 0$. The energy spectrum ($\hbar^2 k_1{}^2/2m$) comprises a bounded continuum.**

This latter conservation of energy statement indicates that the variables ak_1 and $a\kappa$ lie on a circle of radius $g/2$ (Fig. 7.25). The permitted eigen-k_1-values now comprise a positive, bounded continuum, so that the eigenenergies

$$E = \frac{\hbar^2 k_1{}^2}{2m}$$

also comprise a positive, bounded continuum.

The algebra leading to (7.140) remains unaltered so that the transmission coefficient for this case is obtained by making the substitution of (7.144) into (7.142). We also recall that $\sin(iz) = i \sinh z$. There results

$$\frac{1}{T} = 1 + \frac{1}{4}\left(\frac{k_1{}^2 + \kappa^2}{k_1 \kappa}\right)^2 \sinh^2(2\kappa a) \tag{7.145}$$

which, with (7.144), gives

$$\frac{1}{T} = 1 + \frac{1}{4}\frac{V^2}{E(V - E)} \sinh^2(2\kappa a) \tag{7.146}$$

Writing this equation in terms of T,

$$\boxed{T = \frac{1}{1 + \dfrac{1}{4}\dfrac{V^2}{E(V - E)}\sinh^2(2\kappa a)} \qquad E < V} \tag{7.147}$$

indicates that in the domain $E < V$, $T < 1$. The limit that $E \to V$ deserves special attention. With

$$\frac{V - E}{V} = \frac{\hbar^2\kappa^2}{2mV} \equiv \epsilon \to 0$$

one obtains

(7.148)
$$T = \frac{1}{1 + g^2/4} + O(\epsilon)$$
$$g^2 \equiv \frac{2m(2a)^2 V}{\hbar^2}$$

The expression $O(\epsilon)$ represents a sum of terms whose value goes to zero with ϵ. We conclude that for scattering from a potential barrier, the transmission is less than unity at $E = V$ (Fig. 7.26).

Returning to the case $E \neq V$, (7.143) indicates that $T = 1$ when $\sin^2(2k_2 a) = 0$, or equivalently when

(7.149)
$$2ak_2 = n\pi \qquad (n = 1, 2, \ldots)$$

Setting $k_2 = 2\pi/\lambda$, the latter statement is equivalent to

(7.150)
$$2a = n\left(\frac{\lambda}{2}\right)$$

When the barrier width $2a$ is an integral number of half-wavelengths, $n(\lambda/2)$, the barrier becomes transparent to the incident beam; that is, $T = 1$. This is analogous to the case of total transmission of light through thin refracting layers.

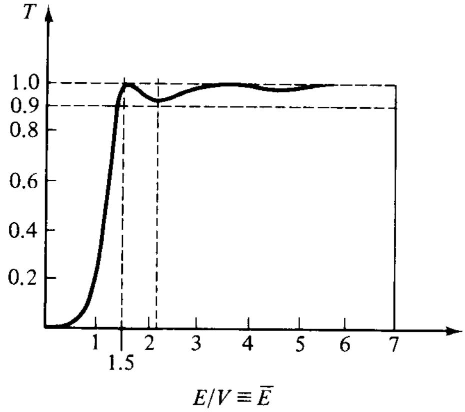


FIGURE 7.26 **Transmission coefficient T versus E/V for scattering from a rectangular barrier with $2m(2a)^2 V/\hbar^2 \equiv g^2 = 16$. The additional lines are in references to Problems 7.50 et seq.**

Written in terms of E and V, the requirement for perfect transmission, (7.149), becomes

$$E - V = n^2\left(\frac{\pi^2\hbar^2}{8a^2m}\right) = n^2 E_1 \tag{7.151}$$

where E_1 is the ground-state energy of a one-dimensional box of width $2a$ (see Eq. 4.14).

Equations (7.143) and (7.146) give the transmission coefficient T, as a function of E, V, and the width of the well $2a$. The former of these indicates that $T \rightarrow 1$ with increasing energy of the incident beam. The transmission is unity for the values of E given by (7.151). Equation (7.146) gives T for $E \leq V$. The transmission is zero for $E = 0$ and is less than 1 for $E = V$. A sketch of T versus $E/V \equiv \bar{E}$ for the case $g^2 = 16$ is given in Fig. 7.26.

