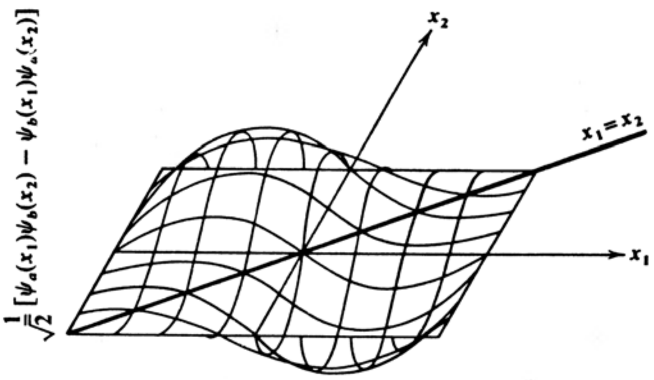
Solutions Supplement to Accompany

QUANTUM PHYSICS

OF ATOMS, MOLECULES, SOLIDS, NUCLEI, AND PARTICLES

Second Edition



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PREFACE

This supplement contains solutions to most of the more-involved problems in the QUANTUM PHYSICS text; with one exception, solutions to problems in the Appendices are not included.

The supplement is directed toward instructors, and this has influenced the presentation. Not every algebraic step is exhibited. The units have not been displayed explicity in every equation. (SI units are adopted in the supplement, mainly because they are briefer than the text notation.) Rules with regard to significant figures have not been strictly observed, although there should be no outlandish violations. Use of symbols and choice of notation is generally obvious and therefore not exhaustively defined for each problem.

It is a pleasure to thank Prof. Richard Shurtleff (Wentworth Institute of Technology) for preparing the solutions to the problems in Chapter 18.

Preparation of the supplement, including choice of problems, was left to the undersigned, who was also his own typist and illustrator. He would appreciate a note, of up to moderate asperity, from those who detect an error and/or mistake.

December 24, 1984

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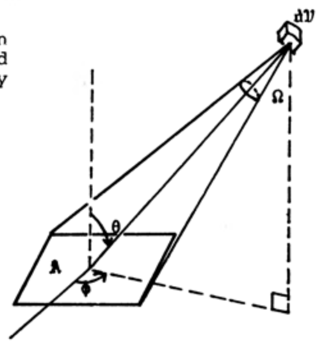


The radiant energy contained in volume dV that is moving toward A at any time, in the frequency interval ν, $\nu+d\nu$ is

$$dE_T(\nu)d\nu = \rho_T(\nu)d\nu \frac{\Omega}{4\pi} dV,$$

where Ω is the solid angle subtended at dV by A. With

$$\Omega = A\cos\theta/r^2$$

and

$$dV = r^2\sin\theta dr d\theta d\phi,$$

the energy becomes

$$dE_T(\nu)d\nu = \frac{1}{4\pi}\rho_T(\nu)d\nu\, A\sin\theta\cos\theta dr d\theta d\phi.$$

The energy in this frequency interval that crosses A in time t from the entire upper hemisphere is

$$E_T(\nu)d\nu = \rho_T(\nu)d\nu \frac{A}{4\pi}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{r=0}^{ct}\sin\theta\cos\theta d\theta d\phi dr$$

$$E_T(\nu)d\nu = \frac{1}{4}\rho_T(\nu)d\nu\, Act.$$

Hence the energy that passes through a unit area in unit time from the upper hemisphere is

$$R_T(\nu)d\nu = E_T(\nu)d\nu/At = \frac{c}{4}\rho_T(\nu)d\nu.$$

1-4

$$P = A\int_{\nu_1}^{\nu_2} R_T(\nu)\,d\nu = \tfrac{1}{4}Ac\int_{\nu_1}^{\nu_2} \rho_T(\nu)\,d\nu \simeq \tfrac{1}{4}Ac\rho_T(\nu_{av})\Delta\nu.$$

$$\nu_1 = c/\lambda_1 = \frac{2.998 \times 10^8}{5.50 \times 10^{-7}} = 5.4509 \times 10^{14}\ \text{Hz},$$

$$\nu_2 = c/\lambda_2 = \frac{2.998 \times 10^8}{5.51 \times 10^{-7}} = 5.4410 \times 10^{14}\ \text{Hz}.$$

Therefore,

$$\nu_{av} = \tfrac{1}{2}(\nu_1 + \nu_2) = 5.46 \times 10^{14}\ \text{Hz};$$

$$\Delta\nu = \nu_2 - \nu_1 = 9.9 \times 10^{11}\ \text{Hz}.$$

Since

$$\rho_T(\nu_{av}) = (8\pi h\nu_{av}^3/c^3)(e^{h\nu_{av}/kT} - 1)^{-1},$$

numerically;

$$8\pi h\nu_{av}^3/c^3 = 1.006 \times 10^{-13},$$

$$h\nu_{av}/kT = 4.37,$$

$$e^{h\nu_{av}/kT} - 1 = 78.04,$$

$$\rho_T(\nu_{av}) = (1.006 \times 10^{-13})(78.04)^{-1} = 1.289 \times 10^{-15}.$$

The area of the hole is

$$A = \pi r^2 = \pi(5 \times 10^{-3})^2 = 7.854 \times 10^{-5}\ \text{m}^2.$$

Hence, finally,

$$P = \tfrac{1}{4}(7.854 \times 10^{-5})(2.998 \times 10^8)(1.289 \times 10^{-15})(9.9 \times 10^{11}),$$

$$P = 7.51\ \text{W}.$$

1-5

(a) $$L = 4\pi R^2\sigma T^4 = 4\pi(7 \times 10^8)^2(5.67 \times 10^{-8})(5700)^4,$$

$$L = 3.685 \times 10^{26}\ \text{W}.$$

$$L = \frac{d}{dt}(mc^2) = c^2\frac{dm}{dt},$$

$$\frac{dm}{dt} = \frac{L}{c^2} = \frac{3.685 \times 10^{26}}{(3 \times 10^8)^2} = 4.094 \times 10^9\ \text{kg/s}.$$

(b) The mass lost in one year is

$$\Delta M = \frac{dm}{dt}t = (4.094 \times 10^9)(3.156 \times 10^7) = 1.292 \times 10^{17}\ \text{kg}.$$

The desired fraction is, then,

$$f = \frac{\Delta M}{M} = \frac{1.292 \times 10^{17}}{2.0 \times 10^{30}} = 6.5 \times 10^{-14}.$$

1-10

(a) The solar constant S is defined by

$$S = \frac{L_{sun}}{4\pi r^2},$$

r = Earth-sun distance, L_{sun} = rate of energy output of the sun. Let R = radius of the earth; the rate P at which energy impinges on the earth is

$$P = \frac{L_{sun}}{4\pi r^2}\pi R^2 = \pi R^2 S.$$

The average rate, per m^2, of arrival of energy at the earth's surface is

$$P_{av} = \frac{P}{4\pi R^2} = \frac{\pi R^2 S}{4\pi R^2} = \tfrac{1}{4}S,$$

$$338\ \text{W/m}^2 \stackrel{?}{=} \tfrac{1}{4}(1353\ \text{W/m}^2) = 338\ \text{W/m}^2.$$

(b)
$$338 = \sigma T^4 = (5.67 \times 10^{-8})T^4,$$
$$T = 280 \text{ K}.$$

<u>1-19</u>

$$R_T(\lambda) = \frac{c}{4}\rho_T(\lambda) = \frac{2\pi k^5T^5}{h^4c^3}\frac{x^5}{e^x - 1},$$

with $x = hc/\lambda kT$. At $\lambda = \lambda_{max}$, $x = 4.965$, by Problem 18. Thus,

$$R_T(\lambda_{max}) = 42.403\pi\ (kT)^5/h^4c^3.$$

Now find x such that $R_T(\lambda) = 0.2R_T(\lambda_{max})$:

$$\frac{2\pi k^5T^5}{h^4c^3}\frac{x^5}{e^x - 1} = (0.2)42.403\pi\frac{k^5T^5}{h^4c^3},$$

$$\frac{x^5}{e^x - 1} = 4.2403,$$

$$x_1 = 1.882, \quad x_2 = 10.136.$$

Numerically,

$$\lambda = \frac{hc}{kT}\frac{1}{x} = \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{(1.38 \times 10^{-23})(3)}\frac{1}{x},$$

$$\lambda = 4.798 \times 10^{-3}/x,$$

so that

$$\lambda_1 = 4.798 \times 10^{-3}/1.882 = 2.55 \text{ mm},$$
$$\lambda_2 = 4.798 \times 10^{-3}/10.136 = 0.473 \text{ mm}.$$

<u>1-20</u>

If $x = hc/\lambda_{max}kT$, then, by Problem 18,

$$e^{-x} + \frac{x}{5} = 1,$$
$$e^x - 1 = \frac{x}{5 - x}.$$

Hence,

$$\rho_T(\lambda_{max}) = \frac{8\pi kT}{\lambda_{max}^4}(5 - x).$$

But,

$$x = 4.965; \quad \frac{1}{\lambda_{max}^4} = \left(4.965\frac{kT}{hc}\right)^4.$$

Upon substitution, these give

$$\rho_T(\lambda_{max}) = 170\pi\frac{(kT)^5}{(hc)^4}.$$

<u>1-21</u>

By Problem 20,

$$\rho_T(\lambda_{max}) = 170\pi\frac{(kT)^5}{(hc)^4},$$

so that the wavelengths sought must satisfy

$$\frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} = \tfrac{1}{2}\cdot 170\pi\frac{(kT)^5}{(hc)^4}.$$

Again let

$$x = hc/\lambda kT.$$

In terms of x, the preceding equation becomes

$$\frac{x^5}{e^x - 1} = \frac{170}{16}.$$

Solutions are

$$x_1 = 2.736; \quad x_2 = 8.090.$$

Since, for λ_{max},

$$x = 4.965,$$

these solutions give

$$\lambda_1 = 1.815\lambda_{max},$$
$$\lambda_2 = 0.614\lambda_{max}.$$

<u>1-24</u>

Let $\lambda' = 200$ nm, $\lambda'' = 400$ nm; then,

$$\frac{1}{\lambda'^5}\frac{1}{e^{hc/\lambda' kT} - 1} = (3.82)\frac{1}{\lambda''^5}\frac{1}{e^{hc/\lambda'' kT} - 1},$$

$$\frac{e^{hc/\lambda'' kT} - 1}{e^{hc/\lambda' kT} - 1} = 3.82\left(\frac{\lambda'}{\lambda''}\right)^5.$$

Numerically,

$$\frac{hc}{\lambda'' k} = \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{(4 \times 10^{-7})(1.38 \times 10^{-23})} = 35987 \text{ K},$$

so that

$$\frac{e^{35987/T} - 1}{e^{71974/T} - 1} = (3.82)(\tfrac{1}{2})^5 = 0.1194.$$

Let $x = e^{35987/T}$; then,

$$\frac{x - 1}{x^2 - 1} = 0.1194 = \frac{1}{x + 1},$$

$$x = 7.375;$$

$$e^{35987/T} = 7.375,$$

$$T = \frac{35987}{\ln 7.375} = 18{,}000 \text{ K}.$$

CHAPTER TWO

<u>1-5</u>

The photoelectric equation is

$$hc = eV_0\lambda + w_0\lambda.$$

With $V_0 = 1.85$ V for $\lambda = 300$ nm, and $V_0 = 0.82$ V for $\lambda = 400$ nm,

$$hc = 8.891 \times 10^{-26} + 3 \times 10^{-7} w_0,$$
$$hc = 5.255 \times 10^{-26} + 4 \times 10^{-7} w_0.$$

Hence,

$$8.891 \times 10^{-26} + 3 \times 10^{-7} w_0 = 5.255 \times 10^{-26} + 4 \times 10^{-7} w_0,$$

(b) $$w_0 = 3.636 \times 10^{-19} \text{ J} = 2.27 \text{ eV}.$$

Therefore,

$$hc = 8.891 \times 10^{-26} + (3 \times 10^{-7})(3.636 \times 10^{-19}),$$
$$hc = 19.799 \times 10^{-26} \text{ J-m},$$

(a) $$h = \frac{19.799 \times 10^{-26}}{2.998 \times 10^8} = 6.604 \times 10^{-34} \text{ J-s}.$$

(c) $$w_0 = hc/\lambda_0,$$

$$3.636 \times 10^{-19} = 19.799 \times 10^{-26}/\lambda_0,$$
$$\lambda_0 = 5.445 \times 10^{-7} \text{ m} = 544.5 \text{ nm}.$$

<u>2-8</u>

In a magnetic field

$$r = mv/eB.$$

$$p = mv = erB = (1.602 \times 10^{-19})(1.88 \times 10^{-4}),$$

$$p = 3.012 \times 10^{-23}\ \text{kg-m/s},$$

$$p = \frac{(3.012 \times 10^{-23})(2.998 \times 10^{8})}{c(1.602 \times 10^{-13})} = \frac{0.05637\ \text{MeV}}{c}.$$

Also,

$$E^2 = p^2c^2 + E_0^2,$$

$$E^2 = (0.05637)^2 + (0.511)^2,$$

$$E = 0.5141\ \text{MeV}.$$

Hence,

(a) $$K = E - E_0 = 0.5141 - 0.5110 = 0.0031\ \text{MeV}.$$

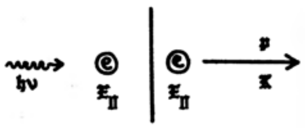
(b) The photon energy is

$$E_{ph}(\text{eV}) = \frac{1240}{\lambda(\text{nm})} = \frac{1240}{0.071} = 0.0175\ \text{MeV};$$

$$w_0 = E_{ph} - K = 17.5 - 3.1 = 14.4\ \text{keV}.$$

2-9

(a) Assuming the process can operate, apply conservation of mass-energy and of momentum:



$$h\nu + E_0 = K + E_0 \rightarrow h\nu = K;$$

$$\frac{h\nu}{c} + 0 = p.$$

These equations taken together imply that

$$p = K/c. \qquad (*)$$

But, for an electron,

$$E^2 = p^2c^2 + E_0^2,$$

$$(K + E_0)^2 = p^2c^2 + E_0^2,$$

$$p = \sqrt{(K^2 + 2E_0K)}/c. \qquad (**)$$

(*) and (**) can be satisfied together only if $E_0 \neq 0$, which is not true for an electron.

(b) In the Compton effect, a photon is present after the collision; this allows the conservation laws to hold without contradiction.

2-14

Let n = number of photons per unit volume. In time t, all photons initially a distance < ct will cross area A normal to the beam direction. Thus,

$$I = \frac{\text{Energy}}{At} = \frac{n(h\nu)A(ct)}{At} = nhc\nu = \frac{nhc^2}{\lambda}.$$

For two beams of wavelengths λ_1 and λ_2 with $I_1 = I_2$,

$$\frac{I_1}{I_2} = 1 = \frac{n_1/\lambda_1}{n_2/\lambda_2} \rightarrow \frac{n_1}{n_2} = \frac{\lambda_1}{\lambda_2},$$

and therefore

$$\frac{n_1/At}{n_2/At} = \frac{\lambda_1}{\lambda_2}.$$

The energy density is $\rho = nh\nu = nhc/\lambda$. Since this differs from I only by the factor c (which is the same for both beams), then if $\rho_1 = \rho_2$, the equation above holds again.

2-26

Set K_i = 20 keV, K_f = 0; K_1 = electron kinetic energy after the first deceleration; then

$$\frac{hc}{\lambda_1} = K_i - K_1; \quad \frac{hc}{\lambda_2} = K_1 - K_f = K_1; \quad \lambda_2 = \lambda_1 + \Delta\lambda,$$

with $\Delta\lambda$ = 0.13 nm. Since

$$hc = 1.2400\ \text{keV-nm},$$

these equations become

$$\frac{1.2400}{\lambda_1} = 20 - K_1; \quad \frac{1.2400}{\lambda_2} = K_1; \quad \lambda_2 = \lambda_1 + 0.13.$$

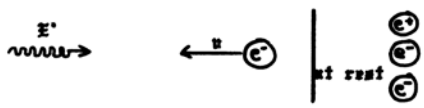
Solving yields,

(a) $$K_1 = 5.720 \text{ keV};$$

(b) $$\lambda_1 = 0.0868 \text{ nm}; \quad \lambda_2 = 0.2168 \text{ nm}.$$

<u>2-28</u>

Apply the laws of conservation of total energy and of momentum.
<u>Center of Mass Frame</u>



$$E' + mc^2 = 3m_0c^2,$$

$$\frac{E'}{c} - mv = 0.$$

Therefore,

$$mv + mc = 3m_0c,$$

$$m(1 + \beta) = 3m_0,$$

$$\frac{m_0}{(1 - \beta^2)^{\frac{1}{2}}}(1 + \beta) = 3m_0,$$

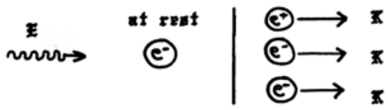
$$\beta = 4/5.$$

Hence, $m = 5m_0/3$ and $E' = mc^2\beta = 4m_0c^2/3$. By the Doppler shift, with $\beta = 4/5$,

$$E' = E\{(1 - \beta)/(1 + \beta)\}^{\frac{1}{2}} = E/3,$$

$$E = 3E' = 3(\tfrac{4}{3}m_0c^2) = 4m_0c^2.$$

<u>Laboratory Frame</u>



$$E + m_0c^2 = 3m_0c^2 + 3K,$$

$$E/c = 3(K^2 + 2m_0c^2K)^{\frac{1}{2}}/c.$$

Therefore,

$$2m_0c^2 + 3K = 3(K^2 + 2m_0c^2K)^{\frac{1}{2}},$$

$$K = \frac{2}{3}m_0c^2,$$

so that

$$E = 2m_0c^2 + 3K = 4m_0c^2.$$

<u>2-29</u>

(a) $$E + M_0c^2 = M_0c^2 + 2m_0c^2 + K,$$

$$E = 2(0.511) + 1 = 2.022 \text{ MeV}.$$

(b) $$p = E/c = 2.022 \text{ MeV/c}; \quad p_+ = 0;$$

$$p_- = \frac{1}{c}(K^2 + 2m_0c^2K)^{\frac{1}{2}} = \frac{1}{c}\{1^2 + 2(0.511)(1)\}^{\frac{1}{2}} = 1.422 \text{ MeV/c}.$$

$$P = (2.022 - 1.422) = 0.600 \text{ MeV/c};$$

$$\% \text{ transferred} = \frac{0.600}{2.022}(100) = 29.7\%.$$

2-31

Use the Doppler shift to convert the given wavelengths to wavelengths as seen in the rest frame of the pair:

$$2\lambda_1' = \lambda_2',$$

$$2\lambda\left(\frac{c-v}{c+v}\right)^{\frac{1}{2}} = \lambda\left(\frac{c+v}{c-v}\right)^{\frac{1}{2}},$$

$$2\lambda\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} = \lambda\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}},$$

$$4\,\frac{1-\beta}{1+\beta} = \frac{1+\beta}{1-\beta},$$

$$3\beta^2 - 10\beta + 3 = 0,$$

$$\beta = \frac{1}{3}; \quad v = \frac{c}{3}.$$

2-33

The number of particles stopped/scattered between distances x and x+dx is $dI(x) = \sigma I(x)\rho dx$. Hence, for a very thick slab that ultimately stops/scatters all the incident particles, the average distance a particle travels is

$$x_{av} = \frac{\int x dI}{\int dI} = \frac{\sigma\rho\int xIdx}{\sigma\rho\int Idx} = \frac{\int xe^{-\sigma\rho x}\,dx}{\int e^{-\sigma\rho x}dx} = \frac{1}{\sigma\rho} = \Lambda,$$

the limits on all x integrals being $x = 0$ to $x = \infty$.

CHAPTER THREE

3-4

$$\frac{\lambda_x}{\lambda_e} = \frac{h/m_x v_x}{h/m_e v_e} = \frac{m_e}{m_x}\frac{v_e}{v_x},$$

$$1.813 \times 10^{-4} = \frac{9.109 \times 10^{-31}\text{ kg}}{m_x}\left(\frac{1}{3}\right),$$

$$m_x = 1.675 \times 10^{-27}\text{ kg};$$

evidently, the particle is a neutron.

3-7

(a) $$E^2 = p^2c^2 + E_0^2; \quad (K + E_0)^2 = p^2c^2 + E_0^2,$$

$$p = \frac{1}{c}(K^2 + 2KE_0)^{\frac{1}{2}} = \frac{\sqrt{(2KE_0)}}{c}\{1 + \frac{K}{2E_0}\}^{\frac{1}{2}}.$$

But $K = eV$ and $E_0 = m_0c^2$, so that

$$\frac{1}{c}\sqrt{(2KE_0)} = \left(\frac{2KE_0}{c^2}\right)^{\frac{1}{2}} = \left\{\frac{2(eV)(m_0c^2)}{c^2}\right\}^{\frac{1}{2}} = \sqrt{(2m_0eV)},$$

and

$$K/2E_0 = eV/2m_0c^2.$$

Therefore,

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{(2m_0eV)}}\{1 + \frac{eV}{2m_0c^2}\}^{-\frac{1}{2}}.$$

(b) Nonrelativistic limit: $eV \ll m_0c^2$; set $1 + eV/2m_0c^2 = 1$ to get

$$\lambda = h/(2m_0eV)^{\frac{1}{2}} = h/(2m_0K)^{\frac{1}{2}} = \frac{h}{m_0v}.$$

3-8

$$\lambda = \frac{h}{mv} = \frac{h(1 - v^2/c^2)^{\frac{1}{2}}}{m_0 v} = \frac{hc(1 - v^2/c^2)^{\frac{1}{2}}}{(m_0c^2)\ \ (v/c)} = \frac{hc}{E_0}\ \frac{(1 - \beta^2)^{\frac{1}{2}}}{\beta}\ .$$

Numerically,

$$hc = \frac{(6.626 \times 10^{-34}\ \text{J-s})\,(2.998 \times 10^{8}\ \text{m/s})}{(1.602 \times 10^{-13}\ \text{J/MeV})\,(10^{-9}\ \text{m/nm})},$$

$$hc = 1.2400 \times 10^{-3}\ \text{MeV-nm};$$

hence,

$$\lambda\,(\text{nm}) = \frac{1.2400 \times 10^{-3}\ \ \text{MeV-nm}}{E_0\,(\text{MeV})}\ \frac{(1 - \beta^2)^{\frac{1}{2}}}{\beta}.$$

3-19

(a)

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34}\ \text{J-s}}{(10^{-11}\ \text{m})\ c}\ \frac{(2.998 \times 10^{8}\ \text{m/s})}{(1.602 \times 10^{-13}\ \text{J/MeV})} = \frac{0.12400\ \text{MeV}}{c}.$$

$$E^2 = p^2c^2 + E_0^2,$$

$$E^2 = (0.1240)^2 + (0.511)^2 \rightarrow E = 0.5258\ \text{MeV};$$

$$K = E - E_0 = 0.5258 - 0.5110 = 0.0148\ \text{MeV} = 14.8\ \text{keV}.$$

(b)

$$p = \frac{0.12400\ \text{MeV}}{c} = \frac{E_{ph}}{c} \rightarrow E_{ph} = 124\ \text{keV}.$$

These are gamma-rays, or hard x-rays.

(c) The electron microscope is preferable: the gamma-rays are difficult to focus, and shielding would be required.

3-28

(a) Set $\Delta x = 10^{-10}$ m.

$$p \simeq \Delta p = \frac{h}{4\pi\Delta x} = \frac{6.626 \times 10^{-34}\ \text{J-s}}{4\pi(10^{-10}\ \text{m})},$$

$$p = \frac{5.2728 \times 10^{-25}\ \text{kg-m/s}}{c}\ \frac{2.998 \times 10^{8}\ \text{m/s}}{1.602 \times 10^{-16}\ \text{J/keV}} = \frac{0.9868\ \text{keV}}{c}.$$

$$E = (p^2c^2 + E_0^2)^{\frac{1}{2}} = \{(0.9868)^2 + (511)^2\}^{\frac{1}{2}} = 511.00095\ \text{keV};$$

$$K = E - E_0 = 511.00095 - 511 = 0.95\ \text{eV}.$$

Atomic binding energies are on the order of a few electron volts so that this result is consistent with finding electrons inside atoms.

(b) $\Delta x = 10^{-14}$ m; hence, p = 9.868 MeV/c, from (a).

$$E = (p^2c^2 + E_0^2)^{\frac{1}{2}} = (9.868^2 + 0.511^2)^{\frac{1}{2}} = 9.8812\ \text{MeV};$$

$$K = E - E_0 = 9.8812 - 0.5110 = 9.37\ \text{MeV}.$$

This is approximately the average binding energy per nucleon, so electrons will tend to escape from nuclei.

(c) For a neutron or proton, p = 9.868 MeV/c, from (b). Using 938 MeV as a rest energy,

$$E = (p^2c^2 + E_0^2)^{\frac{1}{2}} = (9.868^2 + 938^2)^{\frac{1}{2}} = 938.052\ \text{MeV};$$

$$K = E - E_0 = 938.052 - 938 = 0.052\ \text{MeV}.$$

This last result is much less than the average binding energy per nucleon; thus the uncertainty principle is consistent with finding these particles confined inside nuclei.

3-32

(a) Since $p_x \geq \Delta p_x$ and $x \geq \Delta x$, for the smallest E use $p_x = \Delta p_x$ and $x = \Delta x$ to obtain

$$E = \frac{1}{2m}(\Delta p_x)^2 + \tfrac{1}{2}C(\Delta x)^2.$$

With

$$\Delta p_x \Delta x = \tfrac{1}{2}\hbar = \frac{h}{4\pi},$$

the minimum energy becomes

$$E = \frac{1}{2m}\left(\frac{h}{4\pi\Delta x}\right)^2 + \tfrac{1}{2}C(\Delta x)^2 = \frac{h^2}{32\pi^2 m(\Delta x)^2} + \tfrac{1}{2}C(\Delta x)^2.$$

(b) Set the derivative equal to zero:

$$\frac{dE}{d(\Delta x)} = -\frac{h^2}{16\pi^2 m}\frac{1}{(\Delta x)^3} + C(\Delta x) = 0 \rightarrow (\Delta x)^2 = \frac{h}{4\pi\sqrt{(Cm)}}.$$

Substituting this into the expression for E above gives

$$E_{min} = \tfrac{1}{2}h\left\{\frac{1}{2\pi}\left(\frac{C}{m}\right)^{\frac{1}{2}}\right\} = \tfrac{1}{2}h\nu.$$

3-34

(a) Let the crack be of zero width and Δx = horizontal aiming error (i.e., drop point not exactly above crack). This implies an initial horizontal speed v_x given by

$$v_x = \Delta v_x = \frac{\hbar/2}{m\Delta x}.$$

As a result of this, the ball lands a horizontal distance x from the release point, given by

$$x = v_x\left(\frac{2H}{g}\right)^{\frac{1}{2}} = \frac{\hbar}{2m\Delta x}\left(\frac{2H}{g}\right)^{\frac{1}{2}}.$$

Hence, the total horizontal distance X from crack to impact point is

$$X = \Delta x + x = \Delta x + \frac{\hbar}{2m}\left(\frac{2H}{g}\right)^{\frac{1}{2}}\frac{1}{\Delta x}.$$

To minimize, set

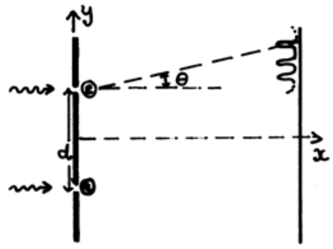
$$\frac{dX}{d(\Delta x)} = 0 \rightarrow \Delta x = \left\{\frac{\hbar}{2m}\left(\frac{2H}{g}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

Therefore,

$$X_{min} = \left\{\frac{2\hbar}{m}\left(\frac{2H}{g}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \simeq \left(\frac{\hbar}{m}\right)^{\frac{1}{2}}\left(\frac{H}{g}\right)^{\frac{1}{4}}.$$

(b) If H = 10 m, H/g = 1, m = 0.001 kg, then $X_{min} = (10^{-34}10^3)^{\frac{1}{2}}$ $= 3 \times 10^{-16}$ m, approximately.

3-35



Put an electron behind each slit and observe any recoil due to its collision with a photon emerging from the slit. To determine which electron recoiled, its observed displacement Δy must satisfy

$$\Delta y << \tfrac{1}{2}d,$$

at least, or even

$$\Delta y << \frac{d}{4\pi}.$$

Due to the collision, the photon's momentum changes. In order not to destroy the interference pattern,

$$\frac{\Delta p_y}{p_x} << \theta \simeq \frac{m\lambda}{d} = \frac{mh}{pd} \simeq \frac{mh}{p_x d},$$

$$\Delta p_y << \frac{mh}{d} \simeq \frac{h}{d},$$

m the order of the fringe. By conservation of momentum, this is also the uncertainty in the electron's momentum. Hence, for the electron, it is required that, in order not to destroy the pattern,

$$(\Delta y)(\Delta p_y) << \left(\frac{d}{4\pi}\right)\left(\frac{h}{d}\right) = \tfrac{1}{2}\hbar.$$

CHAPTER FOUR

4-1

Consider an electron oscillating along a diameter. When at a distance r from the center of the atom, the force on the electron is

$$F = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{4\pi}{3}r^3\rho\right)e/r^2,$$

where $\rho = e/(4\pi R^3/3) > 0$ since the net charge on atom-electron is +e. Therefore,

$$F = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^3}\,r.$$

This force is attractive: i.e., directed toward the center of the atom. Hence,

$$F = ma,$$

$$-\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^3}\,r = m\ddot{r} \rightarrow \omega^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mR^3}.$$

If the electron revolves in a circular orbit of radius R,

$$F = ma_r$$

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m\frac{v^2}{R} = m\frac{(R\omega)^2}{R} = mR\omega^2 \rightarrow \omega^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mR^3}.$$

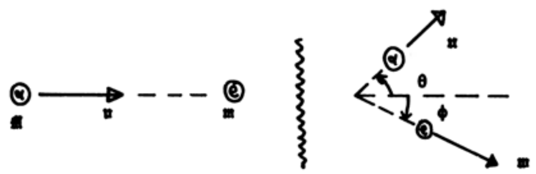
The two frequencies are seen to be equal. The equality applies also to oscillations of amplitude less than R and circular orbits of radius less than R, since the charge exterior to the amplitude or radius exerts zero force on the electron for spherically symmetric charge distributions.

4-4

(a) Momentum conservation:

$$Mv = Mu\cos\theta + mw\cos\phi,$$

$$Mu\sin\theta = mw\sin\phi.$$



Kinetic energy conservation:

$$\tfrac{1}{2}Mv^2 = \tfrac{1}{2}Mu^2 + \tfrac{1}{2}mw^2.$$

The momentum equations give:

$$mw\cos\phi = M(v - u\cos\theta)$$
$$mw\sin\phi = Mu\sin\theta.$$

Hence,

$$m^2w^2 = M^2(v^2 - 2uv\cos\theta + u^2).$$

The energy equation yields

$$m^2w^2 = Mm(v^2 - u^2).$$

Equating the two expressions for m^2w^2:

$$M^2(v^2 - 2uv\cos\theta + u^2) = Mm(v^2 - u^2),$$

$$\cos\theta = \frac{v}{2u}\left(1 - \frac{m}{M}\right) + \frac{u}{2v}\left(1 + \frac{m}{M}\right).$$

(Since $m < M$, $\cos\theta \neq 0$, $\theta \neq 90°$.) For maximum θ, set $d\cos\theta/du=0$ and look for minima:

$$\frac{d(\cos\theta)}{du} = 0 \rightarrow u = v\left(\frac{M - m}{M + m}\right)^{\frac{1}{2}} \quad (u > 0).$$

This gives $d^2\cos\theta/du^2 > 0$, a minimum for $\cos\theta$, a maximum for θ. Substitute this value of u into the equation for $\cos\theta$ to get

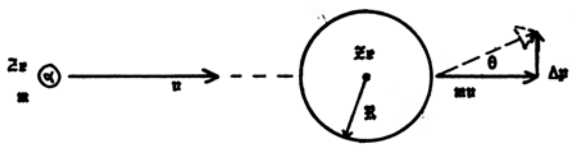
$$\cos\theta_{max} = \left(1 - \frac{m^2}{M^2}\right)^{\frac{1}{2}}.$$

Since m << M, this implies that

$$1 - \tfrac{1}{2}\theta_{max}^2 = 1 - \tfrac{1}{2}m^2/M^2,$$

$$\theta_{max} = \frac{m}{M} = \frac{1}{7400} \simeq 10^{-4} \text{ rad.}$$

(b)



The maximum force felt by the alpha-particle in passage through the atom occurs at the atom's surface:

$$F_m = \frac{1}{4\pi\epsilon_0}\frac{(Ze)(2e)}{R^2}.$$

For maximum deflection, suppose this force imparts momentum Δp perpendicular to the original direction of motion:

$$\Delta p = \int F dt \simeq F_m(\Delta t) \simeq F_m \cdot \frac{2R}{v},$$

$$\Delta p = \frac{1}{4\pi\epsilon_0}\frac{4Ze^2}{Rv}.$$

Then, anticipating a small deflection θ,

$$\frac{\Delta p}{mv} = \frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{R(\frac{1}{2}Mv^2)} = \frac{U_{surf}}{K_\alpha}.$$

For gold, $Z = 79$; suppose $K_\alpha = 5$ MeV; then

$$\theta = (8.988 \times 10^9)\frac{(2)(79)(1.602 \times 10^{-19})^2}{(10^{-10})(5)(1.602 \times 10^{-13})} = 4.55 \times 10^{-4} \text{ rad.}$$

Hence, the deflection due both to the positive and negative charge in a Thomson atom each are about 0.0001 rad, so that the overall deflection is about 0.0001 rad also. Only if all the deflections due to the electrons are in the same direction could a larger deflection, about 0.01 rad, be obtained.

4-10

By Eqs. 4-8,9,

$$dN = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{zZe^2}{2Mv^2}\right)^2 In \frac{1}{\sin^4(\theta/2)} d\Omega.$$

The solid angle of the detector is

$$d\Omega = dA/r^2 = 1.0/(10)^2 = 10^{-2} \text{ strad.}$$

Also,

$$n = (\#\text{ nuclei per cm}^3)(\text{thickness}),$$

$$n = \frac{19.3}{(197)(1.661 \times 10^{-24})}(10^{-5}) = 5.898 \times 10^{21} \text{ m}^{-2}.$$

Hence, by direct numerical substitution,

$$dN = 6.7920 \times 10^{-5} \frac{1}{\sin^4(\theta/2)} \text{ s}^{-1}.$$

The number of counts per hour is

$$\# = (3600)dN = 0.2445 \frac{1}{\sin^4\theta/2}.$$

This gives: $\theta = 10°$: $\# = 4237$;
$\theta = 45°$: $\# = 11.4$.

4-13

$$L = n\hbar = \frac{nh}{2\pi},$$

$$7.382 \times 10^{-34} = \frac{n}{2\pi}(6.626 \times 10^{-34})$$

$$n = 7.$$

4-18

The periods of revolution of electron and proton are equal:

$$\frac{2\pi r_e}{v_e} = \frac{2\pi r_p}{v_p},$$

$$v_p = (\frac{r_p}{r_e})v_e.$$

The motion is about the center of mass of the electron-proton system, so that

$$m_p r_p = m_e r_e \rightarrow \frac{r_p}{r_e} = \frac{m_e}{m_p}.$$

Therefore,

$$v_p = (\frac{m_e}{m_p})v_e \simeq (\frac{m_e}{m_p})(\frac{c}{137}),$$

$$v_p \simeq \frac{1}{1836}\frac{3 \times 10^8}{137} = 1.2 \times 10^3 \text{ m/s}.$$

4-22

(a) Frequency of the first line: $\nu_1 = c/\lambda_1 = cR_H\{\frac{1}{m^2} - \frac{1}{(m+1)^2}\}$.

Frequency of the series limit: $\nu_\infty = c/\lambda_\infty = cR_H\{\frac{1}{m^2} - 0\}$.

Therefore,

$$\Delta\nu = \nu_\infty - \nu_1 = \frac{cR_H}{(m+1)^2}.$$

(b)

$$\frac{\Delta\nu_{Ly}}{\Delta\nu_{Pf}} = \frac{cR_H/(1+1)^2}{cR_H/(5+1)^2} = 9.$$

4-25

(a)

$$E_{ph,2} = hc/\lambda_2 = \frac{1240}{46.6} = 26.6 \text{ eV};$$

$$K = 26.6 - 10.2 = 16.4 \text{ eV}.$$

(b)

$$E_{ph,1} = 13.6 + 16.4 = 30.0 \text{ eV}.$$

4-29

(a) By momentum conservation,

$$\frac{h\nu}{c} = Mv.$$

Combining this with energy conservation gives

$$\Delta E = h\nu_0 = h\nu + \tfrac{1}{2}Mv^2 = h\nu + \tfrac{1}{2}M(\frac{h\nu}{Mc})^2 = h\nu + \frac{(h\nu)^2}{2Mc^2}.$$

$$h\nu_0 \simeq h\nu + \frac{(h\nu)(h\nu_0)}{2Mc^2} = h\nu + \frac{(h\nu)(\Delta E)}{2Mc^2},$$

$$\nu_0 = \nu(1 + \frac{\Delta E}{2Mc^2}),$$

$$\nu = \nu_0(1 - \frac{\Delta E}{2Mc^2}).$$

(b) Since $\nu = c/\lambda$, $\nu_0 = c/\lambda_0$,

$$\lambda = \lambda_0(1 + \frac{\Delta E}{2Mc^2}).$$

$$\Delta E = (13.6)(\frac{1}{1^2} - \frac{1}{3^2}) = 12.089 \text{ eV}.$$

Neglecting recoil:

$$\lambda_0 = \frac{1240}{12.089} = 102.6 \text{ nm}.$$

With recoil:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta E}{2Mc^2} = \frac{(12.089)\ (1.602 \times 10^{-19})}{(2)(1.673 \times 10^{-27})(2.998 \times 10^8)^2} = 6.440 \times 10^{-9},$$

$$\Delta\lambda = 661 \text{ am}.$$

4-34

The kinetic energy of the electron is

$$K = (0.511 \text{ MeV})\left(\frac{1}{\sqrt{1 - \beta^2}} - 1\right).$$

With

$$\beta = \frac{1.2 \times 10^7}{2.998 \times 10^8} = 0.0400,$$

this gives K = 409.3 eV. For helium, the second ionization potential from the ground state is

$$E_{ion} = \frac{13.6\ Z^2}{n^2} = \frac{(13.6)(2)^2}{1^2} = 54.4 \text{ eV}.$$

Hence,

$$E_{ph} = 54.4 + 409.3 = 463.7 \text{ eV},$$

$$\lambda = \frac{1240}{463.7} = 2.674 \text{ nm}.$$

4-38

(a) Hydrogen Hα: $\lambda_H^{-1} = R_H\{\frac{1}{2^2} - \frac{1}{3^2}\}$.

Helium, Z = 2: $\lambda_{He}^{-1} \simeq 4R_H\{\frac{1}{n_f^2} - \frac{1}{n_i^2}\} = R_H\{\frac{1}{(n_f/2)^2} - \frac{1}{(n_i/2)^2}\}$.

If $\lambda_H = \lambda_{He}$, then

$$2 = n_f/2 \rightarrow n_f = 4,$$

$$3 = n_i/2 \rightarrow n_i = 6.$$

(b) Now take into account the reduced mass μ:

$$R_H = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{\mu_H(1)^2e^4}{4\pi\hbar^3c}; \quad R_{He} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{\mu_{He}(2)^2e^4}{4\pi\hbar^3c} = \frac{\mu_{He}}{\mu_H}(4R_H).$$

$$\mu_H = \frac{m_em_p}{m_e+m_p} \simeq m_e\left(1 - \frac{m_e}{m_p}\right); \quad \mu_{He} \simeq \frac{m_e(4m_p)}{(4m_p)+m_e} \simeq m_e\left(1 - \frac{m_e}{4m_p}\right).$$

Therefore,

$$\mu_{He} > \mu_H,$$

so that

$$\frac{1}{\lambda_{He}} = R_{He}\{\frac{1}{n_f^2} - \frac{1}{n_i^2}\} > 4R_H\{\frac{1}{n_f^2} - \frac{1}{n_i^2}\}.$$

Hence, compared to the hydrogen Hα line, the helium 6→4 line wavelength is a little shorter.

(b) Since $\lambda \propto \mu^{-1}$ (the factor Z^2 is combined with $1/n_f^2 - 1/n_i^2$ to give equal values for H and He),

$$\frac{\lambda_H - \lambda_{He}}{\lambda_H} = \frac{\mu_{He} - \mu_H}{\mu_{He}} = 1 - \frac{\mu_H}{\mu_{He}},$$

$$\frac{\Delta\lambda}{\lambda_H} = 1 - \frac{1 - m_e/m_p}{1 - m_e/4m_p} = \frac{3}{4}\frac{m_e}{m_p} = \frac{3}{4}\frac{0.511}{938.3} = 4.084 \times 10^{-4},$$

$$\Delta\lambda = (4.084 \times 10^{-4})(656.3 \text{ nm}) = 0.268 \text{ nm}.$$

4-42

The momentum associated with the angle θ is $L = I\omega$. The total energy E is

$$E = K = \tfrac{1}{2}I\omega^2 = \frac{L^2}{2I}.$$

L is independent of θ for a freely rotating object. Hence, by the Wilson-Sommerfeld rule,

$$\oint L d\theta = nh,$$

$$L\oint d\theta = L(2\pi) = \sqrt{(2IE)}(2\pi) = nh,$$

$$\sqrt{(2IE)} = n\hbar,$$

$$E = \frac{n^2\hbar^2}{2I}.$$

CHAPTER FIVE

5-3

(a) The time-dependent part of the wavefunction is

$$e^{-\frac{1}{2}it\sqrt{(C/m)}} = e^{-iEt/\hbar} = e^{-i2\pi\nu t}.$$

Therefore,

$$\tfrac{1}{2}\left(\frac{C}{m}\right)^{\frac{1}{2}} = 2\pi\nu \rightarrow \nu = \frac{1}{4\pi}\left(\frac{C}{m}\right)^{\frac{1}{2}}.$$

(b) Since $E = h\nu = 2\pi\hbar\nu$,

$$E = \tfrac{1}{2}\hbar\left(\frac{C}{m}\right)^{\frac{1}{2}}.$$

(c) The limiting x can be found from

$$\tfrac{1}{2}Cx^2 = E,$$

$$x = \pm\left(\frac{2E}{C}\right)^{\frac{1}{2}} = \pm\hbar^{\frac{1}{2}}(Cm)^{-\frac{1}{4}}.$$

5-4

According to Example 5-6, the normalizing integral is

$$1 = 2B^2\left(\frac{m}{C}\right)^{\frac{1}{2}}\int_0^{(2E/C)^{\frac{1}{2}}}\frac{dx}{\{(2E/C) - x^2\}^{\frac{1}{2}}} = 2B^2\left(\frac{m}{C}\right)^{\frac{1}{2}}\sin^{-1}\frac{x}{(2E/C)^{\frac{1}{2}}}\Big|_0^{(2E/C)^{\frac{1}{2}}}$$

$$1 = B^2\pi\left(\frac{m}{C}\right)^{\frac{1}{2}} \rightarrow B^2 = (C/m\pi^2)^{\frac{1}{2}}.$$

5-5

Problem 5-3(c) provides the limits on x; the wavefunction is

$$\Psi = \frac{(Cm)^{1/8}}{(\pi\hbar)^{1/4}}\, e^{-\sqrt{(Cm)}x^2/2\hbar}\, e^{-i\omega t}.$$

Hence, the desired probability is given by

$$\text{Prob.} = 2\frac{(Cm)^{\frac{1}{4}}}{(\pi\hbar)^{\frac{1}{2}}}\int_0^{\hbar^{\frac{1}{2}}(Cm)^{-\frac{1}{4}}} e^{-\sqrt{(Cm)}x^2/\hbar}dx.$$

If

$$u = \frac{(4Cm)^{\frac{1}{4}}}{\hbar^{\frac{1}{2}}}x,$$

$$\text{Prob.} = 2\int_0^{\sqrt{2}} \frac{1}{\sqrt{(2\pi)}}e^{-u^2/2}du = 2(0.42) = 0.84.$$

The last integral is the normal probability integral.

