(d) Suppose the earth made a transition to the next lower level ($n - 1$). How much energy (in Joules) would be released? What would the wavelength of the emitted photon (or, more likely, graviton) be? (Express your answer in light years—is the remarkable answer[20] a coincidence?)

4.3 ANGULAR MOMENTUM

As we have seen, the stationary states of the hydrogen atom are labeled by three quantum numbers: n, l, and m. The principal quantum number (n) determines the energy of the state (Equation 4.70); as it turns out, l and m are related to the orbital angular momentum. In the classical theory of central forces, energy and angular momentum are the fundamental conserved quantities, and it is not surprising that angular momentum plays a significant (in fact, even *more* important) role in the quantum theory.

Classically, the angular momentum of a particle (with respect to the origin) is given by the formula

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \qquad [4.95]$$

which is to say,

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \qquad [4.96]$$

The corresponding quantum operators are obtained by the standard prescription $p_x \to -i\hbar\partial/\partial x$, $p_y \to -i\hbar\partial/\partial y$, $p_z \to -i\hbar\partial/\partial z$. In the following section we'll obtain the eigenvalues of the angular momentum operators by a purely algebraic technique reminiscent of the one we used in Chapter 2 to get the allowed energies of the harmonic oscillator; it is all based on the clever exploitation of commutation relations. After that we will turn to the more difficult problem of determining the eigenfunctions.

4.3.1 Eigenvalues

The operators L_x and L_y do not commute; in fact[21]

$$\begin{aligned} [L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\ &= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]. \end{aligned} \qquad [4.97]$$

[20]Thanks to John Meyer for pointing this out.

[21]Note that all the operators we encounter in quantum mechanics (footnote 15, Chapter 1) are *distributive* with respect to addition: $A(B + C) = AB + AC$. In particular, $[A, B + C] = [A, B] + [A, C]$.

From the canonical commutation relations (Equation 4.10) we know that the only operators here that *fail* to commute are x with p_x, y with p_y, and z with p_z. So the two middle terms drop out, leaving

$$[L_x, L_y] = yp_x[p_z, z] + xp_y[z, p_z] = i\hbar(xp_y - yp_x) = i\hbar L_z. \qquad [4.98]$$

Of course, we could have started out with $[L_y, L_z]$ or $[L_z, L_x]$, but there is no need to calculate these separately—we can get them immediately by cyclic permutation of the indices ($x \to y$, $y \to z$, $z \to x$):

$$\boxed{[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y.} \qquad [4.99]$$

These are the fundamental commutation relations for angular momentum; everything else follows from them.

Notice that L_x, L_y, and L_z are *incompatible* observables. According to the generalized uncertainty principle (Equation 3.62),

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\langle i\hbar L_z\rangle\right)^2 = \frac{\hbar^2}{4}\langle L_z\rangle^2,$$

or

$$\sigma_{L_x}\sigma_{L_y} \geq \frac{\hbar}{2}|\langle L_z\rangle|. \qquad [4.100]$$

It would therefore be futile to look for states that are simultaneously eigenfunctions of L_x and L_y. On the other hand, the *square* of the *total* angular momentum,

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2, \qquad [4.101]$$

does commute with L_x:

$$\begin{aligned}[L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\ &= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z \\ &= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z \\ &= 0.\end{aligned}$$

(I used Equation 3.64 to simplify the commutators; note also that *any* operator commutes with *itself*.) It follows, of course, that L^2 also commutes with L_y and L_z:

$$[L^2, L_x] = 0, \quad [L^2, L_y] = 0, \quad [L^2, L_z] = 0, \qquad [4.102]$$

or, more compactly,

$$[L^2, \mathbf{L}] = 0. \qquad [4.103]$$

So L^2 *is* compatible with each component of $\mathbf{L}$, and we *can* hope to find simultaneous eigenstates of L^2 and (say) L_z:

$$L^2 f = \lambda f \quad \text{and} \quad L_z f = \mu f. \qquad [4.104]$$

We'll use a "ladder operator" technique, very similar to the one we applied to the harmonic oscillator back in Section 2.3.1. Let

$$\boxed{L_\pm \equiv L_x \pm i L_y.} \qquad [4.105]$$

The commutator with L_z is

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm i L_y),$$

so

$$[L_z, L_\pm] = \pm\hbar L_\pm. \qquad [4.106]$$

And, of course,

$$[L^2, L_\pm] = 0. \qquad [4.107]$$

I claim that if f is an eigenfunction of L^2 and L_z, so also is $L_\pm f$: Equation 4.107 says