The fact that T does not vanish for $E < V$ is a purely quantum mechanical result. This phenomenon of particles passing through barriers higher than their own incident energy is known as *tunneling*. It allows emission of α particles from a nucleus and field emission of electrons from a metal surface in the presence of a strong electric field.

PROBLEMS

7.44 In terms of the new variables,

$$\alpha_{\pm} \equiv \frac{k_1{}^2 \pm k_2{}^2}{2k_1k_2}, \qquad \beta \equiv 2k_2 a$$

$$\frac{F}{A} = \sqrt{T}\, e^{i\phi_T}, \qquad \frac{B}{A} = \sqrt{R}\, e^{i\phi_R}$$

(7.140) may be rewritten in the simpler form

$$\sqrt{T}\, e^{i\phi_T} = \frac{e^{2iak_1}}{\cos\beta + i\alpha_+ \sin\beta}$$

$$\sqrt{R}\, e^{i\phi_R} = i\alpha_- \sqrt{T}\, e^{i\phi_T} \sin\beta$$

Use these expressions to show:

(a) $T + R = 1$.

(b) $\phi_T = \phi_R - n(\pi/2)$, $n = 1, 2, 3, \ldots$.

(c) $\tan(\phi_T - 2k_1 a) = \alpha_+ \tan\beta$

(d) What is ϕ_R for the infinite potential step: $V(x) = \infty$, $x \geq 0$; $V(x) = 0$, $x < 0$?

Answers (partial)

(a) Solving for $T + R$ from (7.140) gives

$$T + R = \frac{1 + \alpha_-{}^2 \sin^2\beta}{\cos^2\beta + \alpha_+{}^2 \sin^2\beta}$$

Substituting the definitions of $\alpha_{\pm}$ gives the desired result.

(c) From the first of the two given equations above, we obtain

$$\sqrt{T}\, e^{i(\phi_T - 2k_1 a)} = \frac{1}{\cos\beta + i\alpha_+ \sin\beta} = \frac{e^{-i\phi}}{\sqrt{\cos^2\beta + {\alpha_+}^2 \sin^2\beta}}$$

Equating the tangents of the phases of both sides gives the desired result.

7.45 An electron beam is sent through a potential barrier 1 cm long. The transmission coefficient exhibits a third maximum at $E = 100$ V. What is the height of the barrier?

7.46 An electron beam is incident on a barrier of height 10 V. At $E = 10$ V, $T = 3.37 \times 10^{-3}$. What is the width of the barrier?

7.47 Use the correspondence principle with (7.147) to show that $T = 0$ for $E < V$, for the classical case of a beam of particles of energy E incident on a potential barrier of height V.

7.8 THE RAMSAUER EFFECT

The configuration for this case is depicted in Fig. 7.17c. The relevant domains are shown in Fig. 7.27. Once again Eqs. 7.138 et seq. apply with the modification

(7.152)
$$\frac{\hbar^2 {k_2}^2}{2m} = E - V = E + |V|$$

The transmission coefficient (7.143) becomes, for $E \geq 0$,

(7.153)
$$\boxed{\frac{1}{T} = 1 + \frac{1}{4}\frac{V^2}{E(E + |V|)} \sin^2 (2k_2 a)}$$

Again there is perfect transmission when an integral number of half-wavelengths fit the barrier width.

(7.154)
$$2ak_2 = n\pi \qquad (n = 1, 2, \ldots)$$

This condition may also be cast in terms of the eigenenergies of a one-dimensional box of width $2a$:

(7.155)
$$E + |V| = n^2 E_1$$

From (7.153) we see that $T \to 1$ with increasing incident energy. At $E = 0$, $T = 0$. Thus we obtain an idea of the shape of T versus E. It is similar to the curve shown in Fig. 7.26. The transmission is zero for $E = 0$ and rises to the first maximum (unity) at $E = E_1 - |V|$. It has successive maxima of unity at the values given by (7.155), and approaches 1 with growing incident energy E.

