The First Way: Changing the Basis

Consider a system described by the Hamiltonian $\mathbf{H}$

$$\mathbf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and by a second observable operator $\mathbf{\Omega}$

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

At $t = 0$, this system is initially in the state $\mid \psi(t = 0) >$ represented by $N \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

(1) Change the basis so the Hamiltonian is diagonal in the new basis. Then $\mathbf{H}$ becomes

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2) Transform $\mathbf{\Omega}$ into the new basis.

(3) Transform $\mid \psi(t = 0) >$ into the new basis.

(4) Follow the procedure for the case where $\mathbf{H}$ is diagonal illustrated in the old exams.

The Second Way: Without Changing the Basis

Consider a system described by the Hamiltonian $\mathbf{H}$

$$\mathbf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and by a second observable operator $\mathbf{\Omega}$

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

At $t = 0$, this system is initially in the state $\mid \psi(t = 0) >$ represented by $N \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

(a) Calculate the normalization constant N.

$$3^2 + 4^2 = 5^2 \text{ so } N = \frac{1}{5}$$

(b) Find the eigenvalues and the normalized eigenvectors of the Hamiltonian operator.

$$\text{eigenvalue } E_1 = 1 \quad \text{with eigenvector} \mid E_1 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{eigenvalue } E_2 = -1 \quad \text{with eigenvector} \mid E_2 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(c) Find the eigenvalues and the normalized eigenvectors of the Omega operator.

$$\text{eigenvalue } \omega_1 = 1 \quad \text{with eigenvector} \mid \omega_1 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\text{eigenvalue } \omega_2 = -1 \quad \text{with eigenvector} \mid \omega_2 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(d) Write down the $t = 0$ state vector $\mid \psi(0) >$ in the energy eigenbasis

$$\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} = a\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$a + b = 0.6\sqrt{2} \quad a - b = 0.8\sqrt{2} \quad a = 0.7\sqrt{2} \quad b = -0.1\sqrt{2}$$

$$\mid \psi(0) >= 0.7\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 0.1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(e) Write down the corresponding time-dependent state vector $\mid \psi(t) >$.

$$\mid \psi(t) >= 0.7\begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(-i(1)t/\hbar) - 0.1\begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp(-i(-1)t/\hbar)$$

(f) If you were to measure the energy at time t, what results could you obtain?

You would always obtain one of the energy eigenvalues $E = 1$ or $E = -1$

(g) With what probabilities would you obtain them?

$$P(E = 1) = |< E = 1 \mid \psi(t) >|^2$$

$$P(E = -1) = |< E = -1 \mid \psi(t) >|^2$$

(h) What would the state vector be right after an energy measurement?

It would be in the corresponding energy eigenstate $\mid E = 1 >$ or $\mid E = -1 >$

(i) If you were to measure the omega-ness at time t, what results could you obtain?

You would always obtain one of the omega eigenvalues $\omega = 1$ or $\omega = -1$

(j) What would the state vector be right after an omega measurement?

It would be in the corresponding omega eigenstate $\mid \omega = 1 >$ or $\mid \omega = -1 >$

(k) With what probabilities would you obtain them?

$$P(\omega = 1) = |< \omega = 1 \mid \psi(t) >|^2$$

$$P(\omega = -1) = |< \omega = -1 \mid \psi(t) >|^2$$

(l) Example calculation $P(\omega = 1) = |< \omega = 1 \mid \psi(t) >|^2 \ =$

$$= \left| \frac{1}{\sqrt{2}}(1, i)^* \left[0.7 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(-i(1)t/\hbar) - 0.1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp(-i(-1)t/\hbar) \right] \right|^2$$

QUANTITATIVE ASPECTS (130 points)

Consider a system described by the Hamiltonian $\mathbf{H}$ and by a second observable operator $\mathbf{\Lambda}$ with

$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is initially in the state $\mid \psi(t=0) >= N \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(a) Calculate the normalization constant N.

(b) Find the eigenvalues and the eigenvectors of the Hamiltonian operator. Whether you find them by inspection, or by full calculation, show that they work!