5-7

(a) Since

$$\Psi = \left(\frac{2}{a}\right)^{\frac{1}{2}}\cos\frac{\pi x}{a}\, e^{-iEt/\hbar},$$

$$\text{Prob.} = \frac{2}{a}\int_{a/6}^{a/2}\cos^2(\pi x/a)dx = \frac{2}{\pi}\int_{\pi/6}^{\pi/2}\cos^2 u\,du = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.1955,$$

independent of E.

(b) Classically,

$$\text{Prob.} = \frac{a/3}{a} = \frac{1}{3} = 0.3333.$$

5-9

(a),(b) Let V = 0 in the region in which the particle is confined, so that Schrodinger's equation becomes

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t},$$

with

$$\Psi = A\sin(2\pi x/a)e^{-iEt/\hbar}.$$

Putting these into Schrodinger's equation gives

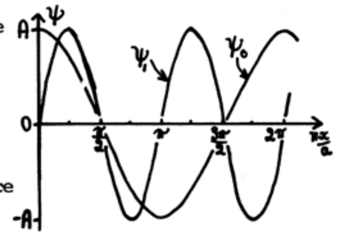
$$\left(-\frac{\hbar^2}{2m}\right)\left(-\frac{4\pi^2}{a^2}\Psi\right) = i\hbar\left(-\frac{iE}{\hbar}\Psi\right) = E\Psi; \quad E = E_1 = \frac{2\pi^2\hbar^2}{2ma^2}.$$

In the ground state, $E = E_0 = \pi^2\hbar^2/2ma^2$, so that $E_1 = 4E_0$.

(c) The space parts of the wave functions are

$$\psi_0 = A\cos(\pi x/a),$$

$$\psi_1 = A\sin(2\pi x/a).$$



ψ_1 oscillates more rapidly, since with $E_1 > E_0$, $\psi_1, \psi_0 \le A$,

$$|-d^2\psi_1/dx^2| = \frac{2m}{\hbar^2}|E_1\psi_1| > |-d^2\psi_0/dx^2| = \frac{2m}{\hbar^2}|E_0\psi_0|,$$

for most x.

5-10

(a) To normalize the wavefunction, evaluate

$$1 = \int_{-a/2}^{+a/2}\Psi^*\Psi dx,$$

($\Psi = 0$ outside this region). With $\Psi = A\sin(2\pi x/a)e^{-iEt/\hbar}$, this becomes

$$1 = 2A^2\int_0^{a/2}\sin^2(2\pi x/a)dx = \frac{a}{\pi}A^2\int_0^{\pi}\sin^2 u\,du = \frac{a}{\pi}A^2\frac{\pi}{2},$$

$$A = \sqrt{\left(\frac{2}{a}\right)}.$$

(b) This equals the value of A for the ground state wavefunction and, in fact, the normalization constant of all the excited states equals this also. Since all of the space wave functions are simple sines or cosines, this equality is understandable.

5-11

The wavefunction is

$$\Psi = \left(\frac{2}{a}\right)^{\frac{1}{2}}\sin(2\pi x/a)\,e^{-iEt/\hbar},$$

and therefore

$$\overline{x} = \frac{2}{a}\int_{-a/2}^{+a/2} x\,\sin^2(2\pi x/a)\,dx = 0.$$

As for x^2:

$$\overline{x^2} = \frac{2}{a}\int_{-a/2}^{+a/2} x^2\,\sin^2(2\pi x/a)\,dx = \frac{a^2}{2\pi^3}\int_0^{\pi} u^2\sin^2 u\,du = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{2\pi^2}\right)a^2,$$

$$\overline{x^2} = 0.07067a^2.$$

5-12

The linear momentum operator is $-i\hbar\,\frac{\partial}{\partial x}$ and therefore

$$\overline{p} = \frac{2}{a}\int_{-a/2}^{+a/2}\sin\frac{2\pi x}{a}\,\{-i\hbar\,\frac{\partial}{\partial x}(\sin\frac{2\pi x}{a})\}\,dx = -\frac{4i\hbar}{a}\int_0^{\pi}\sin u\,\cos u\,du = 0.$$

Similarly,

$$\overline{p^2} = \frac{2}{a}\int_{-a/2}^{+a/2}\sin\frac{2\pi x}{a}\{i^2\hbar^2\,\frac{\partial^2}{\partial x^2}(\sin\frac{2\pi x}{a})\}\,dx,$$

$$\overline{p^2} = -8\pi i^2\left(\frac{\hbar}{a}\right)^2\int_0^{\pi}\sin^2 u\,du = 4\pi^2\left(\frac{\hbar}{a}\right)^2 = \left(\frac{h}{a}\right)^2.$$

5-13

Let $\Delta x = (\overline{x^2})^{\frac{1}{2}};\quad \Delta p = (\overline{p^2})^{\frac{1}{2}}.$

(a) Problems 5-11 and 5-12 yield

$$\Delta x = na,\ n^2 = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{2\pi^2}\right);\quad \Delta p = \frac{h}{a}.$$

Hence,

$$\Delta x\Delta p = (na)\left(\frac{h}{a}\right) = 4\pi n\left(\frac{\hbar}{2}\right) = \left(\frac{4}{3}\pi^2 - 2\right)^{\frac{1}{2}}\,\frac{\hbar}{2} = 3.34\,\frac{\hbar}{2}.$$

(b) In the ground state,

$$\Delta x\Delta p = (0.18a)\left(\frac{h}{2a}\right) = 1.13\,\frac{\hbar}{2}.$$

In the first excited state the uncertainties in position and momentum both increase over the ground state values, due to the higher energy of the particle.

5-14

The normalized wavefunction is

$$\Psi = \frac{(Cm)^{1/8}}{(\pi\hbar)^{1/4}}\,e^{-\sqrt{(Cm)}\,x^2/2\hbar}\,e^{-iEt/\hbar},$$

with $E = \frac{1}{2}\hbar\sqrt{(C/m)}$.

(a) Since the kinetic energy is $p^2/2m$ the corresponding operator is

$$T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}.$$

Therefore,

$$\overline{T} = \frac{(Cm)^{\frac{1}{4}}}{(\pi\hbar)^{\frac{1}{2}}}\left(-\frac{\hbar^2}{2m}\right)\int_{-\infty}^{+\infty} e^{-\sqrt{(Cm)}\,x^2/2\hbar}\,\frac{\partial^2}{\partial x^2}e^{-\sqrt{(Cm)}\,x^2/2\hbar}\,dx,$$

$$\overline{T} = \hbar\left(\frac{C}{\pi m}\right)^{\frac{1}{2}}\int_0^{\infty}(1 - u^2)e^{-u^2}\,du = \frac{\hbar}{4}\left(\frac{C}{m}\right)^{\frac{1}{2}} = \frac{1}{2}E.$$

Similarly for the potential energy $U = \frac{1}{2}Cx^2$:

$$\overline{U} = \frac{(Cm)^{1/2}}{(\pi\hbar)^{1/2}}\frac{C}{2}\int_{-\infty}^{+\infty} x^2 e^{-\sqrt{(Cm)}\,x^2/\hbar}dx = \hbar\left(\frac{C}{\pi m}\right)^{1/2}\int_0^{\infty} u^2 e^{-u^2}du,$$

$$\overline{U} = \hbar\left(\frac{C}{\pi m}\right)^{1/2}\frac{\pi^{1/2}}{4} = \overline{T} = \tfrac{1}{2}E.$$

(b) This same relation, $\overline{U} = \overline{T} = \frac{1}{2}E$, is obeyed by the classical oscillator also.

5-15

Use the notation

$$(xp)_1 = -i\hbar\int_{-\infty}^{+\infty}\Psi^* x\frac{\partial\Psi}{\partial x}dx, \quad (xp)_2 = -i\hbar\int_{-\infty}^{+\infty}\Psi^*\frac{\partial x\Psi}{\partial x}dx.$$

Clearly

$$(xp)_2 = -i\hbar\int_{-\infty}^{+\infty}\Psi^*\left(x\frac{\partial\Psi}{\partial x} + \Psi\right)dx = (xp)_1 - i\hbar,$$

implying that $(xp)_1$ and $(xp)_2$ cannot both be real. Also, by integrating by parts,

$$(xp)_2 = -i\hbar\left\{x\Psi^*\Psi\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}x\Psi\frac{\partial\Psi^*}{\partial x}dx\right\} = i\hbar\int_{-\infty}^{+\infty}\Psi x\frac{\partial\Psi^*}{\partial x}dx.$$

Thus,

$$(xp)_2 = (xp)_1^*.$$

If $(xp)_1$ is real, this last relation says that $(xp)_2$ is real also, which contradicts the first finding above. Hence $(xp)_1$ is complex and therefore so is $(xp)_2$. Now try

$$\overline{xp} = -\tfrac{1}{2}i\hbar\int_{-\infty}^{+\infty}\Psi^*\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right)\Psi\,dx;$$

$$\overline{xp} = \tfrac{1}{2}\{(xp)_1 + (xp)_2\} = \tfrac{1}{2}\{(xp)_1 + (xp)_1^*\},$$

$$\overline{xp} = \mathrm{Re}(xp)_1,$$

so that this new $\overline{xp}$ is real, as desired.

5-21

With $V = 0$, the energy of the photon is

$$E = pc.$$

Replacing the energy E and momentum p by their operators gives

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar c\frac{\partial\Psi}{\partial x}.$$

Now set $\Psi(x,t) = \psi(x)T(t)$ and divide the equation by Ψ to get

$$i\hbar\frac{1}{T}\frac{dT}{dt} = -i\hbar c\frac{1}{\psi}\frac{d\psi}{dx} = K,$$

where K is independent of x and t. Write $K = k\hbar c$ and the two equations directly above become

$$\frac{dT}{dt} = -ikcT \rightarrow T \propto e^{-ikct},$$

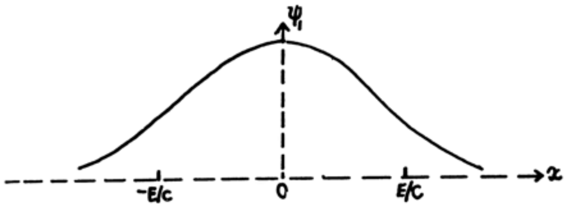
$$\frac{d\psi}{dx} = ik\psi \rightarrow \psi \propto e^{ikx}.$$

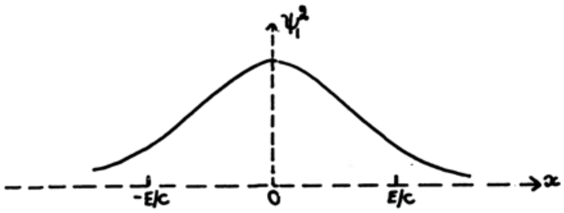
Hence, for the photon,

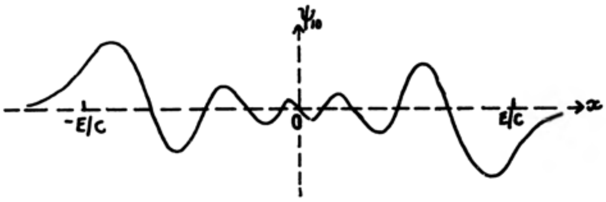
$$\Psi \propto e^{ik(x-ct)}.$$

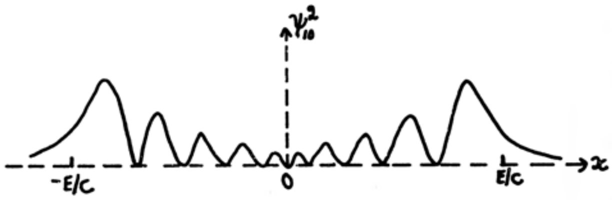
5-22

(a),(b) The curvature of ψ is proportional to $|V - E|$: where $|V - E|$ is large the function oscillates rapidly in x, and where $|V - E|$ is small it oscillates less rapidly (hence, nodes are close together in the former case, farther apart in the latter). In the first state, $|V - E|$ is just large enough to turn ψ over: no nodes. The 10th state will have $10-1 = 9$ nodes, leading to an odd function since V is symmetrical about the origin. The wavefunction decays exponentially wherever $V>E$, the classically forbidden region. For further discussion, see Example 5-12, which treats the similar simple harmonic oscillator potential.









(c) Classically, the probability density function P is given by

$$P = B^2/v,$$

B^2 the normalization constant. Energy conservation gives v:

$$E = \tfrac{1}{2}mv^2 \pm Cx,$$

the upper sign for $x>0$. Using this,

$$P = B^2\left(\frac{m}{2}\right)^{\frac{1}{2}}(E \mp Cx)^{-\frac{1}{2}}.$$

To determine B, use the normalization condition

$$\left(\frac{m}{2}\right)^{\frac{1}{2}}B^2\left\{\int_{-E/C}^{0}\frac{dx}{\sqrt{(E + Cx)}} + \int_{0}^{+E/C}\frac{dx}{\sqrt{(E - Cx)}}\right\} = 1,$$

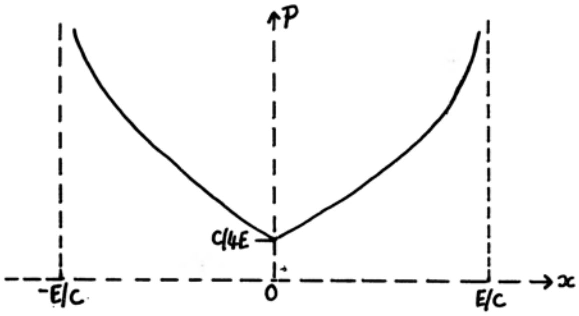
since the turning points (and therefore limits) are given by $v = 0$ or $E = V = \pm Cx$. Evaluating the integrals gives

$$P = \frac{C}{4\sqrt{E}}\,\frac{1}{\sqrt{(E \mp Cx)}}.$$

Particular values are:

$$P(0) = C/4E;\; P(E/C) = P(-E/C) = \infty;\; \frac{dP}{dx}(x=0) = \pm C^2/8E^2.$$

(d) The graph of the classical density function resembles that for the simple harmonic oscillator, the lack of a horizontal tangent at the origin being the main difference on a rough sketch.



5-24

See Problem 5-25.

5-25

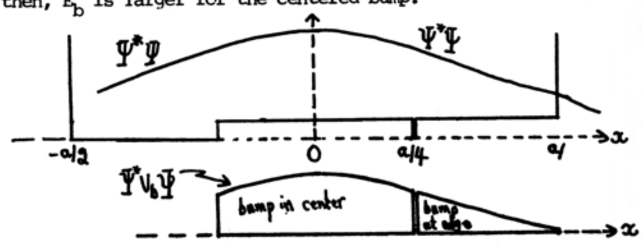
With no bump, the wavefunction will be sinusoidal inside the classical region of motion and a decaying exponential outside. The lowest energy wavefunction will contain no nodes.
In the present situation, in the region of the bump the curvature of the wavefunction will be less than outside the bump, since the curvature is proportional to E - V. This will upset the good behavior of the wavefunction at large x associated with the value E_1 corresponding to the first bound state without the bump. To compensate for this reduced curvature in the region of the bump, a larger curvature (as compared with the no-bump case) is needed outside the bump. Here V = 0 so that the curvature is proportional to E. Hence, a larger E is required: that is, the first eigenvalue with bump is greater than the first eigenvalue without the bump.

5-25

By assumption,

$$E_b = E_1 + \int \Psi^* V_b \Psi dx,$$

V_b = bump potential energy, Ψ = wavefunction with no bump in the potential. The integral is the area under a curve of $\Psi^* V_b \Psi$ vs x. Now $V_b = 0$ except where it is equal to $V_0/10$. Clearly the area will be larger if the bump is located where $\Psi^*\Psi$ is relatively large (i.e., in the center for Ψ_1) than if the bump is placed where $\Psi^*\Psi$ is small, i.e., at the edge in this case. Evidently then, E_b is larger for the centered bump.



5-27

Schrodinger's equation is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0.$$

In the region in question, $V = V_0$ = constant, $E < V_0$, so that

$$q^2 = \frac{2m}{\hbar^2}(V_0 - E) > 0.$$

Hence,

$$\psi = Ae^{-qx} + Be^{qx},$$

is the general solution. However, $\psi(x=\infty) = 0$, requiring B = 0, leaving

$$\psi = Ae^{-qx},$$

as the wavefunction.

5-28

Since ψ is real, the probability density P is

$$P = \psi^*\psi = \psi^2 = A^2e^{-2qx}.$$

Recalling that x is measured from the center of the binding region, the suggested criterion for D gives

$$A^2e^{-2q(\frac{1}{2}a+D)} = e^{-1}A^2e^{-2q(\frac{1}{2}a)},$$

$$e^{-qa - 2qD} = e^{-aq - 1},$$

$$D = \frac{1}{2q} = \frac{\hbar}{2\{2m(V_0 - E)\}^{\frac{1}{2}}}.$$

5-31

Use the scheme suggested in Problem 5-26:

$$E = E_1 + \int \Psi^* V \Psi dx,$$

in which E_1 and Ψ are eigenvalue and eigenfunction of the lowest energy state of the infinite, flat, square well potential. From

Example 5-9,10 the time-independent part of Ψ is

$$\psi = \left(\frac{2}{a}\right)^{\frac{1}{2}}\cos(\pi x/a).$$

Hence,

$$\int\psi^*V\psi dx = \frac{2}{a}\int_{-\frac{1}{2}a}^{+\frac{1}{2}a}\cos^2(\pi x/a)V_0\cos(\pi x/a)\,dx = \frac{4V_0}{a}\int_0^{\frac{1}{2}a}\cos^3(\pi x/a)\,dx,$$

$$\int\psi^*V\psi dx = \frac{4V_0}{\pi}\int_0^{\frac{1}{2}\pi}\cos^3 u\,du = \frac{8V_0}{3\pi}.$$

Thus,

$$E = \frac{\pi^2\hbar^2}{2ma^2} + \frac{8V_0}{3\pi}.$$

5-32

The wavefunctions in question are

$$\Psi_1 = \left(\frac{2}{a}\right)^{\frac{1}{2}}\cos(\pi x/a)e^{-iE_1t/\hbar};\quad \Psi_2 = \left(\frac{2}{a}\right)^{\frac{1}{2}}\sin(2\pi x/a)e^{-iE_2t/\hbar},$$

with $E_2 = 4E_1$. The linear combination is

$$\Psi = c_1\Psi_1 + c_2\Psi_2.$$

Normalizing this last gives

$$1 = \int\Psi^*\Psi dx,$$

$$c_1c_1^*\int\Psi_1^*\Psi_1dx + c_2c_2^*\int\Psi_2^*\Psi_2dx + c_1^*c_2\int\Psi_1^*\Psi_2dx + c_2^*c_1\int\Psi_2^*\Psi_1dx = 1.$$

Since Ψ_1 and Ψ_2 already are normalized,

$$\int\Psi_1^*\Psi_1dx = \int\Psi_2^*\Psi_2dx = 1.$$

The real parts of $\int\Psi_1^*\Psi_2dx$ and $\int\Psi_2^*\Psi_1dx$ each are

$$\int_{-\frac{1}{2}a}^{+\frac{1}{2}a}\cos(\pi x/a)\sin(2\pi x/a)\,dx = \frac{a}{\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos u\,\sin 2u\,du = 0.$$

Therefore, in order that Ψ be normalized, it is required that

$$c_1c_1^* + c_2c_2^* = 1.$$

5-33

(a) The total energy is $E = p^2/2m + V$. But $V = 0$ in the region of motion, so that

$$E = p^2/2m = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}.$$

Hence,

$$\overline{E} = -\frac{\hbar^2}{2m}\int_{-\frac{1}{2}a}^{+\frac{1}{2}a}(c_1^*\Psi_1^* + c_2^*\Psi_2^*)\frac{\partial^2}{\partial x^2}(c_1\Psi_1 + c_2\Psi_2)\,dx.$$

But

$$\partial^2\Psi_1/\partial x^2 = -\left(\frac{\pi}{a}\right)^2\Psi_1;\quad \partial^2\Psi_2/\partial x^2 = -\left(\frac{2\pi}{a}\right)^2\Psi_2.$$

Also, by Problem 5-32, $\int\Psi_1^*\Psi_2dx = \int\Psi_2^*\Psi_1dx = 0$ and therefore

$$\overline{E} = \frac{\hbar^2}{2m}\{\left(\frac{\pi}{a}\right)^2c_1c_1^*\int\Psi_1^*\Psi_1dx + \left(\frac{2\pi}{a}\right)^2\int\Psi_2^*\Psi_2dx\},$$

$$\overline{E} = c_1c_1^*\frac{\pi^2\hbar^2}{2ma^2} + c_2c_2^*\frac{2\hbar^2\pi^2}{ma^2},$$

$$\overline{E} = c_1c_1^*E_1 + c_2c_2^*E_2.$$

(b) Since $c_1c_1^* + c_2c_2^* = 1$,

$$\overline{E} = (1 - c_2c_2^*)E_1 + c_2c_2^*E_2 = E_1 + c_2c_2^*(E_2 - E_1).$$

With $0 \le c_2c_2^* \le 1$, this means that

$$E_1 \le \overline{E} \le E_2.$$

Hence, if the particle can be found either in level 1 or 2, making transitions between them, its average energy, as would be expected, lies between the energies of the two levels.

5-34

(a) The probability density $\Psi^*\Psi$ has a time dependence of

$$e^{-i(E_2 - E_1)t/\hbar},$$

and therefore the frequency is

$$\nu = (E_2 - E_1)/h.$$

But,

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2} = \frac{h^2}{8ma^2} = \frac{(6.626 \times 10^{-34})^2}{8(1.67 \times 10^{-27})(10^{-14})^2(1.602 \times 10^{-13})},$$

$$E_1 = 2.051 \text{ MeV}; \quad E_2 = 4E_1 = 8.204 \text{ MeV}.$$

Hence,

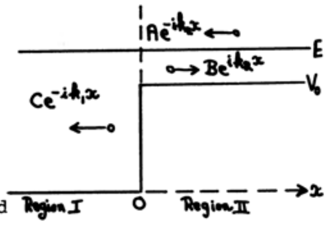
$$\nu = \frac{8.204 - 2.051}{4.136 \times 10^{-21}} = 1.488 \times 10^{21} \text{ Hz}.$$

(b) The frequency of the photon is the same as in (a). The photon's energy is

$$h\nu = 8.204 - 2.051 = 6.153 \text{ MeV}.$$

(c) Photons with this energy lie in the gamma-ray region of the spectrum.

6-2



Assume that

$$\psi_1 = Ce^{-ik_1x},$$

$$\psi_2 = Ae^{-ik_2x} + Be^{ik_2x},$$

where A = amplitude of incident wave, B = amplitude of reflected wave, C = amplitude of the transmitted wave. There is no wave moving in the +x-direction in region I. Also,

$$k_1 = \frac{(2mE)^{1/2}}{\hbar}, \quad k_2 = \frac{\{2m(E - V_0)\}^{1/2}}{\hbar}.$$

Continuity of wavefunction and derivative at x = 0 imply

$$A + B = C, \quad -k_2A + k_2B = -k_1C.$$

These equations may be solved to give the reflection and the transmission amplitudes in terms of the incident amplitude, the results being:

$$B = \frac{k_2 - k_1}{k_2 + k_1} A, \quad C = \frac{2k_2}{k_2 + k_1} A.$$

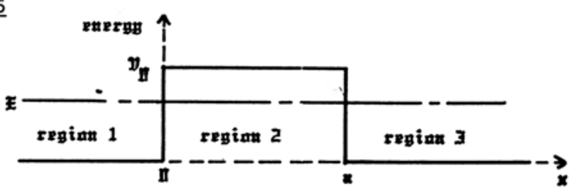
The reflection coefficient R and transmission coefficient T now become

$$R = \frac{B^*B}{A^*A} = \frac{B^2}{A^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2,$$

$$T = \frac{v_1C^*C}{v_2A^*A} = \left(\frac{\hbar k_1}{\hbar k_2}\right)\left(\frac{2k_2}{k_1 + k_2}\right)^2 = \frac{4k_1k_2}{(k_1 + k_2)^2}.$$

These expressions for R and T are the same as those obtained if the incident wave came from the left.

6-5



(a) Assuming a wave incident from the left:

region 1: $\psi = Ae^{ik_1x} + Be^{-ik_1x}$, $k_1 = (2mE)^{1/2}/\hbar$;

region 2: $\psi = Fe^{-k_2x} + Ge^{k_2x}$, $k_2 = \{2m(V_0 - E)\}^{1/2}/\hbar$;

region 3: $\psi = Ce^{ik_1x} + De^{-ik_1x}$, but $D = 0$ since there exists only a wave moving to the right in this region.

Continuity of the wavefunction at $x = 0$ and $x = a$ require that:

$$A + B = F + G, \tag{i}$$

$$Fe^{-k_2a} + Ge^{k_2a} = Ce^{ik_1a}, \tag{ii}$$

Continuity of $d\psi/dx$ at these same points yields

$$ik_1A - ik_1B = -k_2F + k_2G, \tag{iii}$$

$$-k_2Fe^{-k_2a} + k_2Ge^{k_2a} = ik_1Ce^{ik_1a}. \tag{iv}$$

(b) From (i), $\quad A + B - G = F;$

From (ii), $\quad (A + B - G)e^{-k_2a} + Ge^{k_2a} = Ce^{ik_1a},$

$$Ae^{-k_2a} + Be^{-k_2a} + G(e^{k_2a} - e^{-k_2a}) = Ce^{ik_1a}. \tag{iia}$$

From (iii), $\quad Aik_1 - Bik_1 = -k_2(A + B - G) + k_2G,$

$$A(ik_1 + k_2) + B(k_2 - ik_1) = 2Gk_2. \tag{iiia}$$

From (iv),

$$-k_2e^{-k_2a}(A + B - G) + Gk_2e^{k_2a} = ik_1Ce^{ik_1a},$$

$$-Ak_2e^{-k_2a} - k_2Be^{-k_2a} + Gk_2(e^{k_2a} + e^{-k_2a}) = Cik_1e^{ik_1a}. \tag{iva}$$

Now work with (iia), (iiia), (iva). From (iiia),

$$G = \frac{1}{2k_2}\{A(k_2 + ik_1) + B(k_2 - ik_1)\}.$$

Substituting this into (iia) gives

$$\begin{aligned} &A\{(\tfrac{1}{2} + ik_1/2k_2)e^{k_2a} + (\tfrac{1}{2} - ik_1/2k_2)e^{-k_2a}\} \\ &+ B\{(\tfrac{1}{2} - ik_1/2k_2)e^{k_2a} + (\tfrac{1}{2} + ik_1/2k_2)e^{-k_2a}\} = Ce^{ik_1a}, \end{aligned} \tag{iib}$$

and (iva) becomes

$$\begin{aligned} &A\{(\tfrac{1}{2} + ik_1/2k_2)e^{k_2a} - (\tfrac{1}{2} - ik_1/2k_2)e^{-k_2a}\} \\ &+ B\{(\tfrac{1}{2} - ik_1/2k_2)e^{k_2a} - (\tfrac{1}{2} + ik_1/2k_2)e^{-k_2a}\} = iC\frac{k_1}{k_2}e^{ik_1a}. \end{aligned} \tag{ivb}$$

Solve for B in (iib) and substitute into (ivb); if q, q^* are defined by

$$q = 1 + ik_1/k_2; \quad q^* = 1 - ik_1/k_2,$$

the result may be written

$$\frac{q^*e^{k_2a} - qe^{-k_2a}}{q^*e^{k_2a} + qe^{-k_2a}}\{Ce^{ik_1a} - \frac{A}{2}(qe^{k_2a} + q^*e^{-k_2a})\}$$
$$+ \frac{A}{2}(qe^{k_2a} - q^*e^{-k_2a}) = iC\frac{k_1}{k_2}e^{ik_1a}.$$

Now solve for C/A, using the definitions of q, q^*, such as

$$q^2 - q^{*2} = 4ik_1/k_2,$$

etc., to obtain

$$\frac{C}{A} = \frac{4i(k_1/k_2)e^{-ik_1a}}{-q^{*2}e^{k_2a} + q^2e^{-k_2a}}.$$

Hence, the transmission coefficient is

$$T = \frac{v_1 C^*C}{v_1 A^*A} = \frac{-4i(k_1/k_2)e^{ik_1a}}{q^{*2}e^{-k_2a} - q^2e^{k_2a}}\,\frac{4i(k_1/k_2)e^{ik_1a}}{-q^{*2}e^{k_2a} + q^2e^{-k_2a}},$$

$$T = 16(k_1/k_2)^2\{q^2q^{*2}(e^{k_2a} - e^{-k_2a})^2 - (q^2 - q^{*2})^2\}^{-1},$$

$$T = 16(k_1/k_2)^2\{(1 + k_1^2/k_2^2)^2(e^{k_2a} - e^{-k_2a})^2 + 16(k_1/k_2)^2\}^{-1},$$

$$T = \{1 + \frac{(1 + k_1^2/k_2^2)^2}{16(k_1/k_2)^2}(e^{k_2a} - e^{-k_2a})^2\}^{-1}.$$

Finally,

$$(\frac{k_1}{k_2})^2 = \frac{E}{V_0 - E}; \quad 1 + k_1^2/k_2^2 = \frac{V_0}{V_0 - E},$$

so that

$$T = \{1 + \frac{V_0^2/(V_0 - E)^2}{16E/(V_0 - E)}(e^{k_2a} - e^{-k_2a})^2\}^{-1},$$

$$T = \{1 + \frac{(e^{k_2a} - e^{-k_2a})^2}{16\frac{E}{V_0}(1 - \frac{E}{V_0})}\}^{-1}.$$

6-6

If $k_2a \gg 1$, then $e^{k_2a} \gg e^{-k_2a}$ and the transmission coefficient becomes, under these circumstances,

$$T = \{1 + \frac{e^{2k_2a}}{16\frac{E}{V_0}(1 - \frac{E}{V_0})}\}^{-1}.$$

Now $0 < E/V_0 < 1$ and therefore

$$16\frac{E}{V_0}(1 - \frac{E}{V_0}) \le 4,$$

the upper limit occuring at $E/V_0 = 1/2$. Hence, if $e^{2k_2a} > 4$,

$$\frac{e^{2k_2a}}{16\frac{E}{V_0}(1 - \frac{E}{V_0})} > 1.$$

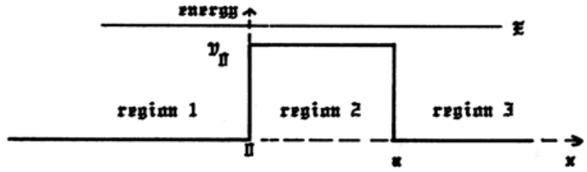
Since, in fact, it is assumed that $k_2a \gg 1$,

$$\frac{e^{2k_2a}}{16\frac{E}{V_0}(1 - \frac{E}{V_0})} \gg 1,$$

and therefore, under these conditions,

$$T = 16\frac{E}{V_0}(1 - \frac{E}{V_0})e^{-2k_2a}.$$

6-7



Region 1: $\psi = Ae^{ik_1x} + Be^{-ik_1x}$,

region 2: $\psi = Fe^{ik_3x} + Ge^{-ik_3x}$,

region 3: $\psi = Ce^{ik_1x}$.

In these equations,

$$k_1 = (2mE)^{1/2}/\hbar, \quad k_3 = \{2m(E - V_0)\}^{1/2}/\hbar.$$

(a) Continuity of the wavefunction at x = 0 and x = a gives

$$A + B = F + G,$$

$$Fe^{ik_3a} + Ge^{-ik_3a} = Ce^{ik_1a}.$$

Continuity of $d\psi/dx$ at x = 0 and x = a gives

$$ik_1A - ik_1B = ik_3F - ik_3G,$$

$$Fik_3e^{ik_3a} - ik_3Ge^{-ik_3a} = ik_1Ce^{ik_1a}.$$

(b) These are the same as the corresponding expressions in Problem 6-5, if in the latter k_2 is replaced with $-ik_3$. Making this alteration in the expression for T in Problem 6-5 yields for the new transmission coefficient,

$$T = \{1 - \frac{(1 - k_1^2/k_3^2)^2}{16(k_1/k_3)^2}(e^{ik_3a} - e^{-ik_3a})^2\}^{-1}.$$

Using the expressions for k_1, k_3 given above reduces this to

$$T = \{1 - \frac{(e^{ik_3a} - e^{-ik_3a})^2}{16\frac{E}{V_0}(\frac{E}{V_0} - 1)}\}^{-1}.$$

6-9

(a) The opacity of a barrier is proportional to $2mV_0a^2/\hbar^2$ and therefore the lower mass particle (proton) has the higher probability of getting through.

(b) With $V_0 = 10$ MeV, $E = 3$ MeV, $a = 10^{-14}$ m, it follows that

$$16\frac{E}{V_0}(1 - \frac{E}{V_0}) = 3.36.$$

The required masses are $m_p = 1.673 \times 10^{-27}$ kg, $m_d \simeq 2m_p$. For the proton $k_2a = 5.803$ and, using the approximate formula,

$$T_p = 3.36e^{-2(5.803)} = 3.06 \times 10^{-5}.$$

Since $m_d \simeq 2m_p$, as noted above, $k_2a \simeq \sqrt{2}(5.803) = 8.207$. Hence, for the deuteron,

$$T_d = 3.36e^{-2(8.207)} = 2.5 \times 10^{-7}.$$

6-10

(a) $$V_0 = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r'} = (9 \times 10^9)\frac{(6)(1)(1.6 \times 10^{-19})^2}{2 \times 10^{-15}},$$

$$V_0 = \frac{6.912 \times 10^{-13}\text{ J}}{1.6 \times 10^{-13}\text{ J/MeV}} = 4.32\text{ MeV}.$$

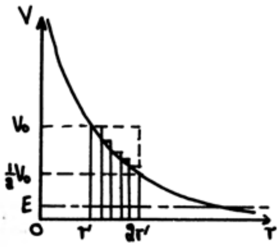
(b) $E = 10kT = (10)(1.38 \times 10^{-23})(10^7) = 1.38 \times 10^{-15}$ J = 8.625×10^{-3} MeV = $0.002V_0$.

(c) Numerically, $a = 2r' - r' = 2 \times 10^{-15}$ m; also,

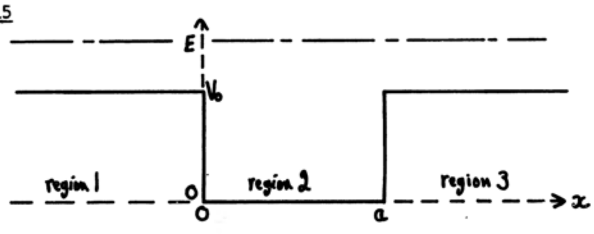
$$16\frac{E}{V_0}(1 - \frac{E}{V_0}) = 0.032; \quad k_2a = \frac{\sqrt{\{2m(V_0-E)\}}}{\hbar}a = 0.91.$$

$$T = \{1 + \frac{(2.484 - 0.403)^2}{0.032}\}^{-1} = 0.0073.$$

(d) The actual barrier can be considered as a series of barriers, each of constant height but the heights decreasing with r; hence V_0-E diminishes with r and the probability of penetration is greater than for an equal width barrier of constant height V_0.



6-15



(a) Assuming a wave incident from the left, the wavefunction in the indicated regions will be

region 1: $\psi = Ae^{ik_1x} + Be^{-ik_1x}$,

region 2: $\psi = Fe^{-ik_2x} + Ge^{ik_2x}$,

region 3: $\psi = Ce^{ik_1x}$.

The expressions for the k's are

$$k_1 = \sqrt{\{2m(E - V_0)\}}/\hbar, \quad k_2 = \sqrt{(2mE)}/\hbar.$$

The equations for the wavefunction are identical with those in Problem 6-5 if in the latter k_2 is replaced with ik_2 (note the different expressions for the k's in the two problems, however) Using $T(k_1,k_2)$ from Problem 6-5 and making the change gives

$$T = \{1 - \frac{(1 - k_1^2/k_2^2)^2}{16(k_1/k_2)^2}(e^{ik_2a} - e^{-ik_2a})^2\}^{-1}.$$

But,

$$(k_1/k_2)^2 = (E - V_0)/E; \quad e^{ik_2a} - e^{-ik_2a} = 2i\sin(k_2a),$$

and therefore,

$$T = \{1 + \frac{\sin^2 k_2a}{4\frac{E}{V_0}(\frac{E}{V_0} - 1)}\}^{-1}.$$

Alternatively, apply the continuity of the wavefunction and of its derivative at $x = 0$ and then at $x = a$ to get

$$A + B = F + G,$$

$$k_1A - k_1B = -k_2F + k_2G,$$

$$Fe^{-ik_2a} + Ge^{ik_2a} = Ce^{ik_1a},$$

$$-k_2Fe^{-ik_2a} + k_2Ge^{ik_2a} = k_1Ce^{ik_1a}.$$

These are four equations for the five amplitudes A,B,F,G,C. Solving relative to C gives

$$A/C = e^{ik_1a}\{\cos k_2a - \tfrac{1}{2}i(\frac{k_1}{k_2} + \frac{k_2}{k_1})\sin k_2a\},$$

$$B/C = \tfrac{1}{2}i(\frac{k_2}{k_1} - \frac{k_1}{k_2})e^{ik_1a}\sin k_2a,$$

$$F/C = \tfrac{1}{2}e^{i(k_1+k_2)a}(1 - \frac{k_1}{k_2}),$$

$$G/C = \tfrac{1}{2}e^{i(k_1-k_2)a}(1 + \frac{k_1}{k_2}).$$

The transmission coefficient is $T = C^*C/A^*A$. Substitution of the appropriate amplitudes given yields the same expression for T as obtained above.

(b) In order that $T = 1$ it is required that $\sin^2 k_2a = 0$ which in turn requires

$$k_2a = n\pi, \quad n = 1, 2, 3, \ldots.$$

In terms of the particle energy E, this is

$$\frac{\sqrt{(2mE)}}{\hbar}a = n\pi,$$

$$E = \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

(c) In the region of the well, that is, in region 2, the probability density is

$$\psi_2^*\psi_2 = (F^*e^{ik_2x} + G^*e^{-ik_2x})(Fe^{-ik_2x} + Ge^{ik_2x}),$$

$$\psi_2^*\psi_2 = C^*C\{\frac{k_1^2}{k_2^2} + (1 - \frac{k_1^2}{k_2^2})\cos^2 k_2(a - x)\},$$

evaluated by using the F,G amplitudes found in (a). The oscillatory part of this probability density has a maximum at

$x = a$. If $k_2a = n\pi$, it also has a maximum at $x = 0$. This implies that an integral number of half-wavelengths fit above the well; i.e., that

$$a = n\ \lambda_2/2;$$

but this is equivalent to

$$k_2a = n\pi,$$

as obtained in (b).
(d) One example of an optical analogue is thin film interference as in the optical coating of lenses.

6-17

Numerically $a = 2(4 \times 10^{-10}\ \text{m})$ and $K = 0.7$ eV. $E = K + V_0$ where

$$E = \frac{n^2h^2}{8ma^2} = n^2 \frac{(6.626 \times 10^{-34})^2}{8(9.11 \times 10^{-31})(8 \times 10^{-10})^2(1.6 \times 10^{-19})},$$

$$E = n^2(0.588\ \text{eV}).$$

Set $n = 1$; then

$$E_1 = 0.588\ \text{eV} < K,$$

which is not possible. Using $n = 2$ gives

$$E_2 = 2^2E_1 = 2.352\ \text{eV},$$

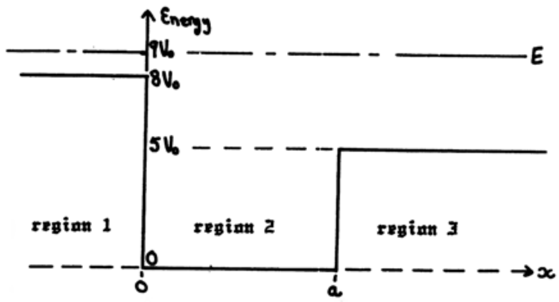
$$V_0 = E - K = 1.65\ \text{eV}.$$

The electron is too energetic for only half its wavelength to fit into the well; this may be verified by calculating the deBroglie wavelength of an electron with a kinetic energy over the well of 2.35 eV.