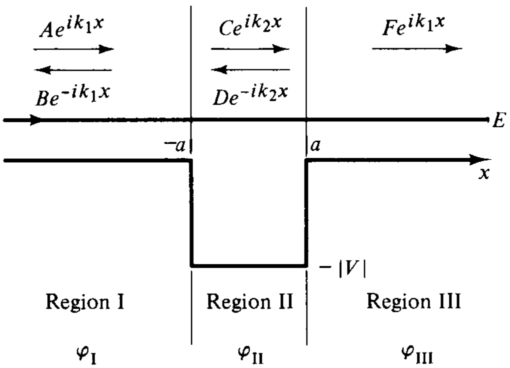


FIGURE 7.27 Domains relevant to the rectangular well scattering problem, $E > 0$.

The preceding theory of scattering of a beam of particles by a potential well has been used as a model for the scattering of low-energy electrons from atoms. The attractive well represents the field of the nucleus, whose positive charge becomes evident when the scattering electrons penetrate the shell structure of the atomic electrons. The reflection coefficient is a measure of the scattering cross section.[1] Experiments in which this cross section is measured (for rare gas atoms) detect a low-energy minimum which is consistent with the first maximum that T goes through for typical values of well depth and width according to the model above, (7.153). This transparency to low-energy electrons of rare gas atoms is known as the Ramsauer effect.

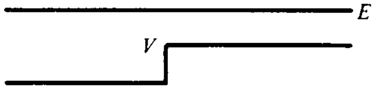
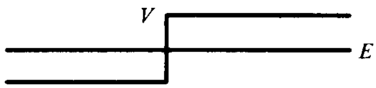
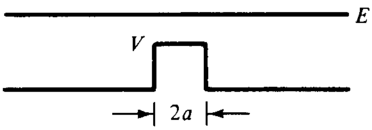
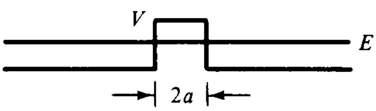
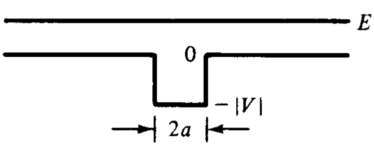
The student should not lose sight of the following fact. For any of the solutions to the scattering problems considered in these last few sections, we have in essence found the eigenfunctions and eigenenergies for the corresponding Hamiltonian. These Hamiltonians are of the form

$$H = \frac{p^2}{2m} + V(x) \tag{7.156}$$

with the potential $V(x)$ depicted by any of the configurations of Fig. 7.17. In each case considered, the spectrum of energies is a continuum, $E = \hbar^2k^2/2m$. For each value of k, a corresponding set of coefficient ratios (B/A, C/A for the simple step and B/A, C/A, D/A, F/A for the rectangular potential) are determined. The coefficient A is fixed by the data on the incident beam. These coefficients then determine the wavefunction, which is an eigenfunction of the Hamiltonian above. All such scattering eigenstates are unbound states. A continuous spectrum is characteristic of unbound states, while a discrete spectrum is characteristic of bound states (e.g., particle in a box, harmonic oscillator).

[1] The notion of scattering cross section is discussed in Chapter 14.

TABLE 7.2 Transmission coefficients for three elementary potential barriers

	$T = \dfrac{4k_2/k_1}{[1 + (k_2/k_1)]^2}$ $\left(\dfrac{k_2}{k_1}\right)^2 = 1 - \dfrac{V}{E}$
	$T = 0, \quad R = 1$
	$\dfrac{1}{T} = 1 + \dfrac{1}{4}\dfrac{V^2}{E(E - V)}\sin^2(2k_2 a)$ $\dfrac{\hbar^2 k_2^2}{2m} = E - V$
	$\dfrac{1}{T} = 1 + \dfrac{1}{4}\dfrac{V^2}{E(V - E)}\sinh^2(2\kappa a)$ $\dfrac{\hbar^2 \kappa^2}{2m} = V - E$
	$\dfrac{1}{T} = 1 + \dfrac{1}{4}\dfrac{V^2}{E(E + \|V\|)}\sin^2(2k_2 a)$ $\dfrac{\hbar^2 k_2^2}{2m} = E - V = E + \|V\|$

The transmission coefficients corresponding to the one-dimensional potential configurations considered above are summarized in Table 7.2.