(c) Find the eigenvalues and the eigenvectors of the Lambda operator. Whether you find them by inspection, or by full calculation, show that they work!

(d) Calculate the 2×2 matrices that represent the two projection operators $\mathbf{P}_{E_i} =\mid E_i >< E_i \mid$ and $\mathbf{P}_{\lambda_i} =\mid \lambda_i >< \lambda_i \mid$ for $i=1$ and for $i=2$.

(e) Show that the eigenvectors of $\mathbf{H}$ and the eigenvectors of $\mathbf{\Lambda}$ both form a basis for the two dimensional space by showing that the sum of each of their projection operators is the identity operator, *i.e.*, that $\mathbf{I} = \sum \mathbf{P}_{E_i} = \sum \mathbf{P}_{\lambda_i}$.

(f) Do $\mathbf{H}$ and $\mathbf{\Lambda}$ commute? I'm not looking for a yes or no answer: please show that they do commute, or that they do not commute.

(g) If you were to measure the energy at time $t=0$, what results would you obtain, and with what probabilities would you obtain them?

(h) If instead you were to measure the lambda-ness at time $t=0$, what results would you obtain, and with what probabilities would you obtain them?

(i) Calculate the expectation values of $\mathbf{H}$ and $\mathbf{\Lambda}$ at $t=0$ and show that they agree with the probabilities and the eigenvalues you obtained above, *i.e.*, that $< \mathbf{H} >= \sum E_i \, P(E_i)$ and that $< \mathbf{\Lambda} >= \sum \lambda_i \, P(\lambda_i)$.

(j) Calculate $\Delta\mathbf{H}$ and $\Delta\mathbf{\Lambda}$ and show that your results agree with the values you obtained above, *i.e.*, that $\Delta\mathbf{\Omega} = \sqrt{\sum P(\omega_i) \, (\omega_i - < \mathbf{\Omega} >)^2}$ for $\mathbf{\Omega} = \mathbf{H}$ and $\mathbf{\Lambda}$.

(k) Sketch $P(E_i)$ versus E and $P(\lambda_i)$ versus λ_i. Indicate your values of $< \mathbf{H} >$ and $\Delta\mathbf{H}$ and your values of $< \mathbf{\Lambda} >$ and $\Delta\mathbf{\Lambda}$ on the respective sketches.

(l) Now expand $\mid \psi(0) >$ in the energy eigenbasis and calculate the time evolution of $\mid \psi(t) >$.

(m) Finally, calculate the results and the probabilities that would be obtained for energy measurements and for λ measurements at time t. Explain why the energy measurements are time independent, but the λ measurements are not.

SJB

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$\boxed{a}$ $\psi = N\begin{pmatrix}3\\4\end{pmatrix}$ $\langle\psi|\psi\rangle = 1$ $N^2(9+16) = 1$ $\boxed{N = \frac{1}{5}}$ +10

$\boxed{b}$ $H = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ Since this is already diagonalized, I can read off the eigen values, 1 & -1 w/ corresponding eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, respectively. +10

e.g. $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -1\begin{pmatrix}0\\1\end{pmatrix}$

$\boxed{c}$ $\Lambda = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \lambda\begin{pmatrix}a\\b\end{pmatrix}$

$b = \lambda a$ Sol's $\lambda = 1$ & $b = a$ or

$a = \lambda b$ $\lambda = -1$ & $b = -a$

So $1, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $-1, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ +10

$\boxed{d}$ $P_{E_1} = |E_1\rangle\langle E_1| = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ ✓ +10

$P_{E_2} = \begin{pmatrix}0\\1\end{pmatrix}(0\ 1) = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$ ✓

$P_{\lambda_1} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1\ 1) = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$ ✓ $P_{\lambda_2} = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1\ -1) = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$ ✓

$\boxed{e}$ $\sum_i P_{E_i} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I$ ✓ +10