6-18

$$\hbar k_1 = (2mV_0)^{1/2};\quad \hbar k_2 = \{2m(9V_0)\}^{1/2};\quad \hbar k_3 = \{2m(4V_0)\}^{1/2};$$

These relations can be summarized as



$$k_1 = k,\quad k_2 = 3k,\quad k_3 = 2k.$$

Therefore,

$$\psi_1 = Ae^{ikx} + Be^{-ikx},$$

$$\psi_2 = Fe^{-3ikx} + Ge^{3ikx},$$

$$\psi_3 = Ce^{2ikx}.$$

Matching ψ and $d\psi/dx$ at $x = 0$ gives

$$A + B = F + G,$$

$$A - B = 3(G - F).$$

At $x = a$ the same conditions yield

$$Fe^{-3ika} + Ge^{3ika} = Ce^{2ika},$$

$$-3Fe^{-3ika} + 3Ge^{3ika} = 2Ce^{2ika}.$$

Writing

$$z = e^{ika}$$

these last equations become

$$Fz^{-3} + Gz^3 = Cz^2,$$

$$-3Fz^{-3} + 3Gz^3 = 2Cz^2.$$

In terms of C, the transmitted wave amplitude, the solutions are

$$A = \frac{10 - z^6}{6z} C,$$

$$B = \frac{2z^6 - 5}{6z} C,$$

$$F = \frac{1}{6} z^5 C,$$

$$G = \frac{5}{6}\frac{1}{z} C.$$

The desired transmission probability is

$$T = \frac{v_3 C^*C}{v_1 A^*A} = \frac{(\hbar k_3)C^*C}{(\hbar k_1)A^*A} = 2\frac{C^*C}{A^*A}.$$

$$C^*C = \left(\frac{6z^*}{10 - z^{*6}} A^*\right)\left(\frac{6z}{10 - z^6} A\right) = \frac{36z^*z}{(10 - z^{*6})(10 - z^6)} A^*A.$$

Now,

$$z^*z = e^{-ika}e^{ika} = 1;$$

$$(10 - z^{*6})(10 - z^6) = 100 - 10(e^{-6ika} + e^{6ika}) + 1,$$

$$(10 - z^{*6})(10 - z^6) = 101 - 20\cos 6ka.$$

Hence,

$$T = \frac{72}{101 - 20\cos 6ka}.$$

6-20

(a) In the lowest energy state $n = 1$, ψ has no nodes. Hence ψ_I must correspond to $n = 2$, ψ_{II} to $n = 3$. Since $E_n \propto n^2$ and $E_I = 4$ eV,

$$E_{II}/E_I = 3^2/2^2; \quad E_{II} = 9 \text{ eV}.$$

(b) By the same analysis,

$$E_0/E_I = 1^2/2^2; \quad E_0 = 1 \text{ eV}.$$

6-23

(a) The energy in question is

$$E_n = n^2 \frac{\pi^2\hbar^2}{2ma^2},$$

and therefore the energy of the adjacent level is

$$E_{n+1} = (n + 1)^2 \frac{\pi^2\hbar^2}{2ma^2},$$

so that

$$\frac{\Delta E_n}{E_n} = \frac{E_{n+1} - E_n}{E_n} = \frac{(n + 1)^2 - n^2}{n^2} = \frac{2n + 1}{n^2}.$$

(b) In the classical limit $n \to \infty$; but

$$\lim_{n\to\infty} \frac{\Delta E_n}{E_n} = \lim_{n\to\infty} \frac{2n + 1}{n^2} = 0,$$

meaning that the energy levels get so close together as to be indistinguishable. Hence, quantum effects are not apparent.

6-24

The eigenfunctions for odd n are

$$\psi_n = B_n \cos(n\pi x/a).$$

For normalization,

$$1 = \int \psi_n^2 dx = B_n^2 \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \cos^2(n\pi x/a)\,dx = 2B_n^2(a/n\pi)\int_0^{n\pi/2} \cos^2 u\,du,$$

$$1 = 2B_n^2(a/n\pi)(n\pi/4) = \frac{a}{2} B_n^2,$$

$$B_n = \sqrt{\frac{2}{a}},$$

for all odd n and, therefore, for $n = 3$.

6-25

By virtue of Problem 6-24, the normalized eigenfunctions are

$$\psi_n = \sqrt{\frac{2}{a}}\cos(n\pi x/a).$$

(a)

$$\overline{x} = \frac{2}{a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} x\cos^2(n\pi x/a)\,dx = 0.$$

(b)

$$\overline{p} = \frac{2}{a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\cos(n\pi x/a)\{-i\hbar\frac{\partial}{\partial x}\cos(n\pi x/a)\}dx = \frac{2}{a}(i\hbar)\int_{-\frac{1}{2}n\pi}^{\frac{1}{2}n\pi}\cos u\sin u\,du = 0$$

These results are expected from the symmetry of the potential, and from the fact that the function being averaged takes on both the positive and negative values.

(c)

$$\overline{x^2} = \frac{2}{a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a}x^2\cos^2(n\pi x/a)\,dx = \frac{4}{a}(a/n\pi)^3\int_0^{\frac{1}{2}n\pi}u^2\cos^2u\,du,$$

$$\overline{x^2} = \frac{4a^2}{n^3\pi^3}\left(\frac{n^3\pi^3}{48} - \frac{n\pi}{8}\right) = \frac{1}{2}a^2\left(\frac{1}{6} - \frac{1}{n^2\pi^2}\right),$$

$$\overline{x^2}(n{=}3) = 0.0777a^2.$$

This changes little with n since the size of the box is fixed.

(d)

$$\overline{p^2} = \frac{2}{a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\cos(n\pi x/a)\{i^2\hbar^2\frac{\partial^2}{\partial x^2}\cos(n\pi x/a)\}dx,$$

$$\overline{p^2} = \frac{2}{a}\hbar^2(n\pi/a)\int_{-\frac{1}{2}n\pi}^{\frac{1}{2}n\pi}\cos^2u\,du = \left(\frac{\hbar}{a}\right)^2 n^2\pi^2,$$

$$\overline{p^2}(n{=}3) = 88.83\left(\frac{\hbar}{a}\right)^2.$$

This increases sharply with n since $E \propto n^2$, $E \propto p^2$, the particle moving faster at higher energies.

6-26

(a) Using the results of the previous problem,

$$\Delta x = \sqrt{\overline{x^2}} = \frac{a}{\sqrt{12}}\left(1 - \frac{6}{n^2\pi^2}\right)^{\frac{1}{2}},$$

$$\Delta p = \sqrt{\overline{p^2}} = n\pi\left(\frac{\hbar}{a}\right).$$

Hence, for n = 3,

$$\Delta x\Delta p = \frac{a}{\sqrt{12}}\left(1 - \frac{2}{3\pi^2}\right)^{\frac{1}{2}}3\pi\left(\frac{\hbar}{a}\right) = 2.627\hbar.$$

(b) The other results are

$$n = 1,\quad \Delta x\Delta p = 0.57\hbar,$$

$$n = 2,\quad \Delta x\Delta p = 1.67\hbar.$$

The increase with n is due mainly to the uncertainty in p: see Problem 6-25.

(c) From (a), the limits as $n \to \infty$ are

$$\Delta x \to \frac{a}{\sqrt{12}};\quad \Delta p \to \infty.$$

6-27

$$\int_{-\infty}^{+\infty}\psi_1\psi_3\,dx = \frac{2}{a}\int_{-\frac{1}{2}a}^{+\frac{1}{2}a}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)dx = \frac{1}{a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\{\cos\left(\frac{4\pi x}{a}\right) - \cos\left(\frac{2\pi x}{a}\right)\}dx,$$

$$\int_{-\infty}^{\infty}\psi_1\psi_3\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos 2u - \cos u)\,du = 0,$$

the integrand being an even function of u.

6-28

The Schrodinger equation in three rectangular coordinates is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0.$$

Inside the cubical region where $V = 0$,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2} E\psi = 0.$$

Assume that

$$\psi(x,y,z) = X(x)Y(y)Z(z).$$

Then, if ' denotes the derivative of a function with respect to its independent variable,

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + 2mE/\hbar^2 = 0. \qquad (*)$$

This gives

$$\frac{X''}{X} = -k_x^2; \quad X = A\sin(k_x x) + B\cos(k_x x),$$

k_x = real constant. Similarly

$$Y = C\sin(k_y y) + D\cos(k_y y); \quad Z = E\sin(k_z z) + F\cos(k_z z).$$

Also, from (*),

$$k_x^2 + k_y^2 + k_z^2 = 2mE/\hbar^2.$$

Since $V = \infty$ outside the cubical region, $\psi = 0$ at the boundary:

$$0 = X(0) = Y(0) = Z(0) \rightarrow B = D = F = 0;$$

$$0 = X(a) = Y(a) = Z(a) \rightarrow k_x a = n_x\pi,\ k_y a = n_y\pi,\ k_z a = n_z\pi;$$

with $n_x, n_y, n_z = 1,2,3, \ldots$. Hence

$$\psi = (CAE)\sin(n_x\pi x/a)\sin(n_y\pi y/a)\sin(n_z\pi z/a),$$

$$E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2) = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2).$$

6-29

(a) Let M = mass of wing. The zero-point energy is

$$E_0 = (0 + \tfrac{1}{2})\hbar\omega = \tfrac{1}{2}h\nu = h/2T,$$

T = period of oscillation. The actual energy of oscillation is

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}M\omega^2A^2 = 2\pi^2MA^2/T^2.$$

Thus, the value of M at which $E = E_0$ is

$$M = \frac{hT}{4\pi^2A^2} = \frac{(6.626 \times 10^{-34})(1)}{4\pi^2(10^{-1})^2} = 1.68 \times 10^{-33}\ \text{kg}.$$

This is less than the mass of an electron. Hence $E \gg E_0$ and the observed vibration is not the zero-point motion.

(b) Clearly then, $n \gg 1$ and therefore

$$E = nh\nu = 2\pi^2MA^2/T^2 \rightarrow n = 2\pi^2MA^2/hT.$$

As an example, take M = 2000 kg:

$$n = \frac{2\pi^2(2000)(10^{-1})^2}{(6.626 \times 10^{-34})(1)} = 6 \times 10^{35}.$$

6-30

The zero-point energy is

$$E_0 = \tfrac{1}{2}\hbar\omega = \tfrac{1}{2}\hbar(C/m)^{1/2}.$$

Therefore,

$$E_0 = \tfrac{1}{2}(1.055 \times 10^{-34})\left(\frac{10^3}{4.1 \times 10^{-26}}\right)^{1/2}(1.6 \times 10^{-19})^{-1},$$

$$E_0 = 0.051\ \text{eV}.$$

6-31

(a) Using $E_0 = 0.051$ eV, the level spacing will be

$$\Delta E = \Delta(n + \tfrac{1}{2})\hbar\omega = \hbar\omega = 0.102 \text{ eV} = 2E_0.$$

(b) The energy $\mathcal{E}$ of the photon $= \Delta E = 0.102$ eV.

(c) For the photon,

$$\mathcal{E} = \hbar\omega_{ph}.$$

But

$$\mathcal{E} = \Delta E = \hbar\omega,$$

$$\omega_{ph} = \omega,$$

where ω = classical oscillation frequency. Thus,

$$\nu = \frac{\mathcal{E}}{h} = \frac{(0.102)(1.6 \times 10^{-19})}{(6.626 \times 10^{-34})} = 2.5 \times 10^{13} \text{ Hz}.$$

(d) Photons of this frequency are in the infrared spectrum, $\lambda = 12{,}000$ nm.

6-32

(a)
$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{1}} = 3.13 \text{ rad/s},$$

$$\nu = \frac{\omega}{2\pi} = 0.498 \text{ Hz}.$$

(b)
$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}\frac{mg}{L}A^2 \quad \rightarrow \quad E = 0.049 \text{ J}.$$

(c) Since $n >> 1$,

$$n = \frac{E}{h\nu} \approx \frac{0.0490}{(6.626 \times 10^{-34})(0.498)} = 1.5 \times 10^{32}.$$

(d) Since $\Delta n = 1$, $\Delta E = h\nu = 3.3 \times 10^{-34}$ J.

(e) A polynomial of degree n has n nodes; hence, the distance between "bumps" = distance between adjacent nodes = $2A/n$ = $2(0.1)/(1.5 \times 10^{32}) = 1.3 \times 10^{-33}$ m.

CHAPTER SEVEN

7-1

The time-dependent equation is

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}.$$

Let

$$\Psi(x,y,z,t) = \psi(x,y,z)T(t).$$

Putting this into the first equation gives

$$-\frac{\hbar^2}{2\mu}T(t)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V(x,y,z)\psi(x,y,z)T(t) = i\hbar\frac{dT}{dt},$$

assuming that V does not depend on t explicitly. Dividing the above by the wave function yields

$$-\frac{\hbar^2}{2\mu}\frac{1}{\psi}\nabla^2\psi + V = i\hbar\frac{1}{T}\frac{dT}{dt} = \text{constant} = E.$$

There are two equations:

$$\nabla^2\psi + \frac{2\mu}{\hbar^2}(E - V)\psi = 0,$$

for the space dependent part of the wave function, and

$$\frac{i\hbar}{T}\frac{dT}{dt} = E; \quad \frac{dT}{T} = \frac{E}{i\hbar}dt = -\frac{iE}{\hbar}dt; \quad T = e^{-iEt/\hbar},$$

for the time dependent part.

7-3

The ground state energy of a hydrogen like atom is

$$E = -\frac{\mu Z^2 e^4}{(4\pi\epsilon_0)^2 2\hbar^2}; \quad E \propto \mu Z^2.$$

The reduced mass is

$$\mu = \frac{m_{nuc}m_e}{m_{nuc} + m_e} \simeq m_e(1 - \frac{m_e}{m_{nuc}}),$$

since each nucleus to be considered is surrounded by one electron. The charges and masses of the nuclei are

$$m_H = m_p; \quad m_D = 2m_p; \quad m_{He} = 4m_p; \; Z_H = Z_D = 1; \quad Z_{He} = 2,$$

m_p = proton mass. The mass relations are approximate. Since

$$\frac{m_e}{m_p} \simeq \frac{1}{1836},$$

it follows that

$$E_H \propto \mu_H(1)^2 = m_e\{1 - \frac{1}{1836}\}(1)^2 = 0.9995m_e;$$

$$E_D \propto \mu_D(1)^2 = m_e\{1 - \frac{1}{2(1836)}\}(1)^2 = 0.9997m_e;$$

$$E_{He} \propto \mu_{He}(2)^2 = m_e\{1 - \frac{1}{4(1836)}\}(2)^2 = 3.9995m_e.$$

These give the ratios:

$$E_D/E_H = 1.0002; \quad E_{He}/E_H = 4.0015.$$

7-7

(a) Since $R_{21} = re^{-r/2a_0}$, $P(r) = (r^2e^{-r/a_0})r^2$. P has a maximum where $dP/dr = 0$:

$$\frac{dP}{dr} = (4 - r/a_0)r^3e^{-r/a_0} \rightarrow r = 4a_0,$$

$r = 0$ and $r = \infty$ yielding minima.

(b) Direct application of Eq.7-29 with $Z = 1$, $n = 2$, $\ell = 1$ gives $\overline{r} = 5a_0$.

(c) There is a lower limit (zero) to r, but no upper limit: thus it is expected that $\overline{r} > 4a_0$.

7-8

(a) The potential energy and the ground state wave function are:

$$V = -e^2/4\pi\epsilon_0 r, \quad \psi_{100} = \pi^{-\frac{1}{2}}a_0^{-3/2}e^{-r/a_0}.$$

Therefore,

$$\overline{V} = \frac{1}{\pi a_0^3}(-\frac{e^2}{4\pi\epsilon_0})\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} e^{-r/a_0}(\frac{1}{r})e^{-r/a_0}r^2\sin\theta drd\theta d\phi;$$

$$\overline{V} = -\frac{e^2}{4\pi\epsilon_0 a_0}\int_0^{\infty} xe^{-x}dx = -\frac{e^2}{4\pi\epsilon_0 a_0}.$$

(b) In the ground state,

$$E = -\mu e^4/(4\pi\epsilon_0)^2 2\hbar^2.$$

Since $a_0 = 4\pi\epsilon_0\hbar^2/\mu e^2$,

$$\overline{V} = -2\frac{\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2} = 2E; \quad E = \tfrac{1}{2}\overline{V}.$$

(c) As for the kinetic energy,

$$E = K + V; \quad E = \overline{K} + \overline{V}; \quad \tfrac{1}{2}\overline{V} = \overline{K} + \overline{V}; \quad \overline{K} = -\tfrac{1}{2}\overline{V}.$$

7-9

(a) For the state with $m_\ell = 0$,

$$\psi_{210} = \frac{1}{4\sqrt{2}}a_0^{-3/2}(r/a_0)e^{-r/2a_0}\cos\theta,$$

so that

$$\overline{V}_{210} = -\frac{e^2}{32\pi a_0^3(4\pi\epsilon_0)}\int(r^3/a_0^2)e^{-r/a_0}\cos^2\theta\sin\theta drd\theta d\phi,$$

$$\overline{V}_{210} = -\frac{e^2}{24a_0^3(4\pi\epsilon_0)}a_0^2\int_0^\infty x^3e^{-x}dx. \qquad (*)$$

For the limits on r, θ, ϕ see Problem 7-8. Now,

$$\int_0^\infty x^3e^{-x}dx = 6,$$

so that

$$\overline{V}_{210} = -\frac{e^2}{4\pi\epsilon_0 a_0}\frac{1}{2^2} = -\frac{e^4}{(4\pi\epsilon_0)^2\hbar^2 2^2} = 2E_2.$$

For the states with $m_\ell = \pm1$:

$$\psi_{21\pm1} = \frac{1}{8\sqrt{\pi}}a_0^{-3/2}(r/a_0)e^{-r/2a_0}\sin\theta\, e^{\pm im\phi},$$

and therefore

$$\overline{V}_{21\pm1} = -\frac{e^2}{64\pi a_0^3(4\pi\epsilon_0)}\int\frac{r^3}{a_0^2}e^{-r/a_0}\sin^3\theta drd\theta d\phi,$$

$$\overline{V}_{21\pm1} = -\frac{e^2}{24a_0(4\pi\epsilon_0)}\int_0^\infty x^3e^{-x}dx.$$

This is the same as (*) above. Hence, regardless of the value of m_ℓ,

$$\overline{V}_{21} = 2E_2.$$

(b) In the case of $\ell = 0$,

$$\psi_{200} = \frac{1}{4\sqrt{(2\pi)}}a_0^{-3/2}(2 - r/a_0)e^{-r/2a_0},$$

giving

$$\overline{V}_{20} = \frac{1}{32\pi a_0^3}\left(-\frac{e^2}{4\pi\epsilon_0}\right)\int\frac{1}{r}(2 - r/a_0)^2e^{-r/a_0}r^2\sin\theta drd\theta d\phi,$$

$$\overline{V}_{20} = -\frac{e^2}{8a_0(4\pi\epsilon_0)}\int_0^\infty(2 - x)^2xe^{-x}dx = -\frac{e^2}{8a_0(4\pi\epsilon_0)}(2) = 2E_2.$$

(c) These results are expected since, with $V \propto r^{-1}$, the average potential energy seen by the electron is the same for all $n = 2$ states, regardless of ℓ. Thus, the expectation value of an energy will be the same for these states.

<u>7-10</u>

$R(r)$ must satisfy Eq.7-17:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{2\mu}{\hbar^2}(E - V)R = \ell(\ell + 1)\frac{R}{r^2}.$$

If $R = r^\ell$,

$$\frac{dR}{dr} = \ell r^{\ell-1};\quad \frac{d^2R}{dr^2} = \ell(\ell - 1)r^{\ell-2}.$$

Substituting these into the radial equation gives

$$\ell(\ell + 1)r^{\ell-2} + \frac{2\mu}{\hbar^2}\{Er^\ell - Vr^\ell\} \stackrel{?}{=} \ell(\ell + 1)r^{\ell-2}.$$

Now E is a constant independent of r, and $V = k/r$; thus the two terms in {} are proportional to r^ℓ, $r^{\ell-1}$. As r approaches zero, $r^{\ell-2} \gg r^\ell, r^{\ell-1}$; hence, $\{\} \to 0$, and the equation is satisfied.

<u>7-11</u>

(a) To avoid infinities, integrate radially to a finite limit R:

$$P = \frac{1}{4\pi R^3/3}\int dV = \frac{1}{4\pi R^3/3}\int_{r=0}^{R}\int_{\theta=0}^{\theta}2\pi r^2\sin\theta drd\theta = \tfrac{1}{2}(1 - \cos\theta),$$

$$P(23.5°) = 4.147\%.$$

(b) For this state:

$$\psi_{210} = \frac{1}{4\sqrt{(2\pi)}}a_0^{-5/2}re^{-r/2a_0}\cos\theta.$$

Since this is already normalized,

$$P = \int\psi^2 dV = \frac{2\pi}{16(2\pi)}a_0^{-5}\int_0^{\infty}\int_0^{\theta} r^2 e^{-r/a_0}\cos^2\theta r^2 \sin\theta dr d\theta,$$

$$P = \tfrac{1}{2}(1 - \cos^3\theta),$$

$$P(23.5°) = 11.44\%.$$

7-12

(a) See sketch, following page.
(b) $P \propto (3\cos^2\theta - 1)^2$; hence $P_{min} = 0$ at $\cos\theta = \pm 1/\sqrt{3}$, giving $\theta = 54.7°$, $125.3°$.
(c)

$$P_{max} = 4 \quad (\theta = 0°),$$

$$\tfrac{1}{4}P_{max} = 1,$$

$$(3\cos^2\theta - 1)^2 = 1,$$

$$3\cos^2\theta - 1 = \pm 1,$$

$$\theta = 35.3°,\ 90°,\ 144.7°.$$

7-14

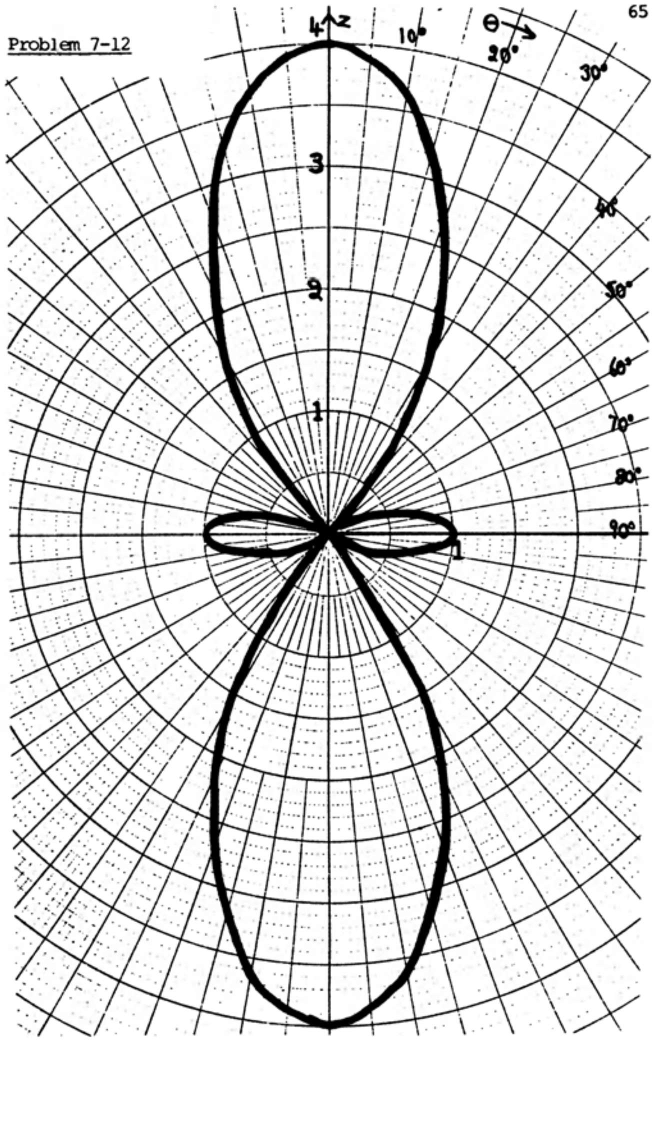
Let (3,2,-1) represent $\psi(n=3,\ell=2,m_\ell=-1)$; (2,0,0)* represent $\psi^*(n=2,\ell=0,m_\ell=0)$ etc. Then it is required to show that

$$\overline{\psi_3^*\psi_3} = \frac{1}{9}\{(3,0,0)^*(3,0,0) + (3,1,0)^*(3,1,0) + (3,1,-1)^*(3,1,-1) + (3,1,1)^*(3,1,1) + (3,2,0)^*(3,2,0) + (3,2,1)^*(3,2,1) + (3,2,-1)^*(3,2,-1) + (3,2,2)^*(3,2,2) + (3,2,-2)^*(3,2,-2)\}$$

is independent of θ,ϕ. Now

$$(3,2,-2)^*(3,2,-2) = (3,2,2)^*(3,2,2),$$
$$(3,2,-1)^*(3,2,-1) = (3,2,1)^*(3,2,1),$$
$$(3,1,-1)^*(3,1,-1) = (3,1,1)^*(3,1,1),$$

Problem 7-12



and therefore

$$\overline{\psi_3^*\psi_3} = \frac{1}{9}\{(3,0,0)^*(3,0,0) + (3,1,0)^*(3,1,0) + 2(3,1,1)^*(3,1,1)$$
$$+ (3,2,0)^*(3,2,0) + 2(3,2,1)^*(3,2,1) + 2(3,2,2)^*(3,2,2)\}.$$

Now substitute the specific expressions for the various wavefunctions appearing in the above.

$\ell = 2$ terms:

$$2\{(3,2,1)^*(3,2,1) + (3,2,2)^*(3,2,2)\} =$$
$$\frac{1}{2(81)^2\pi a_0^7} r^4 e^{-2r/3a_0}(1 + 2\cos^2\theta - 3\cos^4\theta);$$
$$(3,2,0)^*(3,2,0) = \frac{1}{6(81)^2\pi a_0^7} r^4 e^{-2r/3a_0}(3\cos^2\theta - 1)^2.$$

Hence, the sum of these terms is

$$\frac{2}{3(81)^2\pi a_0^7} r^4 e^{-2r/3a_0},$$

independent of θ,ϕ.

$\ell = 1$ terms:

$$2(3,1,1)^*(3,1,1) + (3,1,0)^*(3,1,0) =$$
$$\frac{2}{(81)^2\pi a_0^3}(6 - \frac{r}{a_0})^2(\frac{r}{a_0})^2 e^{-2r/3a_0},$$

independent of θ,ϕ. The $\ell = 0$ terms depend on r only. Thus, all terms in $\overline{\psi_3^*\psi_3}$ have been accounted for and their sum found to be independent of direction, so that $\overline{\psi_3^*\psi_3}$ is spherically symmetric since it depends on r only.

7-16

(a)
$$L_{x,op} = i\hbar(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi \frac{\partial}{\partial\phi}),$$
$$\psi_{21-1} = \frac{1}{8\sqrt{\pi}} a_0^{-5/2} r e^{-r/2a_0} \sin\theta e^{-i\phi}.$$

$$\frac{\partial\psi_{21-1}}{\partial\theta} = \psi_{21-1}\cot\theta; \quad \frac{\partial\psi_{21-1}}{\partial\phi} = -i\psi_{21-1}.$$

Therefore,

$$L_{x,op}\psi_{21-1} = \hbar(i\cot\theta\sin\phi + \cot\theta\cos\phi)\psi_{21-1},$$
$$L_{x,op}\psi_{21-1} = \hbar\cot\theta(\cos\phi + i\sin\phi)\psi_{21-1} = \hbar\cot\theta e^{i\phi}\psi_{21-1}.$$

(b) This result cannot be put into the form

$$L_{x,op}\psi_{21-1} = C\psi_{21-1},$$

with C independent of r,θ,ϕ.

7-17

The operator is question is given by

$$L_{op}^2 = -\hbar^2\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\}.$$

By Eq.7-13 this may be written

$$L_{op}^2 = -\hbar^2\{r^2\nabla^2 - \frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r})\}.$$

But

$$\frac{\partial}{\partial r}(r^2 \frac{\partial\psi}{\partial r}) = \Theta\Phi\frac{d}{dr}(r^2 \frac{dR}{dr}) = \Theta\Phi\{\ell(\ell + 1)R - \frac{2\mu}{\hbar^2}(E - V)Rr^2\},$$
$$\frac{\partial}{\partial r}(r^2 \frac{\partial\psi}{\partial r}) = \ell(\ell + 1)\psi - \frac{2\mu}{\hbar^2}(E - V)r^2\psi,$$

by Eq.7-17. Schrodinger's equation is

$$\nabla^2\psi = \frac{2\mu}{\hbar^2}(V - E)\psi.$$

Substituting these last two results into the expression for L_{op}^2 gives

$$L_{op}^2\psi = \ell(\ell + 1)\hbar^2\psi.$$

7-18

(a) Since

$$p_{op} = -i\hbar \frac{\partial}{\partial x}, \quad \psi = e^{ikx},$$

$$p_{op}\psi = -i\hbar \frac{\partial e^{ikx}}{\partial x} = (-i\hbar)(ik)e^{ikx} = \hbar k\psi,$$

and thus the eigenvalue is $\hbar k$.

(b) Using $\psi = e^{-ikx}$, replace k in (a) with -k to get the eigenvalue $-\hbar k$.

(c) These results indicate that measurements of momentum will yield $\pm\hbar k$ precisely.

(d) ψ = sinkx, coskx are not eigenfunctions since $\partial/\partial x$ converts sine to cosine and cosine to sine.

(e) These states, not being eigenfunctions of the momentum operator, will not yield precise values of the momentum upon measurement.

7-20

(a) With R a constant and V = 0, the total energy E is

$$E = K + V = K.$$

But the kinetic energy K is simply

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}I(L/I)^2 = L^2/2I,$$

where I = rotational inertia about the z-axis. Hence

$$E = K = L^2/2I.$$

(b)

$$L_{op} = L_{z,op} = -i\hbar \frac{\partial}{\partial \phi}; \quad L^2_{op} = (-i\hbar)^2 \frac{\partial^2}{\partial \phi^2} = -\hbar^2 \frac{\partial^2}{\partial \phi^2};$$

$$E_{op} = -i\hbar \frac{\partial}{\partial t}.$$

Also, $\Psi = \Psi(r,\theta,\phi,t)$; but $r = R$ = constant, $\theta = \pi/2$ = constant, so that $\Psi = \Psi(\phi,t)$. Substituting the operators into (a) gives, then,

$$L^2/2I = E \rightarrow -\frac{\hbar^2}{2I}\frac{\partial^2\Psi(\phi,t)}{\partial\phi^2} = i\hbar\frac{\partial\Psi(\phi,t)}{\partial t}.$$

7-21

Let $\Psi(\phi,t) = \Phi(\phi)T(t)$.

Substitute this into the energy equation of Problem 7-20(b) and divide by Ψ to obtain

$$-\frac{\hbar^2}{2I} T \frac{d^2\Phi}{d\phi^2} = i\hbar \frac{dT}{dt}; \quad -\frac{\hbar^2}{2I}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = i\hbar\frac{1}{T}\frac{dT}{dt} = E,$$

where E is the separation constant. Thus two equations emerge:

(a) $$-\frac{\hbar^2}{2I}\frac{d^2\Phi}{d\phi^2} = E\Phi,$$

and

(b) $$i\hbar\frac{dT}{dt} = ET; \quad \frac{dT}{dt} = \frac{E}{i\hbar}T = -\frac{iE}{\hbar}T.$$

7-22

(a) From the preceding problem,

$$\frac{dT}{dt} = -\frac{iE}{\hbar}T; \quad \frac{dT}{T} = -\frac{iE}{\hbar}dt; \quad T = e^{-iEt/\hbar}.$$

The normalization constant will be incorporated into Φ.

(b) The solution above represents an oscillation of frequency ω given by

$$E = \hbar\omega.$$

But this is the de Broglie-Einstein relation. Hence E is the total energy.

7-23

The equation for Φ is, from Problem 7-21(a):

$$\frac{d^2\Phi}{d\phi^2} + \frac{2IE}{\hbar^2}\Phi = 0.$$

This is analogous to the classical simple harmonic oscillator equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

The complex forms of the solution will therefore be

$$\Phi = ce^{im\phi} + c^*e^{-im\phi}, \quad m = \sqrt{(2IE)}/\hbar.$$

7-24

(a) The particular solution being considered is

$$\Phi = e^{im\phi} = \cos(m\phi) + i\sin(m\phi).$$

Single-valuedness requires that

$$\Phi(0) = \Phi(2\pi) \rightarrow 1 = \cos(2\pi m) + i\sin(2\pi m).$$

Hence,

$$\sin(2\pi m) = 0; \; \cos(2\pi m) = 1; \quad m = 0, \pm 1, \pm 2, \ldots$$

(b) By Problem 7-23

$$m = \frac{\sqrt{(2IE)}}{\hbar}; \quad E = \frac{m^2\hbar^2}{2I}, \quad m = 0, \pm 1, \pm 2, \ldots$$

(c) Solving the problem via the old quantum theory gives

$$E = \frac{L^2}{2I} = \frac{(n\hbar)^2}{2I} = \frac{n^2\hbar^2}{2I}, \quad n = 1, 2, 3, \ldots$$

Evidently the new version, in contrast to the old theory, introduces an $m = 0$ $(E = 0)$ state; also, the excited levels are now two-fold degenerate.

(d) Apparently there is no room for zero-point energy since R is assumed constant. In actuality, the masses, on a microscopic scale, would be atoms which oscillate slightly, so that R cannot be assumed to be fixed.

7-25

The complete wave function is, from the preceding problems,

$$\Psi = Ne^{im\phi}e^{iEt/\hbar}.$$

The wavefunction must be normalized at any time t: thus,

$$\int\Phi^*\Phi d\phi = 1,$$

$$1 = \int_0^{2\pi}(N^*e^{-im\phi})(Ne^{im\phi})d\phi = N^*N(2\pi),$$

$$N^*N = \frac{1}{2\pi}.$$

7-23

Use
$$\Phi = (2\pi)^{-\frac{1}{2}}e^{im\phi}; \quad L_z = -i\hbar\frac{\partial}{\partial\phi}.$$

(a)

$$\overline{L} = \int_0^{2\pi}\Phi^*(-i\hbar\frac{\partial\Phi}{\partial\phi})d\phi = -\frac{i\hbar}{2\pi}\int_0^{2\pi}e^{-im\phi}(im)e^{im\phi}d\phi = \frac{m\hbar}{2\pi}\int_0^{2\pi}d\phi;$$

$$\overline{L} = m\hbar.$$

(b)

$$\overline{L^2} = \frac{1}{2\pi}\int_0^{2\pi}e^{-im\phi}(-i\hbar)^2\frac{\partial^2e^{im\phi}}{\partial\phi^2}d\phi = m^2\hbar^2.$$

Also, from (a)

$$\overline{L}^2 = m^2\hbar^2.$$

Since $\overline{L^2} = \overline{L}^2$, measurements of L_z will yield $m\hbar$ exactly.

CHAPTER EIGHT

8-2

(a) Let the area of the ellipse be A; then,

$$\mu_\ell = iA = \frac{e}{T}A,$$

T the period of revolution. The angular momentum is $L = mr^2 d\theta/dt$; also, $dA = \frac{1}{2}r^2 d\theta$, so that

$$L = 2m\frac{dA}{dt} = 2m\frac{A}{t},$$

since dA/dt equals a constant in classical mechanics if the force is central. Therefore,

$$\mu_\ell = \frac{e}{T}\frac{LT}{2m} = \frac{eL}{2m}; \quad \frac{\mu_\ell}{L} = \frac{e}{2m}.$$

(b) This result is identical to Eq.8-5, derived assuming a circular orbit.

8-4

The first apparatus produces two beams, one with spin parallel (in quantum mechanical terms) to the direction of the field (+z), the other with spin antiparallel. This latter beam is blocked by the first diaphragm. Hence, a "polarized" beam of atoms enter the second apparatus, field direction +z'. This second magnet produces a new space quantization along z'. In analogy with the passing of polarized light through polaroid (except that the angle for no transmission is 90° in the optical case, 180° in the atomic), the second magnet allows only the projection of the entering spins along +z' (not -z') to pass. Thus, if I' is the intensity of the beam entering the second apparatus and I_0 the intensity of the unpolarized beam entering the first,

$$I = \tfrac{1}{2}I'(1 + \cos\alpha) = \tfrac{1}{4}I_0(1 + \cos\alpha).$$

8-5

The deflecting force is

$$F = \mu_z dB_z/dz,$$

where

$$\mu_z = g_s\mu_b m_s,$$

since $\ell = 0$. If D is the deflection and F is constant,

$$D = \tfrac{1}{2}at^2 = \tfrac{1}{2}(F/m)(L/v)^2,$$

L = length of magnet and v = speed of the atoms. Thus,

$$D = \tfrac{1}{2}(g_s\mu_b m_s/m)(dB_z/dz)(L/v)^2; \quad \frac{dB_z}{dz} = \frac{2mDv^2}{L^2 g_s\mu_b m_s}.$$

For atoms emitted from the oven, $\frac{1}{2}mv^2 = 2kT$ with T = 1233 K. Hence,

$$\frac{dB_z}{dz} = \frac{8kTD}{L^2 g_s\mu_b m_s} = \frac{8(1.381\times10^{-23})(1233)(0.0005)}{(0.5)^2(2)(9.27\times10^{-24})(\frac{1}{2})} = 29\ \text{T/m}.$$

8-6

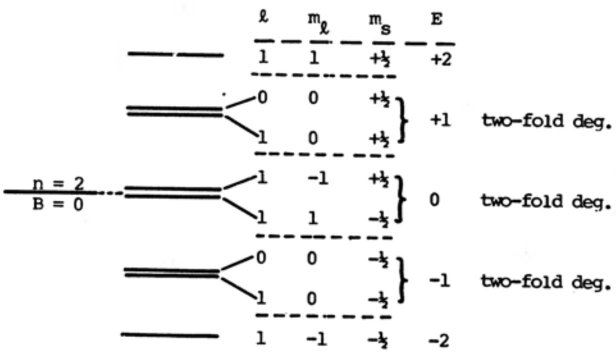
(a) The orbit and spin energies are $(g_\ell\mu_b m_\ell)B$ and $(g_s\mu_b m_s)B$. Hence, with respect to B = 0,

$$E = g_\ell\mu_b m_\ell B + g_s\mu_b m_s B = (g_\ell m_\ell + g_s m_s)\mu_b B = (m_\ell + 2m_s)\mu_b B.$$

(b) For n = 2, ℓ = 0,1 giving the result:

ℓ	m_ℓ	m_s	ΔE (units of $\mu_b B$)
0	0	$+\frac{1}{2}$	+1
		$-\frac{1}{2}$	-1
1	-1	$+\frac{1}{2}$	0
		$-\frac{1}{2}$	-2
	0	$+\frac{1}{2}$	+1
		$-\frac{1}{2}$	-1
	+1	$+\frac{1}{2}$	+2
		$-\frac{1}{2}$	0

Thus the energy level diagram appears as follows:



(c) The maximum separation is

$$\Delta E_{max} = 4\mu_b B = 10.2 \text{ eV},$$

$$4(9.27 \times 10^{-24})B = (10.2)(1.6 \times 10^{-19}),$$

$$B = 4.4 \times 10^4 \text{ T}.$$

8-8

Since $\ell, j \geq 0$ and $s = \frac{1}{2}$ the relation

$$\sqrt{\{j(j+1)\}} \geq |\sqrt{\{\ell(\ell+1)\}} - \sqrt{\{s(s+1)\}}|,$$

becomes

$$j(j+1) \geq \ell(\ell+1) + \frac{3}{4} - \sqrt{\{3\ell(\ell+1)\}}. \qquad \text{(A)}$$

(i) $\ell = 0$. In this case (A) reduces to

$$j(j+1) \geq 3/4.$$

But for $\ell = 0$, $j = \frac{1}{2}$ (the only possibility), so that the relation in question is satisfied.

(ii) $\ell \neq 0$. (a) $j = \ell + \frac{1}{2}$. Putting this into (A) gives

$$\ell \geq -\sqrt{\{3\ell(\ell+1)\}},$$

which clearly is satisfied for all $\ell > 0$.

(b) $j = \ell - \frac{1}{2}n$, $n = 1, 3, 5, \ldots$ Putting this into (a) yields

$$-n\ell + \tfrac{1}{4}(n^2 - 2n - 3) \geq -\sqrt{\{3\ell(\ell+1)\}}. \qquad \text{(B)}$$

n = 1 (B) becomes

$$0 \leq (2\ell - 1)(\ell + 1),$$

which is satisfied for $\ell \geq \frac{1}{2}$ (i.e., $j \geq 0$), so the relation is obeyed here also.

n = 3 In this event, (B) gives

$$0 \leq 3\ell(-2\ell + 1).$$

Evidently this is not satisfied for $\ell > \frac{1}{2}$, $\ell < 0$, but is for $0 \leq \ell \leq \frac{1}{2}$. But then $j = \ell - 3/2 < 0$, which is impossible.

n = 5 (B) now reduces to

$$0 \leq -22\ell^2 + 33\ell - 9.$$

This is satisfied for some ℓ, e.g. $\ell = 1$, but then $j = \ell - 5/2 < 0$. In fact, put $\ell = j + 5/2$ to get

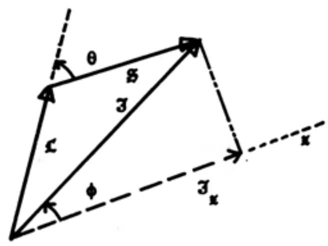
$$0 \leq -22j^2 - 77j - 64,$$

which does not hold for $j \geq 0$.

Results similar to the last apply to $n = 7, 9, \ldots$ etc. Hence, since $j \geq 0$ the inequality is restricted to the values of j given in the problem.

8-10

(a) Largest $j = 4 + \frac{1}{2} = 9/2$; largest $m_j = j = 9/2$. The magnitudes of the vectors are:



$$J = \sqrt{\{j(j+1)\}}\hbar = \sqrt{(99)}\hbar/2,$$

$$L = \sqrt{\{\ell(\ell+1)\}}\hbar = \sqrt{(20)}\hbar,$$

$$S = \sqrt{\{s(s+1)\}}\hbar = \sqrt{3}\hbar/2.$$

Apply the law of cosines to the L,S,J triangle:

$$J^2 = L^2 + S^2 - 2LS\cos(180° - \theta),$$

$$\frac{99}{4} = 20 + \frac{3}{4} + 2\sqrt{\{20(\tfrac{3}{4})\}}\cos\theta,$$

$$\theta = \cos^{-1}(\frac{2}{\sqrt{15}}) = 58.91°.$$

(b) Since $\vec{\mu}_\ell$ is antiparallel to $\vec{L}$ and $\vec{\mu}_s$ is antiparallel to $\vec{S}$, the angle between $\vec{\mu}_\ell, \vec{\mu}_s = 58.91°$.