PROBLEMS

7.48 The scattering cross section for the scattering of electrons by a rare gas of krypton atoms exhibits a low-energy minimum at $E \simeq 0.9$ V. Assuming that the diameter of the atomic well seen by the electrons is 1 Bohr radius, calculate its depth.

7.49 Show that the transmission coefficient for the rectangular barrier may be written in the form

$$T = T(g, \bar{E})$$

where

$$g^2 \equiv \frac{2m(2a)^2 V}{\hbar^2}$$

$$\bar{E} \equiv \frac{E}{V}$$

Answer (partial)

For $\bar{E} \geq 1$,

$$T^{-1} = 1 + \frac{1}{4}\frac{1}{\bar{E}(\bar{E} - 1)}\sin^2\sqrt{g^2(\bar{E} - 1)}$$

7.50 Using your answer to Problem 7.49, derive an equation for an approximation to the curve on which minimum values of T fall.

$$T_{\text{min}} = T_{\text{min}}(\bar{E})$$

Show that the values of T and $\bar{E}$ at the first minimum in the sketch of T versus $\bar{E}$ depicted in Fig. 7.26 ($g^2 = 16$) agree with your equation. [*Hint:* The minima of T fall at the values of $\bar{E}$ where T^{-1} is maximum. From Problem 7.49,

$$T^{-1} \leq 1 + \frac{1}{4}\frac{1}{\bar{E}(\bar{E} - 1)}.\Big]$$

7.51 For the rectangular barrier:

(a) Write the values of $\bar{E}$ for which $T = 1$ as a function of g.

(b) Using your answer to part (a) and the two preceding problems, make a sketch of T versus $\bar{E}$ in the two limits $g \gg 1$, $g \ll 1$. Cite two physical situations to which these limits pertain.

(c) Show that for an electron, $g^2/V \equiv 2m(2a)^2/\hbar^2 = 0.26(2a)^2(eV)^{-1}$, where a is in angstroms.

7.52 For the case depicted in Fig. 7.26, show that the first maximum falls at a value consistent with your answer to part (a) of Problem 7.51.

7.53 Write the transmission coefficient for the rectangular well as a function of g and $\bar{E}$.

Answer

$$T^{-1} = 1 + \frac{1}{4}\frac{1}{\bar{E}(\bar{E} + 1)}\sin^2\sqrt{g^2(\bar{E} + 1)}$$

7.54 In the limit $g^2 \gg 1$, show that the minima of T for the rectangular well fall on a curve which is well approximated by

$$T_{\text{min}} = 4\bar{E}$$

Use this result together with (7.155) for the values of $\bar{E}$ where $T = 1$ to obtain a sketch of T versus $\bar{E}$ for the case $g^2 = 10^5$.

Answer

See Fig. 7.28.

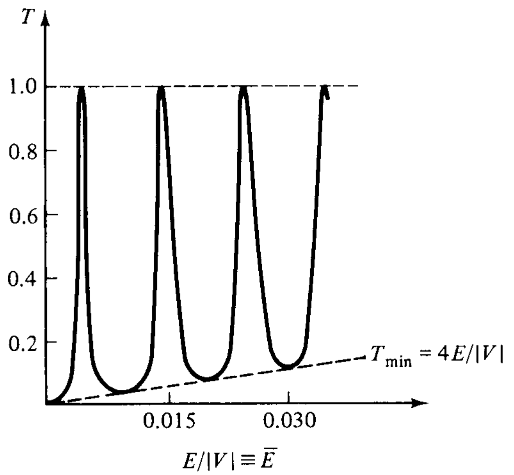


FIGURE 7.28 Resonances in the transmission coefficient for scattering by a potential well for $g^2 = 10^5$. (See Problems 7.54 et seq.)

7.55 Show that the spaces between resonances in T for the case of scattering from a potential well grow with decreasing g.

7.56 (a) Calculate the transmission coefficient T for the double potential step shown in Fig. 7.29a.