$$L^2(L_\pm f) = L_\pm(L^2 f) = L_\pm(\lambda f) = \lambda(L_\pm f), \qquad [4.108]$$

so $L_\pm f$ is an eigenfunction of L^2, with the same eigenvalue λ, and Equation 4.106 says

$$L_z(L_\pm f) = (L_z L_\pm - L_\pm L_z) f + L_\pm L_z f = \pm\hbar L_\pm f + L_\pm(\mu f)$$
$$= (\mu \pm \hbar)(L_\pm f), \qquad [4.109]$$

so $L_\pm f$ is an eigenfunction of L_z with the *new* eigenvalue $\mu \pm \hbar$. We call L_+ the "raising" operator, because it *increases* the eigenvalue of L_z by $\hbar$, and L_- the "lowering" operator, because it *lowers* the eigenvalue by $\hbar$.

For a given value of λ, then, we obtain a "ladder" of states, with each "rung" separated from its neighbors by one unit of $\hbar$ in the eigenvalue of L_z (see Figure 4.8). To ascend the ladder we apply the raising operator, and to descend, the lowering operator. But this process cannot go on forever: Eventually we're going

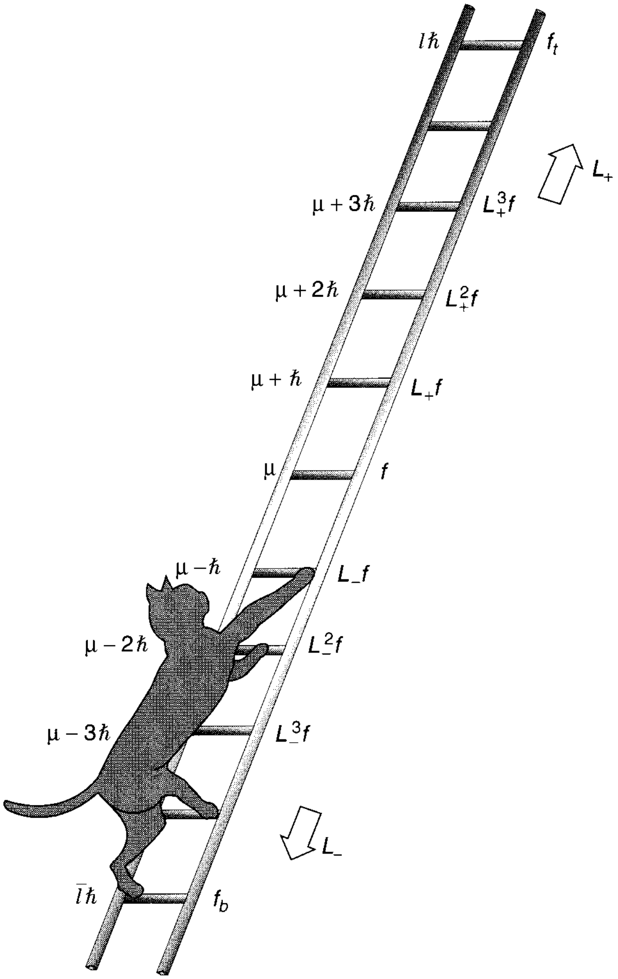

FIGURE 4.8: The "ladder" of angular momentum states.

to reach a state for which the z-component exceeds the *total*, and that cannot be.[22] There must exist a "top rung," f_t, such that[23]

$$L_+ f_t = 0. \qquad [4.110]$$

---

[22]Formally, $\langle L^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle$, but $\langle L_x^2 \rangle = \langle f | L_x^2 f \rangle = \langle L_x f | L_x f \rangle \geq 0$ (and likewise for L_y), so $\lambda = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \mu^2 \geq \mu^2$.

[23]Actually, all we can conclude is that $L_+ f_t$ is *not normalizable*—its norm could be *infinite*, instead of zero. Problem 4.18 explores this alternative.

Let $\hbar l$ be the eigenvalue of L_z at this top rung (the appropriateness of the letter "l" will appear in a moment):

$$L_z f_t = \hbar l f_t; \quad L^2 f_t = \lambda f_t. \tag{4.111}$$

Now,

$$\begin{aligned} L_\pm L_\mp &= (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) \\ &= L^2 - L_z^2 \mp i(i\hbar L_z), \end{aligned}$$

or, putting it the other way around,

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z. \tag{4.112}$$

It follows that

$$L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t = (0 + \hbar^2 l^2 + \hbar^2 l) f_t = \hbar^2 l(l+1) f_t,$$

and hence

$$\lambda = \hbar^2 l(l+1). \tag{4.113}$$

This tells us the eigenvalue of L^2 in terms of the *maximum* eigenvalue of L_z.