(c)

$$\cos\phi = \frac{J_z}{J} = \frac{9}{\sqrt{99}}; \quad \phi = 25.24°.$$

8-12

Define the relativistic energy as

$$E_{rel} = K + V.$$

Now

$$K = mc^2 - m_0c^2 = m_0c^2\{\frac{1}{\sqrt{(1 - \beta^2)}} - 1\},$$

$\beta = v/c$. The relativistic momentum p is

$$p = mv = m_0c\beta(1 - \beta^2)^{-\frac{1}{2}}.$$

To express K in terms of p note that

$$p' = \frac{p}{m_0c} = \beta(1 - \beta^2)^{-\frac{1}{2}}; \quad (1 - \beta^2)^{-1} = 1 + p'^2,$$

so that

$$K = m_0c^2\{(1 + p'^2)^{\frac{1}{2}} - 1\} = m_0c^2\{(1 + \tfrac{1}{2}p'^2 - \tfrac{1}{8}p'^4 + \ldots) - 1\},$$

$$K \simeq m_0c^2(\tfrac{1}{2}p'^2 - \tfrac{1}{8}p'^4) = \frac{p^2}{2m_0} - \frac{p^4}{8m_0^3c^2}.$$

Hence,

$$E_{rel} = K + V = \frac{p^2}{2m_0} - \frac{p^4}{8m_0^3c^2} + V,$$

and therefore

$$\Delta E_{rel} = E_{rel} - E_{cl} = (\frac{p^2}{2m_0} - \frac{p^4}{8m_0^3c^2} + V) - (\frac{p_{cl}^2}{2m_0} + V).$$

If $p \simeq p_{cl}$, then

$$\Delta E_{rel} \simeq -p^4/8m_0^3c^2.$$

Now using classical expressions, in the spirit of the approximation,

$$\frac{p^2}{2m} + V = E; \quad p^4 = 4m^2(E - V)^2,$$

so that

$$\Delta E_{rel} = -\frac{E^2 - 2EV + V^2}{2mc^2}.$$

With E = constant and $V = -e^2/4\pi\varepsilon_0$, the above yields the final quoted result directly.

8-15

(a) The integrals to examine are

$$\int\psi_f^*(e\vec{r})\psi_f d\tau; \quad \int\psi_i^*(e\vec{r})\psi_i d\tau.$$

Since both ψ_i, ψ_f are single electron eigenfunctions, each has the form $\psi_{n\ell m_\ell} = (n,\ell,m_\ell)$. Hence each integral may be written

$$e\int(n,\ell,m_\ell)^*(\vec{r})(n,\ell,m_\ell)d\tau.$$

Now the parity of $\vec{r}$ is odd: $P(\vec{r}) = -\vec{r}$; the parity of (n,ℓ,m_ℓ) is $(-1)^\ell$ so that the parity of $(n,\ell,m_\ell)^*(n,\ell,m_\ell)$ is $(-1)^{2\ell}$ and therefore is even regardless of whether ℓ is odd or even. Thus the parity of the integrand above is odd, and the integral over all space vanishes.

(b) Electric dipole moments constant in time do not exist, since the governing integral above is zero. Only integrals preceded by $e^{\pm i(E_i-E_f)t/\hbar}$, depending on time, may be non-zero.

8-16

For n = 2: $\ell = 0$, $m_\ell = 0$; $\ell = 1$, $m_\ell = 1, 0, -1$. Hence the integrals to be considered are

(i) $\int(1,0,0,)^*(e\vec{r})(2,1,\pm1)d\tau$; (ii) $\int(1,0,0)^*(e\vec{r})(2,1,0)d\tau$;

(iii) $\int(1,0,0)^*(e\vec{r})(2,0,0)d\tau$,

where $\psi_{n\ell m_\ell} = (n,\ell,m_\ell)$. Also,

$$\vec{r} = (r\sin\theta\sin\phi)\vec{i} + (r\sin\theta\cos\phi)\vec{j} + r\cos\theta\vec{k}.$$

Substituting the explicit expressions for the wavefunctions gives the following for the integrals above.

(i)

$$e\int e^{-r/a_0}(\vec{r})e^{-r/2a_0}e^{\pm i\phi}r^2\sin\theta dr d\theta d\phi.$$

Insert the expression for r given above. The integrals over r, θ, and φ that appear are

$$\int_0^\infty e^{-3r/2a_0}r^4dr \neq 0$$

$$\int_0^\pi \sin^3\theta d\theta = \frac{4}{3}; \quad \int_0^\pi \sin^2\theta\cos\theta d\theta = 0;$$

$$\int_0^{2\pi} \begin{matrix}\sin\phi\\ \cos\phi\end{matrix}\ e^{\pm i\phi}d\phi = \int_0^{2\pi} \begin{matrix}\sin\phi\\ \cos\phi\end{matrix}\ (\cos\phi + i\sin\phi)d\phi \neq 0.$$

Hence $\int(1,0,0)^*(e\vec{r})(2,1,\pm)d\tau \neq 0$ and a transition is permitted; in this case $\Delta\ell = -1$, in accord with the selection rule.

(ii) The integral here becomes

$$e\int e^{-r/a_0}(\vec{r})re^{-r/2a_0}r^2\sin\theta\cos\theta dr d\theta d\phi.$$

In the $\vec{i},\vec{j}$ terms, the integrals $\int\sin\phi d\phi = \int\cos\phi d\phi = 0$. In the $\vec{k}$ terms remaining, however,

$$\int_0^{2\pi} d\phi \neq 0; \quad \int_0^\infty r^4e^{-3r/2a_0}dr \neq 0; \quad \int_0^\pi \cos^2\theta\sin\theta d\theta = \frac{2}{3} \neq 0.$$

Thus here also $\int(1,0,0)^*(e\vec{r})(2,1,0)d\tau \neq 0$ and a transition is allowed, again in accord with the selection rule since here $\Delta\ell = 1$.

(iii) Finally, this integral gives

$$e\int e^{-r/a_0}(\vec{r})(2 - r/a_0)e^{-r/2a_0}r^2\sin\theta dr d\theta d\phi.$$

The $\vec{i},\vec{j}$ terms vanish for the same reason as in (ii). But this time the θ integral in the $\vec{k}$ term is

$$\int_0^\pi \sin\theta\cos\theta d\theta = 0$$

This time, then, $\int(1,0,0)^*(e\vec{r})(2,0,0)d\tau = 0$ and the transition is forbidden. The selection rule is obeyed since $\Delta\ell = 2$ between the two states and the selection rule is $\Delta\ell = \pm1$.

8-17

It is desired to check the selection rule $\Delta n = \pm1$ by evaluating

$$I \propto e\int_{-\infty}^\infty \psi_{n_f}^* u\psi_{n_i}du \propto \int_{-\infty}^\infty \psi_{n_f}u\psi_{n_i}du,$$

since ψ(u) is real and $u \propto x$.

(i) $n_i = 3$, $n_f = 0$. In this case,

$$I \propto \int_{-\infty}^{+\infty}(3u - 2u^3)e^{-\frac{1}{2}u^2}e^{-\frac{1}{2}u^2}udu = 2\int_0^\infty(3u^2 - 2u^4)e^{-u^2}du,$$

$$I \propto 2\{3(\tfrac{1}{4}\pi^{\frac{1}{2}}) - 2(\tfrac{3}{8}\pi^{\frac{1}{2}})\} = 0.$$

But $\Delta n = 3$, so the selection rule is not violated.

(ii) $n_i = 2$, $n_f = 0$:

$$I \propto \int_{-\infty}^{+\infty} (1 - 2u^2)e^{-u^2} u\,du = 0,$$

since the integrand is of odd parity; $\Delta n = 2$ in this instance.

(iii) $n_i = 1$, $n_f = 0$.

$$I \propto \int_{-\infty}^{+\infty} u^2 e^{-u^2} du = 2(\tfrac{1}{4}\pi^{\frac{1}{2}}) \neq 0,$$

and $\Delta n = 1$.
Thus the selection rule $\Delta n = \pm 1$ is obeyed in these three cases.

8-18

(a) From Eq. 8-43, the transition rate R is

$$R = \frac{16\pi^3\nu^3}{3\epsilon_0 hc^3}\left|\int_{-\infty}^{+\infty}\psi_0^* ex\psi_1 dx\right|^2,$$

with

$$\nu = \frac{1}{2\pi}(C/m)^{\frac{1}{2}}.$$

The integral in the expression for R is, with $u = (Cm)^{\frac{1}{4}}x/\hbar^{\frac{1}{2}}$,

$$P = 2eA_0A_1\int_0^{\infty} e^{-\frac{1}{2}u^2} xue^{-\frac{1}{2}u^2} dx = 2eA_0A_1 \frac{\hbar}{\sqrt{(Cm)}}\int_0^{\infty} u^2 e^{-u^2} du,$$

$$P = \tfrac{1}{2}eA_0A_1\hbar\left(\frac{\pi}{Cm}\right)^{\frac{1}{2}}.$$

The normalization constants A_0 and A_1 are determined from:

$$1 = A_0^2\int_{-\infty}^{+\infty} e^{-u^2} dx = A_0^2\,\frac{(\hbar\pi)^{\frac{1}{2}}}{(Cm)^{\frac{1}{4}}},$$

$$1 = A_1^2\int_{-\infty}^{+\infty} u^2 e^{-u^2} dx = \tfrac{1}{2}A_1^2\,\frac{(\hbar\pi)^{\frac{1}{2}}}{(Cm)^{\frac{1}{4}}}.$$

Solving these last for A_0, A_1, the expression for P becomes

$$P = \tfrac{1}{2}e\,\frac{(2\hbar)^{\frac{1}{2}}}{(Cm)^{\frac{1}{4}}}.$$

Putting this and the expression for ν into R gives finally, for $m = 10m_p$,

$$R = e^2C/\tfrac{3}{2}(4\pi\epsilon_0)c^3m^2,$$

$$R = \frac{(1.6 \times 10^{-19})^2(9 \times 10^9)(10^3)}{1.5(3 \times 10^8)^3(1.67 \times 10^{-26})^2} = 20\ s^{-1}.$$

(b) The lifetime is

$$\tau = R^{-1} = 0.05\ s.$$

8-19

The infinite square well eigenfunctions, apart from the not-needed normalization constant, are

$$\psi_n(x) = \sin(n\pi x/a),\ n\ \text{even};\quad \psi_n(x) = \cos(n\pi x/a),\ n\ \text{odd}.$$

It is required to evaluate

$$P = \int \psi_f^*(ex)\psi_i dx \propto \int_{-\frac{1}{2}a}^{+\frac{1}{2}a}\psi_f x\psi_i dx.$$

In the following, $u = n\pi x/a$; also, n,m denote the two levels.

(i) Transitions between even-n levels:

$$P \propto \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} u\sin(nu)\sin(mu)\,du = 0$$

since the integrand is odd.

(ii) Transitions between odd-n levels:

$$P \propto \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} u\cos(nu)\cos(mu)\,du = 0$$

again since the integrand is odd.

(iii) Transitions between an odd and an even-n level:

$$P \propto \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} u\sin(nu)\cos(mu)\,du = 2\int_0^{+\frac{1}{2}\pi} u\sin(nu)\cos(mu)\,du,$$

$$P \propto \frac{1}{2}\left\{\frac{\sin\frac{1}{2}\pi(n-m)}{(n-m)^2} + \frac{\sin\frac{1}{2}\pi(n+m)}{(n+m)^2}\right\} - \frac{\pi}{4}\left\{\frac{\cos\frac{1}{2}\pi(n-m)}{n-m} + \frac{\cos\frac{1}{2}\pi(n+m)}{n+m}\right\},$$

evaluating the final integral. Clearly the above is not zero for n even, m odd. Hence, transitions are permitted only between levels such that one is of odd n, the other even n; i.e., the selection rule is

$$\Delta n = \pm 1, \pm 3, \pm 5, \ldots .$$

8-20

The eigenfunctions are

$$\Phi(\phi) = \frac{1}{\sqrt{(2\pi)}}e^{im\phi}, \quad m = 0, 1, 2, \ldots$$

and

$$P_{fi} = \left|\int_0^{2\pi} \Phi_f^*(e\vec{r})\Phi_i\,d\phi\right|.$$

But $R\cos\phi = r_x$, $R\sin\phi = r_y$. Dropping the absolute value signs,

$$I_x = \frac{eR}{2\pi}\int_0^{2\pi} e^{-im_f\phi}(\cos\phi)e^{im_i\phi}d\phi = \frac{eR}{2\pi}\int_0^{2\pi} e^{i(m_i-m_f)\phi}\cos\phi\,d\phi,$$

$$I_x = \frac{eR}{4\pi}\int_0^{2\pi}\{e^{i(\Delta m+1)\phi} + e^{i(\Delta m-1)\phi}\}d\phi \quad \Delta m = m_i - m_f,$$

since

$$\cos\phi = \tfrac{1}{2}(e^{i\phi} + e^{-i\phi}).$$

Now,

$$\int_0^{2\pi} e^{in\theta}d\theta = \int_0^{2\pi}\cos(n\theta)\,d\theta + i\int_0^{2\pi}\sin(n\theta)\,d\theta = \frac{1}{n}(\sin 2\pi n - i\cos 2\pi n + i).$$

Hence, if n = 1, 2, 3, ...

$$\int_0^{2\pi} e^{in\theta}d\theta = 0.$$

On the other hand, if n = 0, then

$$\int_0^{2\pi} e^{in\theta}d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

Thus, $I_x \neq 0$ only if either (i) $\Delta m + 1 = 0$, or (ii) $\Delta m - 1 = 0$; that is, only if $\Delta m = \pm 1$.

Since

$$\sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}),$$

$$I_y = \frac{eR}{2\pi}\int_0^{2\pi} e^{i\Delta m\phi}\sin\phi\,d\phi = \frac{eR}{4\pi i}\int_0^{2\pi}\{e^{i(\Delta m+1)\phi} - e^{i(\Delta m-1)\phi}\}d\phi.$$

The integrand is similar to the one for I_x. Therefore, the selection rule is $\Delta m = \pm 1$.

8-21

By Eq. 8-43,

$$\frac{R_{12}}{R_{01}} = \frac{\nu_{12}^3 P_{12}^2}{\nu_{01}^3 P_{01}^2}.$$

But P_{fi} depends only on Δm; since $\Delta m = m_i - m_f = +1$ for both transitions, $P_{12} = P_{01}$. Hence,

$$\frac{R_{12}}{R_{01}} = \frac{\nu_{12}^3}{\nu_{01}^3} = \left(\frac{E_1 - E_2}{E_0 - E_1}\right)^3,$$

since $\nu = \frac{1}{2\pi}(\Delta E/\hbar)$. But,

$$E_m = \frac{m^2\hbar^2}{2I},$$

and therefore

$$\frac{R_{12}}{R_{01}} = \left(\frac{1^2 - 2^2}{0^2 - 1^2}\right)^3 = 27.$$

CHAPTER NINE

9-3

The probability densities are

$$\psi^*_{\substack{S\\A}}\psi_{\substack{S\\A}} = \frac{1}{2}\{\overset{\mathrm{I}}{\psi^*_\alpha(1)\psi^*_\beta(2)\psi_\alpha(1)\psi_\beta(2)} \pm \overset{\mathrm{II}}{\psi^*_\alpha(1)\psi^*_\beta(2)\psi_\beta(1)\psi_\alpha(2)}$$
$$\pm \overset{\mathrm{III}}{\psi^*_\beta(1)\psi^*_\alpha(2)\psi_\alpha(1)\psi_\beta(2)} + \overset{\mathrm{IV}}{\psi^*_\beta(1)\psi^*_\alpha(2)\psi_\beta(1)\psi_\alpha(2)}\}.$$

Making the switch interchanging 1 and 2, terms I and IV interchange, as do II and III. Thus,

$$\psi^*_{\substack{S\\A}}\psi_{\substack{S\\A}} = \frac{1}{2}\{\mathrm{I} \pm \mathrm{II} \pm \mathrm{III} + \mathrm{IV}\} \rightarrow \frac{1}{2}\{\mathrm{IV} \pm \mathrm{III} \pm \mathrm{II} + \mathrm{I}\} = \psi^*_{\substack{S\\A}}\psi_{\substack{S\\A}}.$$

9-4

From Example 9-2:

$$\psi_A = \frac{1}{\sqrt{3!}}\{\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3) + \psi_\beta(1)\psi_\gamma(2)\psi_\alpha(3) + \psi_\gamma(1)\psi_\alpha(2)\psi_\beta(3)$$
$$- \psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3) - \psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3) - \psi_\alpha(1)\psi_\gamma(2)\psi_\beta(3)\}.$$

As an example, interchange particles 1 and 3 to get

$$\psi'_A = \frac{1}{\sqrt{3!}}\{\psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3) + \psi_\alpha(1)\psi_\gamma(2)\psi_\beta(3) + \psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3)$$
$$- \psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3) - \psi_\gamma(1)\psi_\alpha(2)\psi_\beta(3) - \psi_\beta(1)\psi_\gamma(2)\psi_\alpha(3)\},$$

$$\psi'_A = -\psi_A.$$

The same result is achieved if, instead, particles 1 and 2, or 2 and 3 are interchanged.

9-6

The antisymmetric function for three particles is

$$\psi_A = \frac{1}{\sqrt{3!}}\{\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3) + \psi_\beta(1)\psi_\gamma(2)\psi_\alpha(3) + \psi_\gamma(1)\psi_\alpha(2)\psi_\beta(3)$$
$$- \psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3) - \psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3) - \psi_\alpha(1)\psi_\gamma(2)\psi_\beta(3)\}.$$

Upon forming $\int\psi_A^*\psi_A d\tau$ there appears the following terms:

(i) Six terms, each the square of those above; for example,

$$\int\psi_\alpha^*(1)\psi_\beta^*(2)\psi_\gamma^*(3)\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)d\tau_1 d\tau_2 d\tau_3$$

$$= \{\int\psi_\alpha^*(1)\psi_\alpha(1)d\tau_1\}\{\int\psi_\beta^*(2)\psi_\beta(2)d\tau_2\}\{\int\psi_\gamma^*(3)\psi_\gamma(3)d\tau_3\},$$

$$= \{1\}\{1\}\{1\} = 1,$$

assuming that each wave function is normalized. Hence, these terms add to 6.

(ii) Cross terms; for example,

$$\int\psi_\alpha^*(1)\psi_\beta^*(2)\psi_\gamma^*(3)\psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3)d\tau_1 d\tau_2 d\tau_3$$

$$= \{\int\psi_\alpha^*(1)\psi_\gamma(1)d\tau_1\}\{\int\psi_\beta^*(2)\psi_\beta(2)d\tau_2\}\{\int\psi_\gamma^*(3)\psi_\alpha(3)d\tau_3\},$$

$$= \{0\}\{1\}\{0\} = 0,$$

due to the orthogonality of the eigenfunctions. All of these cross terms vanish.

Thus, the total integral equals 6 and, since $3! = 6$ also, ψ_A is normalized as originally written.

9-9

By assumption

$$\psi_A = \psi_{space}\psi_{spin}.$$

Look first at the symmetric space functions; since both of the electrons are in the same (ground) state,

$$\psi_{space} = \frac{1}{\sqrt{2}}\{\psi_{100}(1)\psi_{100}(2) + \psi_{100}(2)\psi_{100}(1)\},$$
$$= \frac{2}{\sqrt{2}}\{\frac{1}{\sqrt{\pi}}(\frac{2}{a_0})^{3/2}e^{-2r_1/a_0}\}\{\frac{1}{\sqrt{\pi}}(\frac{2}{a_0})^{3/2}e^{-2r_2/a_0}\}.$$

ψ_{spin} must be antisymmetric since the space function was chosen to be symmetric (electrons in the same level). The coulomb energy is

$$V = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_{12}},$$

where $r_{12} \neq r_1 - r_2$ is the distance between the electrons. Thus

$$\bar{V} = \int\psi_A^* V\psi_A d\tau_1 d\tau_2 d\sigma_1 d\sigma_2,$$

in which σ_1, σ_2 are the spin variables. Now V is independent of the spin of the electrons, so that if the spin wave function is normalized, then

$$\bar{V} = \int\psi_{space}^* V\psi_{space} d\tau_1 d\tau_2.$$

Putting in the wave function gives

$$\bar{V} = \frac{8e^2}{\pi^3 a_0^6\epsilon_0}\int e^{-4(r_1+r_2)/a_0}\frac{1}{r_{12}}r_1^2 dr_1 r_2^2 dr_2 \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\phi_1 d\phi_2,$$

in which $r_{12} = r_{12}(r_1, r_2, \theta_1, \theta_2, \phi_1, \phi_2)$.

Now suppose that the antisymmetric space function had been chosen. With both electrons in the ground state, this will be

$$\psi_{space} = \frac{1}{\sqrt{2}}\{\psi_{100}(1)\psi_{100}(2) - \psi_{100}(2)\psi_{100}(1)\} = 0.$$

It may be concluded then that with both electrons in the ground state, the electron spin must be in the antisymmetric (singlet) state. The coulomb interaction, being positive, will increase the ground state energy over that calculated by ignoring it.

9-10

Consider two eigenfunctions ψ_j, ψ_i, solutions of

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_j}{dx^2} + V\psi_j = E_j\psi_j \quad (1); \quad -\frac{\hbar^2}{2m}\frac{d^2\psi_i}{dx^2} + V\psi_i = E_i\psi_i \quad (2).$$

Take the complex conjugate of (2) to get

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_i^*}{dx^2} + V\psi_i^* = E_i\psi_i^*. \qquad (3)$$

Multiply (1) by ψ_i^*, (3) by ψ_j and subtract; this gives

$$-\frac{\hbar^2}{2m}\left(\psi_i^*\frac{d^2\psi_j}{dx^2} - \psi_j\frac{d^2\psi_i^*}{dx^2}\right) = (E_j - E_i)\psi_i^*\psi_j.$$

Now integrate over x:

$$-\frac{2m}{\hbar^2}(E_j - E_i)\int_{-\infty}^{+\infty}\psi_i^*\psi_j dx = \int_{-\infty}^{+\infty}\left(\psi_i^*\frac{d^2\psi_j}{dx^2} - \psi_j\frac{d^2\psi_i^*}{dx^2}\right)dx$$

$$= \int_{-\infty}^{+\infty}\frac{d}{dx}\left(\psi_i^*\frac{d\psi_j}{dx} - \psi_j\frac{d\psi_i^*}{dx}\right)dx = \psi_i^*\frac{d\psi_j}{dx} - \psi_j\frac{d\psi_i^*}{dx}\Big|_{-\infty}^{+\infty}.$$

(i) If $E_j \neq E_i$ and the system is bound, the wavefunctions approach zero at both infinities. Thus the integral vanishes and

$$\int_{-\infty}^{+\infty}\psi_i^*\psi_j dx = 0.$$

(ii) In the continuum region (unbound system), the wave function remains finite at large positive and negative x. In practice, however, box normalization is invoked and the wavefunction vanishes at the surface of the box, so that the result above is achieved here also.

(iii) If $E_j = E_i$ (degenerate case), construct

$$\psi_a = a_i\psi_i + a_j\psi_j,$$

also a solution for this potential. If a_i, a_j are chosen properly ψ_a can be made orthogonal to ψ_j:

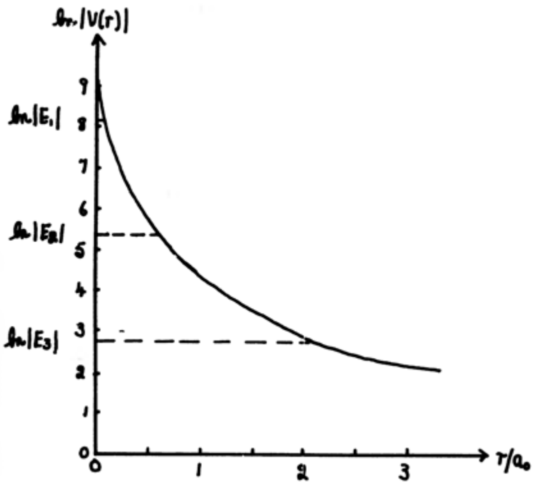
$$\int\psi_a^*\psi_j dx = 0 = a_i^*\int\psi_i^*\psi_j dx + a_j^*\int\psi_j^*\psi_j dx,$$

$$a_i^*\int\psi_i^*\psi_j dx + a_j^* = 0,$$

and so choose

$$\frac{a_j^*}{a_i^*} = -\int\psi_i^*\psi_j dx.$$

9-13



(a)

$$V(r) = -\frac{Z(r)e^2}{4\pi\epsilon_0 r} = -\frac{Z(r)e^2}{4\pi\epsilon_0 a_0 (r/a_0)} = -(2E_{1H})\,\frac{Z(r)}{r/a_0},$$

$$V(r) = -(27.2\text{ eV})\,\frac{Z(r)}{r/a_0},$$

where E_{1H} = ground state energy of the hydrogen atom. Values of $Z(r)$ may be taken (not easily) from Fig.9-11.

(b) The results are shown on the preceding page. The first three levels of argon are

$$E_1 = -3500\text{ eV};\quad E_2 = -220\text{ eV};\quad E_3 = -16\text{ eV}.$$

9-14

(a) Eq.9-27 is, with E_H referring to hydrogen,

$$E = -\frac{\mu Z_n^2 e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2} = -Z_n^2 E_H.$$

In the ground state (n = 1) of helium, Fig.9-6 gives $E_1 \simeq -80$ eV and therefore

$$-80 \simeq -Z_1^2 E_{1H} = -Z_1^2(13.6),$$

$$Z_1 = 2.4.$$

(b) With so few electrons, it is not clear whether an inner or outer shell is being described. If $Z_1 \simeq n = 1$ an outer shell is indicated; for an inner shell, $Z_1 \simeq Z - 2 = 2 - 2 = 0$.

(c) The fact that Z_1 equals (or roughly equals) neither n = 1 (outer shell) nor Z - 2 = 0 (inner shell) implies that the Hartree method is not applicable to helium. This is not very surprising since a statistical method cannot be expected to work well with so few particles (two electrons).

9-15

(a) From Fig.9-6, $E_{coul} = +30$ eV;

$$E_{coul} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r},$$

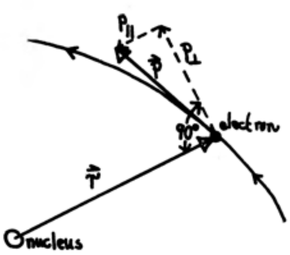
$$30 = (9\times10^9)\frac{(1.6\times10^{-19})^2}{r(1.6\times10^{-19})},$$

$$r = 0.048\text{ nm}.$$

(b) $$E_{coul} = +9\text{ eV} \rightarrow r = 0.16\text{ nm}.$$

9-17

For the electron $\vec{L} = \vec{r}\times\vec{p}$ and $\vec{L}$ is perpendicular to the plane of the orbit. Now $\vec{v} \neq 0$ anywhere in the orbit and therefore $\vec{p} \neq 0$. If $L \neq 0$, then $r \neq 0$ everywhere and the electron avoids the nucleus ($r \simeq 0$). If L = 0, the electron would move on a straight line through the nucleus.



(a)

$$E = K + V = \frac{p^2}{2m} + V,$$

$$E = \frac{1}{2m}(p_\parallel^2 + p_\perp^2) + V.$$

But $L = rp_\perp$ (since $\vec{r}\times\vec{p}_\parallel = 0$); hence,

$$p_\perp^2 = \frac{L^2}{r^2};\quad E = \frac{p_\parallel^2}{2m} + \left(\frac{L^2}{2mr^2} + V\right).$$

(b) In one dimension $p_\perp = 0$, L = 0 and $E = p_\parallel^2/2m + V'$. This and the preceding equation are formally identical if

$$V' = L^2/2mr^2 + V.$$

(c) If the electron is bound, $V(r) < 0$. Clearly $L^2/2mr^2 > 0$ (recall that L is a constant for central forces). For small enough r, $L^2/2mr^2 >> |V(r)|$ and $V' > 0$, indicating a repulsive core in the one-dimensional formalism. Only if $V \propto r^{-n}$, $n \geq 2$ will this core disappear (unless $L = 0$).

9-18

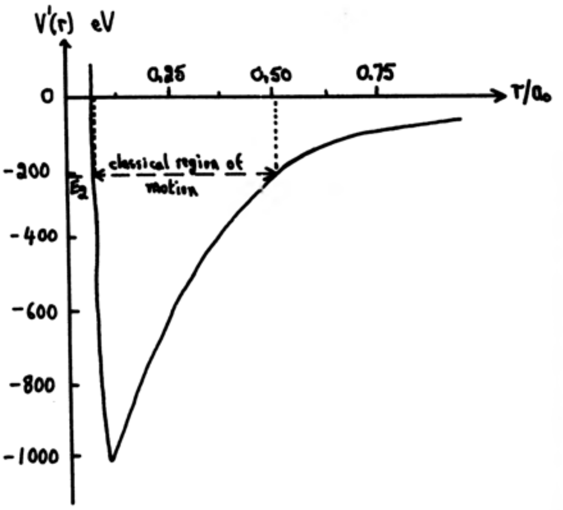
(a) The potential in question is V':

$$V' = V + L^2/2mr^2.$$

Now, in electron volts,

$$\frac{L^2}{2mr^2} = \frac{\ell(\ell+1)\hbar^2}{2mr^2} = \frac{\hbar^2}{2ma_0^2}\frac{\ell(\ell+1)}{(r/a_0)^2} = (13.6)\frac{\ell(\ell+1)}{(r/a_0)^2}.$$

Clearly, for $\ell = 0$, $V' = V$; see Problem 9-13. For $\ell = 1$,



$$V' = V + \frac{27.2}{(r/a_0)^2} = -(27.2)\frac{Z(r)}{(r/a_0)} + (27.2)(r/a_0)^{-2}.$$

(b) $E_2 = -220$ eV.

(c) The classical region of motion is shown on the sketch: it falls within the range for which $P_{21}(r)$, Fig.9-10, is large ($\ell = 1$).

(d) For $\ell = 0$ see Fig.9-13. The classically permitted region there falls within $r = 0.2a_0$, a bit smaller than for $\ell = 1$. This result also corresponds roughly to Fig.9-10, where P_{20} is large at $r = 0.5a_0$. There is qualitative agreement between classical and quantum results.

9-22

(a) From Fig.9-15, the ionization energy for the first electron is 24 eV. In the ground state the energy of the atom, from Fig.9-6, is -78 eV. Thus the energy after the first electron is removed is -78 + 24 = -54 eV. The energy with both electrons removed is zero; thus the energy needed to remove the remaining electron is 54 eV.

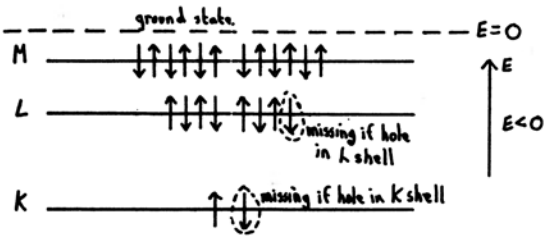
(b) With the first electron gone, the helium atom resembles a hydrogen atom with Z = 2. For such an atom the ground state energy will be

$$E_1 = Z^2(E_{1H}) = 2^2(-13.6) = -54.4 \text{ eV},$$

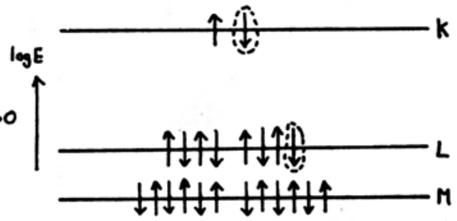
and therefore 54.4 eV are required to ionize it. Agreement with (a) is excellent.

9-23

(a), (b)



(c) In the x-ray diagram the levels are inverted, the ground state is taken as zero energy, and the energy is plotted on a logarithmic scale.



(d) When the hole is in an inner shell the energy differences for likely transitions are large; thus the x-ray diagram, plotting logE, is easier to handle than the standard diagram.

(e) When the hole is in an outer shell, the transitions are more likely to be optical, and the associated energy differences are small. Hence, the standard diagram is adequate.

9-24

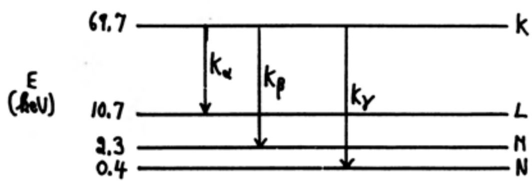
The photon energies are

$$E = \frac{hc}{\lambda}; \quad E(\text{keV}) = \frac{1.2400}{\lambda(\text{nm})}.$$

Use of the last expression gives the following:

K_α (L→K): $\lambda = 0.0210$ nm, $E = 59.0$ keV;
K_β (M→K): $\lambda = 0.0184$ nm, $E = 67.4$ keV;
K_γ (N→K): $\lambda = 0.0179$ nm, $E = 69.3$ keV.

For the absorption edge, $E = 1.2400/0.0178 = 69.7$ keV = energy needed to ionize the atom by removing an electron from the K shell. Hence, $E_K = 0$ (ground state) + 69.7 = 69.7 keV = energy of the atom with a hole in the K shell. Then $E_L = 69.7 - 59.0 = 10.7$ keV; similarly, $E_M = 69.7 - 67.4 = 2.3$ keV and $E_N = 69.7 - 69.3 = 0.4$ keV.



9-25

(a) The L_α line is emitted when a hole jumps from the n = 2 to an n = 3 level. The energy required for this is approximately the energy needed to ionize the atom by removing an n = 2 electron. Using the one-electron formula with $Z_2 = Z - 10$,

$$E_2 = -\left(\frac{Z_n}{n}\right)^2(13.6) = -\left(\frac{26-10}{2}\right)^2(13.6) = -870 \text{ eV}.$$

Thus the required voltage is about 870 V.

(b) The wavelength is obtained from

$$\frac{hc}{\lambda} = E_M - E_L = E_3 - E_2 \simeq -E_2 = 870 \text{ eV}; \quad \lambda = 1.4 \text{ nm}.$$

9-26

(a) The empirical formula is

$$\lambda^{-1} = C(Z-a)^2; \; \lambda^{-1/2} = C^{1/2}(Z-a).$$

Thus a plot of $\lambda^{-1/2}$ vs. Z is a straight line with a Z-intercept of Z = a and a slope of $\sqrt{C}$. From Fig.9-18, the Z-intercept = a ≃ 1.7. Also

$$\text{slope} = C^{1/2} = \frac{1.0 - 0.5}{34 - 17}(10^5); \quad C = 8.65 \times 10^6 \text{ m}^{-1}.$$

(b) For a: a = 1.7; $C \simeq R_M = 11 \times 10^6 \text{ m}^{-1}$.

9-27

(a) The K absorption edge (n = 1) should be given by

$$E_{edge} = (13.6 \text{ eV})(Z_n/n)^2.$$

Thus,

$$\text{Cobalt: } E_K = (13.6)\left(\frac{27-2}{1}\right)^2 = 8.5 \text{ keV;}$$

$$\text{Iron: } E_K = (13.6)\left(\frac{26-2}{1}\right)^2 = 7.83 \text{ keV.}$$

(b) For photon energies greater than 7.83 keV, the probability that iron will absorb the photon diminishes sharply. With 8.5 greater than 7.83, and since 8.5 keV falls near the K edge for cobalt, where the probability for absorption is high, a photon energy near 8.5 keV would be best.

9-28

For the inner electrons, the wave functions are essentially hydrogenic, with an appropriate effective Z. For the K_α line, use

$$\psi_{100} = \frac{1}{\sqrt{\pi}}(Z/a_0)^{3/2}e^{-Zr/a_0},$$

$$\psi_{210} = \frac{1}{4\sqrt{(2\pi)}}(Z/a_0)^{5/2}re^{-Zr/2a_0}\cos\theta,$$

with the selection rule $\Delta\ell = \pm 1$. The matrix element is

$$p_{fi} = |\int\psi_f^* e\vec{r}\psi_i d\tau| = \frac{e}{4\pi\sqrt{2}}\frac{Z^4}{a_0^4}\left|\int r^3 e^{-3Zr/2a_0}\vec{r}\cos\theta\sin\theta d\theta d\phi dr\right|.$$

Since

$$\vec{r} = r\sin\theta\cos\phi\vec{i} + r\sin\theta\sin\phi\vec{j} + r\cos\theta\vec{k},$$

the Z-dependence follows from

$$Z^4\int_0^\infty r^4 e^{-3Zr/2a_0}dr = Z^4\frac{4!}{(3Z/2a_0)^5},$$

$$p_{fi} \propto Z^{-1}.$$

The lifetime becomes

$$T = \frac{1}{R} = \frac{3\epsilon_0 hc^3}{16\pi^3\nu^3 p_{fi}^2} \rightarrow \frac{T_{Pb}}{T_H} = \frac{\nu_H^3 p_{fi,H}^2}{\nu_{Pb}^3 p_{fi,Pb}^2}.$$

By Example 9-8,

$$h\nu \approx E_K \propto Z_{eff}^2.$$

Therefore,

$$\frac{T_{Pb}}{T_H} = \frac{Z_{H,eff}^6}{Z_{Pb,eff}^6}\frac{Z_{Pb,eff}^2}{Z_{H,eff}^2} = \left(\frac{Z_{H,eff}}{Z_{Pb,eff}}\right)^4,$$

$$\frac{T_{Pb}}{T_H} = \left(\frac{1}{82-2}\right)^4 = 2.44 \times 10^{-8},$$

$$T_H \approx 10^{-8} \text{ s} \rightarrow T_{Pb} \approx 2.44 \times 10^{-16} \text{ s.}$$

CHAPTER TEN

10-1

(a) From Fig.10-1,

$$E = E_{2p} - E_{2s} = -3.50 - (-5.35) = 1.85 \text{ eV}.$$

For photons,

$$\lambda(\text{nm}) = \frac{1240}{E(\text{eV})},$$

so that

$$\lambda = \frac{1240}{1.85} = 670.3 \text{ nm} = 6703 \text{ Å}.$$

(b) By Example 10-1,

$$d\lambda = \frac{hc\,dE}{E^2} = \frac{hc}{E}\frac{dE}{E} = \lambda\frac{dE}{E}.$$

From Table 10-1, $dE = 0.42 \times 10^{-4}$ eV, and therefore

$$d\lambda = (670.3)\frac{0.42 \times 10^{-4}}{1.85} = 0.0152 \text{ nm} = 0.152 \text{ Å}.$$

10-3

(a) The ground state configuration is $1s^2 2s^2 2p^6 3s^1$, the first three shells being closed; the possible excited states will be those with the optical electron in the 3p,3d,4s,4p,4d,4f,5s etc. levels. But 4d,4f lie above 5s and 4s lies below 3d. Also, for any n, the level with maximum permitted ℓ ($=n - 1$) corresponds to the hydrogen atom level of that n. That is, for 3d ($n = 3$, $\ell = 2$),

$$E = E_{3H} = -1.5 \text{ eV}.$$

(b) Each level, except s-states, is split into two levels, with the state of smaller j being more negative in energy. The energy splitting is small compared to the energy of the degenerate states. This spin-orbit splitting is given by

$$\Delta E = \overline{\frac{1}{r}\frac{dV}{dr}}\,\frac{\hbar^2}{4m^2c^2}\{j(j + 1) - \ell(\ell + 1) - s(s + 1)\},$$

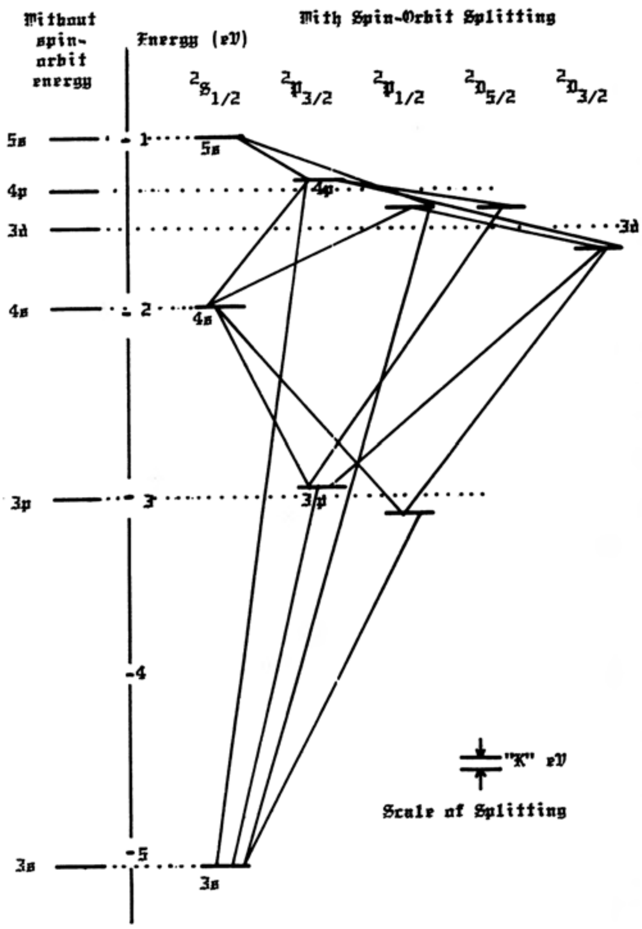
$$\Delta E = K\{j(j + 1) - \ell(\ell + 1) - s(s + 1)\}.$$

Assume $K = K(r=0)$ = constant. In all states $s = \frac{1}{2}$, $2s + 1 = 2$, $j = \ell \pm \frac{1}{2}$, except $\ell = 0$ where $j = s$. Putting all this together gives the results below.

$$
\begin{array}{lll}
3s: & \ell = 0, & j = 1/2; \quad \Delta E = 0; \quad {}^2S_{1/2}. \\
3p: & \ell = 1, & \begin{cases} j = 3/2; \quad \Delta E = +K; \quad {}^2P_{3/2}. \\ j = 1/2; \quad \Delta E = -2K; \quad {}^2P_{1/2}. \end{cases} \\
4s: & \ell = 0; & j = 1/2; \quad \Delta E = 0; \quad {}^2S_{1/2}. \\
3d: & \ell = 2; & \begin{cases} j = 5/2; \quad \Delta E = 2K; \quad {}^2D_{5/2}. \\ j = 3/2; \quad \Delta E = -3K; \quad {}^2D_{3/2}. \end{cases} \\
4p: & \ell = 1; & \begin{cases} j = 3/2; \quad \Delta E = K; \quad {}^2P_{3/2}. \\ j = 1/2; \quad \Delta E = -2K; \quad {}^2P_{1/2}. \end{cases} \\
5s: & \ell = 0; & j = 1/2; \quad \Delta E = 0; \quad {}^2S_{1/2}.
\end{array}
$$

(c) The selection rules are: $\Delta\ell = \pm 1$, $\Delta j = 0, \pm 1$. Using these gives for the permitted transitions:

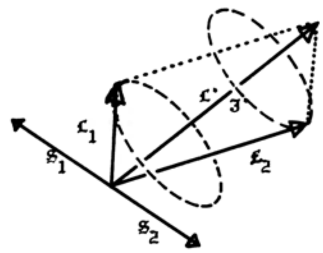
${}^2S_{1/2} \rightarrow {}^2P_{1/2}, {}^2P_{3/2}.$	Each of the 3s,4s,5s levels can make a transition to each of the two levels of the 3p,4p states.
${}^2P_{1/2} \rightarrow {}^2D_{3/2}.$	But not to ${}^5D_{5/2}$ since then $\Delta j = 2$. Also to the S-state above.
${}^2P_{3/2} \rightarrow {}^2D_{3/2}, {}^2D_{5/2}.$	These in addition to the ${}^2S_{1/2}$ transition.
${}^2D_{3/2}, {}^2D_{5/2}.$	These are included above.