(b) If we call T_1 the transmission coefficient appropriate to the single potential step V_1, and T_2 that appropriate to the single potential step V_2, show that

$$T \leq T_1, \qquad T \leq T_2$$

Offer a physical explanation for these inequalities.

(c) What are the three sets of conditions under which T is maximized? What do these conditions correspond to physically?

(d) A student argues that T is the product $T_1 T_2$ on the following grounds. The particle current that penetrates the V_1 barrier is $T_1 \mathbb{J}_{\text{inc}}$. This current is incident on the V_2 barrier so that $T_2(T_1 \mathbb{J}_{\text{inc}})$ is the current transmitted through the second barrier. What is the incorrect assumption in his argument?

Answer (partial)

Applying boundary conditions to the wavefunctions

$$\begin{aligned} \varphi_{\text{I}} &= Ae^{ik_1x} + Be^{-ik_1x} && \text{(region I)} \\ \varphi_{\text{II}} &= Ce^{ik_2x} + De^{-ik_2x} && \text{(region II)} \\ \varphi_{\text{III}} &= Fe^{ik_3x} && \text{(region III)} \end{aligned}$$

at $x = 0$ and $x = a$, respectively, and solving for $T = (k_3/k_1)|F/A|^2$ gives the desired result:

$$T = \frac{4k_1k_3k_2^2}{k_2^2(k_1 + k_3)^2 + (k_3^2 - k_2^2)(k_1^2 - k_2^2)\sin^2(k_2a)} \qquad (k_1 \geq k_2 \geq k_3)$$

7.57 Calculate the transmission coefficient for the potential configuration and energy of incident particles depicted in Fig. 7.30. (*Note:* T is easily obtained from the answer given to Problem 7.56.)

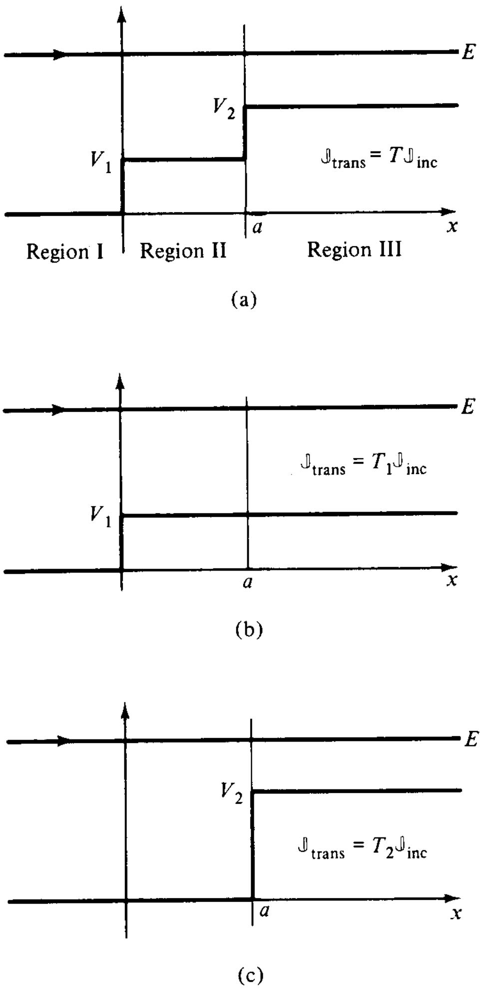


FIGURE 7.29 **(a) Double potential step showing three regions discussed in Problem 7.56. (b) and (c) Two related single potential steps: $T_1 \geq T$ and $T_2 \geq T$.**

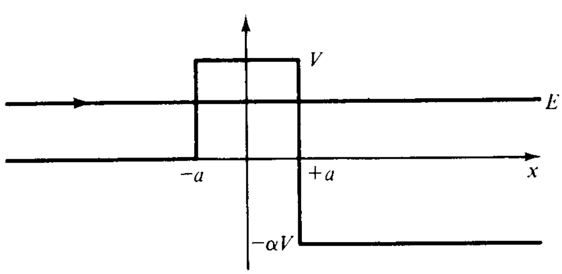


FIGURE 7.30 **Tunneling configuration for Problem 7.57. The constant α is real and greater than zero.**