Meanwhile, there is also (for the same reason) a *bottom* rung, f_b, such that

$$L_- f_b = 0. \tag{4.114}$$

Let $\hbar \bar{l}$ be the eigenvalue of L_z at this bottom rung:

$$L_z f_b = \hbar \bar{l} f_b; \quad L^2 f_b = \lambda f_b. \tag{4.115}$$

Using Equation 4.112, we have

$$L^2 f_b = (L_+ L_- + L_z^2 - \hbar L_z) f_b = (0 + \hbar^2 \bar{l}^2 - \hbar^2 \bar{l}) f_b = \hbar^2 \bar{l}(\bar{l} - 1) f_b,$$

and therefore

$$\lambda = \hbar^2 \bar{l}(\bar{l} - 1). \tag{4.116}$$

Comparing Equations 4.113 and 4.116, we see that $l(l+1) = \bar{l}(\bar{l} - 1)$, so either $\bar{l} = l + 1$ (which is absurd—the bottom rung would be higher than the top rung!) or else

$$\bar{l} = -l. \tag{4.117}$$

Evidently the eigenvalues of L_z are $m\hbar$, where m (the appropriateness of this letter will also be clear in a moment) goes from $-l$ to $+l$ in N integer steps. In particular, it follows that $l = -l + N$, and hence $l = N/2$, so l must be *an integer or a half-integer*. The eigenfunctions are characterized by the numbers l and m:

$$\boxed{L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m,} \qquad [4.118]$$

where

$$l = 0,\ 1/2,\ 1,\ 3/2,\ \ldots; \quad m = -l,\ -l+1,\ \ldots,\ l-1,\ l. \qquad [4.119]$$

For a given value of l, there are $2l + 1$ different values of m (i.e., $2l + 1$ "rungs" on the "ladder").

Some people like to illustrate this result with the diagram in Figure 4.9 (drawn for the case $l = 2$). The arrows are supposed to represent possible angular momenta—in units of $\hbar$ they all have the same length $\sqrt{l(l+1)}$ (in this case $\sqrt{6} = 2.45$), and their z components are the allowed values of m (-2, -1, 0, 1, 2). Notice that the magnitude of the vectors (the radius of the sphere) is *greater* than the maximum z component! (In general, $\sqrt{l(l+1)} > l$, except for the "trivial" case $l = 0$.) Evidently you can't get the angular momentum to point perfectly along the z direction. At first, this sounds absurd. "Why can't I just *pick* my axes so that z points along the direction of the angular momentum vector?" Well, to do this you would have to know all three components simultaneously, and the

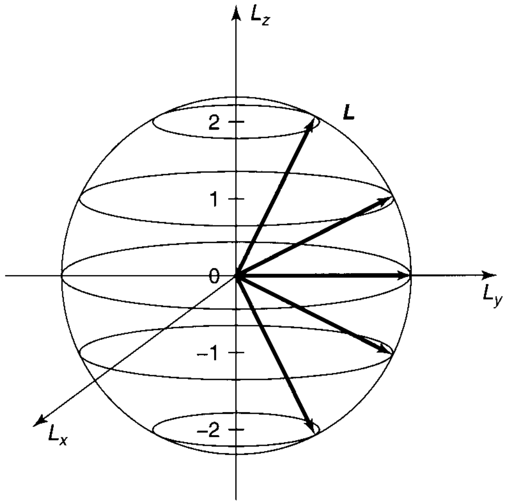

FIGURE 4.9: Angular momentum states (for $l = 2$).

uncertainty principle (Equation 4.100) says that's impossible. "Well, all right, but surely once in a while, by good fortune, I will just *happen* to aim my z-axis along the direction of **L**." No, no! You have missed the point. It's not merely that you don't *know* all three components of **L**; there simply *aren't* three components—a particle just cannot *have* a determinate angular momentum vector, any more than it can simultaneously have a determinate position and momentum. If L_z has a well-defined value, then L_x and L_y do *not*. It is misleading even to *draw* the vectors in Figure 4.9—at best they should be smeared out around the latitude lines, to indicate that L_x and L_y are indeterminate.