10-4

(a) $\ell_1 = 1$, $\ell_2 = 2$; $s_1 = s_2 = \frac{1}{2}$. Thus the possible values of ℓ' are $\ell' = 3,2,1$; possible $s' = 1,0$. The smallest ℓ',s' is the state of maximum energy: i.e., $\ell' = 1$, $s' = 0$. With $\ell' = 1$, $s' = 0$, there is only one possible j': to wit, $j' = 1$.

(b) Since $\vec{S}' = 0$, $\vec{L}'$ lies along $\vec{J}'$.



10-5

$\ell_1 = 2$, $\ell_2 = 3$; $s_1 = s_2 = \frac{1}{2}$. The possible $\ell' = 5,4,3,2,1$; the possible $s' = 1,0$. For j':

$$j' = \ell' + s',\ \ell' + s' - 1,\ \ldots\ |\ell' - s'|.$$

Therefore, the possible configurations are:

ℓ'	s'	j'
5	1	6,5,4
5	0	5
4	1	5,4,3
4	0	4
3	1	4,3,2
3	0	3
2	1	3,2,1
2	0	2
1	1	2,1,0
1	0	1

10-7

For the configuration 4s3d, $\ell_1 = 0$, $\ell_2 = 2$, $s_1 = s_2 = \frac{1}{2}$. Hence, $\ell' = 2$ only, $s' = 1,0$. With $\ell' = 2$, $s' = 1$, $j' = 3,2,1$ giving ${}^3D_{3,2,1}$ levels. For $\ell' = 2$, $s' = 0$, $j' = 2$ only, resulting in a ${}^1D_2$ state. By the Lande interval rule, the separation is

$$({}^3D_3 - {}^3D_2)/({}^3D_2 - {}^3D_1) = 3/2.$$

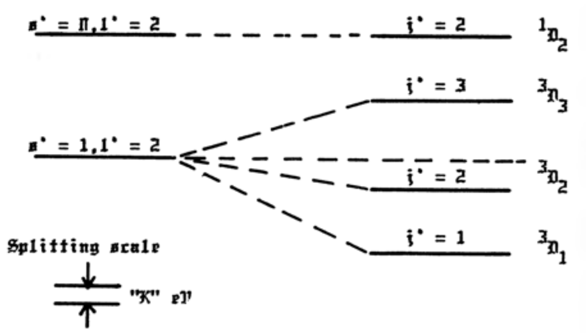
The energy shifts themselves are

$$\Delta E = K\{j'(j' + 1) - \ell'(\ell' + 1) - s'(s' + 1)\},$$

giving

$${}^1D_2\text{:}\quad \Delta E = 0;$$
$${}^3D_3\text{:}\quad \Delta E = 4K;$$
$${}^3D_2\text{:}\quad \Delta E = -2K;$$
$${}^3D_1\text{:}\quad \Delta E = -6K.$$

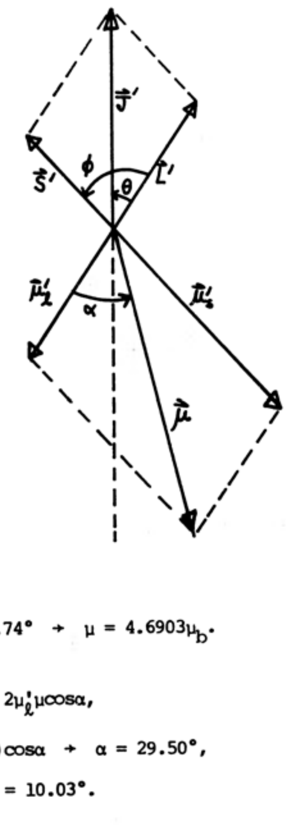
The latter three shifts obey the Lande rule.



10-8

The magnitudes of the vectors are:

$J' = \sqrt{12}\hbar$; $L' = \sqrt{6}\hbar$; $S' = \sqrt{2}\hbar$.



Applying the cosine law:

$$S'^2 = J'^2 + L'^2 - 2J'L'\cos\theta,$$

$$2 = 12 + 6 - 2\sqrt{72}\cos\theta,$$

$$\theta = 19.47°.$$

Again resorting to the cosine rule:

$$J'^2 = S'^2 + L'^2 + 2S'L'\cos\phi,$$

$$12 = 2 + 6 + 2\sqrt{12}\cos\phi,$$

$$\phi = 54.74°.$$

Turning to the magnetic moments:

$$\mu_s' = \frac{2\mu_b}{\hbar}S' = 2\sqrt{2}\mu_b,$$

$$\mu_\ell' = \frac{\mu_b}{\hbar}L' = \sqrt{6}\mu_b.$$

$$\mu^2 = \mu_s'^2 + \mu_\ell'^2 + 2\mu_s'\mu_\ell'\cos\phi,$$

$$\mu^2 = 8\mu_b^2 + 6\mu_b^2 + 4\sqrt{12}\mu_b^2\cos 54.74° \rightarrow \mu = 4.6903\mu_b.$$

Finally,

$$\mu_s'^2 = \mu_\ell'^2 + \mu^2 - 2\mu_\ell'\mu\cos\alpha,$$

$$8 = 6 + 21.999 - 2\sqrt{6}(4.6903)\cos\alpha \rightarrow \alpha = 29.50°,$$

so that $\measuredangle(\vec{\mu}, -\vec{J}') = 29.50° - 19.47° = 10.03°$.

10-9

(a) On Fig.9-13, the columns reveal the last shell being filled, the row the numbers of electrons in that shell. Therefore,

$$^{12}\mathrm{Mg}:\ 1s^2,\ 2s^2,\ 2p^6,\ 3s^2;$$

$$^{13}\mathrm{Al}:\ 1s^2,\ 2s^2,\ 2p^6,\ 3s^2,\ 3p^1;$$

$$^{14}\mathrm{Si}:\ 1s^2,\ 2s^2,\ 2p^6,\ 3s^2,\ 3p^2.$$

(b) ^{12}Mg; the configuration represents a filled shell, and thus all the angular momenta are zero, leading to $^1S_0$.
^{13}Al: there is a single valence electron ($s = s' = \frac{1}{2}$); thus $2s' + 1 = 2$; $\ell' = 1$ giving a P-state; $j' = 3/2,\ 1/2$ with the smaller j' lying lower, leading to $^2P_{\frac{1}{2}}$.
^{14}Si: here there are two $\ell = 1$ electrons; $s' = 1,0$ and $\ell' = 2,1,0$. For the lower energy pick the larger s'. This gives as possibilities:

$$\ell' = 2;\quad j' = 3,2,1;\quad {}^3D_{3,2,1}$$

$$\ell' = 1;\quad j' = 2,1,0;\quad {}^3P_{2,1,0}$$

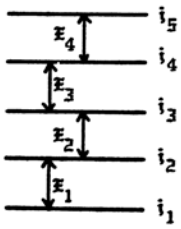
$$\ell' = 0;\quad j' = 1;\quad {}^3S_1.$$

The $^3D_1$ and $^3S_1$ states are, however, prohibited by the Exclusion principle. Of the $^3P_{2,1,0}$ states, the smallest j' lies lowest; hence the ground state configuration should be $^3P_0$.

10-11

For a single multiplet s' and ℓ' have the same value for each level. By the interval rule,

$$\mathcal{E}_4 = 2Kj_5,$$
$$\mathcal{E}_3 = 2Kj_4,$$
$$\mathcal{E}_2 = 2Kj_3,$$
$$\mathcal{E}_1 = 2Kj_2.$$



Therefore

$$\mathcal{E}_4/\mathcal{E}_3 = \frac{4}{3} = j_5/j_4 = (j_4 + 1)/j_4;\quad j_4 = 3.$$

Since $j = 1$, $j_5 = 4$, $j_4 = 3$, $j_3 = 2$, $j_2 = 1$, $j_1 = 0$. But

$$j' = \ell'+s',\ \ell'+s'-1,\ \ldots\ |\ell'-s'|,$$

so that

$$\ell' + s' = 4,\quad \ell' - s' = 0;\quad \ell' = s' = 2,$$

and hence the results are

$$\ell' = s' = 2;\ j' = 4,3,2,1,0.$$

10-14

(a) The g-factor is

$$g = 1 + \frac{j'(j'+1) + s'(s'+1) - \ell'(\ell'+1)}{2j'(j'+1)}.$$

(i) For $g > 2$,

$$j'(j'+1) < s'(s'+1) - \ell'(\ell'+1). \qquad (*)$$

If $j' = \ell' + s'$, this becomes

$$0 < -(\ell'^2 + \ell' + \ell's'),$$

which is impossible. But if $j' = \ell' - s'$ the relation gives

$$s' > \ell',$$

so that $j' = s' - \ell'$, a contradiction. So try $j' = s' - \ell'$ in (*), which will now reduce to

$$\ell' < s',$$

which is acceptable. For example, $s' = 2$, $\ell' = 1$, $j' = 2 - 1 = 1$, giving $g = 5/2 > 2$, as required.
(ii) For the case $g < 1$, the formula for g requires that

$$j'(j'+1) + s'(s'+1) < \ell'(\ell'+1). \qquad (**)$$

If $j' = \ell' + s'$, (**) becomes

$$0 > s'^2 + s' + \ell' s',$$

which is impossible. But if $j' = \ell' - s'$, (**) reduces to

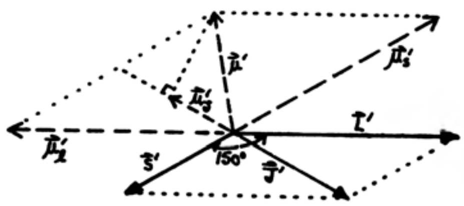
$$\ell' > s'.$$

For example, $\ell' = 2$, $s' = 1$, $j' = 2 - 1 = 1$, $g = \frac{1}{2} < 1$.

(b) Consider this second case: $\ell' = 2$, $s' = 1$, $j' = 1$, $g = \frac{1}{2}$ Then,

$$L' = \sqrt{6}\hbar,\quad S' = \sqrt{2}\hbar,\quad J' = \sqrt{2}\hbar,\quad \measuredangle(\vec{L}',\vec{S}') = 150°.$$

Since $g_\ell = 1$, $g_s = 2$, draw $\vec{\mu}_\ell'$ equal in length to $\vec{L}'$, $\vec{\mu}_s'$ twice as long as $\vec{S}'$; it is seen that $\vec{\mu}_J'$ is about half as long as $\vec{J}'$, indicating that $g = \frac{1}{2}$.



10-17

(a) The $^1P_1$ yields three levels, the $^3P_2$ five, $^3P_1$ three, and the $^3P_0$ gives one; thus the total number is 12.

(b) For the 3s electron $\ell = 0$; $2(2\ell+1) = 2$;
For the 3p electron $\ell = 1$; $2(2\ell+1) = 6$.

Clearly, with $(6)(2) = 12$, the field has removed the degeneracy completely.

10-18

For a singlet $s = 0$, so there is only orbital angular momentum to consider. The potential energy of orientation is

$$\Delta E = -\vec{\mu}\cdot\vec{B}.$$

For orbital angular momentum $g = 1$ so that

$$\frac{\mu}{L} = \frac{e}{2m};\quad \mu = \frac{\mu_b}{\hbar} L.$$

If $\vec{B}$ is in the z-direction,

$$\Delta E = -\frac{\mu_b}{\hbar}\vec{L}\cdot\vec{B} = -\frac{\mu_b}{\hbar} L_z B = -\frac{\mu_b B}{\hbar}(m_\ell \hbar) = -\mu_b B m_\ell.$$

giving rise to $2\ell+1$ levels. Since $s = 0$, $\ell = j$; making this substitution, and inserting a factor $g = 1$, leads to

$$\Delta E = -\mu_b g B m_j,$$

which agrees with Eq.10-22 in the case $g = 1$.

10-19

In the classical model, picture the magnetic field building up to a value B from zero in time T. Faraday's law requires an induced electric field E,

$$E = \tfrac{1}{2} r \frac{dB}{dt} = \tfrac{1}{2} r \frac{B}{T}.$$

This imparts an additional velocity Δv to the electrons, which circulate either clockwise or counterclockwise. Hence,

$$\Delta v = aT = \frac{F}{m} T = \frac{eE}{m} T = \frac{e}{m}\left(\frac{r}{2}\frac{B}{T}\right)T = \frac{erB}{2m}.$$

The new speeds are

$$v = v_0 \pm \frac{erB}{2m},$$

yielding frequencies ν given by

$$\nu = \frac{v}{2\pi r} = \nu_0 \pm \frac{eB}{4\pi m},$$

$$E = h\nu = h\nu_0 \pm \frac{eh}{4\pi m} B,$$

$$\Delta E = \mu_b B,$$

corresponding to Eq.10-22 with $g = 1$, $\Delta m_j' = 1$.

10-20

(a) For $^1P_1$, $j' = 1$ so there are three levels; the $^1D_2$ has $j' = 2$ giving rise to five levels with the field. For both these states $s' = 0$ so that $g = 1$ and $\Delta E = \mu_b B m_j'$. Hence, the level spacings are the same for each state.

(b) The selection rules are $\Delta m_j' = 0, \pm 1$, allowing zero to zero since $\Delta j' \neq 0$ between these states. The group I of transitions give the same wavelength as when $B = 0$; for II, $\Delta E < \Delta E_{B=0}$ so that $\lambda_{II} > \lambda_0$; in III, $\Delta E > \Delta E_{B=0}$ and therefore $\lambda_{III} < \lambda_0$. Also, with all the level spacings the same, all wavelengths in II are equal, as are all in III. Group I has wavelengths equal to those with $B = 0$ and so are all equal. Hence, three lines appear.

(c) The wavelength of a line is given by

$$\lambda = \frac{hc}{\Delta E}.$$

Considering two transitions whose energies differ by $\Delta^2 E$, the wavelengths of these lines differ by

$$\Delta\lambda = \frac{\partial\lambda}{\partial(\Delta E)}\Delta(\Delta E) = \frac{hc}{(\Delta E)^2}\Delta^2 E.$$

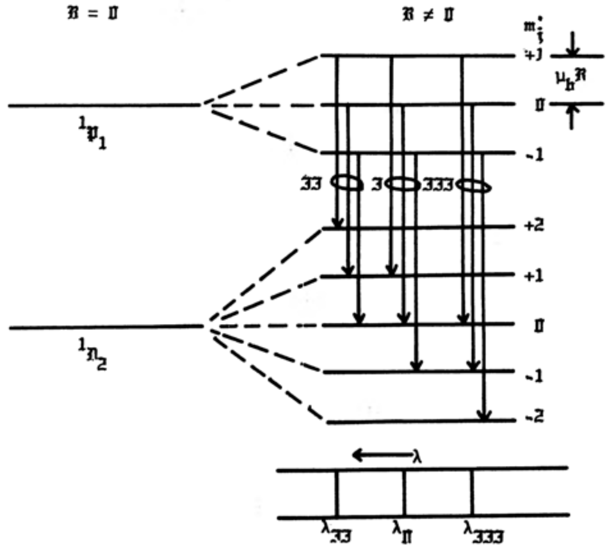
Now clearly,

$$\lambda_{II} - \lambda_0 = \lambda_0 - \lambda_{III},$$

and consequently $\Delta\lambda_{II} = \Delta\lambda_{III} = \Delta\lambda = \lambda_{II} - \lambda_0$. The energy ΔE, corresponding to $\Delta m_j' = 0$, gives wavelength λ_0, which is identical to that for $^1P_1 \rightarrow {}^1D_2$ when $B = 0$. From Fig.10-8, this is

$$\Delta E \simeq 10 - 3.6 = 6.4 \text{ eV} = 10.24 \times 10^{-19} \text{ J}.$$

Since $\Delta m_j' = 1$, $\Delta^2 E = \mu_b B(\Delta m_j') = \mu_b B = (9.27 \times 10^{-24})(0.1) = 9.27 \times 10^{-25}$ J. Therefore



$$\Delta\lambda = \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{(10.24 \times 10^{-19})^2}(9.27 \times 10^{-25}) = 0.000176 \text{ nm}.$$

(d) As given in (c), the energy difference between the degenerate $B = 0$ levels is 6.4 eV. A photon emitted in a transition between levels separated by this energy has a wavelength

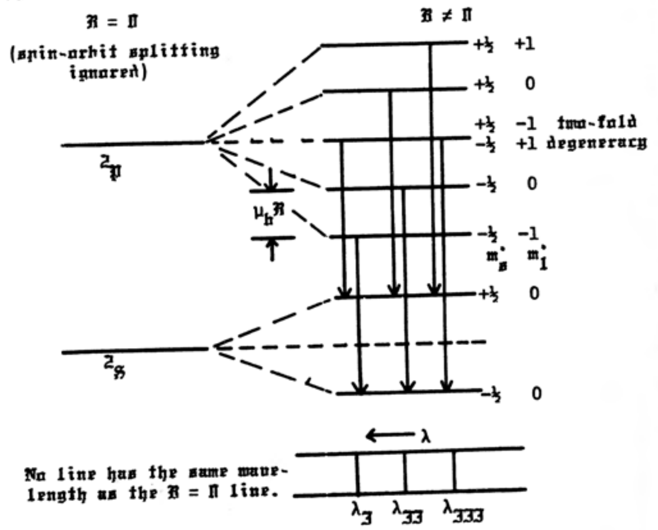
$$\lambda = \frac{1240}{6.4} = 194 \text{ nm}.$$

10-21

(a),(b) From Eq.10-25,

$$\Delta E = \mu_b B(m_\ell' + 2m_s').$$

Now j' is no longer a good quantum number; thus the levels are identified only as ^2P, ^2S.



(i) ^2P: $s' = \frac{1}{2}$, $m_s' = \pm\frac{1}{2}$, $\ell' = 1$, $m_\ell' = 1,0,-1$. Using the relation for ΔE, these give:

m_ℓ'	m_s'	ΔE (units of $\mu_b B$)
1	$+\frac{1}{2}$	+2
1	$-\frac{1}{2}$	0
0	$+\frac{1}{2}$	+1
0	$-\frac{1}{2}$	-1
-1	$+\frac{1}{2}$	0
-1	$-\frac{1}{2}$	-2

(ii) ^2S: Again $m_s' = \pm\frac{1}{2}$, but $\ell' = 0$ and therefore $m_\ell' = 0$ only. $\Delta E = \pm\mu_b B$, and no level exists at the B = 0 position. The selection rules are $\Delta m_s' = 0$, $\Delta m_\ell' = 0, \pm 1$.

10-22

(a) From Fig.10-8, the approximate energies for the $2p^2$ levels are:

$$^3P_{2,1,0}\text{: } -11.3 \text{ eV} \quad (s' = 1,\ \ell' = 1);$$

$$^1D_2\text{: } -10 \text{ eV} \quad (s' = 0,\ \ell' = 2);$$

$$^1S_0\text{: } -8.6 \text{ eV} \quad (s' = 0,\ \ell' = 0).$$

Now the $^1D_2$ and $^1S_0$ levels both have $s' = 0$. Thus their energy difference corresponds to lining up $\vec{\ell}_1$ and $\vec{\ell}_2$ "parallel" in one case and "antiparallel" in the other. This energy difference is 10 - 8.6 = 1.4 eV. In the $^3P_{2,1,0}$ and $^1D_2$ levels, $\vec{\ell}_1$ and $\vec{\ell}_2$ are roughly parallel in one and antiparallel in the other. Hence, the energy difference is due mostly to different spin coupling, or 11.3 - 10 = 1.3 eV. Therefore, the aligning energy, spin or orbital momentum, is about 1.4 eV.

(b) The difference in "parallel-antiparallel" energy gives

$$B = \frac{\Delta E}{2\mu} = \frac{(1.4)(1.6 \times 10^{-19})}{2(9.27 \times 10^{-24})} = 10^4 \text{ T}.$$

(c) The largest laboratory fields are about 100 T.

CHAPTER ELEVEN

11-7

(a) Eq.11-25 is

$$E = \frac{3Rh\nu}{k}\frac{1}{e^{h\nu/kT} - 1}.$$

By definition,

$$c_v = \frac{\partial E}{\partial T} = 3R\frac{e^{h\nu/kT}}{(e^{h\nu/kT} - 1)^2}\left(\frac{h\nu}{kT}\right)^2.$$

(b) Let $x = kT/h\nu$; $T \to 0$ implies $x \to 0$ also. Then, in terms of x

$$c_v = 3R\frac{e^{1/x}}{(e^{1/x} - 1)^2}x^{-2}.$$

As $x \to 0$, $e^{1/x} >> 1$; therefore $c_v \to 3Re^{-1/x}x^{-2}$. Hence, for small x,

$$c_v = 3Re^{-1/x}x^{-2}.$$

Now,

$$x^2e^{1/x} = x^2\left(1 + \frac{1}{x} + \frac{1}{2!x^2} + \frac{1}{3!x^3} + ..\right) = x^2 + x + \frac{1}{2!} + \frac{1}{3!x} + \ldots$$

Hence,

$$\lim_{x \to 0} x^2e^{1/x} = \infty,$$

and

$$\lim_{T \to 0}(c_v) = 0.$$

For small T,

$$c_v = 3Re^{-h\nu/kT}(h\nu/kT)^2.$$

11-8

The Debye specific heat is, with $y = \theta/T$,

$$c_v = 9R\{4y^{-3}\int_0^y\frac{x^3}{e^x - 1}dx - y\frac{1}{e^y - 1}\}.$$

For $y << 1$, the second term becomes

$$\frac{y}{e^y - 1} = \frac{y}{(1 + y + y^2/2! + \ldots) - 1} = \frac{1}{1 + y/2! + \ldots},$$

implying that

$$\lim_{y\to 0}\frac{y}{e^y - 1} = 1.$$

The first term is

$$4y^{-3}\int_0^y\frac{x^3}{e^x - 1}dx.$$

If y is small then, over the range of integration, so is x. Expanding the integrand,

$$4y^{-3}\int_0^y\frac{x^3}{e^x - 1}dx = 4y^{-3}\int_0^y\frac{x^3}{(1 + x + \ldots) - 1}dx = \frac{4}{y^3}\int_0^y x^2dx = \frac{4}{3}.$$

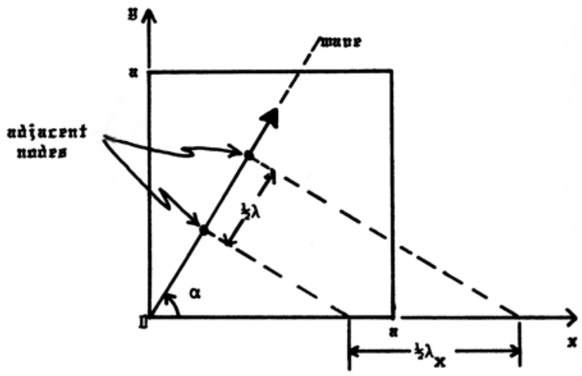
Therefore

$$\lim_{T/\theta>>1}(c_v) = 9R\left(\frac{4}{3} - 1\right) = 3R,$$

the Dulong-Petit result.

11-10

(a) Let the sample be a square of sides a, so that $A = a^2$, and oriented as shown, p.114. If standing waves are set up in the material, then,



$$E(x,t) = E_{0x}\sin(2\pi x/\lambda_x)\sin 2\pi\nu t,$$

$$E(y,t) = E_{0y}\sin(2\pi y/\lambda_y)\sin 2\pi\nu t.$$

There must be nodes at the boundary: hence,

$$2\pi a/\lambda_x = n_x\pi;\quad 2\pi a/\lambda_y = n_y\pi;\quad n_x,n_y = 0,1,2,\ \ldots.$$

If the wave makes an angle α with the x-axis, β with the y-axis, and has wavelength λ, then

$$\lambda_x = \lambda/\cos\alpha;\quad \lambda_y = \lambda/\cos\beta,$$

and therefore

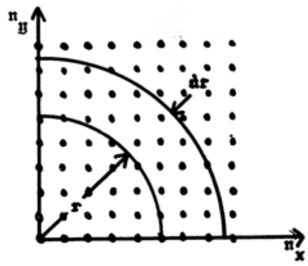
$$n_x^2 + n_y^2 = \left(\frac{2a}{\lambda}\right)^2(\cos^2\alpha + \cos^2\beta) = \left(\frac{2a}{\lambda}\right)^2,$$

$$\frac{2a}{\lambda} = (n_x^2 + n_y^2)^{\frac{1}{2}}.$$

Hence the frequency is

$$\nu = \frac{v}{\lambda} = \frac{v}{2a}(n_x^2 + n_y^2)^{\frac{1}{2}}.$$

Each frequency is represented by a set or sets of points n_x,n_y. If $(n_x^2 + n_y^2)^{\frac{1}{2}} = 2a\nu/v$, then the points n_x,n_y represent ν. It follows that all points on a circle of radius $2a\nu/v$ stand for frequency ν. The density of the points is 1/unit area. Thus with $r = (n_x^2 + n_y^2)^{\frac{1}{2}}$, the number of points between r and r+dr is



$$N(r)dr = \frac{2\pi r}{4}dr = \tfrac{1}{2}\pi r dr,$$

and this is the allowed number of frequencies between ν and $\nu+d\nu$:

$$N(\nu)d\nu = \frac{\pi}{2}\left(\frac{2a}{v}\nu\right)\left(\frac{2a}{v}d\nu\right) = \frac{2a^2\pi}{v^2}\nu d\nu = \frac{2\pi A}{v^2}\nu d\nu.$$

(b) There are still $3N_0$ modes per mole, so that

$$\int_0^{\nu_m} N(\nu)d\nu = 3N_0 = \int_0^{\nu_m}\frac{2\pi A}{v^2}\,\nu d\nu,$$

where A is chosen so that there are N_0 atoms in the sample:

$$3N_0 = \frac{2\pi A}{v^2}(\nu_m^2/2);\quad \nu_m = v(3N_0/\pi A)^{\frac{1}{2}}.$$

The average energy of each oscillator is

$$\bar{E} = \frac{h\nu}{e^{h\nu/kT} - 1},$$

so that the total energy of each mole will be

$$E = \bar{E}N(\nu)d\nu = \int_0^{\nu_m} \frac{h\nu}{e^{h\nu/kT} - 1} \cdot \frac{2\pi A}{v^2}\nu d\nu.$$

If $x = h\nu/kT$,

$$E = \frac{2\pi Ah}{v^2}\left(\frac{kT}{h}\right)^3 \int_0^{x_m} \frac{x^2}{e^x - 1}dx.$$

But

$$A = 3N_0 v^2/\pi\nu_m^2;$$

using this gives

$$E = \frac{6N_0 h}{\nu_m^2}\left(\frac{kT}{h}\right)^3 \int_0^{x_m} \frac{x^2}{e^x - 1}dx.$$

Finally, observe that $x_m = h\nu_m/kT$ is dimensionless; hence, $h\nu_m/k$ has the dimensions of a temperature; let $\nu_m/k = \theta/h$, giving

$$E = 6RT^3/\theta^2 \int_0^{\theta/T} \frac{x^2}{e^x - 1}\, dx; \quad \theta = \frac{h\nu}{k}(3N_0/\pi a)^{\frac{1}{2}}.$$

(c) The specific heat per mole is

$$c_v = \frac{\partial E}{\partial T} = 6R\{3\left(\frac{T}{\theta}\right)^2 \int_0^{\theta/T} \frac{x^2}{e^x - 1}dx - \frac{\theta}{T}\frac{1}{e^{\theta/T} - 1}\}.$$

If $T/\theta \ll 1$, $\theta/T \gg 1$; but

$$\lim_{\theta/T \to \infty} \frac{\theta}{T}\frac{1}{e^{\theta/T} - 1} = 0,$$

and

$$\lim_{\theta/T \to \infty} \int_0^{\theta/T} \frac{x^2}{e^x - 1}dx = \int_0^{\infty} \frac{x^2}{e^x - 1}dx = \text{a finite number.}$$

Thus,

$$\lim_{\theta/T \to \infty} (c_v) \propto T^2.$$

<u>11-11</u>

(a) Since the atoms are distinguishable, use Boltzmann statistics:

$$n_1 = Ae^{-\mathcal{E}_1/kT} \qquad n_1 = A; \quad n_2 = Ae^{-\mathcal{E}/kT},$$

since $\mathcal{E}_1 = 0$, $\mathcal{E}_2 = \mathcal{E}$. With $\mathcal{N}$ = total number of atoms,

$$n_1 + n_2 = \mathcal{N}; \quad \mathcal{N} = A(1 + e^{-\mathcal{E}/kT}); \quad A = \frac{\mathcal{N}}{1 + e^{-\mathcal{E}/kT}}.$$

The total energy is

$$E = n_1\mathcal{E}_1 + n_2\mathcal{E}_2 = n_2\mathcal{E} = Ae^{-\mathcal{E}/kT}\mathcal{E} = \frac{\mathcal{N}\mathcal{E}e^{-\mathcal{E}/kT}}{1 + e^{-\mathcal{E}/kT}}$$

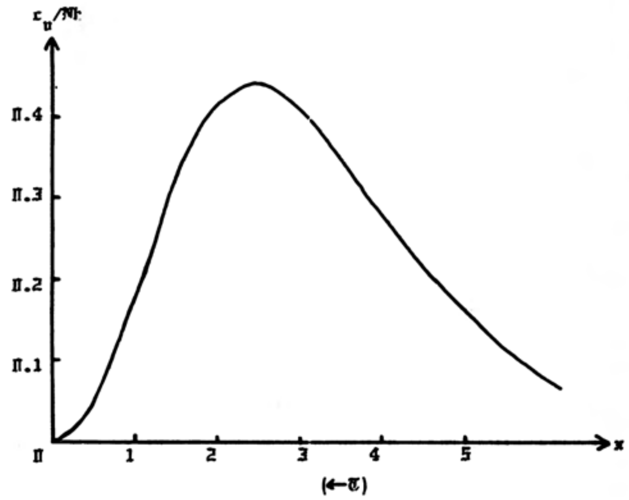
(b) As usual, the specific heat is

$$c_v = \frac{\partial E}{\partial T} = \frac{\mathcal{N}k\left(\frac{\mathcal{E}}{kT}\right)^2 e^{-\mathcal{E}/kT}}{(1 + e^{-\mathcal{E}/kT})^2}.$$

(c) Let $x = \mathcal{E}/kT$; then,

$$c_v/\mathcal{N}k = \frac{x^2 e^{-x}}{(1 + e^{-x})^2}.$$

For $x \ll 1$, T large, $e^{-x} \simeq 1$ and $c_v/\mathcal{N}k = \frac{1}{4}x^2$. On the other hand, with $x \gg 1$, T small, $e^{-x} \simeq 0$, and $c_v/\mathcal{N}k = x^2e^{-x}$. A sketch of the specific heat vs. T is on the next page.



11-15

The population of the levels is

$$n_1 = Ae^{-\mathcal{E}_1/kT}, \quad n_2 = Ae^{-\mathcal{E}_2/kT},$$

A = constant. Hence, the fractional difference is

$$\Delta f = \frac{\Delta n}{N} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{n_1/n_2 - 1}{n_1/n_2 + 1} = \frac{e^{\Delta\mathcal{E}/kT} - 1}{e^{\Delta\mathcal{E}/kT} + 1},$$

with $\Delta\mathcal{E} = \mathcal{E}_2 - \mathcal{E}_1$. Now, for $\vec{\mu}$ aligned parallel or antiparallel to the field, $\Delta\mathcal{E} = 2\mu B$:

$$\Delta\mathcal{E} = 2(1.4 \times 10^{-26})(1) = 2.8 \times 10^{-26}\ \mathrm{J}.$$

(a) $kT = (1.38 \times 10^{-23})(300) = 4.14 \times 10^{-21}$ J. Hence,

$$\frac{\Delta\mathcal{E}}{kT} = 6.76 \times 10^{-6}.$$

Then, to a good approximation,

$$\frac{\Delta n}{N} = \tfrac{1}{2}\frac{\Delta\mathcal{E}}{kT} = 3.4 \times 10^{-6}.$$

(b) $T = 4$ K, $kT = 5.52 \times 10^{-23}$ J. This gives $\Delta\mathcal{E}/kT = 5.07 \times 10^{-4}$, and since this is so small, the same approximation as in (a) applies here also; therefore, $\Delta n/N = 0.00025$.

11-17

(a) At $T = 300$ K, $kT = 2.585 \times 10^{-2}$ eV. For hydrogen,

$$\Delta E = (13.6)\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = 10.2\ \mathrm{eV},$$

giving $\Delta E/kT = 395$. Thus, ignoring the degeneracy of the states,

$$n_2/n_1 = e^{-\Delta E/kT} = e^{-395} = 10^{-170}.$$

(b) The degeneracy of the levels is $2n^2$; therefore,

$$n_2/n_1 = 0.01 = \frac{8}{2}e^{-\Delta E/kT},$$

$$e^{-\Delta E/kT} = 0.0025.$$

Therefore,

$$\Delta E/kT = -\ln(0.0025) = 6; \quad T = \Delta E/6k; \quad T = 20{,}000\ \mathrm{K},$$

since $\Delta E = 10.2$ eV.

11-19

The desired ratio is

$$R = \frac{\text{probability of spontaneous emission}}{\text{probability of stimulated emission}} = \frac{A_{21}}{B_{21}\rho(\nu)}.$$

But,

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}; \quad \rho(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1};$$

therefore

$$R = e^{h\nu/kT} - 1.$$

At room temperature, $kT \simeq 1/40$ eV.

(a) X-ray region: $\lambda \simeq 0.1$ nm; $\nu \simeq 3 \times 10^{18}$ Hz, $E = h\nu \simeq 12.4$ keV. Thus, $h\nu >> kT$, $R \simeq e^{h\nu/kT} \simeq \infty$.

(b) Visible: $\lambda \simeq 600$ nm, $\nu = 5 \times 10^{14}$ Hz, $E \simeq 2$ eV, giving $h\nu/kT \simeq 80$, or $R \simeq \infty$, once again.

(c) Microwave: $\lambda \simeq 1$ cm, $\nu = 3 \times 10^{10}$ Hz, $E = 12 \times 10^{-5}$ eV; $e^{h\nu/kT} \simeq e^{0.005} = 1.005$, so that $R = 0.005$.

11-20

(a) Assuming no degeneracy,

$$n_2/n_1 = e^{-\Delta E/kT}.$$

Since

$$\Delta E = \frac{12400}{5800} = 2.138 \text{ eV}; \quad kT = 0.0259 \text{ eV},$$

$$n_2 = (4 \times 10^{20})e^{-2.138/0.0259} = 5.65 \times 10^{-14} \text{ atoms},$$

i.e., none.

(b) A laser pulse operates until the population inversion is destroyed; hence

$$E = (1.5 \times 10^{20})(2.138)(1.6 \times 10^{-19}) = 51.4 \text{ J}.$$

11-21

For a beam of radiation, applicable to a laser (stimulated emission),

$$\rho = I/c = \frac{4}{2.998 \times 10^8} = 1.334 \times 10^{-8} \text{ J/m}^3\cdot\text{s}.$$

The transition rate is

$$R = nB\rho = (3 \times 10^{18})(3.2 \times 10^5)(1.334 \times 10^{-8}),$$

$$R = 1.28 \times 10^{16} \text{ s}^{-1}.$$

11-23

Since
$$N(\mathcal{E})d\mathcal{E} = \frac{4\pi V}{h^3}(2m^3)^{1/2}\mathcal{E}^{1/2}d\mathcal{E},$$

the total number $\mathcal{N}$ of particles is

$$\mathcal{N} = \int_0^\infty \frac{N(\mathcal{E})}{e^\alpha e^{\mathcal{E}/kT} - 1}d\mathcal{E} = \frac{4\pi V}{h^3}(2m^3)^{1/2}\int_0^\infty \frac{\mathcal{E}^{1/2}}{e^\alpha e^{\mathcal{E}/kT} - 1}d\mathcal{E}.$$

Let $x = \mathcal{E}/kT$; then,

$$\mathcal{N} = \frac{2\pi V}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{x^{1/2}}{e^\alpha e^x - 1}dx.$$

Since $\alpha > 0$,

$$\frac{1}{e^\alpha e^x - 1} = \frac{1}{e^{\alpha+x}(1 - e^{-\alpha-x})} = \frac{e^{-\alpha}}{e^x}(1 + e^{-\alpha-x} + e^{-2\alpha-2x} + ..),$$

$$\frac{1}{e^\alpha e^x - 1} = e^{-\alpha}(e^{-x} + e^{-\alpha-2x} + e^{-2\alpha-3x} + \ldots).$$

Hence,

$$\mathcal{N} = \frac{2\pi V}{h^3}(2mkT)^{3/2}A\int_0^\infty x^{1/2}(e^{-x} + Ae^{-2x} + A^2e^{-3x} + \ldots)dx,$$

with $A = e^{-\alpha}$. Now,

$$\int_0^\infty x^{1/2}e^{-nx}dx = n^{-3/2}\int_0^\infty x^{1/2}e^{-x}dx = n^{-3/2}\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}n^{-3/2}\pi^{1/2}.$$

Therefore,

$$\mathcal{N} = \frac{2\pi V}{h^3}(2mkT)^{3/2}A\pi^{1/2}\tfrac{1}{2}(1 + 2^{-3/2}A + 3^{-3/2}A^2 + \ldots),$$

$$\mathcal{N} = \frac{(2\pi mkT)^{3/2}V}{h^3}A(1 + 2^{-3/2}A + 3^{-3/2}A^2 + \ldots).$$

11-24

The energy of a Bose system is, with $x = \varepsilon/kT$,

$$E = \int_0^\infty \varepsilon n(\varepsilon) N(\varepsilon) d\varepsilon = \frac{4\pi V}{h^3}(2m^3)^{\frac{1}{2}}(kT)^{5/2}\int_0^\infty \frac{x^{3/2}}{e^\alpha e^x - 1}dx.$$

As in Problem 11-23, put

$$\frac{1}{e^\alpha e^x - 1} = e^{-\alpha}(e^{-x} + e^{-\alpha-2x} + e^{-2\alpha-3x} + \ldots);$$

with $A = e^{-\alpha}$, the energy becomes

$$E = \frac{4\pi V}{h^3}(2m^3)^{\frac{1}{2}}(kT)^{5/2}A\int_0^\infty x^{3/2}(e^{-x} + Ae^{-2x} + A^2e^{-3x} + \ldots)dx.$$

But,

$$\int_0^\infty x^{3/2}e^{-nx}dx = n^{-5/2}\int_0^\infty x^{3/2}e^{-x}dx = n^{-5/2}\Gamma(\tfrac{5}{2}) = n^{-5/2}\frac{3\pi^{\frac{1}{2}}}{4}.$$

Therefore,

$$E = \frac{(2\pi mkT)^{3/2}}{h^3}V(\tfrac{3}{2}kT)A(1 + 2^{-5/2}A + 3^{-5/2}A^2 + \ldots).$$

11-25

The average energy per particle in a Fermi gas is, approximately,

$$\overline{E} = \tfrac{3}{2}kT\{1 + 2^{-5/2}(\tfrac{N}{V})\frac{h^3}{(2\pi mkT)^{3/2}}\}.$$

Since the average energy in a classical gas is $3kT/2$, quantum degeneracy occurs when the second term in the above is not negligible compared to the first; i.e., certainly when

$$\frac{N}{V}\frac{h^3}{(2\pi mkT)^{3/2}} > 1.$$

Now if $kT \ll \varepsilon_F$, then

$$\varepsilon_F = \frac{h^2}{8m}(3N/\pi V)^{2/3} \rightarrow \frac{N}{V} = \frac{\pi}{3}(8m\varepsilon_F/h^2)^{3/2}.$$

Substituting this into the inequality above yields

$$\tfrac{8}{3}\pi^{-\frac{1}{2}}(\varepsilon_F/kT)^{3/2} > 1,$$

which clearly is satisfied. Thus, when $\varepsilon_F \gg kT$, quantum effects must be considered.

11-26

At $T = 0$ K,

$$n(\varepsilon) = \begin{cases} 1, & 0 \le \varepsilon \le \varepsilon_F, \\ 0, & \varepsilon > \varepsilon_F. \end{cases}$$

Hence, the total number of particles is

$$N = \int_0^\infty n(\varepsilon)N(\varepsilon)d\varepsilon = \int_0^{\varepsilon_F} N(\varepsilon)d\varepsilon = \frac{8\pi V(2m^3)^{\frac{1}{2}}}{h^3}\int_0^{\varepsilon_F}\sqrt{\varepsilon}d\varepsilon = \frac{16\pi V(2m^3)^{\frac{1}{2}}}{3h^3}\varepsilon_F^{3/2}.$$

Thus, the average energy per fermion is

$$\overline{E} = \frac{1}{N}\int_0^\infty \varepsilon n(\varepsilon)N(\varepsilon)d\varepsilon = \frac{1}{N}\int_0^{\varepsilon_F}\varepsilon N(\varepsilon)d\varepsilon,$$

$$\overline{E} = \frac{1}{N}\frac{8\pi V(2m^3)^{\frac{1}{2}}}{h^3}\int_0^{\varepsilon_F}\varepsilon^{3/2}d\varepsilon = \frac{1}{N}\frac{16}{5}\frac{\pi V(2m^3)^{\frac{1}{2}}}{h^3}\varepsilon_F^{3/2} = 3\varepsilon_F/5,$$

using the expression for N found first.

<u>11-28</u>

The depth of the well is $V_0 = w_0 + \mathcal{E}_F = 4.8\ \text{eV} + \mathcal{E}_F$. Hence, it is only necessary to compute the Fermi energy. If $kT \ll \mathcal{E}_F$, then,

$$\mathcal{E}_F = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}.$$

The electron density is

$$n = \frac{(19.3 \times 10^6)(6.02 \times 10^{23})}{197} = 5.9 \times 10^{28}\ \text{m}^{-3}.$$

Using this and $m = 9.11 \times 10^{-31}$ kg gives

$$\mathcal{E}_F = 5.53\ \text{eV};\quad V_0 = 4.8 + 5.53 = 10.3\ \text{eV}.$$

<u>11-29</u>

It is given that

$$N(\mathcal{E}) = \frac{\ell}{h}(2m/\mathcal{E})^{1/2};\quad n(\mathcal{E}) = \begin{cases} 2, & \mathcal{E} \le \mathcal{E}_F, \\ 0, & \mathcal{E} > \mathcal{E}_F. \end{cases}$$

(a) The number of particles is

$$N = \int_0^\infty n(\mathcal{E})N(\mathcal{E})\,d\mathcal{E} = \int_0^{\mathcal{E}_F} (2)\frac{\ell}{h}(2m/\mathcal{E})^{1/2}\,d\mathcal{E} = 4(2m)^{1/2}\,\frac{\ell\,\mathcal{E}_F^{1/2}}{h},$$

so that

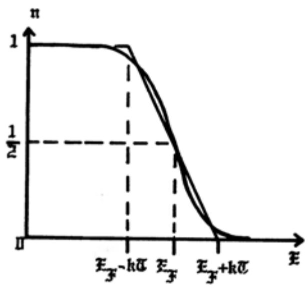
$$\mathcal{E}_F = N^2h^2/32m\ell^2.$$

(b) The average energy is

$$\overline{E} = \frac{1}{N}\int_0^\infty \mathcal{E}\,n(\mathcal{E})N(\mathcal{E})\,d\mathcal{E} = \frac{1}{N}\int_0^{\mathcal{E}_F} \mathcal{E}(2)\frac{\ell}{h}(2m/\mathcal{E})^{1/2}\,d\mathcal{E} = \frac{4}{3}\,\frac{\ell(2m)^{1/2}}{hN}\,\mathcal{E}_F^{3/2}.$$

Using the expression for N found in (a) gives $\overline{E} = \mathcal{E}_F/3$.

<u>11-30</u>



For silver, $\mathcal{E}_F = 5.5$ eV at T = 300 K (T << 10^5 K, above which the material is classical in behavior). Since $kT \simeq 1/40$ eV, $kT \ll \mathcal{E}_F$, and the Fermi distribution is close to the T = 0 K distribution. Approximate the distribution by n = 1 for energies less than $\mathcal{E}_F - kT$, n = 0 for energies greater than $\mathcal{E}_F + kT$, and in the transition region the straight line

$$n = -\frac{\mathcal{E}}{2kT} + \frac{\mathcal{E}_F + kT}{2kT};\quad \mathcal{E}_F - kT \le \mathcal{E} \le \mathcal{E}_F + kT.$$

(Any reasonable approximation will yield the same final result.) The number # of particles with energies greater than the Fermi energy is

$$\#(\mathcal{E}>\mathcal{E}_F) = \int_{\mathcal{E}_F}^\infty N(\mathcal{E})\,n(\mathcal{E})\,d\mathcal{E} = \frac{8\pi V}{h^3}(2m^3)^{1/2}\int_{\mathcal{E}_F}^{\mathcal{E}_F+kT} \sqrt{\mathcal{E}}\left(-\frac{\mathcal{E}}{2kT} + \frac{\mathcal{E}_F+kT}{2kT}\right)d\mathcal{E},$$

$$\# = \frac{4\pi V(2m^3)^{1/2}}{h^3kT}\,\frac{4}{15}\,\mathcal{E}_F^{5/2}\left\{\left(1 + \frac{kT}{\mathcal{E}_F}\right)^{5/2} - \left(1 + \frac{5}{2}\frac{kT}{\mathcal{E}_F}\right)\right\},$$

$$\# \simeq 2\pi V(2m^3)^{1/2}kT\mathcal{E}_F^{1/2}/h^3.$$

For the total number of particles, the value calculated at T = 0 K may be used: see Problem 11-26. Hence

$$\frac{\#}{N} = \frac{3}{8}(kT/\mathcal{E}_F) = \frac{3}{8}\left(\frac{1/40\ \text{eV}}{5.5\ \text{eV}}\right) = 0.0017 \simeq 10^{-3}.$$

CHAPTER TWELVE

12-1

The potential energy of K^+ and Cl^- separated by distance r is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_K q_{Cl}}{r} = \frac{1}{4\pi\varepsilon_0}\frac{(+e)(-e)}{r} = -\frac{(9\times 10^9)(1.6\times 10^{-19})^2}{r\ (\text{in meters})},$$

$$V = \frac{(9\times 10^9)(1.6\times 10^{-19})}{r\ (\text{in nm})\times 10^{-9}}\ \text{eV} = \frac{1.44}{r}\ \text{eV}\quad (r\ \text{in nm}).$$

The required dissociation energy is the negative of the energy required to assemble a KCl molecule from neutral atoms of K and Cl, initially an infinite distance apart. This latter process involves

(i) removing an electron from the K atom; energy needed = 4.34 eV;

(ii) attaching the electron to the Cl atom; energy required = -3.82 eV (i.e., energy is released);

(iii) moving the newly created ions from infinity to their equilibrium positions at a separation of 0.279 nm; this requires

$$-\frac{1.44}{0.279} = -5.161\ \text{eV}$$

of energy (energy is liberated).

Hence, the energy needed to form the molecule is 4.34 - 3.82 - 5.16 = -4.64 eV. Thus, to dissociate a KCl molecule requires +4.64 eV of energy.

12-2

A bound KBr molecule must have negative total energy. Hence, the possible separation distances are bounded by that for which $E_T = 0$. By Problem 12-1, the total energy of a KBr molecule is

$$E_T = \left(\frac{1.44}{R_0} + 3.5 - 4.3\right)\ \text{eV},$$

R_0 in nm. Hence, $R_{0,max} = R^*$ is given by

$$\frac{1.44}{R^*} - 0.8 = 0;\quad R^* = 1.8\ \text{nm}.$$

12-5

(a) If $a = \hbar^2/2IkT$, then

$$n_r = n_0(2r+1)e^{-a(r+r^2)}.$$

At the desired level,

$$dn_r/dr = n_0 e^{-a(r+r^2)}\{2 - a(2r+1)^2\} = 0,$$

$$r = \left(\frac{1}{2a}\right)^{1/2} - \tfrac{1}{2} = \left(\frac{IkT}{\hbar^2}\right)^{1/2} - \tfrac{1}{2}.$$

(b) From p.426, for HCl, $I = 2.66\times 10^{-47}\ \text{kg}\cdot\text{m}^2$; also,

$$kT = \frac{(1.381\times 10^{-23})(600)}{1.602\times 10^{-19}} = 0.05172\ \text{eV};$$

$$\frac{\hbar^2}{I} = \frac{(1.055\times 10^{-34})^2}{(2.66\times 10^{-47})(1.602\times 10^{-19})} = 0.00261\ \text{eV}.$$

Therefore, from (a),

$$r = \left(\frac{0.05172}{0.00261}\right)^{1/2} - \tfrac{1}{2} = 3.95 \rightarrow r = 4.$$

12-6

The rotational energies are

$$E_r = r(r+1)\frac{\hbar^2}{2I}.$$

From Table 12-1, p.429, for H_2, $\hbar^2/2I = 7.56\times 10^{-3}$ eV. Thus,

$$\Delta E = E_1 - E_0 = 2\frac{\hbar^2}{2I} - 0 = 0.015 \text{ eV}.$$

At temperature T, the average translational kinetic energy is $3kT/2$. Since $k = 8.617 \times 10^{-5}$ eV/K, these energies are equal at T = 117 K.
Therefore, at room temperature $T \simeq 300$ K, some molecules will be in excited rotational states, as the moving molecules have energies of translation sufficient to excite rotation upon collision.

12-9

The highest rotational level that can fit into $\Delta E_{vib} = 0.04$ eV is given by

$$r'(r' + 1)\frac{\hbar^2}{2I} = \Delta E_{vib},$$

$$r'^2(2.36 \times 10^{-5}) \simeq 0.04,$$

$$r' = 40.$$

12-10

(a) The internuclear distance R_0 is given by $\partial V/\partial R = 0$; hence, $R_{01} > R_{02}$.

(b) $I = \mu R_0^2$; with $\mu_1 = \mu_2$, $I_1 > I_2$.

(c) $E_r = r(r + 1)\hbar^2/2I$; since $I_1 > I_2$, $\Delta E_{r2} > \Delta E_{r1}$.

(d) The energy $E = 0$ separates the bound and unbound states. Since curve 2 becomes more negative, $E_{B2} > E_{B1}$.

(e) The zero-point energy is

$$E_0 = \frac{h}{4\pi\sqrt{\mu}}\{(\partial^2V/\partial R^2)_{R_0}\}^{\frac{1}{2}}.$$

But $\mu_2 = \mu_1$; also, classically, $\partial^2V/\partial R^2|_{R_0} = k$, where $V = \frac{1}{2}kx^2$. A larger k yields a sharper curve of V vs. x. Hence, $k_2 > k_1$ and therefore $E_{02} > E_{01}$.

(f) The vibrational energy is $E_v = (v + \frac{1}{2})h\nu_0$, so that

$$\Delta E_v = h\nu_0 = 2E_0.$$

Thus, from (e), $\Delta E_{v2} > \Delta E_{v1}$.

12-11

(a) The vibrational states are not degenerate, so that

$$n_1/n_0 = e^{-(E_1-E_0)/kT},$$

with

$$E_v = (v + \tfrac{1}{2})h\nu_0.$$

Therefore,

$$E_1 - E_0 = h\nu_0 = 5.9 \times 10^{-20} \text{ J},$$

by Example 12-3(c). For T = 1000 K, $kT = 1.381 \times 10^{-20}$ J, giving $(E_1-E_0)/kT = 4.27$ and

$$n_1/n_0 = e^{-4.27} = 0.014.$$

(b) Let n_0 = number of molecules in both the ground rotational and ground vibrational states. For rotation,

$$n_r/n_0 = (2r + 1)e^{-(E_r-E_0)/kT}.$$

For r = 1,

$$n_1/n_0 = 3e^{-E_1/kT},$$

since $E_0 = 0$. But

$$E_r = r(r + 1)\frac{\hbar^2}{2I}; \quad E_1 = 2\frac{\hbar^2}{2I} = 2.64 \times 10^{-3} \text{ eV}.$$

Also, with T = 1000 K, $kT = 8.617 \times 10^{-2}$ eV giving $n_1/n_0 = 3e^{-0.0306} = 2.91$. Thus,

$$n_{1r}/n_{1v} = \frac{n_{1r}/n_0}{n_{1v}/n_0} = \frac{2.91}{0.014} = 210.$$

12-14

Let $x = h\nu_0/kT$. Then,

$$\frac{n_1}{n_0} = e^{-x};\ \frac{n_2}{n_0} = e^{-2x};\ \text{etc.}$$

Therefore,

$$R = \frac{n_1 + n_2 + \ldots}{n_0} = e^{-x} + e^{-2x} + e^{-3x} + \ldots$$

$$R = e^{-x}(1 + e^{-x} + e^{-2x} + \ldots),$$

$$R = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^{x} - 1} = \frac{1}{e^{h\nu_0/kT} - 1}.$$

12-15

$$E = \tfrac{1}{2}CA^2 = \tfrac{3}{2}h\nu_0.$$

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{C}{\mu}} = \frac{1}{2\pi}\left(\frac{C}{35m_H/36}\right)^{\frac{1}{2}} = \frac{3}{\pi}\left\{\frac{470}{(35)(1.673 \times 10^{-27})}\right\}^{\frac{1}{2}},$$

$$\nu_0 = 8.555 \times 10^{13}\ \text{Hz}.$$

Therefore,

$$CA^2 = 3h\nu_0,$$

$$(470)A^2 = 3(6.626 \times 10^{-34})(8.555 \times 10^{13}),$$

$$A = 0.0190\ \text{nm}.$$

12-17

The classical vibration frequency ($\simeq \sqrt{k/m}$) is proportional to $\mu^{-\frac{1}{2}}$ and therefore,

$$\Delta\nu_0/\nu_0 = -\tfrac{1}{2}\,\Delta\mu/\mu,$$

ν_0 = classical frequency. For $Cl_2^{37,35}$ the reduced mass is, in u,

$$\mu_{37,35} = \frac{37 \cdot 35}{37 + 35} = 17.986.$$

For $Cl_2^{35,35}$, $\mu_{35,35} = \frac{1}{2}(35) = 17.500$ u. Putting these numbers into the first equation above,

$$\Delta\nu_0/\nu_0 = \frac{\nu_{35,35} - \nu_{37,35}}{\nu_{35,35}} = -\tfrac{1}{2}\,\frac{\mu_{35,35} - \mu_{37,35}}{\mu_{35,35}},$$

$$\frac{2940.8 - \nu_{37,35}}{2940.8} = -\tfrac{1}{2}\,\frac{17.5 - 17.986}{17.5},$$

$$\nu_{37,35} = 2900\ \text{cm}^{-1} \rightarrow \Delta\nu = 40\ \text{cm}^{-1}.$$

12-30

The vibrational levels are separated by $h\nu_0$ in energy. But $h\nu_0 \propto 1/\sqrt{\mu}$, so that

$$h\nu_0(H_2) = C\sqrt{2} = 2(0.265),$$

$$C = 0.3748.$$

Using this,

$$\Delta E_{D_2} = (0.3748)/\sqrt{1} = 0.375\ \text{eV}.$$

For HD the reduced mass $\mu = (1)(2)/(1 + 2) = 2/3$, so that

$$\Delta E_{HD} = (0.3748)/\sqrt{(2/3)} = 0.460\ \text{eV}.$$

12-21

For air to be opaque to visible light, visible light photons must be energetic enough to excite the oxygen or nitrogen molecules; i.e., carry over 3 eV of energy. But in the visible region of the spectrum, $\lambda > 400$ nm, this shortest wavelength characterizing the most energetic visible photons. Their energy is

$$E_{ph} = \frac{hc}{\lambda} = \frac{1240}{400} = 3.1 \text{ eV},$$

barely sufficient (but close). All other visible light photons are less energetic.

12-23

For the laser,

$$\lambda = 694.3 \text{ nm}; \quad \nu = 4.321 \times 10^{14} \text{ Hz},$$

using $c = 3 \times 10^8$ m/s.

(a) $E_r = r(r + 1)\hbar^2/2I$. Now $\Delta r = 2$; $E_2 = 6\hbar^2/2I = 14.88 \times 10^{-4}$ eV. $E_0 = 0$, so $h\nu' = E_2 - E_0 = E_2$ giving $\nu' = 3.593 \times 10^{11}$ Hz. The Raman lines have frequency $\nu - \nu' = 4.321 \times 10^{14} - 4 \times 10^{11} = 0.4317 \times 10^{15}$ Hz, or $\lambda_{20} = 694.9$ nm.

If there are enough molecules in the $r = 1$ state, the $r = 1$ to $r = 3$ transition can be observed. But $E_3 - E_1 = 10\hbar^2/2I = 24.8 \times 10^{-4}$ eV, giving $\nu' = 5.99 \times 10^{11}$ Hz. Again, the line frequency is $\nu - \nu' = (0.4321 - 0.0006) \times 10^{15} = 0.4315 \times 10^{15}$ Hz, indicating that $\lambda_{31} = 695.2$ nm.

(b) The intensity ratio equals the level population ratio. Since the λ_{20} line originates at $r = 0$ and λ_{31} with $r = 1$,

$$n_r/n_0 = (2r + 1)e^{-E_r/kT},$$

$$n_1/n_0 = I_{31}/I_{20} = 3e^{-E_1/kT}.$$

With $E_1 = 2(2.48 \times 10^{-4} \text{ eV})$, $kT = 1/40$ eV, this intensity ratio is 2.94.

(c) For vibrations, $\Delta E = h\nu_0$, giving $\nu' = \nu_0$. For N_2, ν_0 = "2360 cm^{-1}" which translates into $\nu' = 7.08 \times 10^{13}$ Hz. Hence, the line frequencies are

$$\left.\begin{matrix} \nu_{10} = \nu + \nu' \\ \nu_{01} = \nu - \nu' \end{matrix}\right\} = 0.4321 \times 10^{15} \text{ Hz} \pm 0.0708 \times 10^{15} \text{ Hz},$$

$$\left.\begin{matrix} \nu_{10} = 0.5029 \\ \nu_{01} = 0.3613 \end{matrix}\right\} \times 10^{15} \text{ Hz}; \quad \begin{matrix} \lambda_{10} = 596.5 \text{ nm} \\ \lambda_{01} = 830.3 \text{ nm}. \end{matrix}$$

(d) As in (b),

$$I_{10}/I_{01} = n_1/n_0 = e^{-(E_1-E_0)/kT} = e^{-h\nu_0/kT}.$$

Since $\nu_0 = 7.08 \times 10^{13}$ Hz, $h\nu_0 = 0.293$ eV. At room temperature $kT = 1/40$ eV, and these numbers give

$$I_{10}/I_{01} = e^{-(0.293)(40)} = 8.1 \times 10^{-6}.$$

12-24

The energy of the rotational levels, above $r = 0$, are given by

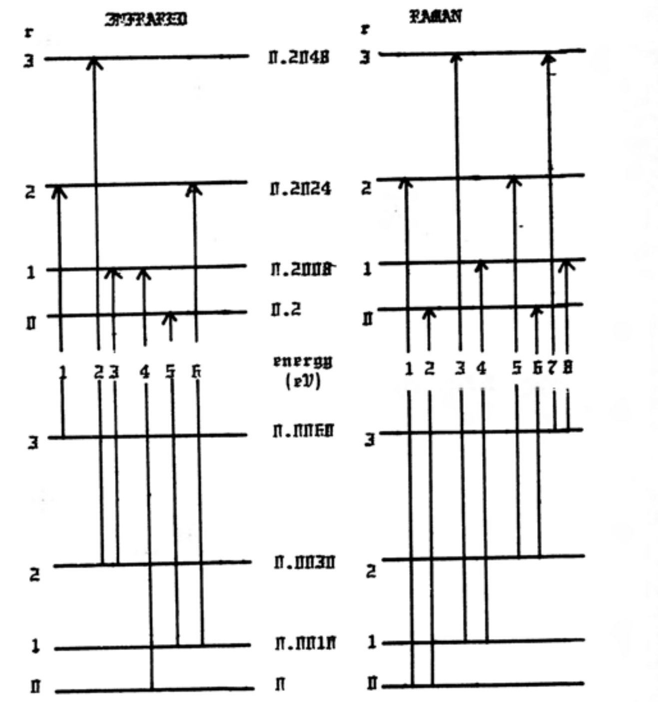
$$E = r(r + 1)\hbar^2/2I.$$

For $r = 1$:

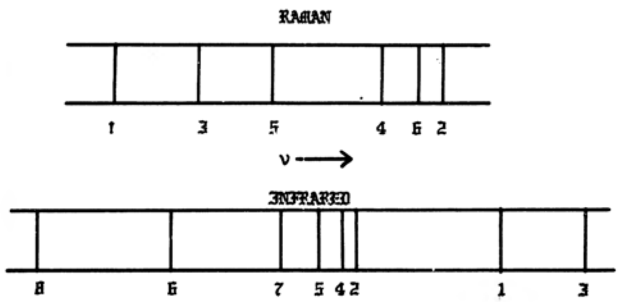
$$E_1 = \hbar^2/I \rightarrow \begin{cases} \hbar^2/2I = 0.0005 \text{ eV} & (v'' = 0), \\ \hbar^2/2I = 0.0004 \text{ eV} & (v' = 1). \end{cases}$$

Thus the energies of all the levels can now be assigned, and are shown on the diagram.

For rotation-vibration transitions, the selection rules are $\Delta r = \pm 1$, $\Delta v = \pm 1$ (rotation transitions with $\Delta v = 0$ are in the far infrared and are not considered here). Assuming that only the $v = 0$ band is occupied, the allowed transitions are shown. The photon energies are the difference between the energies of the two appropriate states.



These infrared transitions are not seen in molecules with identical nuclei. These molecules display only the Raman spectrum. The Raman spectrum is seen in non-identical nuclei molecules also. For these transitions the selection rules are $\Delta v = \pm 1$, $\Delta r = 0, \pm 2$. The resulting transitions are shown above. If the nuclei have identical spin each equal to zero, then one set of the Raman lines will be missing: i.e., those that originate either on even-r or odd-r levels.



CHAPTER THIRTEEN

13-1

(a) Metallic: charge density uniform, like an electron gas.

(b) Ionic: alternating positive and negative distribution.

(c) Molecular: molecules retain identities, charge zero between them.

(d) Covalent: electrons shared, highest charge density between molecules.

13-4

From the text, construct the following table.

Type of Solid	Transparent or opaque in visible?	Melting Point	Malleability	Elect. & Th. Conductivity
molecular		low	soft	poor
ionic	transparent	high	hard	poor
covalent*	most opaque	high	hard	fair
metallic	reflective	high	hard	excellent

(*) Properties vary, depending on bond energy.

Also, by Problem 13-8, for a metal the resistivity increases linearly with T near room temperature. From p.497, the conductivity of a semiconductor increases with T.

Use of these considerations gives the following results:

(a) metal;
(b) covalent, since the conductivity increases with T;
(c) covalent semiconductor;
(d) ionic (if covalent, conductivity may increase with T);
(e) molecular, by virtue of low melting point.

13-5

(a) $V = -\vec{p}'\cdot\vec{F} = -(\alpha\vec{E})\cdot\vec{F}$. Since $\hat{r}\cdot\vec{p} = rp\cos\theta$,

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\vec{p}}{r^3} - 3\frac{p\cos\theta}{r^4}\,\hat{r}\right).$$

Hence,

$$\vec{E}\cdot\vec{E} = \left(\frac{1}{4\pi\epsilon_0}\right)^2\left\{\frac{1}{r^6}\,\vec{p}\cdot\vec{p} + \left(\frac{3p\cos\theta}{r^4}\right)^2\vec{r}\cdot\vec{r} - \frac{6p\cos\theta}{r^4}\,\frac{1}{r^3}\,\vec{p}\cdot\vec{r}\right\},$$

$$\vec{E}\cdot\vec{E} = \left(\frac{1}{4\pi\epsilon_0}\right)^2(1 + 9\cos^2\theta - 6\cos^2\theta)\frac{p^2}{r^6};$$

$$V = -\frac{\alpha}{(4\pi\epsilon_0)^2}(1 + 3\cos^2\theta)\frac{p^2}{r^6}.$$

(b) The force is derived from the potential energy.

$$\vec{F} = -\vec{\nabla}V = -\frac{\partial V}{\partial r}\,\hat{r} = -\frac{1}{r}\frac{\partial V}{\partial r}\,\hat{u}_\theta,$$

$$\vec{F} = -\frac{\alpha}{(4\pi\epsilon_0)^2}\,\frac{p^2}{r^7}\{(1 + 3\cos^2\theta)\hat{r} + 3\sin 2\theta\,\hat{u}_\theta\} \propto r^{-7};$$

since $\alpha > 0$, $F_r < 0$, indicating an attractive force.

13-6

From Fig.12-1, $E_b = 5.1$ eV, $r = 0.24$ nm. Since

$$E_b = eEr,$$

$$5.1 = (1)E(0.24 \times 10^{-9}),$$

$$E = 2.1 \times 10^{10}\ \text{N/C}.$$

13-8

(a) By Eq.13-1(a),

$$\rho = \frac{m\bar{v}}{ne^2\lambda}.$$

By analogy with the classical Boltzmann gas,

$$\tfrac{1}{2}m\overline{v^2} = \tfrac{3}{2}kT;\quad \bar{v} \propto \sqrt{T}.$$

Due to lattice vibrations, each ion has a cross section that is proportional to πA^2, A the amplitude of oscillation. The electron can be treated as a point particle. Thus the collision cross section also is proportional to πA^2. In time t the electron collides with $\pi A^2(\bar{v}t)n$ ions, n = ion number density. The distance travelled in this time is $\bar{v}t$ and therefore

$$(\pi A^2 n\bar{v}t)\lambda = \bar{v}t;\quad \lambda^{-1} = n\pi A^2.$$

But

$$kA^2 \propto kT;\ A^2 \propto T;\quad \lambda^{-1} \propto T,$$

so that

$$\rho \propto \bar{v}\lambda^{-1} \propto T^{\frac{1}{2}}T = T^{3/2}.$$

(b) For $kT \ll E_F$, the Fermi function changes slowly with T, which indicates that $\bar{v}$ is independent of T. Thus $\rho \propto T$ in this event.

13-9

(a) The current density j is

$$j = i/A = nev_d;\quad v_d = i/neA.$$

The Fermi energy is

$$E_F = \frac{h^2}{8m}(3n/\pi)^{2/3} \rightarrow n = \frac{\pi}{3}(8mE_F/h^2)^{3/2},$$

so that

$$v_d = i/\{\tfrac{\pi}{3}(8mE_F/h^2)^{3/2}\}(e)(\tfrac{1}{4}\pi d^2) = 12ih^3/(8mE_F)^{3/2}e\pi^2d^2.$$

Put i = 5 A, E_F = 7.1 eV, d = 0.001 m to get $v_d = 4.63 \times 10^{-4}$ m/s.

(b) $\bar{v} = (3kT/m)^{\frac{1}{2}}$; for T = 300 K, $\bar{v} = 1.17 \times 10^5$ m/s; this is the root-mean-square velocity.

(c) $E_F = \frac{1}{2}mv_F^2$; $v_F = (2E_F/m)^{\frac{1}{2}} = 1.58 \times 10^6$ m/s.

13-10

Eq.13-1(a) gives

$$\rho = \frac{m\bar{v}}{ne^2\lambda},$$

for the resistivity, which is defined by

$$R = \rho L/A,$$

R = resistance of length L of wire. Therefore

$$R = \frac{mL}{ne^2A}\frac{\bar{v}}{\lambda}.$$

But

$$\bar{v}T = \lambda,$$

so that

$$R = \frac{mL}{ne^2AT}.$$

13-13

(a) The current is set up by the electric field E directed along the negative x-axis; hence the resultant current density j_x is

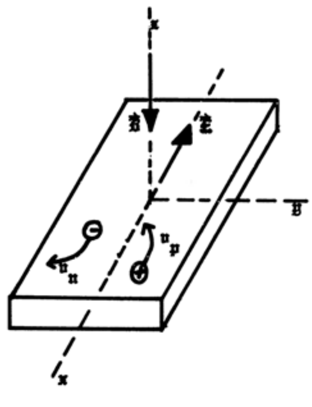
$$j_x = nev_n + pev_p.$$

The mobilities are given by $v = E\mu$, so that

$$j_x = (n\mu_n + p\mu_p)eE.$$

The Hall coefficient is defined from

$$E_H = (\text{coeff})j_xB,$$



where E_H is the transverse (y) Hall electric field. To find E_H, consider the transverse current j_y. It is the sum of two parts: (i) From the Hall electric field E_H, which sends the positive and negative charge carriers in opposite directions; the current from this is

$$(n\mu_n + p\mu_p)eE_H;$$

(ii) by the magnetic field, which sends the positive and negative carriers in the same direction; its contribution is

$$nev_{yn} - pev_{yp} = ne\mu_n E_n - pe\mu_p E_p,$$

with E_n, E_p defined by

$$ev_pB = eE_p; \quad ev_nB = eE_n.$$

Hence, this transverse current become

$$ii = ne\mu_n(v_nB) - pe\mu_p(v_pB) = ne\mu_n\{(\mu_nE)B\} - pe\mu_p\{(\mu_pE)B\},$$

$$ii = (n\mu_n^2 - p\mu_p^2)eEB.$$

In equilibrium, the total transverse current $j_y = 0$; that is, taking (i)+(ii) = 0 gives

$$(n\mu_n + p\mu_p)eE_H + (n\mu_n^2 - p\mu_p^2)eEB = 0,$$

$$E_H = \frac{p\mu_p^2 - n\mu_n^2}{p\mu_p + n\mu_n}EB.$$

But $j_x = (p\mu_p + n\mu_n)eE$ and Hall coeff. $= E_H/j_xB$ which yields

$$\text{Hall coeff.} = \frac{p\mu_p^2 - n\mu_n^2}{e(p\mu_p + n\mu_n)^2}.$$

(b) If $E_H = 0$, Hall coefficient = 0 and from (a),

$$p\mu_p^2 = n\mu_n^2.$$

The fraction of the current carried by holes is

$$f = \frac{pe\mu_pE}{(p\mu_p + n\mu_n)eE} = \frac{p\mu_p}{p\mu_p + n\mu_n}.$$

But $\mu_n = \mu_p(p/n)^{\frac{1}{2}}$ under these conditions, so that

$$f = \frac{\sqrt{p}}{\sqrt{p} + \sqrt{n}}.$$

13-14

(a) The Fermi energy is

$$\mathcal{E}_F = \frac{h^2}{8m}(3n/\pi)^{2/3},$$

where n = number density of electrons. If each atom contributes a single electron, n = number density of atoms also. With A = 64 and $\rho = 8$ g/cm^3, $n = N_{Av}/8 = 7.525 \times 10^{28}$ m^{-3}. This gives $\mathcal{E}_F = 6.52$ eV.

(b) The band width is

$$\mathcal{E}_{max} = \frac{\hbar^2\pi^2}{2ma^2}.$$

For the internuclear spacing, use $a \simeq n^{-1/3} = 2.37 \times 10^{-10}$ m, using (a). Then, $\mathcal{E}_{max} = 6.7$ eV.

13-15

(a) For the Fermi energy use

$$\mathcal{E}_F = \frac{h^2}{8m}(3n/\pi)^{2/3}.$$

By Problem 13-14, 100% copper has $n = 7.525 \times 10^{28}$ m^{-3} = number density of electrons. In an alloy with 10 zinc atoms each contributing 2 atoms, to every 90 copper atoms providing 1 each, the average number of electrons per atom is

$$\frac{(10)(2) + (90)(1)}{10 + 90} = 1.1; \quad n_{alloy} = 1.1 n_{Cu}.$$

Therefore, the formula for the Fermi energy of the alloy gives

$$\mathcal{E}_{F,alloy} = (1.1)^{2/3}\mathcal{E}_{F,Cu} = (1.066)(6.52) = 6.95 \text{ eV}.$$

(b) The band width is $\hbar^2\pi^2/2ma^2$ and depends solely on the internuclear spacing. This, by assumption, is unchanged, and therefore there is no difference in band width between copper and the alloy.

13-17

(a) From Table 11-2, p.408, $\mathcal{E}_F = 3.1$ eV for Na. By definition,

$$\mathcal{E}_F = kT_F; \quad T_F = 3.6 \times 10^4 \text{ K}.$$

(b) Room temperature is about 300 K, much less than the Fermi temperature above. Hence, to a good approximation, put T = 0 K instead of 300 K and use F-D statistics (classical methods not applicable since $T \ll T_F$).

(c) By Example 13-2,

$$\frac{N_{cond}}{N} \approx \frac{kT}{\mathcal{E}_F} = \frac{T}{T_F} = \frac{300}{36000} = 0.008.$$

13-18

(a) By definition,

$$v_F = (2\mathcal{E}_F/m)^{1/2} = 1.29 \times 10^6 \text{ m/s},$$

since $\mathcal{E}_F = 4.72$ eV.

(b) The de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{mv_F} = \frac{6.626 \times 10^{-34}}{(9.11 \times 10^{-31})(1.29 \times 10^6)} = 0.564 \text{ nm}.$$

This is comparable to the interatomic spacing a, which can be estimated as follows: from Table 13-1, p.451,

$$(nq)^{-1} = 1.70 \times 10^{-10} \text{ m}^3/\text{C},$$

so that

$$n^{-1} = (1.70 \times 10^{-10})(1.602 \times 10^{-19}) = 2.723 \times 10^{-29} \text{ m}^3,$$

$$a \approx n^{-1/3} = 0.30 \text{ nm}.$$

13-20

The Fermi distribution is smudged over a region ≈ 2kT wide about the Fermi energy. Of the N_{Av} electrons (i.e., one mole of Li, one valence electron per atom; see Table 13-1), $N_{Av}T/T_F$ have their energy increased upon heating (by Example 13-2, since $T_F \gg T$). The average energy increase is approximately kT, so that

$$U \approx (N_{Av}\frac{T}{T_F})(kT) + K,$$

where K = the constant energy of the other electrons. Hence,

$$C_{el} = \frac{\partial U}{\partial T} = 2N_{Av}kT/T_F.$$

For the lattice,

$$C_{lat} = 3R,$$

and since $k = R/N_{Av}$,

$$\frac{C_{el}}{C_{lat}} = \frac{2}{3}\frac{T}{T_F} = \frac{2}{3}\frac{300}{36000} = 0.0055,$$

the Fermi temperature from Problem 13-17(a).

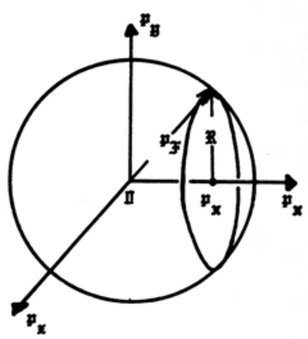
13-22

In momentum space, two electrons occupy each point, out to the surface of a sphere of radius p_F; exterior points are empty. Electrons with x-momentum = p_x lie on a circle, oriented as shown on the sketch, next page. The number # of electrons with x-momentum = p_x is proportional to the area of the circle: i.e.,

$$\# \propto 2\pi R^2 = 2\pi(p_F^2 - p_x^2),$$

$$\# \propto 2\pi p_F^2(1 - p_x^2/p_F^2),$$

$$\# \propto 1 - (p_x/p_F)^2.$$



13-24

The ratio of the number ΔN of electrons in the conduction band to the total number N is, by Example 13-6,

$$\Delta N/N = (\frac{kT}{\mathcal{E}_F})^{3/2}e^{-\mathcal{E}_g/2kT}.$$

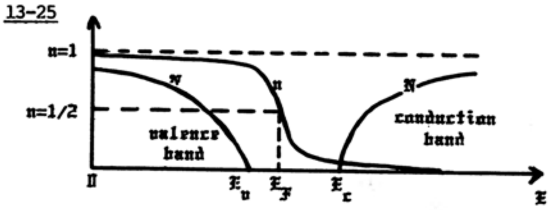
Comparing conditions at the two temperatures T_1 and T_2:

$$\Delta N_1/\Delta N_2 = (T_1/T_2)^{3/2}\exp\{-\frac{\mathcal{E}_g}{2k}(\frac{1}{T_1} - \frac{1}{T_2})\}.$$

With T_2 = 300 K, the number ratio = 20, $\mathcal{E}_g$ = 0.67 eV, this gives

$$0.248 = T_1^{3/2}e^{-3884/T_1}; \quad T_1 = 377 \text{ K}.$$

13-25



(a) The Fermi distribution is

$$n(\mathcal{E}) = \{e^{(\mathcal{E}-\mathcal{E}_F)/kT} + 1\}^{-1},$$

and the density of states is

$$N(\mathcal{E}) = A(\mathcal{E} - \mathcal{E}_C)^{\frac{1}{2}}; \quad A = \frac{8\pi V(2m^3)^{\frac{1}{2}}}{h^3}.$$

Hence, the number of conduction electrons is

$$n_C = \int_{\mathcal{E}_C}^{\infty} A(\mathcal{E} - \mathcal{E}_C)^{\frac{1}{2}}\{e^{(\mathcal{E}-\mathcal{E}_F)/kT} + 1\}^{-1}d\mathcal{E}.$$

If the material is a semiconductor or insulator, it is expected that, over the range of integration, $(\mathcal{E}_C - \mathcal{E}_F) >> kT$ and thus also $(\mathcal{E} - \mathcal{E}_F) >> kT$, so that

$$n_C \simeq A\int_{\mathcal{E}_C}^{\infty}(\mathcal{E} - \mathcal{E}_C)^{\frac{1}{2}}e^{-(\mathcal{E}-\mathcal{E}_F)/kT}d\mathcal{E},$$

$$n_C = Ae^{(\mathcal{E}_F-\mathcal{E}_C)/kT}\int_{\mathcal{E}_C}^{\infty}(\mathcal{E} - \mathcal{E}_C)^{\frac{1}{2}}e^{-(\mathcal{E}-\mathcal{E}_C)/kT}d\mathcal{E},$$

$$n_C = Ae^{(\mathcal{E}_F-\mathcal{E}_C)/kT}(kT)^{3/2}\int_0^{\infty}x^{\frac{1}{2}}e^{-x}dx.$$

The value of the integral is $\sqrt{\pi}/2$ and therefore, inserting the expression for A,

$$n_C/V = \frac{2}{h^3}(2\pi mkT)^{3/2}e^{-(\mathcal{E}_C-\mathcal{E}_F)/kT}.$$

(b) For the valence band,

$$N(\mathcal{E}) = A(\mathcal{E}_V - \mathcal{E})^{\frac{1}{2}}.$$

The number n_v of holes is given from

$$\mathcal{N} - n_v = \int_0^{\mathcal{E}_V} nN d\mathcal{E} = A\int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2}\{e^{(\mathcal{E}-\mathcal{E}_F)/kT} + 1\}^{-1} d\mathcal{E}.$$

Since

$$\mathcal{N} = \int_0^{\infty} nN d\mathcal{E},$$

it follows that

$$n_v = \int_0^{\infty} nN d\mathcal{E} - \int_0^{\mathcal{E}_V} nN d\mathcal{E}.$$

In the first term on the right use $n(T = 0)$. This equals zero above $\mathcal{E}_F$; but then, since in effect in the integration $N = N_{val}$ and $\mathcal{E} < \mathcal{E}_V < \mathcal{E}_F$, as in the second integral which does not extend above the valence band. Hence,

$$n_v = \int_0^{\mathcal{E}_V} N d\mathcal{E} - \int_0^{\mathcal{E}_V} nN d\mathcal{E} = \int_0^{\mathcal{E}_V} (1 - n) N d\mathcal{E},$$

since $N_{val} = 0$ above $\mathcal{E}_V$. But

$$1 - n(0 \le \mathcal{E} \le \mathcal{E}_v) \simeq n(\mathcal{E}_c \le \mathcal{E} \le \infty),$$

if $\mathcal{E}_C = \mathcal{E}_V$; only approximately equal above since the energy cuts off at zero but has no upper bound. Making the switch indicated here and noting that N_{val} and N_{cond} are identical curves, it follows that the resulting integral is essentially the same as for the number of electrons in the conduction band; i.e.,

$$n_v/V = \frac{2}{h^3}(2\pi mkT)^{3/2} e^{-(\mathcal{E}_F - \mathcal{E}_V)/kT}.$$

To obtain this result from direct integration, note that if $(\mathcal{E}_F - \mathcal{E}_V) >> kT$, then $(\mathcal{E}_F - \mathcal{E}) >> kT$ over the range of integration.

Therefore,

$$\int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2}\{e^{(\mathcal{E}-\mathcal{E}_F)/kT} + 1\}^{-1} d\mathcal{E} \simeq \int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2}\{1 - e^{-(\mathcal{E}_F - \mathcal{E})/kT}\} d\mathcal{E},$$

$$= \int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2} d\mathcal{E} - e^{-(\mathcal{E}_F - \mathcal{E}_V)/kT} \int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2} e^{-(\mathcal{E}_V - \mathcal{E})/kT} d\mathcal{E}.$$

The first integral is

$$\int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2} d\mathcal{E} = \frac{2}{3}\mathcal{E}_v^{3/2}.$$

Write the second as

$$\int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2} e^{-(\mathcal{E}_V - \mathcal{E})/kT} d\mathcal{E} = -(kT)^{3/2} \int_{\mathcal{E}_V/kT}^{0} x^{1/2} e^{-x} dx.$$

But $(\mathcal{E}_F - \mathcal{E}_V)/kT >> 1$, so it may be assumed that $\mathcal{E}_V/kT >> 1$; then, replace the lower limit with infinity with little error, and this second integral becomes

$$(kT)^{3/2} \int_0^{\infty} x^{1/2} e^{-x} dx = \frac{x^{1/2}}{2}(kT)^{3/2}.$$

Thus,

$$\mathcal{N} - n_v = A\{\frac{2}{3}\mathcal{E}_v^{3/2} - e^{-(\mathcal{E}_F - \mathcal{E}_V)/kT}(kT)^{3/2}\frac{\sqrt{\pi}}{2}\}.$$

For $\mathcal{N}$ use the $T = 0$ value:

$$\mathcal{N} = A\int_0^{\mathcal{E}_V} (\mathcal{E}_v - \mathcal{E})^{1/2}(1) d\mathcal{E} = \frac{2}{3} A\mathcal{E}_v^{3/2}.$$

Therefore, the same expression as obtained above is derived here also: i.e., for n_v/V.

13-26

Under conditions of charge neutrality, $n_c/V = n_v/V$:

$$\frac{2}{h^3}(2\pi mkT)^{3/2}e^{-(\mathcal{E}_C-\mathcal{E}_F)/kT} = \frac{2}{h^3}(2\pi mkT)^{3/2}e^{-(\mathcal{E}_F-\mathcal{E}_V)/kT};$$

this gives directly

$$\mathcal{E}_C - \mathcal{E}_F = \mathcal{E}_F - \mathcal{E}_V,$$

and therefore the Fermi level lies midway between the top of the valence band and the bottom of the conduction band.

13-27

(a) By Problem 13-25, the densities are (with an obvious change in notation),

$$n_c = \frac{2}{h^3}(2\pi mkT)^{3/2}e^{-(\mathcal{E}_C-\mathcal{E}_F)/kT}; \quad n_v = \frac{2}{h^3}(2\pi mkT)^{3/2}e^{-(\mathcal{E}_F-\mathcal{E}_V)/kT}.$$

The product is

$$n_c n_v = 4h^{-6}(2\pi mkT)^3 e^{-(\mathcal{E}_C-\mathcal{E}_V)/kT} = \frac{4}{h^6}(2\pi mkT)^3 e^{-\mathcal{E}_g/kT},$$

which depends on T and the energy gap $\mathcal{E}_g$ only.