I hope you're impressed: By *purely algebraic means*, starting with the fundamental commutation relations for angular momentum (Equation 4.99), we have determined the eigenvalues of L^2 and L_z—without ever seeing the eigenfunctions themselves! We turn now to the problem of constructing the eigenfunctions, but I should warn you that this is a much messier business. Just so you know where we're headed, I'll begin with the punch line: $f_l^m = Y_l^m$—the eigenfunctions of L^2 and L_z are nothing but the old spherical harmonics, which we came upon by a quite different route in Section 4.1.2 (that's why I chose the letters l and m, of course). And I can now tell you why the spherical harmonics are orthogonal: They are eigenfunctions of hermitian operators (L^2 and L_z) belonging to distinct eigenvalues (Theorem 2, Section 3.3.1).

***Problem 4.18** The raising and lowering operators change the value of m by one unit:

$$L_\pm f_l^m = (A_l^m) f_l^{m\pm 1}, \tag{4.120}$$

where A_l^m is some constant. *Question:* What *is* A_l^m, if the eigenfunctions are to be *normalized*? *Hint:* First show that $L_\mp$ is the hermitian conjugate of $L_\pm$ (since L_x and L_y are *observables*, you may assume they are hermitian ... but *prove* it if you like); then use Equation 4.112. *Answer:*

$$A_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)} = \hbar\sqrt{(l \mp m)(l \pm m + 1)}. \tag{4.121}$$

Note what happens at the top and bottom of the ladder (i.e., when you apply L_+ to f_l^l or L_- to f_l^{-l}).

***Problem 4.19**

(a) Starting with the canonical commutation relations for position and momentum (Equation 4.10), work out the following commutators:

$$\begin{aligned} &[L_z, x] = i\hbar y, \quad [L_z, y] = -i\hbar x, \quad [L_z, z] = 0, \\ &[L_z, p_x] = i\hbar p_y, \quad [L_z, p_y] = -i\hbar p_x, \quad [L_z, p_z] = 0. \end{aligned} \tag{4.122}$$

(b) Use these results to obtain $[L_z, L_x] = i\hbar L_y$ directly from Equation 4.96.

(c) Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$ (where, of course, $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$).

(d) Show that the Hamiltonian $H = (p^2/2m) + V$ commutes with all three components of $\mathbf{L}$, provided that V depends only on r. (Thus H, L^2, and L_z are mutually compatible observables.)

∗ ∗Problem 4.20

(a) Prove that for a particle in a potential $V(\mathbf{r})$ the rate of change of the expectation value of the orbital angular momentum $\mathbf{L}$ is equal to the expectation value of the torque:

$$\frac{d}{dt}\langle \mathbf{L} \rangle = \langle \mathbf{N} \rangle,$$

where

$$\mathbf{N} = \mathbf{r} \times (-\nabla V).$$

(This is the rotational analog to Ehrenfest's theorem.)

(b) Show that $d\langle \mathbf{L} \rangle / dt = 0$ for any spherically symmetric potential. (This is one form of the quantum statement of **conservation of angular momentum**.)

4.3.2 Eigenfunctions

First of all we need to rewrite L_x, L_y, and L_z in spherical coordinates. Now, $\mathbf{L} = (\hbar/i)(\mathbf{r} \times \nabla)$, and the gradient, in spherical coordinates, is:[24]

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}; \qquad [4.123]$$

meanwhile, $\mathbf{r} = r\hat{r}$, so

$$\mathbf{L} = \frac{\hbar}{i}\left[r(\hat{r} \times \hat{r})\frac{\partial}{\partial r} + (\hat{r} \times \hat{\theta})\frac{\partial}{\partial \theta} + (\hat{r} \times \hat{\phi})\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right].$$

[24] George Arfken and Hans-Jurgen Weber, *Mathematical Methods for Physicists*, 5th ed., Academic Press, Orlando (2000), Section 2.5.