(b) The conductivity σ is

$$\sigma = n_c q_n \mu_n + n_v q_p \mu_p.$$

With $n_c = n_v$,

$$\sigma = (n_c n_v)^{\frac{1}{2}}(q_n\mu_n + q_p\mu_p),$$

$$\sigma = \frac{2}{h^3}(2\pi mkT)^{3/2}e^{-\mathcal{E}_g/2kT}(q_n\mu_n + q_p\mu_p).$$

Therefore,

$$\ln(\sigma) = -\frac{\mathcal{E}_g}{2k}\left(\frac{1}{T}\right) + \ln\{\frac{2}{h^3}(2\pi mkT)^{3/2}(q_n\mu_n + q_p\mu_p)\},$$

so that a plot of $\ln(\sigma)$ vs. $1/T$ is a straight line with a slope proportional to $\mathcal{E}_g$, provided that T is not too large (in which case the second term above cannot be considered as roughly constant).

13-28

The electron of an ionized donor is at the donor level, in energy. If contributions from the valence band are ignored, then

$$N_d^0 = N_d\{e^{(\mathcal{E}_d-\mathcal{E}_F)/kT} + 1\}^{-1}$$

since the Fermi energy distribution gives the fractional occupancy at any given level. Also,

$$N_d^+ = N_d - N_d^0 = N_d\{e^{(\mathcal{E}_F-\mathcal{E}_d)/kT} + 1\}^{-1}.$$

13-31

(a) Current p to n: The mechanism is thermal excitation of electrons from the top of the p-valence band to the conduction band. By Problem 13-25, the number of electrons per unit volume in the conduction band is

$$\frac{2}{h^3}(2\pi mkT)^{3/2}e^{-(\mathcal{E}_C^p-\mathcal{E}_F)/kT} \propto e^{-(\mathcal{E}_g-\mathcal{E}_F^p)/kT},$$

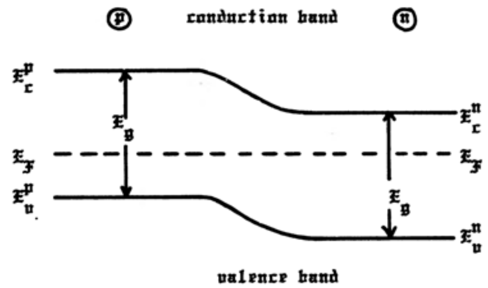
where $\mathcal{E}_F^p$ = Fermi energy measured from the top of the p-valence band.

Current n to p: The mechanism is that (i) there are some electrons in the n-conduction band and (ii) some of these may be thermally excited to above the bottom of the p-conduction band. Hence, the current is proprtional to the number of electrons per unit volume times the probability of surmounting a barrier of height $\mathcal{E}_C^p-\mathcal{E}_C^n$: i.e., taking the sum over $\mathcal{E} \geq \mathcal{E}_C^n$,

$$\text{current} \propto \Sigma n_c(\mathcal{E})e^{-(\mathcal{E}_C^p-\mathcal{E})/kT} \propto e^{-(\mathcal{E}_C^n-\mathcal{E}_F)/kT}e^{-(\mathcal{E}_C^p-\mathcal{E}_C^n)/kT},$$

$$\text{current} \propto e^{-(\mathcal{E}_g-\mathcal{E}_F^n)/kT}e^{-\{(\mathcal{E}_g-\mathcal{E}_F^p)-(\mathcal{E}_g-\mathcal{E}_F^n)\}/kT},$$

where $\mathcal{E}_F^n$ = Fermi energy measured from the top of the n-valence band. It follows that



UNBIASED JUNCTION

$$\text{current n to p} \propto e^{-(\mathfrak{E}_g-\mathfrak{E}_F^n)/kT}e^{-(\mathfrak{E}_F^n-\mathfrak{E}_F^p)/kT} = e^{-(\mathfrak{E}_g-\mathfrak{E}_F^p)/kT},$$

which agrees with current p to n (proportionality constants are equal and depend only on T).

(b) If a forward bias is impressed, then the potential barrier seen by the electrons in the n-conduction band is reduced, in effect, by eV, the energy gained by the electrons in moving through the field. (For reverse bias, the barrier is increased by eV). The current n to p now is proportional to

$$e^{-(\mathfrak{E}_C^n-\mathfrak{E}_F)/kT}e^{-(\mathfrak{E}_C^p-\mathfrak{E}_C^n-eV)/kT} = e^{-(-eV+\mathfrak{E}_g-\mathfrak{E}_F^p)/kT}.$$

The net current is (n to p) - (p to n) and is proportional to

$$e^{-(\mathfrak{E}_g-\mathfrak{E}_F^p)/kT}e^{eV/kT} - e^{-(\mathfrak{E}_g-\mathfrak{E}_F^p)/kT} = e^{-(\mathfrak{E}_g-\mathfrak{E}_F^p)/kT}(e^{eV/kT} - 1).$$

<u>13-33</u>

The current is

$$I = \left(\frac{n}{t}\right)eT,$$

where T is the barrier penetration probability. For this, use Eq.6-50:

$$k_{II}a = \frac{a}{\hbar}\{2m^*(V_0 - E)\}^{\frac{1}{2}} = 12.55,$$

$$T = 16\,\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2k_{II}a} = 1.145 \times 10^{-11}.$$

Hence,

$$I = (1.00 \times 10^{25})(1.602 \times 10^{-19})(1.145 \times 10^{-11}),$$

$$I = 1.834 \times 10^{-5}\ \text{A}.$$

CHAPTER FOURTEEN

14-1

At T = 0 K, $E_g = 3kT_c$ so that

$$3kT_c = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}.$$

By Example 14-1, $T_c = 4.2$ K; hence,

$$r = \frac{1}{4\pi\epsilon_0}\frac{e^2}{3kT_c} = (8.988 \times 10^9)\frac{(1.602 \times 10^{-19})^2}{3(1.381 \times 10^{-23})(4.2)},$$

$$r = 1300 \text{ nm}.$$

14-2

The relevant equations are

$$\int \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}; \quad \vec{j} = \sigma\vec{E}; \quad \vec{E} = \rho\vec{j}.$$

(a) The first of the equations above implies that

$$\oint \rho\vec{j}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} = -A\frac{dB}{dt}.$$

If $\rho = 0$, then $dB/dt = 0$ giving B = constant, not necessarily zero.

(b) If B = 0,

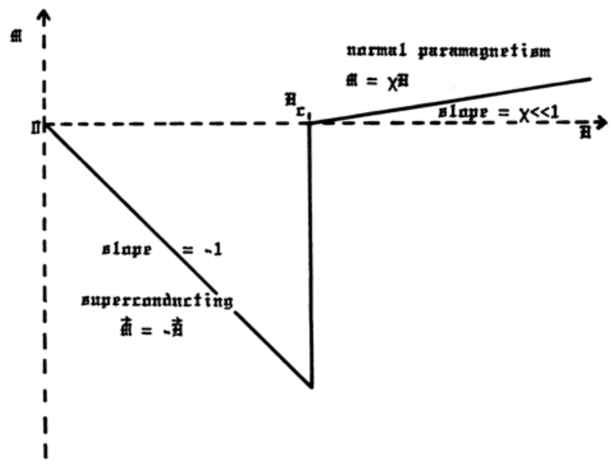
$$\rho\int \vec{j}\cdot d\vec{\ell} = 0.$$

But a current exists on the surface and outer layers of the superconductor; i.e., $j \neq 0$. Hence $\rho = 0$ unless $\int \vec{j}\cdot d\vec{\ell} = 0$.

14-3

Consider the total field inside the superconductor as the sum of an external field and an internal field due to the currents set up in the material as the external field changes. By Lenz's law, if the total field is zero and one tries to change the external field, the induced currents flow so as to set up internal fields oppositely directed, so that the total field remains zero. If the material can do this, the currents must be able to respond precisely to the changing external field, i.e., $\rho = 0$. Hence, Meissner effect implies $\rho = 0$.
Lenz's law states that an induced current will flow in a sense so as to oppose the change that produced it, but this carries no implication that the original change will be annulled completely even if $\rho = 0$ (above, it was assumed that complete annullment occurs). For example, if $dB_{ext}/dt = f(t)$, the current set up may be independent of t, in which case $B_T = 0$ (Meissner effect) cannot be satisfied for all time.

14-4



14-8

The isotope effect is that

$$M^{\frac{1}{2}}T_C = \text{constant}.$$

For naturally occuring vanadium, in atomic mass units,

$$M = 0.9976(50.9440) + 0.0024(49.9472) = 50.9416.$$

Hence,

$$(50.9416)^{\frac{1}{2}}(5.300) = (49.9472)^{\frac{1}{2}}T_C,$$

$$T_C = 5.352 \text{ K}.$$

14-10

Eq.'s 14-2, 14-3 are

$$\vec{B} = \mu_0\vec{M} + \mu_0\vec{H}; \quad \vec{M} = \chi\vec{H}.$$

Inside a superconductor B = 0 and therefore the above imply that

$$\mu_0\vec{M} = -\mu_0\vec{H}; \quad \vec{M} = -\vec{H} = \chi\vec{H}; \quad \chi = -1.$$

From Eq.14-4,

$$\vec{M} = \frac{\chi}{1+\chi}(\vec{B}/\mu_0); \quad \mu_0(1+\chi)\vec{M} = \chi\vec{B}.$$

The last of these with $\chi = -1$ gives B = 0, consistent with the above.

14-11

(a) Since

$$\vec{M} = \frac{\chi}{1+\chi}(\vec{B}/\mu_0),$$

$$M = \frac{(2.1 \times 10^{-6})(5 \times 10^{-5})}{(4\pi \times 10^{-7})(1)} = 8.4 \times 10^{-5} \text{ A/m}.$$

(b) For one kg-mole, $n = 6.022 \times 10^{26}$; then,

$$M_{sat} = \frac{n\mu}{V} = \frac{(6.022 \times 10^{26})\{(2.8)(9.27 \times 10^{-24})\}}{22.4} = 700 \text{ A/m}.$$

14-12

(a) With $x = \mu B/kT$,

$$M = n\mu\frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Set $M = \frac{1}{2}M_s = \frac{1}{2}n\mu$ to get

$$\frac{1}{2} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$x = \ln\sqrt{3} = 0.549.$$

(b) By Fig.14-6, at $M = \frac{1}{2}M_s$, $B/T \simeq 0.53$ T/K. Therefore,

$$x = \frac{\mu B}{kT},$$

$$0.549 = \frac{\mu}{1.381 \times 10^{-23}}(0.53),$$

$$\mu = 1.43 \times 10^{-23} \text{ J/T}.$$

14-13

The magnetization is

$$M = \chi H \simeq \chi B/\mu_0 = \frac{C}{\mu_0}\frac{B}{T},$$

assuming Curie's law ($\chi = C/T$). Hence,

$$B_1/T_1 = B_2/T_2 \rightarrow T_2 = 0.01 \text{ K}.$$

14-14

The magnetization is

$$M = n\mu\,\frac{e^x - e^{-x}}{e^x + e^{-x}},$$

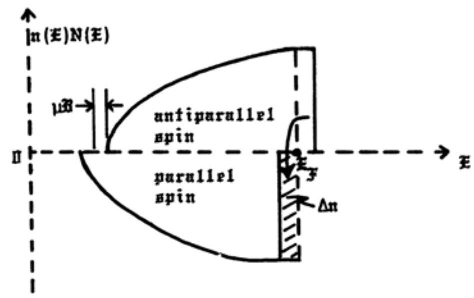
with $x = \mu B/kT$. If $\mu B/kT >> 1$, then $e^x >> 1$, $e^{-x} \simeq 1$, so that $e^x >> e^{-x}$. Thus

$$M \simeq n\mu\,\frac{e^x}{e^x} = n\mu,$$

the saturation value.

14-15

(a)



After some antiparallel-spin electrons make the transition to parallel-spin, as indicated, the excess Δn of electrons with parallel spin over those with antiparallel spin is represented by twice the shaded area, which may be approximated by a rectangle. Thus,

$$\Delta n = 2(\mu B)\{n(E_F)N(E_F)\} = 2(\mu B)(1)\{\tfrac{1}{2}\,\frac{8\pi V(2m^3)^{1/2}}{h^3}E_F^{1/2}\},$$

in which the factor of ½ occurs since the upper curve stands for only half the electrons. The total number of particles N of a Fermi gas is

$$N = \left(\frac{8m}{h^3}\right)^{3/2}\frac{\pi}{3}E_F^{3/2}V,$$

so that the excess per unit volume is

$$\frac{\Delta n}{V} = 3nB\mu/2E_F,$$

where n = number of electrons per unit volume. Now $E_F = kT_F$ and $\mu = \mu_z = g_s\mu_b m_s = (2)\mu_b(\tfrac{1}{2}) = \mu_b$ giving

$$M = \left(\frac{\Delta n}{V}\right)\mu = 3nB\mu_b^2/2E_F \simeq \chi B/\mu_0,$$

$$\chi = \frac{3}{2}\,\frac{n\mu_0\mu_b^2}{kT_F}.$$

(b) For copper $E_F = kT_F = 7.1$ eV (Table 11-2) and $n = 1.1364 \times 10^{29}\ m^{-3}$ (Table 13-1). Substitution of these into the formula above for the susceptibility gives $\chi = 1.62 \times 10^{-5}$.

14-16

(a) The number of dipoles per unit volume, aligned parallel and antiparallel to the field is

$$n_- = cne^{\mu B/kT};\quad n_+ = cne^{-\mu B/kT}.$$

The energy associated with parallel alignment is $-\mu B$, with antiparallel alignment $+\mu B$. Hence, the total energy is

$$U = V\{n_-(-\mu B) + n_+(+\mu B)\} = \mu BcNe^{-\mu B/kT}(1 - e^{2\mu B/kT}).$$

Now, with $x = \mu B/kT$,

$$n_+ + n_- = n = cn(e^x + e^{-x}),$$

so that

$$c = (e^x + e^{-x})^{-1} = e^x(1 + e^{2x})^{-1}.$$

Put this into the expression for U above to get

$$U = \mu B N \frac{1 - e^{2x}}{1 + e^{2x}}.$$

The specific heat is

$$c_B = \frac{\partial U}{\partial B} = (2x)^2 kN \frac{e^{2x}}{(1 + e^{2x})^2}.$$

(b), (c) See Problem 11-11 replacing E with $-2\mu B$.

<u>14-17</u>

(a) $$kT_C = \mu B_W = (2.2\mu_b)B_W; \quad B_W = 677 \text{ T},$$

since $T_C = 1000$ K. Then the internal field H_W is

$$H_W = B_W/\mu_0 = 5.4 \times 10^8 \text{ A/m}.$$

(b) The number of atoms in 1 cm^3 is

$$N = \frac{7.9}{56}(6.02 \times 10^{23}) = 8.492 \times 10^{22} \text{ cm}^{-3}.$$

Since all the dipoles are aligned,

$$M = N\mu = (8.492 \times 10^{28})(2.2)(9.27 \times 10^{-24}) = 1.73 \times 10^6 \text{ A/m}.$$

(c) $$U = NkT_C = 1200 \text{ J}.$$

<u>14-18</u>

(a) If H = external field, H_W the molecular field,

$$M = \frac{C}{T}(H + H_W),$$

assuming Curie's law. If $H_W = \lambda M$, then

$$M = \frac{C}{T}(H + \lambda M).$$

With susceptibility defined by $M = \chi H$,

$$M(1 - \frac{C\lambda}{T}) = \frac{C}{T}H; \quad M = \frac{C}{T - T_C}H,$$

so that

$$\chi = \frac{C}{T - T_C}; \quad T_C = \lambda C.$$

(b) Using results from Problem 14-17,

$$\lambda = H_W/M = \frac{5.39 \times 10^8}{1.73 \times 10^6} = 310.$$

CHAPTER FIFTEEN

15-1

The number of levels equals 2i+1 if $j > i$ and 2j+1 if $i > j$. In this case the number of levels is 4 and $2j+1 = 5$ ($j = 2$), so that $j > i$. Therefore, $4 = 2i+1$, or $i = 3/2$.

15-2

Boron: $Z = 5$, $A = 10$, $N = 5$, $i = 3$, symmetric.

(a) Assume 5 protons, 5 neutrons.
Mass: $m_p \simeq m_n$, mass $\simeq 10$ u.
Charge: $q_p = +1$, $q_n = 0$, charge = +5.
Spin: $s_p = s_n = \frac{1}{2}$; 10 particles each of spin $\frac{1}{2}$; integral spin.
Symmetry: protons and neutrons are fermions; 10 fermions: $(-1)^{10} = +1$, symmetric nucleus.

All of these agree with observation.

(b) Assume 10 protons, 5 electrons.

Mass: $m_e \simeq 0$, mass $\simeq 10$ u; agrees with observation.
Charge: $q_p = +1$, $q_e = -1$; total charge = +5; agrees.
Spin: odd number of fermions (15), total spin half-integral; disagrees.
Symmetry: $(-1)^{15} = -1$, antisymmetric nucleus; disagrees.

15-4

(a) From Fig.15-6, p.518, $r_{\frac{1}{2}} = 6.6$ F.

(b) $\theta = \lambda/r \simeq \lambda/r_{\frac{1}{2}}$. The kinetic energy $K = 1000$ MeV >> 0.511 MeV = electron rest energy, so that

$$\lambda = \frac{h}{p} \simeq \frac{h}{E/c} = \frac{hc}{E} \simeq \frac{hc}{K},$$

$$\lambda = \frac{(6.626 \times 10^{-34})(2.998 \times 10^{8})}{(10^{9})(1.602 \times 10^{-19})} = 1.240 \text{ F}.$$

With this,

$$\theta \simeq \frac{1.240}{6.6} = 0.188 \text{ rad} = 11°.$$

15-5

With $\rho(0) \simeq$ central density,

$$\rho = \frac{\rho(0)}{1 + e^{(r-a)/b}}.$$

At $r = r_9$,

$$1 + e^{(r_9-a)/b} = 1/0.9,$$

and at $r = r_1$,

$$1 + e^{(r_1-a)/b} = 1/0.1.$$

These give

$$r_9 = -2.197b + a; \quad r_1 = 2.197b + a;$$

$$\Delta r = r_1 - r_9 = 4.394b = 2.4 \text{ F}.$$

15-6

Since

$$\rho = \frac{\rho(0)}{1 + e^{(r-a)/b}},$$

the central density is

$$\rho_c = \frac{\rho(0)}{1 + e^{-a/b}}.$$

At $\rho = \frac{1}{2}\rho_c$,

$$\frac{\rho(0)}{1 + e^{(r-a)/b}} = \frac{1}{2}\frac{\rho(0)}{1 + e^{-a/b}},$$

$$r = a + b\ln(1 + 2e^{-a/b}).$$

$$\left.\begin{array}{l} a = 1.07A^{1/3} = 1.07(12)^{1/3} \\ b = 0.55 \end{array}\right\} \rightarrow \frac{a}{b} = 4.454.$$

Thus,

$$r = a + (0.55)\ln(1 + 2e^{-4.454}) = a + 0.0126.$$

15-9

Conservation of mass-energy and of momentum (classical form is used) require that

$$K_a + m_a c^2 + m_A c^2 = K_B + m_B c^2 + K_b + m_b c^2,$$

$$m_a v_a = m_B v_B \cos\phi + m_b v_b \cos\theta; \quad m_B v_B \sin\phi = m_b v_b \sin\theta.$$

Using classical expressions for K also, this last gives

$$\sin\phi = \frac{(2K_b m_b)^{\frac{1}{2}} \sin\theta}{(2K_B m_B)^{\frac{1}{2}}}.$$

Therefore,

$$(2K_a m_a)^{\frac{1}{2}} - (2K_b m_b)^{\frac{1}{2}} \cos\theta = (2K_B m_B)^{\frac{1}{2}} \cos\phi,$$

$$(2K_B m_B)\cos^2\phi = 2K_a m_a + 2K_b m_b \cos^2\theta - 2(4K_a K_b m_a m_b)^{\frac{1}{2}} \cos\theta.$$

But,

$$\sin^2\phi = \frac{2K_b m_b \sin^2\theta}{2K_B m_B} = 1 - \cos^2\phi,$$

$$\cos^2\phi = \frac{2K_B m_B - 2K_b m_b \sin^2\theta}{2K_B m_B}.$$

Equating the two expressions for $\cos^2\phi$ gives

$$K_a m_a + K_b m_b - K_B m_B - 2(K_a K_b m_a m_b)^{\frac{1}{2}} \cos\theta = 0.$$

Finally,

$$K_B = Q - K_b + K_a.$$

Using this to eliminate K_B gives

$$Q = K_b\left(1 + \frac{m_b}{m_B}\right) - K_a\left(1 - \frac{m_a}{m_B}\right) - \frac{2}{m_B}(K_a K_b m_a m_b)^{\frac{1}{2}} \cos\theta.$$

15-11

$$E_0(Cr^{52}) + Q = 2E_0(Mg^{26}),$$

$$Q = \{2M(Mg^{26}) - M(Cr^{52})\}c^2,$$

$$Q = \{2(25.98260) - 51.94051\}uc^2,$$

$$Q = (0.02469)(931.5 \text{ MeV}) = 23.0 \text{ MeV}.$$

15-12

(a)

$$Q = (m_{H2} + m_{H3} - m_{He} - m_n)c^2,$$

$$Q = (2.0141022 + 3.0160500 - 4.0026033 - 1.0086654)(uc^2),$$

$$Q = (0.0188835)(uc^2) = (0.0188835)(931.5 \text{ MeV}) = 17.59 \text{ MeV}.$$

(b) In Eq.15-16, let $a = H^2$, $b = n$, $A = H^3$, $B = He^4$; then,

$$m_b/m_B = \frac{1}{4}; \; m_a/m_B = \frac{1}{2}; \; (m_a m_b/m_B^2)^{\frac{1}{2}} = \sqrt{2}/4; \; K_a = 0.5 \text{ MeV}.$$

With these, Eq.15-16 becomes

$$35.68 = 5K_b/2 - K_b^{\frac{1}{2}}; \quad K_b = 15.9 \text{ MeV}.$$

15-13

The binding energy ΔE is

$$\Delta E = \{Zm_H + Nm_n - M(\text{atom})\}(c^2),$$

$$\Delta E = \{6(1.0078252) + 6(1.0086654) - 12.0000000\}uc^2,$$

$$\Delta E = (0.098943)(931.5) = 92.165404 \text{ MeV}.$$

The average binding energy per nucleon becomes

$$\frac{\Delta E}{A} = \frac{92.165404}{12} = 7.68 \text{ MeV}.$$

15-14

(a) The energy release is

$$E = \{2m_{H2} - m_{He}\}c^2 = \{2(2.0141022) - (4.0026033)\}uc^2,$$

$$E = (0.0256011)(931.5) = 23.8 \text{ MeV}.$$

(b) With the nuclei just touching, their centers are 3 F apart:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \frac{(9 \times 10^9)(1.6 \times 10^{-19})^2}{(3 \times 10^{-15})(1.6 \times 10^{-13})} = 0.48 \text{ MeV}.$$

15-15

(a) With $r' = 1.1A^{1/3} \times 10^{-15}$ m,

$$V = \frac{3}{5}\frac{Z^2e^2}{4\pi\varepsilon_0 r'} = \frac{3}{5}\frac{e^2}{(4\pi\varepsilon_0)(1.1 \times 10^{-15})}Z^2A^{-1/3} = aZ^2A^{-1/3};$$

this has the same form as the Coulomb term of the mass formula.

(b) The energy coefficient in V above is

$$a = \frac{3}{5}(8.988 \times 10^9)\frac{(1.602 \times 10^{-19})^2}{(1.1 \times 10^{-15})(1.602 \times 10^{-13})} = 0.7854 \text{ MeV}.$$

In mass units,

$$a = \frac{0.7854}{931.5} = 0.000843 \text{ u},$$

compared to $a_3 = 0.000763$ u of the mass formula.

15-16

(a) The binding energy of ${}_5B^{11}$ is, with masses in u,

$$\Delta E_B = (5m_H + 6m_n - m_B)c^2 = (11.091118 - 11.009305)(931.5 \text{ MeV}),$$

$$E_B = 76.209 \text{ MeV}.$$

Similarly, the binding energy of ${}_6C^{11}$ is

$$\Delta E_C = (6m_H + 5m_n - m_C)c^2 = (11.090278 - 11.011432)(931.5 \text{ MeV}),$$

$$E_C = 73.445 \text{ MeV}.$$

The difference is 2.764 MeV.

(b) From Problem 15-15,

$$V = (0.7854 \text{ MeV})\frac{Z^2}{r'}.$$

For ${}_6C^{11}$, Z = 6 and for ${}_5B^{11}$ Z = 5. Assuming the same r', it is required that

$$\frac{0.7854}{r'}(6^2 - 5^2) = 2.764; \quad r' = 3.13 \text{ F}.$$

(c) This is somewhat larger than the mean radius ($\simeq$ 2.5 F) of the charge distribution of ${}_6C^{12}$ given in Fig.15-6.

15-17

(a) For ${}_{26}Fe^{56}$, Z = 26, A = 56, N = 30. The mass formula is Eq. 15-30 and the terms are the following:

Mass of separate parts: 1.007825(26) + 1.008665(30) = 56.4634 u.

Volume term: $-a_1A = -0.01691(56) = -0.94696$ u.

Surface term: $+a_2A^{2/3} = (0.01911)(56)^{2/3} = 0.279717$ u.

Coulomb term: $a_3Z^2A^{-1/3} = (0.000763)(26)^2(56)^{-1/3} = 0.134816$ u.

Asymmetry term: $a_4(Z - \frac{1}{2}A)^2/A = (0.10175)(2)^2/56 = 0.007268$ u.

Pairing term: Z even, N even so this term is $-a_5A^{-\frac{1}{2}} = -(0.012)/\sqrt{56} = -0.001604$ u.

(b) To convert to energy, multiply by 931.5 MeV/u. To form the average binding energy per nucleon, omit the first term, add and divide by A:

$$\frac{\Delta E}{A} = |-15.7517 + 4.6528 + 2.24252 + 0.1209 - 0.02668|.$$

volume surface coulomb asymmetry pairing

There is good agreement with Fig.15-12, except that the pairing term is too small to discriminate from the graph.

(c) The atomic mass is the sum of the terms in (a): to wit, 55.93664 u.

(d) From (b), $\Delta E/A$ = 8.762 MeV, and agrees well with Fig.15-10.

15-18

(a) The binding energy of ${}^{6}C^{12}$ is

$$(6m_H + 6m_n - m_C)(931.5\text{ MeV/u}) = 92.1660\text{ MeV},$$

and that of ${}^{2}He^{4}$ is

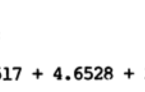
$$(2m_H + 2m_n - m_{He})(931.5\text{ MeV/u}) = 28.2970\text{ MeV}.$$

The difference, BE(C) - 3BE(He) = 7.275 MeV, and equals the binding energy of C on the alpha-particle model.

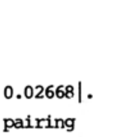
(b) The binding energy of ${}^{8}O^{16}$, calculated as above, is just 127.62463 MeV. Then, BE(O) - 4BE(He) = 14.44 MeV, and is the binding energy of O on the alpha-particle model.

(c) The number of bonds is $N(N-1)/2!$ = 3 for ${}^{6}C^{12}$ (N = 3) and 6 for ${}^{8}O^{16}$ (N = 4).

${}^{6}C^{12}$: 3 bonds



${}^{8}O^{16}$: 6 bonds



(d) The energy per bond for ${}^{6}C^{12}$ is 7.275/3 = 2.43 MeV; ${}^{8}O^{16}$ gives 14.44/6 = 2.41 MeV, virtually identical.

15-21

(a) ${}^{6}C^{11}$. Protons: an even number; j = 0, P even.
Neutrons: 5, 3 in $1p_{3/2}$; i = 3/2, $P = \{(-1)^1\}^3 = -1$, odd; hence, (3/2, odd).

${}^{20}Ca^{44}$. Even N, even Z: (0, even).

${}^{28}Ni^{61}$. Z even, j = 0, P even. N = 33, last single neutron in $1f_{5/2}$; j = 5/2, $P = (-1)^3 = -1$, giving (5/2, odd).

${}^{32}Ge^{73}$. Z even, j = 0, P even. N = 41, last single neutron in $1g_{9/2}$; i = 9/2, $P = (-1)^4 = +1$, predicting (9/2, even).

(b) Discrepancy is ${}^{28}Ni^{61}$: observed is (3/2, odd), and predicted is (5/2, odd). The $1f_{5/2}$, $2p_{3/2}$ levels are very close together, so one neutron in $2p_{3/2}$ couples with the single neutron in $1f_{5/2}$, leaving a nucleon in $2p_{3/2}$, giving i = 3/2 rather than 5/2. Parity is unaffected since $(-1)^{\ell} = (-1)^1 = (-1)^3 = -1$.

15-23

(a) By Example 15-10, spin i = 5/2.

(b) By Example 15-11, parity $(-1)^{\ell} = (-1)^2$ is even.

(c) Unpaired neutron is in a $1d_{5/2}$ state: $\ell = 2$, $j = 5/2 \rightarrow j = \ell + \frac{1}{2}$. By Fig.15-19, the lower Schmidt line gives $-1.8\mu_n$, i.e., negative.

(d) Z = 8, which is a magic number; hence, q = 0.

15-24

(a) ${}^{23}V^{51}$ has odd Z, even N; V^{51} lies close to the upper ($j = \ell + \frac{1}{2}$) line, so that on this basis $\ell = j - \frac{1}{2} = 7/2 - 1/2 = 3$ is expected.

(b) By the shell model, an even number of neutrons couple in pairs; the last proton is in $1f_{7/2}$; the f-level has $\ell = 3$, which agrees with (a).

15-25

(a) From Fig.15-20, $q/Zr'^2 = 0.09 > 0$. The desired ratio is $1 + q/Zr'^2 = 1.09$.

(b) From Eq.15-6, $a = 1.07A^{1/3} = 1.07(181)^{1/3} = 6.0526$ F.

(c) $a'/a'' = 1.09$; $\frac{1}{2}(a' + a'') = 6.0526$. Solving these two equations gives $a' = 6.31$ F, $a'' = 5.79$ F.

(d) The cross section is an ellipse of small eccentricity: $e^2 = 0.16$.

15-26

The quadrupole moment q is defined by

$$q = \int \rho\{3z^2 - (x^2 + y^2 + z^2)\}d\tau.$$

The origin of coordinates is at the center of the nucleus. For the cylinder, put the z-axis along the axis of the cylinder and use cylindrical coordinates; then,

$$d\tau = 2\pi r dr dz,$$

$$r^2 = x^2 + y^2.$$

Hence,

$$q = 2\pi\rho \int_{z=-\frac{1}{2}L}^{z=+\frac{1}{2}L}\int_{r=0}^{R}(2z^2 - r^2)r dr dz,$$

$$q = 2\pi\rho\{2(\frac{L^3}{12})(\frac{R^2}{2}) - (L)(\frac{R^4}{4})\},$$

$$q = \frac{1}{2}\pi\rho LR^2(\frac{1}{3}L^2 - R^2).$$

Therefore, $q > 0$ for $L/R > \sqrt{3}$.

CHAPTER SIXTEEN

16-1

(a) The decay energy is

$$E = \{M({}^{83}Bi^{210}) - M({}^{81}Tl^{206}) - M({}^{2}He^{4})\}c^2.$$

	${}^{83}Bi^{210}$	${}^{81}Tl^{206}$
1.007825Z	83.649475	81.633825
1.008665(A-Z)	128.10046	126.08313
$-a_1A$	-3.5511	-3.48346
$a_2A^{2/3}$	0.6751615	0.6665605
$a_3Z^2A^{-1/3}$	0.8843179	0.8476299
$a_4(Z-A/2)^2/A$	0.2345095	0.2390631
$\pm a_5A^{-\frac{1}{2}}$	+0.0008281	0.0008361
	209.993652	205.9875846

Use, from Table 15-1, $M({}^{2}He^{4}) = 4.0026033$ u. Then, since 1 u is equivalent to 931.5 MeV, E = 3.23 MeV.

(b) By Fig.16-1, E ≃ 5 MeV.

16-2

By the definition of average,

$$T = \{\int_0^\infty t dN\}/\{\int_0^\infty dN\},$$

where t = nuclear lifetime and dN = change in the number of atoms due to those that decay between t and t+dt. Now,

$$N = N(0)e^{-Rt}; \quad dN = -N(0)Re^{-Rt}dt.$$

Since $N(0)$ = number of undecayed atoms at $t = 0$,

$$\int_0^\infty dN = N(0),$$

and therefore,

$$T = R\int_0^\infty te^{-Rt}dt = \frac{1}{R}.$$

16-3

Let τ = half-life; the number of atoms that decay in the time interval $0 \le t \le \tau$ must equal the number that decay in the interval $\tau \le t \le \infty$:

$$\int_0^\tau NRdt = \int_\tau^\infty NRdt.$$

Substituting $N = N(0)e^{-Rt}$ and integrating gives

$$e^{-R\tau} - 1 = e^{-R\tau},$$

$$\tau = \frac{\ln 2}{R} = T\ln 2.$$

16-4

Let $N(t)$ = number of nuclei present at time t. Then $-NRdt$ = the number added in the time from t to t+dt due to decay, and Idt = number added from the cyclotron. If dN = total change in the number of nuclei in this same time interval, then

$$dN = -NRdt + Idt; \quad dt = \frac{dN}{I - NR},$$

$$-\frac{1}{R}\ln(I - NR) = t + K.$$

But $N(0) = 0$ so that $K = -\ln(I)/R$ and therefore

$$-\frac{1}{R}\ln(I - NR) = t - \frac{\ln I}{R}; \quad N(t') = \frac{I}{R}(1 - e^{-Rt'}).$$

16-7

(a) Currently,

$$\text{\# atoms of Thorium} = N = \frac{10^3}{232}N_{Av} = 2.596 \times 10^{24};$$

$$\text{\# atoms of lead} = N_{Pb}' = \frac{200}{208}N_{Av} = 5.791 \times 10^{23}.$$

In the above, $N_{Av} = 6.02 \times 10^{23}$. If few nuclei are "on the way" (i.e. have left $^{90}Th^{232}$ but not yet arrived as $^{82}Pb^{208}$), the original number $N(0)$ of thorium atoms is

$$N(0) = (2.596 + 0.579) \times 10^{24} = 3.175 \times 10^{24}.$$

Then,

$$N = N(0)e^{-Rt} = N(0)e^{-t\ln 2/T_{1/2}},$$

$$t = T_{1/2}(\ln 2)^{-1}\ln\frac{N(0)}{N} = \frac{1.4 \times 10^{10}}{\ln 2}\ln\left(\frac{3.175}{2.596}\right),$$

$$t = 4 \times 10^9 \text{ yr.}$$

(b) The # alpha-particles = (# nuclei that have decayed)(6) = (6)(5.791×10^{23}) = 3.475×10^{24}. The atomic weight of the helium is 4, and therefore,

$$\text{\# grams of } {}^2He^4 = \frac{3.475 \times 10^{24}}{6.023 \times 10^{23}}(4) = 23 \text{ g}.$$

(c) Assuming radioactive equilibrium,

$$N_{Th}R^{Th} = N_{Ra}R^{Ra}; \quad N_{Ra} = (2.596 \times 10^{24})\frac{5.7}{1.4 \times 10^{10}},$$

$$N_{Ra} = 10.57 \times 10^{14}.$$

16-8

For A atoms,

$$\frac{dN_A}{dt} = - R_A N_A \rightarrow N_A = N_{A0} e^{-R_A t}.$$

The number of B atoms increases due to additional A atoms that decay to B atoms, but decreases as B atoms decay to C atoms; hence,

$$\frac{dN_B}{dt} = -R_B N_B + R_A N_A = -R_B N_B + N_{A0} R_A e^{-R_A t}.$$

Multiply by $e^{R_B t}$ and integrate:

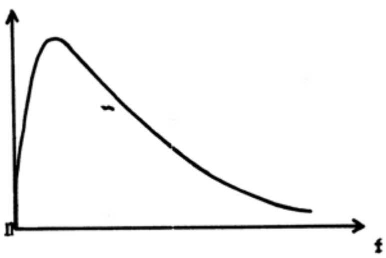
$$\int_0^t (R_B N_B + \frac{dN_B}{dt}) e^{R_B t} dt = \int_0^t N_{A0} R_A e^{(R_B - R_A)t} dt,$$

$$\int_0^t \frac{d}{dt}(N_B e^{R_B t}) dt = \int_0^t N_{A0} R_A e^{(R_B - R_A)t} dt,$$

$$N_B e^{R_B t} - N_{B0} = N_{A0} R_A \left\{\frac{e^{(R_B - R_A)t}}{R_B - R_A} - \frac{1}{R_B - R_A}\right\}.$$

But $N_{B0} = 0$ by assumption. Therefore,

$$N_B = \frac{N_{A0} R_A}{R_B - R_A}(e^{-R_A t} - e^{-R_B t}).$$



16-10

(a)

	Z=12,N=15	Z=13,N=14	Z=14,N=13
1.007825 Z	12.0939	13.101725	14.10955
1.008665(A-Z)	15.129975	14.12131	13.112645
$-a_1 A$	−0.45657	−0.45657	−0.45657
$a_2 A^{2/3}$	0.17199	0.17199	0.17199
$a_3 Z^2 A^{-1/3}$	0.0366239	0.0429823	0.0498493
$a_4 (Z-A/2)^2/A$	0.00847916	0.00094213	0.00094213
$(-1,0,+1)a_5/A^{1/2}$	0	0	0

Summing gives: $M_{12,27} = 26.984397$ u; $E_{12,27} = 25135.965$ MeV;
$M_{13,27} = 26.982379$ u; $E_{13,27} = 25134.086$ MeV;
$M_{14,27} = 26.988406$ u; $E_{14,27} = 25139.700$ MeV.

(b) The smallest M is the most stable: this belongs to Z = 13.

(c) Electron rest mass $m_e = 0.0005486$ u; rest energy $= m_e c^2 =$ 0.511 MeV. Masses in (a) are atomic masses, and so the various decay possibilities are:

(i) Electron emission by Z = 12; $E = (M_{12,27} - M_{13,27})c^2 =$ 25135.965 − 25134.086 = 1.88 MeV.
(ii) Electron capture by Z = 14; $E = (M_{14,27} - M_{13,27})c^2 =$ 25139.7 − 25134.086 = 5.61 MeV.
(iii) Positron emission by Z = 14; $E = (M_{14,27} - M_{13,27} - 2m_e)c^2$ = 5.614 − 2(0.511) = 4.59 MeV.

16-11

(a) By the conservation of momentum, with the initial momentum equal to zero,

$$mV = p_\nu = \frac{E}{c},$$

with E = neutrino energy = (0.00093) (931.5 MeV) = 0.8663 MeV, ignoring the kinetic energy $\frac{1}{2}mV^2$ of the nucleus. Evidently, then,

$$V = \frac{E}{mc} = \frac{(0.8663)(1.602 \times 10^{-13})}{(7)(1.661 \times 10^{-27})(2.998 \times 10^{8})} = 3.98 \times 10^{4} \text{ m/s}.$$

(b) The process may be monitored by detecting the x-ray emission as another electron drops into the hole created by absorption of the K-shell electron, and other transitions.

16-12

(a) For the electron,

$$E^2 = p^2c^2 + m^2c^4.$$

The kinetic energy $K = E - mc^2$ so that

$$K^2 + 2Kmc^2 - p^2c^2 = 0; \quad K = (m^2c^4 + p^2c^2)^{\frac{1}{2}} - mc^2.$$

If $p = nmc$, then

$$K = mc^2\{\sqrt{(1 + n^2)} - 1\}.$$

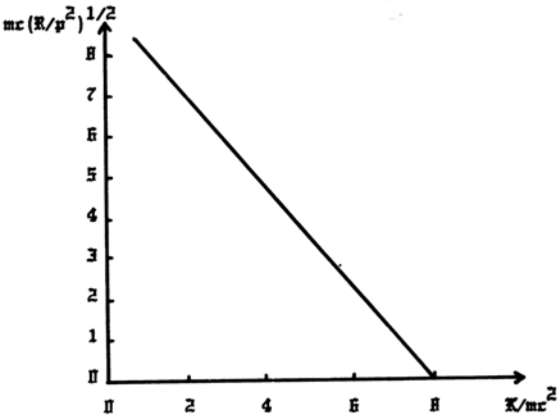
Putting in the indicated numbers gives the following:

p/mc	2.8	4.9	6.9
R	375	500	250
$mc(R/p^2)^{\frac{1}{2}}$	6.916	4.563	2.292
K/mc^2	1.973	4	5.972

(b) From the graph, p.175,

$$K_{max} = (mc^2)(8) = (\tfrac{1}{2} \text{ MeV})(8) = 4 \text{ MeV}.$$

When K, the kinetic energy of the electron, $= K_{max}$ the energy of the antineutrino is zero, so that the decay energy E = 4 MeV.



16-14

(a) From Fig.16-12, $K_{e,max} \simeq 0.8$ MeV. Fig.16-13 has logF = 0.3; Hence,

$$F = 2; \quad T = 1000 \text{ s}; \quad FT = 2000 \text{ s}.$$

In actuality, FT = 2340 s.

(b) The decay is a little slower tha that of $_1H^3$, for which FT = 1200 s.

16-15

(a) Use FT = 2000 s and assume M' = 1. Example 16-5 with FT = 1200 s gets $\beta = 3.7 \times 10^{-62}$ J·m³. Since

$$\beta^2 \propto 1/FT,$$

the value of β for the present case will be 2.9×10^{-62} J·m³.

(b) The result for this process is a little less than for Example 16-5 due to the larger FT.

(c) The shell structure of the initial nucleus, a neutron, is identical with that of the final nucleus, a proton (Coulomb energy is absent with only a single proton as the final nucleus) and thus the eigenfunctions are identical.

16-16

$$\frac{U(\text{magnetic})}{U(\text{electric})} \simeq \frac{-\vec{\mu}\cdot\vec{B}}{-\vec{p}\cdot\vec{E}} \simeq \frac{\mu B}{pE}.$$

Let a = characteristic distance between charges, j = current density and ρ = charge per unit volume,

$$\mu \simeq iA \simeq ia^2 \simeq (ja^2)a^2 = ja^4,$$

$$p \simeq ea \simeq (\rho a^3)a = \rho a^4,$$

and thus

$$\frac{\mu B}{pE} \simeq \frac{ja^4 B}{\rho a^4 E} = \frac{j}{\rho}\frac{B}{E} \simeq \left(\frac{\rho v}{\rho}\right)\frac{B}{cB} = \frac{v}{c},$$

where E = cB for a plane electromagnetic wave in a vacuum.

16-17

$$U_{quad}/U_{dip} = qr'^2 \frac{\partial E}{\partial x}/qr'E.$$

If $E = E_0 \sin 2\pi(\frac{x}{\lambda} - \nu t) \simeq E_0$,

$$\frac{\partial E}{\partial x} = (2\pi E_0/\lambda)\cos 2\pi(\frac{x}{\lambda} - \nu t) \simeq E_0/\lambda,$$

implying that

$$U_{quad}/U_{dip} \simeq qr'^2 E_0/\lambda qr'E_0 = r'/\lambda.$$

16-18

Follow the reasoning on p.580.

(2,even) to (0,even): $|i_i - i_f| = 2 = L$; since parity does not change and L is even, radiation is electric quadrupole (L = 2).

(1,odd) to (0,even): $|i_i - i_f| = 1 = L$; L is odd and the parity changes, indicating electric dipole.

(1,odd) to (2,even); $|i_i - i_f| = 1 = L$; L is odd and the parity changes; as above, this indicates electric dipole radiation.

16-20

The integral in Eq.16-26 is

$$\int_{-\infty}^{+\infty} \psi_f^* x^2 \psi_i d\tau.$$

The parity of x^2 is even; if $\psi_f^*{}_i$ is of odd parity, the parity of the integrand is odd and the integral vanishes. For the integral to be different from zero, ψ_f^* must have the same parity as ψ_i.

The other integral being considered is

$$\int_{-\infty}^{+\infty} \psi_f^* \left(y \frac{\partial z}{\partial t} - z \frac{\partial y}{\partial t}\right)\psi_i d\tau.$$

These parities are:

$$P(y) = -y, \text{ odd}; \quad P\left(\frac{\partial y}{\partial t}\right) = \frac{\partial(-z)}{\partial t} = -\frac{\partial z}{\partial t}, \text{ also odd.}$$

Thus the parity of the operator, involving products of those above, is even. As a consequence, ψ_f^* must have the same parity as ψ_i or the integral vanishes.

16-25

The net reaction is

$$3\,{}_1H^2 + {}_{15}P^{31} \to {}_{14}Si^{31} + 2\,{}_1H^1 + {}_2He^4; \quad Q_{total} = 20.91 \text{ MeV}.$$

That is,

$$\{3M({}_1H^2) + M({}_{15}P^{31}) - 2M({}_1H^1) - M({}_2He^4) - M({}_{14}Si^{31})\}c^2 = 20.91$$

in MeV. Using values from Table 15-1, this becomes

$$\{M(^{15}P^{31}) - M(^{14}Si^{31}) + 0.0240529\}c^2 = 20.91 \text{ MeV}.$$

In terms of rest energies,

$$E(^{15}P^{31}) - E(^{14}Si^{31}) = -1.4953 \text{ MeV}.$$

With the rest energy of the Si isotope greater than that of the P isotope, electron emission can occur, with energy = 1.5 MeV.

16-26

(a) Using the indicated notation,

$$k = \frac{1}{\hbar}[2m\{E - (V_R+iV_I)\}]^{\frac{1}{2}} = \frac{1}{\hbar}\{2m(E-V_R) - i(2mV_I)\}^{\frac{1}{2}},$$

$$k = (a + ib)^{\frac{1}{2}},$$

with

$$a = 2m(E - V_R)/\hbar^2;\ b = -2mV_I/\hbar^2.$$

Therefore,

$$k = (a + ib)^{\frac{1}{2}} = k_R + ik_I;\quad a = k_R^2 - k_I^2;\quad b = 2k_Rk_I.$$

k_R and k_I can be determined in terms of a and b from these equations.

(b) With $k = k_R + ik_I$,

$$\psi^*\psi = e^{ikx} = e^{i(k_R+ik_I)x} = (e^{-k_Ix})e^{ik_Rx},$$

the term in parenthesis being an "amplitude" that decreases exponentially.

(c) From (b),

$$\psi^*\psi = e^{-2k_Ix}.$$

If L = distance for $\psi^*\psi$ to decrease by a factor of 1/e,

$$2k_IL = 1;\quad L = 1/2k_I.$$

16-27

The probability of fission is given by

$$P = \sigma n,$$

σ = cross section per atom = 10^{-28} m^2, n = number of atoms per unit area. If m = mass of one atom = 235 u, then the mass per unit area = nm and

$$P = \sigma\frac{\text{mass per unit area}}{m},$$

$$P = (10^{-28})\frac{0.10}{(235)(1.66 \times 10^{-27})} = 3 \times 10^{-5}.$$

16-28

The solid angle of the detector, at the nucleus, is

$$d\Omega = dA/r^2 = 10^{-5}/(1)^2 = 10^{-5} \text{ sr},$$

$$d\sigma = \frac{d\sigma}{d\Omega}\,d\Omega = 10^{-5}\,\frac{d\sigma}{d\Omega}.$$

The probability of any one proton being scattered into the detector is

$$dP = n\,d\sigma = (9.562 \times 10^{22})\,d\sigma,$$

since

$$n = \frac{0.01}{(63)(1.66 \times 10^{-27})} = 9.562 \times 10^{22} \text{ m}^{-2}.$$

The number of protons incident each second on the target is $(10^{-8}$ C/s$)/(1.6 \times 10^{-19}$ C$) = 0.625 \times 10^{11}$ s^{-1}. Since the number scattered to the detector = 240 min^{-1} = 4 s^{-1},

$$dP = \frac{4}{6.25 \times 10^{10}} = 6.4 \times 10^{-11},$$

and

$$d\sigma = \frac{dP}{n} = \frac{6.4 \times 10^{-11}}{9.562 \times 10^{22}} = 6.7 \times 10^{-34} \text{ m}^2.$$

Finally,

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{10^{-5}} = 6.7 \times 10^{-29}\ m^2/sr = 0.67\ bn/sr.$$

16-29

The number # of decays in 12 hours is

$$\# = RNt,$$

$$\# = \frac{\ln 2}{(1600)(24)(365\tfrac{1}{4})}\{\frac{0.005}{226}(6 \times 10^{23})\}(12),$$

$$\# = 7.87 \times 10^{12}.$$

The energy absorbed is

$$E = (0.9)(7.87 \times 10^{12})(4.87)(1.6 \times 10^{-13}) = 5.52\ J.$$

Hence the dose is

$$\text{Dose} = \frac{5.52 \times 10^2}{75} = 7.36\ \text{rad}.$$

16-30

(a) The probability of decaying via a specific channel $= \Gamma_r/\Gamma$; the total probability of decaying by means of all the other channels is Γ_n/Γ. Clearly,

$$\Gamma = \Gamma_n + \Gamma_r; \quad \Gamma_r = 0.140 - 0.005 = 0.135\ MeV.$$

(b) The cross section is

$$\sigma_r(E_i) = \pi(\frac{\lambda}{2\pi})^2 \frac{\Gamma_n\Gamma_r}{(E-E_i)^2 + \Gamma^2/4} = \pi(\frac{\lambda}{2\pi})^2 \frac{4\Gamma_n\Gamma_r}{2} = 0.01096\lambda^2.$$

But $E_i = 0.29$ eV, and as this is definitely nonrelativistic,

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{(2mE_i)}} = 5.339 \times 10^{-11}\ m.$$

Substituting this into the previous equation yields

$$\sigma_r = 3.12 \times 10^{-23}\ m^2.$$

(c) The Uncertainty principle gives an estimate of the lifetime:

$$\tau = \frac{\hbar}{\Gamma_r} = 4.9 \times 10^{-15}\ s,$$

using the result from (a).

16-31

(a) Eq.16-33 is

$$E = \frac{i(i+1)}{2\mathcal{I}}\hbar^2, \quad i = 0, 2, 4, \ldots$$

The theoretical and measured ratios are

$$\frac{E_4}{E_2} = \frac{10}{3} \simeq \frac{0.309}{0.093} = 3.323,$$

$$\frac{E_6}{E_4} = \frac{21}{10} \simeq \frac{0.641}{0.309} = 2.074,$$

$$\frac{E_8}{E_6} = \frac{72}{42} \simeq \frac{1.085}{0.641} = 1.693.$$

Evidently, agreement is good in all cases.

(b) Using E_4 as an example:

$$E_4 = \frac{10\hbar^2}{\mathcal{I}}; \quad \mathcal{I} = \frac{10\hbar^2}{E_4} = 2.25 \times 10^{-54}\ kg{\cdot}m^2,$$

since $E_4 = 0.309$ MeV.

16-32

Into Eq.15-16,

$$Q = K_b(1 + \frac{m_b}{m_B}) - K_a(1 - \frac{m_a}{m_B}) - 2(K_aK_bm_am_b/m_B^2)^{\frac{1}{2}}\cos\theta,$$

put

$$Q = 0,\ m_B = 12,\ m_a = m_b = 1,\ \theta = 90°,\ K_a = ?\ \text{MeV},$$

to get

$$0 = K_b(1 + \frac{1}{12}) - K_a(1 - \frac{1}{12});\quad \Delta K = K_b - K_a = -\frac{2}{13}K_a.$$

(a) $K_a = 1$ MeV; $\Delta K = 0.154$ MeV.

(b) $K_a = 0.001$ MeV; $\Delta K = 154$ eV.

(c) $E = 3kT/2 = 3(8.617 \times 10^{-5})(500)/2 = 0.06463$ eV.

(d) Since

$$K_{b1} = K_{a2} = \frac{11}{13}K_{a1};\ K_{b2} = K_{a3} = \frac{11}{13}K_{a2} = (\frac{11}{13})^2 K_{a1}\quad \text{etc.,}$$

it follows that

$$K_f = 0.06463\ \text{eV} = (\frac{11}{13})^n(1\ \text{MeV}),$$

$$-16.5546 = n\ln(\frac{11}{13}) = -0.1671n;\quad n = 99.$$

16-34

(a) $$Q = \{2M(_1H^2) - M(_2He^3) - m_n\}(uc^2),$$

$$Q = (0.00351)(931.5) = 3.27\ \text{MeV}.$$

(b) One Megaton yields 2.6×10^{28} MeV; hence 48 Megatons gives 1.248×10^{30} MeV. The number of required fusions is

$$\# = \frac{1.248 \times 10^{30}}{3.27} = 0.38165 \times 10^{30}.$$

Since two hydrogen atoms are required for each fusion, the minimum mass of hydrogen needed is

$$m = (0.38165 \times 10^{30})(2)\{(2)(1.66 \times 10^{-27})\}$$

$$m = 2534\ \text{kg}.$$

CHAPTER SEVENTEEN

17-1

(a) Eq.7-17, with $\ell = 0$ is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{2\mu}{\hbar^2}(E - V)R = 0.$$

(b) With $R(r) = u(r)/r$,

$$\frac{dR}{dr} = -\frac{u}{r^2} + \frac{1}{r}\frac{du}{dr};\quad \frac{d^2R}{dr^2} = \frac{2u}{r^3} - 2\frac{1}{r^2}\frac{du}{dr} + \frac{1}{r}\frac{d^2u}{dr^2}.$$

Substituting these into the equation for $R(r)$ given in (a) yields

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + Vu = Eu.$$

(c) Eq.5-43, the time independent one-dimensional Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi,$$

identical in form to the equation of (b).

(d) Since $u^*u = (r^*R^*)(rR) = r^2(R^*R)$,

$$\int_{r_1}^{r_2} u^*u\,dr = \int_{r_1}^{r_2}(R^*R)\frac{4\pi}{4\pi}r^2\,dr,$$

$= \frac{1}{4\pi}$(probability of the neutron-proton separation being between r_1 and r_2).

(e) The reduced mass is

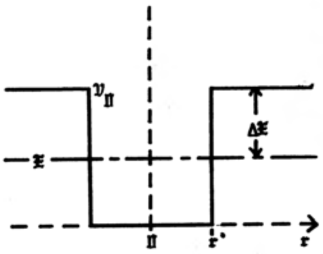
$$\mu = \frac{m_p m_n}{m_p + m_n} \simeq \tfrac{1}{2} m_p.$$

Use of the reduced mass reduces the problem to one of relative motion; i.e., one nucleon is at the center or origin of the coordinates.

17-2

(a) Assume a bound state, $E<V_0$. For $r \leq r'$, $V = 0$ and

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} = Eu.$$



In the region $r > r'$, $V = V_0$ so that

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + V_0 u = Eu.$$

(b) Replace E with $V_0 - \Delta E$; the equations above and their solutions will be:

$$r \leq r': \quad \frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}(V_0 - \Delta E)u = 0; \quad u = A\sin k_1 r + B\cos k_1 r;$$

$$r > r': \quad \frac{d^2u}{dr^2} - \frac{2\mu}{\hbar^2}(\Delta E)u = 0; \quad u = Ce^{-k_2 r} + De^{k_2 r}.$$

(c) From (b)

$$r \leq r': \quad k_1^2 = \frac{2\mu}{\hbar^2}(V_0 - \Delta E); \quad r > r': \quad k_2^2 = \frac{2\mu}{\hbar^2}(\Delta E).$$

17-3

(a),(b) In the solutions given in Problem 17-2,

(i) $D = 0$, otherwise $u \to \infty$ as $r \to \infty$.

(ii) $B = 0$, otherwise $R(r) = u(r)/r \to \pm\infty$ as $r \to 0$.

For the remaining, the conditions at $r = r'$ are

(iii) u continuous at r': $A\sin k_1 r' = Ce^{-k_2 r'}$;

(iv) $\frac{du}{dr}$ continuous at r': $k_1 A\cos k_1 r' = -k_2 Ce^{-k_2 r'}$.

These last two conditions thus imply that

$$k_1 \cot(k_1 r') = -k_2.$$

Substituting the expressions for k_1 and k_2 from Problem 17-2(c) gives the desired relation.

17-4

Making the substitutions indicated in Problems 17-1(e) and 17-3 results in

$$\cot\{\frac{1}{\hbar}(m_p V_0)^{\frac{1}{2}} r'(1 - x)^{\frac{1}{2}}\} = -(\frac{x}{1 - x})^{\frac{1}{2}},$$

with $x = \Delta E/V_0$. Numerically, with $r' = 2$ F $= 2 \times 10^{-15}$ m and $V_0 = 36$ MeV,

$$\frac{1}{\hbar}(m_p V_0)^{\frac{1}{2}} r' = 1.86.$$

Hence, the equation becomes

$$\cot\{1.86(1 - x)^{\frac{1}{2}}\} = -(\frac{x}{1 - x})^{\frac{1}{2}}.$$

The solution found by trial and error is $x = 0.055$, which gives $\Delta E = 2.0$ MeV. Evidently, to obtain 2.2 MeV, either the depth of the well and/or the range of the potential must be altered slightly from the text values, or greater precision is needed in the calculations. In any event, 2.2 MeV will be used in the following problem to maintain conformity with the text.

<u>17-5</u>

(a) With $\Delta E = 2.2$ MeV, $V_0 = 36$ MeV, $r' = 2$ F, the constants k_1 and k_2 are

$$k_1 = 0.90\ F^{-1};\quad k_2 = 0.23\ F^{-1}.$$

Hence,

$$u = A\sin(0.90r),\quad r \le r' = 2\ F;$$
$$u = Ce^{-0.23r},\quad r > r' = 2F.$$

These expressions must be equal at r'; that is,

$$A\sin(1.80) = Ce^{-0.46};\quad C/A = 1.54.$$

Also, it is necessary that

$$\int_0^\infty 4\pi r^2 R^2(r)\,dr = \int_0^\infty 4\pi u^2 dr = 1.$$

Putting in the expressions for u, and C in terms of A yields

$$4\pi A^2\{\int_0^2 \sin^2(0.90r)\,dr + \int_2^\infty (1.54)^2 e^{-0.46r} dr\} = 1.$$

Evaluating the integrals,

$$\int_0^2 \sin^2(0.90r)\,dr = 1.123;\quad \int_2^\infty e^{-0.46r} dr = 0.866.$$

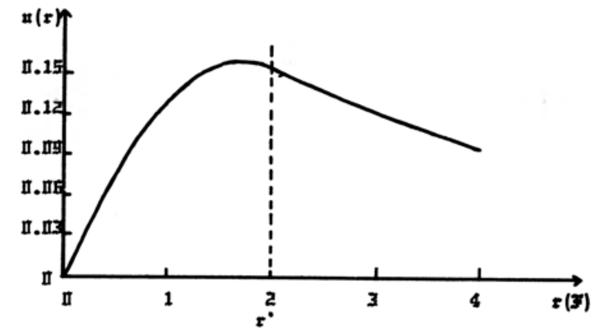
Therefore,

$$4\pi A^2\{(1.123) + (1.54)^2(0.866)\} = 1;\quad A = 0.16.$$

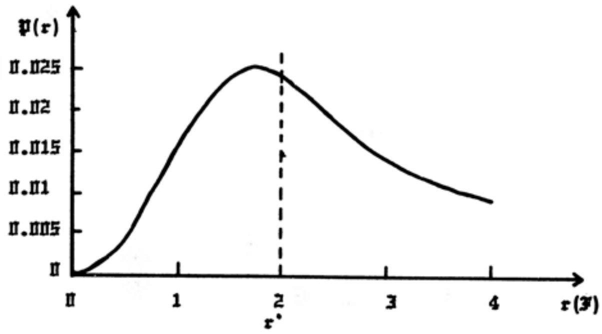
Thus, the final results are, with r in F,

$$u(r) = (0.16)\sin(0.90r),\quad r \le 2,$$
$$u(r) = (0.24)e^{-0.23r},\quad r > 2.$$

(b) For V(r), see Problem 17-2.



The radial probability density $P(r) = u^*u = u^2(r)$.



17-6

In the Lab frame,

$$K_{lab} = \tfrac{1}{2}mv^2 + 0 = \tfrac{1}{2}mv^2.$$

The speed of the center of mass in the Lab frame is

$$v_{cm} = \frac{mv + m(0)}{2m} = \frac{v}{2}.$$

Therefore,

$$K_{cm} = \tfrac{1}{2}m(v - \tfrac{v}{2})^2 + \tfrac{1}{2}m(0 - \tfrac{v}{2})^2 = \tfrac{1}{4}mv^2 = \tfrac{1}{2}(\tfrac{1}{2}mv^2),$$

$$K_{cm} = \tfrac{1}{2}K_{lab} = \tfrac{1}{2}K.$$

17-7

By Example 17-1,

$$K = \frac{\ell(\ell + 1)\hbar^2}{2Mr'^2}; \quad \frac{\hbar^2}{2Mr'^2} \simeq 5 \text{ MeV}.$$

It follows that

(a) $K \geq 30$ MeV gives $\ell \geq 2$, so that for $K < 30$ MeV, $\ell_{max} = 1$.

(b) $K \geq 60$ MeV, $\ell \geq 3$, and if $30 \leq K \leq 60$ MeV, $\ell_{max} = 2$.

17-8

(a)

$$K = \frac{L^2}{2mr'^2} = \frac{\ell(\ell + 1)\hbar^2}{2mr'^2},$$

$$r' = 1.07(100)^{1/3} + 2 \simeq 7 \text{ F}.$$

Taking some numbers from Example 17-1,

$$\frac{\hbar^2}{2mr'^2} = (5 \text{ MeV})\frac{4}{49} = \frac{20}{49} \text{ MeV}.$$

Hence,

$$50 = \ell(\ell + 1)\frac{20}{49}; \quad \ell(\ell + 1) = 122.5; \quad \ell_{max} = 10.$$

(b) The angle is

$$\theta = \frac{\lambda}{r'} = \frac{h/p}{r'}.$$

The kinetic energy $K = 50$ MeV is much less than the rest energy 931.5 MeV and therefore classical expressions can be used. Put $p = \sqrt{(2mK)}$ and substitute into the expression for θ to get

$$\theta = h/r'\sqrt{(2mK)}.$$

With $r' = 7$ F and $K = 50$ MeV the formula gives $\theta = 0.58 = 33°$. From Fig.16-26, $\theta = 30°$, so agreement is good.

17-10

(a) The uncertainty principle implies that

$$\Delta p \Delta x \simeq \tfrac{1}{2}\hbar; \quad \Delta p \simeq \hbar/r',$$

and therefore

$$K_{min} \simeq \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2mr'^2} \propto \frac{1}{r'^2}.$$

(b) It may be anticipated that V is proportional to the product of the number of nucleons within a sphere of radius about 2 F and the nucleon-nucleon potential V_n; that is,

$$V \propto \{\frac{A}{(4/3)\pi r'^3}\frac{4}{3}\pi(2)^3\}V_n,$$

A = number of nucleons. Now V_n depends on the distance between nucleons and not on r', for it is observed that the interior mass density is about the same for all nuclei and therefore so is the average nucleon-nucleon separation. Also, $V_n < 0$ for attractive forces. This indicates that

$$V \propto \frac{1}{r'^3}.$$

(c) The total energy E is

$$E = K + V = \frac{a}{r'^2} - \frac{b}{r'^3},$$

a,b positive. As r' goes to zero, E becomes negative regardless of the relative magnitudes of a,b, since r'^3 approaches zero faster than r'^2. The nucleus would collapse.

17-11

(a) $$T_z = \frac{Z - N}{2}.$$

For ${}^1H^3$, Z = 1, N = 2 so that $T_z = -\frac{1}{2}$. For ${}^2He^3$, Z = 2, N = 1 giving $T_z = +\frac{1}{2}$. T must be half-integral. Since A = 3, T = 1/2, 3/2. If T = 3/2, states with $T_z = \pm 3/2$ would have about the same energy as ${}^1H^3$ and ${}^2He^3$; these would be ${}^0n^3$, ${}^3p^3$ or ${}^3Li^3$. But these nuclei are not observed (either stable or as beta-emitters). Hence, T = $\frac{1}{2}$.

(b) ${}^3Li^7$: Z = 3, N = 4; $T_z = -\frac{1}{2}$. ${}^4Be^7$: Z = 4, N = 3; $T_z = +\frac{1}{2}$. No other nuclei with A = 7 are observed, so T = $\frac{1}{2}$.

17-12

(a) Use the uncertainty principle to obtain an estimate:

$$\Delta E \Delta t = \tfrac{1}{2}h, \quad \Delta E = m_\pi c^2,$$

$$\Delta t = \frac{h}{2m_\pi c^2} = \frac{4.136 \times 10^{-21}}{(2)(140)} = 1.477 \times 10^{-23} \text{ s}.$$

(b) Assume the meson speed to be close to the speed of light. Then a time $\Delta t/2$ takes the meson a distance $r = c(\frac{1}{2}\Delta t) = 2.2$ F. Hence, within 2 F only one meson is expected.

(c) Clearly, up to one-fourth this distance, four may be found.

17-13

$$K^+ \to \pi^0 + \pi^+.$$

Since the rest energies of these particles are 493.8 MeV, 135.0 MeV and 139.6 MeV, the kinetic energy of the pions is, in total, 493.8 - 135.0 - 139.6 = 219.2 MeV. Using 137 MeV as an average rest mass and dividing the kinetic energy equally, for an estimate, then,

$$K = mc^2 - m_0c^2,$$

$$110 = (137)\{(1 - \beta^2)^{-\frac{1}{2}} - 1\},$$

$$\beta = 0.83; \quad v = \beta c = 2.5 \times 10^8 \text{ m/s}.$$

The distance travelled to decay is $(2.5 \times 10^8 \text{ m/s})(8 \times 10^{-17} \text{ s})$ = 20 nm.

17-14

(a) In the LAB frame,

$$v_{cm} = \frac{m_1(0) + m_2v_2}{m_1 + m_2} = \frac{p_2}{m_1 + m_2} = \frac{p_2c^2}{E_1 + E_2}.$$

Thus, the speed of the frame in which the center of mass is at rest is

$$v = v_{cm} = c\,\frac{cp_2}{E_1 + E_2}.$$

(b) $E_{LAB} = E_1 + E_2$. In the CM frame the particle energies are

$$E_{CM1} = \frac{E_1 - \beta cp_1}{(1 - \beta^2)^{\frac{1}{2}}} = \frac{E_1}{(1 - \beta^2)^{\frac{1}{2}}}; \quad E_{CM2} = \frac{E_2 - \beta cp_2}{(1 - \beta^2)^{\frac{1}{2}}}.$$

Thus,

$$E_{CM} = E_{CM1} + E_{CM2} = (1 - \beta^2)^{-\frac{1}{2}}\{(E_1 + E_2) - \beta cp_2\},$$

$$E_{CM} = (1 - \beta^2)^{-\frac{1}{2}}(E_{LAB} - \beta cp_2).$$

By (a),

$$\beta = \frac{v}{c} = cp_2/E_{LAB},$$

and therefore

$$E_{LAB} - \beta cp_2 = E_{LAB}(1 - \beta^2).$$

Since $E_1 = m_0c^2$ and $E_2^2 = c^2p_2^2 + m_0^2c^4$,

$$1 - \beta^2 = 1 - \frac{c^2p_2^2}{E_{LAB}^2} = \frac{(E_1 + E_2)^2 - (E_2^2 - E_1^2)}{E_{LAB}^2} = \frac{2E_1^2 + 2E_1E_2}{E_{LAB}^2},$$

$$1 - \beta^2 = \frac{2E_1}{E_{LAB}} = \frac{2m_0c^2}{E_{LAB}}.$$

These give,

$$E_{CM} = E_{LAB}(2m_0c^2/E_{LAB})^{\frac{1}{2}},$$

$$E_{CM} = (2m_0c^2E_{LAB})^{\frac{1}{2}}.$$

17-15

The reaction under consideration is

$$p + p \rightarrow p + p + p + \bar{p}.$$

In the CM frame, immediately after the reaction the four protons could all be at rest. The total energy after reaction in this event is $4m_0c^2$ and therefore this must be the total energy before reaction in the CM frame. Hence,

$$E_{CM} = 4m_0c^2 = (2m_0c^2E_{LAB})^{\frac{1}{2}},$$

$$E_{LAB} = 8m_0c^2.$$

The kinetic energy required in the LAB frame is just $8m_0c^2 - 2m_0c^2 = 6m_0c^2 = 5630$ MeV.

17-16

(a) The neutrino de Broglie wavelength is $\lambda = h/(E/c)$. The time interval Δt during which the neutrino-proton distance is less than the de Broglie wavelength is

$$\Delta t \simeq \frac{2\lambda}{c} = \frac{2h}{E}.$$

With T = characteristic time for the reaction, the probability P of reaction is

$$P = \Delta t/T = 2h/ET,$$

so that

$$\sigma \simeq P\lambda^2 = 2h^3c^2/E^3T = 10^{-43}\ \text{cm}^2.$$

(b) For lead, Z = 82, and $\sigma \simeq (100)(10^{-43}\ \text{cm}^2) = 10^{-41}\ \text{cm}^2$. If ℓ = mean free path,

$$\ell = \frac{1}{\sigma n},$$

n = number of nuclei per unit volume. But the density of lead is 11.36 g/cm^3 and hence the mean free path becomes about 10^{18} cm.

17-18

(a) Forbidden: baryon number not conserved.

(b) Forbidden: lepton number not conserved.

(c) Forbidden: lepton number not conserved.

(d) Forbidden: $\Delta T_z = +1$; $\Delta S = -2$.

(e) Satisfies all conservation laws; strong interaction would be fastest.

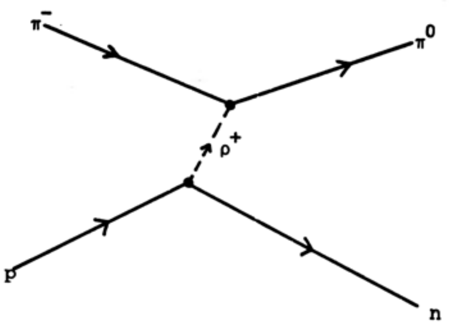
(f) Forbidden: $\Delta T_z = -1$; $\Delta S = 2$.

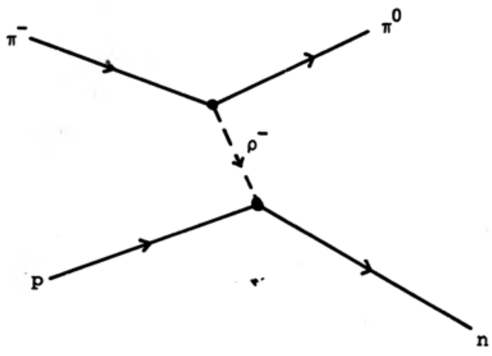
(g) Forbidden: rest energy of daughters exceeds that of K^0.

CHAPTER EIGHTEEN

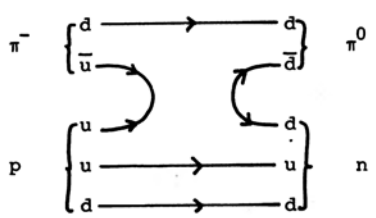
18-3

(a) ρ^+, ρ^- are antiparticles. The absorption of a ρ^+ is equivalent to the emission of a ρ^-.





(b)



18-4

q	u	d	s	$\bar{u}$	ds	$\bar{d}$	us	$\bar{s}$	ud
T_z	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
Y	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

Thus $\bar{q}_j$ and $q_i q_k$ have the same T_z and $Y = S + B$ quantum numbers. Then the meson octet $q_j\bar{q}_j$ and the baryon octet $q_i q_j q_k$ have the same T_z and Y quantum numbers.

18-5

(a)

	$\bar{p}$	$\bar{u}$	$\bar{u}$	$\bar{d}$	Σ^+	u	u	s	ρ^-	d	$\bar{u}$
Q	-1	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$+1$	$\frac{2}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	-1	$-\frac{1}{3}$	$-\frac{2}{3}$
T_z	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	1	$\frac{1}{2}$	$+\frac{1}{2}$	$+0$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
B	-1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$
S	0	0			-1	0	$+0$	-1	0	0	$+0$

(b) The π is composed of a quark and antiquark in the anti-parallel spin state $^1S_0$. The quark and antiquark in a ρ have parallel spin: $^3S_1$.

18-6

$$I = (Ne)\left(\frac{c}{2\pi r}\right) = 6.87 \times 10^{11},$$

electrons in each beam. By Problem 18-2,

$$\frac{dI}{I} = -n\sigma dx = -n\sigma c\ dt,$$

where $n = N/2\pi rA$ = number of electrons per unit volume. Hence,

$$L = nc = \frac{Nf}{A},$$

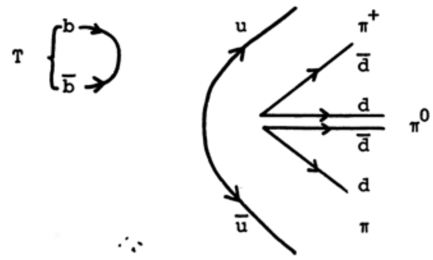
with f = frequency of revolution = $c/2\pi r$. Since $A = 10^{-6}\ m^2$,

$$L = Nf/A = 9.37 \times 10^{22}\ /m^2\cdot s.$$

18-9

Expect the s-quark in a Σ to be replaced by a c-quark in Σ_c: Σ_c^{++} = uuc; Σ_c^0 = ddc; Σ_c^+ = udc.

18-11



Energy conservation prevents decay into a particle with b quarks, leaving Zweig forbidden decays possible.

18-13

The substitution of $\Psi = e^{-i\theta}\Psi'$ implies first derivatives of the form

$$\frac{\partial\Psi}{\partial u} = e^{-i\theta}\left(\frac{\partial}{\partial u} - i\frac{\partial\theta}{\partial u}\right)\Psi',$$

for u = x and t. Then the one-dimensional time-dependent Schroedinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t},$$

implies

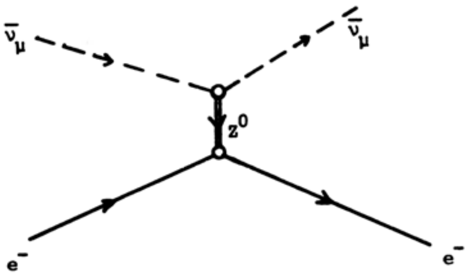
$$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x} - i\frac{\partial\theta}{\partial x}\right)^2\Psi' = i\hbar\left(\frac{\partial}{\partial t} - i\frac{\partial\theta}{\partial t}\right)\Psi'.$$

18-14

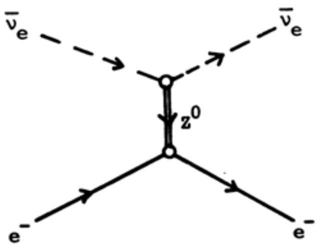
The three combinations of the type rr → rr, normalization 1/6, each have, by Fig. 18-21(c), color charge product $2\chi^2/3$, implying a total contribution $3\left(\frac{1}{6}\right)(2\chi^2/3) = \frac{\chi^2}{3}$. Expanding $(rb + br)^2$, there are two square terms of the form Fig.18-21(b), rb → rb, giving color charge $2\left(-\frac{\chi^2}{3}\right) + 2(\chi^2) = \frac{4}{3}\chi^2$. The ry and yb terms of the same form contribute the same color charge. Since the normalization is $\frac{1}{12}$, the contribution is $3\left(\frac{1}{12}\right)\left(\frac{4}{3}\chi^2\right) = \frac{1}{3}\chi^2$. Thus the total non binding potential is $\frac{1}{3}\frac{\chi^2}{r} + \frac{1}{3}\frac{\chi^2}{r} = \frac{2}{3}\frac{\chi^2}{r}$.

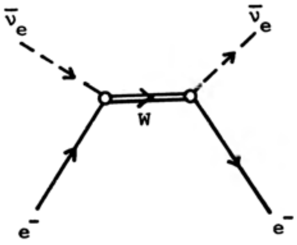
18-17

In $\bar{\nu}_\mu + e^- \to e^- + \bar{\nu}_\mu$, μ and e lepton conservation prevent charge exchange.

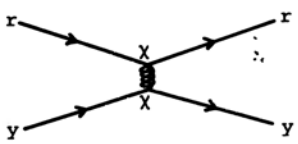


$\bar{\nu}_e + e^- \rightarrow e^- + \bar{\nu}_e$ may proceed by either of the following schemes.





18-18



$$(r\bar{r} - y\bar{y})/\sqrt{2}$$
$$+$$
$$(r\bar{r} + y\bar{y} - 2b\bar{b})/\sqrt{6}$$

The $(r\bar{r} - y\bar{y})/\sqrt{2}$ gluon couplings to the top, $\chi/\sqrt{2}$, and bottom vertex, $-\chi/\sqrt{2}$, contribute $-\chi^2/2$ to the color charge product. The other gluon couples to the top, $\chi/\sqrt{6}$, and bottom, $\chi/\sqrt{6}$, contributing $\chi^2/6$ for a total $-\chi^2/3$ for both gluons.

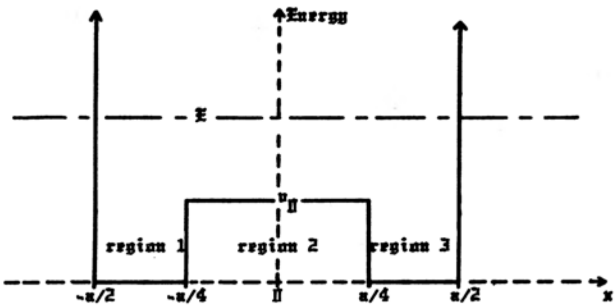
18-19

Given $d_C = d\cos\theta_C + s\sin\theta_C$ and $s_C = s\cos\theta_C - d\sin\theta_C$, the strangeness changing part of $u\bar{u} + c\bar{c} + d_C\bar{d}_C + s_C\bar{s}_C$ is contained in the two terms:

$$d_C\bar{d}_C + s_C\bar{s}_C = (d\bar{d} + s\bar{s})\cos^2\theta_C + (s\bar{s} + d\bar{d})\sin^2\theta_C$$
$$+ (s\bar{d} + \bar{s}d - \bar{s}d - s\bar{d})\sin\theta_C\cos\theta_C.$$

In this last term, the unwanted $\Delta S = 1$ neutral currents cancel.

H-3



For the flat infinite well, $E_1 = \pi^2\hbar^2/2ma^2$ and this is greater than V_0. By Problem 5-25, a bump increases the energy of the eigenfunctions. Hence, in this problem, take $E > V_0$. The form of the wavefunction in the various regions will be:

region 1: $\psi = A\sin k_1x + B\cos k_1x$,

region 2: $\psi = C\sin k_2x + D\cos k_2x$,

region 3: $\psi = A'\sin k_1x + B'\cos k_1x$,

with $k_1 = (2mE)^{1/2}/\hbar$,

$k_2 = \surd\{2m(E-V_0)\}/\hbar$.

Since $V(a/2) = V(-a/2) = \infty$, $\psi(a/2) = \psi(-a/2) = 0$, giving

$$-A\sin(k_1a/2) + B\cos(k_1a/2) = 0,$$

$$A'\sin(k_1a/2) + B'\cos(k_1a/2) = 0,$$

so that

$$A'B + AB' = 0.$$

At the points $x = a/4$ and $x = -a/4$ the wavefunction and its derivative with respect to x must be continuous. Using the expressions for the wavefunctions in the appropriate regions as given above, these requirements yield the following:

$$A'\sin(k_1a/4) + B'\cos(k_1a/4) = C\sin(k_2a/4) + D\cos(k_2a/4),$$

$$-A\sin(k_1a/4) + B\cos(k_1a/4) = -C\sin(k_2a/4) + D\cos(k_2a/4),$$

$$k_1A'\cos(k_1a/4) - k_1B'\sin(k_1a/4) = k_2C\cos(k_2a/4) - k_2D\sin(k_2a/4),$$

$$k_1A\cos(k_1a/4) + k_1B\sin(k_1a/4) = k_2C\cos(k_2a/4) + k_2D\sin(k_2a/4).$$

Write $s_1 = \sin(k_1a/4)$, $c_2 = \cos(k_2a/4)$ etc.; then the sum and difference of the first two equations above, and the sum and difference of the last two equations above are

$$(A' - A)s_1 + (B' + B)c_1 = 2Dc_2, \tag{1}$$

$$(A' + A)s_1 + (B' - B)c_1 = 2Cs_2, \tag{2}$$

$$(A' + A)c_1 - (B' - B)s_1 = 2(k_2/k_1)Cc_2, \tag{3}$$

$$(A' - A)c_1 - (B' + B)s_1 = -2(k_2/k_1)Ds_2. \tag{4}$$

Divide equation (1) by (4) to get

$$\frac{(A' - A)s_1 + (B' + B)c_1}{(A' - A)c_1 - (B' + B)s_1} = -\frac{k_1}{k_2}\frac{c_2}{s_2}.$$

If $B' + B \neq 0$, this becomes,

$$\frac{1 + \frac{A' - A}{B' + B}\tan_1}{\frac{A' - A}{B' + B} - \tan_1} = -\frac{k_1}{k_2}\cot_2.$$

Let

$$\frac{A' - A}{B' + B} = \tan\theta;$$

then, in terms of θ, the previous expression becomes

$$\cot(\theta - k_1a/4) = -\frac{k_1}{k_2}\cot(k_2a/4). \tag{I}$$

Similarly, divide equation (2) by (3); if $A' + A \neq 0$ and the angle ϕ is defined by

$$\frac{B' - B}{A' + A} = \cot\phi,$$

then the result will be

$$\cot(\phi - k_1a/4) = \frac{k_1}{k_2}\tan(k_2a/4). \qquad \text{(II)}$$

Now equations (I) and (II) cannot be satisfied simultaneously for, if they were, they would imply

$$-(k_1/k_2)^2 = \cot(\theta - k_1a/4)\cot(\phi - k_1a/4).$$

But

$$\cot\theta - \cot\phi = \frac{B' + B}{A' - A} - \frac{B' - B}{A' + A} = 2\,\frac{AB' + A'B}{A'^2 - A^2},$$

(see conditions at $x = a/2, -a/2$) and therefore

$$\theta = \phi; \quad -(k_1/k_2)^2 = \cot^2(\theta - k_1a/4).$$

Now k_1, k_2, a are real; hence if θ is real (I) and (II) cannot hold together. Hence, <u>either</u>

$$A' = -A,\ B' \neq -B; \quad \cot(\theta - k_1a/4) = -\frac{k_1}{k_2}\cot(k_2a/4), \qquad \text{(I)}$$

<u>or</u>

$$A' \neq -A,\ B' = -B; \quad \cot(\theta - k_1a/4) = \frac{k_1}{k_2}\tan(k_2a/4). \qquad \text{(II)}$$

Also, from the relation $A'B + AB' = 0$,

$$B = B',\ \text{(I)};\quad A = A',\ \text{(II)};$$

$$C = 0,\ \text{(I)};\quad D = 0,\ \text{(II)}.$$

Hence, the solutions are

$$\left.\begin{array}{ll} \text{region 1:} & \psi = A\sin(k_1x) + B\cos(k_1x), \\ \text{region 2:} & \psi = D\cos(k_2x), \\ \text{region 3:} & \psi = -A\sin(k_1x) + B\cos(k_1x). \end{array}\right\} \text{(I), symmetric.}$$

$$\begin{array}{ll} \text{region 1:} & \psi = A\sin(k_1x) + B\cos(k_1x), \\ \text{region 2:} & \psi = C\sin(k_2x), \qquad \text{(II), antisymmetric.} \\ \text{region 3:} & \psi = A\sin(k_1x) - B\cos(k_1x). \end{array}$$

For the lowest energy take the symmetric case (fewer nodes); for (I),

$$\cot\theta = \frac{B' + B}{A' - A} = -\frac{B}{A}.$$

But,

$$\psi(a/2) = -A\sin(k_1a/2) + B\cos(k_1a/2) = 0,$$

$$-\frac{B}{A} = -\tan(k_1a/2) = \cot\theta,$$

$$\theta = \frac{\pi}{2} + k_1\frac{a}{2}.$$

Using this for θ the relation

$$\cot(\theta - k_1a/4) = -\frac{k_1}{k_2}\cot(k_2a/4),$$

becomes

$$-\tan(k_1a/4) = -\frac{k_1}{k_2}\cot(k_2a/4),$$

$$\tan^2(k_1a/4)\tan^2(k_2a/4) = (k_1/k_2)^2.$$

Using the definitions of k_1, k_2, given earlier and letting $z = E/v_0$, the last equation may be written

$$\tan^2(\tfrac{\pi}{8}\sqrt{z})\tan^2\{\tfrac{\pi}{8}\sqrt{(z - 1)}\} = \frac{z}{z - 1}.$$

The solution is $z = 4.8$, approximately, and hence $E = 4.8v_0$.