But $(\hat{r} \times \hat{r}) = 0$, $(\hat{r} \times \hat{\theta}) = \hat{\phi}$, and $(\hat{r} \times \hat{\phi}) = -\hat{\theta}$ (see Figure 4.1), and hence

$$\mathbf{L} = \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right). \qquad [4.124]$$

The unit vectors $\hat{\theta}$ and $\hat{\phi}$ can be resolved into their cartesian components:

$$\hat{\theta} = (\cos\theta\cos\phi)\hat{\imath} + (\cos\theta\sin\phi)\hat{\jmath} - (\sin\theta)\hat{k}; \qquad [4.125]$$

$$\hat{\phi} = -(\sin\phi)\hat{\imath} + (\cos\phi)\hat{\jmath}. \qquad [4.126]$$

Thus

$$\mathbf{L} = \frac{\hbar}{i}\left[(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\frac{\partial}{\partial\theta} - (\cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k})\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right].$$

Evidently

$$L_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right), \qquad [4.127]$$

$$L_y = \frac{\hbar}{i}\left(+\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right), \qquad [4.128]$$

and

$$\boxed{L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.} \qquad [4.129]$$

We shall also need the raising and lowering operators:

$$L_{\pm} = L_x \pm iL_y = \frac{\hbar}{i}\left[(-\sin\phi \pm i\cos\phi)\frac{\partial}{\partial\theta} - (\cos\phi \pm i\sin\phi)\cot\theta\frac{\partial}{\partial\phi}\right].$$

But $\cos\phi \pm i\sin\phi = e^{\pm i\phi}$, so

$$L_{\pm} = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right). \qquad [4.130]$$

In particular (Problem 4.21(a)):

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right), \qquad [4.131]$$

and hence (Problem 4.21(b)):

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]. \qquad [4.132]$$

We are now in a position to determine $f_l^m(\theta, \phi)$. It's an eigenfunction of L^2, with eigenvalue $\hbar^2 l(l+1)$:

$$L^2 f_l^m = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] f_l^m = \hbar^2 l(l+1) f_l^m.$$

But this is precisely the "angular equation" (Equation 4.18). And it's also an eigenfunction of L_z, with the eigenvalue $m\hbar$:

$$L_z f_l^m = \frac{\hbar}{i}\frac{\partial}{\partial\phi} f_l^m = \hbar m f_l^m,$$

but this is equivalent to the azimuthal equation (Equation 4.21). We have already solved this system of equations: The result (appropriately normalized) is the spherical harmonic, $Y_l^m(\theta, \phi)$. *Conclusion:* Spherical harmonics *are* eigenfunctions of L^2 and L_z. When we solved the Schrödinger equation by separation of variables, in Section 4.1, we were inadvertently constructing simultaneous eigenfunctions of the three commuting operators H, L^2, and L_z:

$$H\psi = E\psi, \quad L^2\psi = \hbar^2 l(l+1)\psi, \quad L_z\psi = \hbar m\psi. \qquad [4.133]$$

Incidentally, we can use Equation 4.132 to rewrite the Schrödinger equation (Equation 4.14) more compactly:

$$\frac{1}{2mr^2}\left[-\hbar^2 \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + L^2 \right]\psi + V\psi = E\psi.$$

There is a curious final twist to this story, for the *algebraic* theory of angular momentum permits l (and hence also m) to take on *half*-integer values (Equation 4.119), whereas separation of variables yielded eigenfunctions only for *integer* values (Equation 4.29). You might suppose that the half-integer solutions are spurious, but it turns out that they are of profound importance, as we shall see in the following sections.

∗Problem 4.21

(a) Derive Equation 4.131 from Equation 4.130. *Hint:* Use a test function; otherwise you're likely to drop some terms.

(b) Derive Equation 4.132 from Equations 4.129 and 4.131. *Hint:* Use Equation 4.112.

∗Problem 4.22

(a) What is $L_+ Y_l^l$? (No calculation allowed!)

(b) Use the result of (a), together with Equation 4.130 and the fact that $L_z Y_l^l = \hbar l Y_l^l$, to determine $Y_l^l(\theta, \phi)$, up to a normalization constant.

(c) Determine the normalization constant by direct integration. Compare your final answer to what you got in Problem 4.5.

Problem 4.23 In Problem 4.3 you showed that

$$Y_2^1(\theta, \phi) = -\sqrt{15/8\pi}\, \sin\theta \cos\theta e^{i\phi}.$$

Apply the raising operator to find $Y_2^2(\theta, \phi)$. Use Equation 4.121 to get the normalization.

Problem 4.24 Two particles of mass m are attached to the ends of a massless rigid rod of length a. The system is free to rotate in three dimensions about the center (but the center point itself is fixed).

(a) Show that the allowed energies of this **rigid rotor** are

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2}, \quad \text{for} \quad n = 0,\ 1,\ 2, \ldots$$

Hint: First express the (classical) energy in terms of the total angular momentum.

(b) What are the normalized eigenfunctions for this system? What is the degeneracy of the nth energy level?