$\sum_i P_{\lambda_i} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I$

$\boxed{f}$ $[H, \Lambda] = H\Lambda - \Lambda H$

$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 2\\-2 & 0\end{pmatrix}$ Nope, they don't commute. +10

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g $|\psi\rangle = \frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix} = \frac{1}{5}\left[3|E=1\rangle + 4|E=-1\rangle\right]$

You could measure $E=1$ w/ a probability of $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$

or you could get -1 w/ a prob. of $\left(\frac{4}{5}\right)^2 = \frac{16}{25}$. +10

h You could get $\lambda=1$ w/ prob. $|\langle\lambda=1|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}3\\4\end{pmatrix}\frac{1}{5}\right|^2 = \frac{1}{25}\cdot\frac{1}{2}\cdot(3+4)^2$

$= \boxed{\frac{49}{50} \text{ for } 1}$ ✓

or -1 w/ $P = |\langle\lambda=-1|\psi\rangle|^2 = \frac{1}{50}\left[(1\ -1)\begin{pmatrix}3\\4\end{pmatrix}\right]^2$

$= \frac{1}{50}(3-4)^2 = \boxed{\frac{1}{50} \text{ for } -1}$ ✓ +10

i $\langle H\rangle = \langle\psi|H|\psi\rangle$

$= \frac{1}{25}(3\ 4)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \frac{1}{25}(3\ 4)\begin{pmatrix}3\\-4\end{pmatrix} = \frac{1}{25}(9-16) = \boxed{-\frac{7}{25}}$ ✓

$\sum E_i P(E_i) = 1\cdot\frac{9}{25} + -1\cdot\frac{16}{25} = \frac{1}{25}(9-16) = -\frac{7}{25}$ ✓

$\langle\lambda\rangle = \frac{1}{25}(3\ 4)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \frac{1}{25}(3\ 4)\begin{pmatrix}4\\3\end{pmatrix} = \frac{1}{25}(12+12) = \boxed{\frac{24}{25}}$ ✓ +10

$\sum \lambda_i P(\lambda_i) = \frac{49}{50}\cdot 1 + \frac{1}{50}\cdot -1 = \frac{48}{50} = \frac{24}{25}$ ✓

j $\langle H^2\rangle = \frac{1}{25}(3\ 4)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix} = \frac{1}{25}(3\ 4)\begin{pmatrix}3\\4\end{pmatrix} = \frac{1}{25}(9+16) = 1$

$\Delta H^2 = \langle H^2\rangle - \langle H\rangle^2 = 1 - \left(\frac{7}{25}\right)^2 = \frac{576}{625}$

$\boxed{\Delta H = \frac{24}{25}}$ ✓

$\sqrt{\sum P(E_i)(E_i - \langle H\rangle)^2} = \frac{9}{25}\left(1+\frac{7}{25}\right)^2 + \frac{16}{25}\left(-1+\frac{7}{25}\right)^2 =$

$= \frac{1}{25}\left[9\left(\frac{32}{25}\right)^2 + 16\left(\frac{-18}{25}\right)^2\right]$

$= \frac{9216}{15625} + \frac{5184}{15625} = \frac{14400}{15625}$

$\sqrt{\ }$ of that $= 0.96 = \frac{24}{25}$ ✓

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$\boxed{j}$ (cont.)

$\langle\Lambda^2\rangle$

$= \frac{1}{25}(3\ 4)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} = \frac{1}{25}(3\ 4)\begin{pmatrix}3\\4\end{pmatrix} = 1$

$\Delta\Lambda^2 = \langle\Lambda^2\rangle - \langle\Lambda\rangle^2 = 1 - \left(\frac{24}{25}\right)^2 = \frac{49}{625}$

$\boxed{\cancel{\Delta\Lambda = \frac{1}{5}}}$ $\boxed{\Delta\Lambda = \frac{7}{25}}$ ✓

$\sum P(\lambda_i)\cdot(\lambda_i - \langle\Lambda\rangle)^2 = \frac{49}{50}\left(1-\frac{24}{25}\right)^2 + \frac{1}{50}\left(-1-\frac{24}{25}\right)^2$

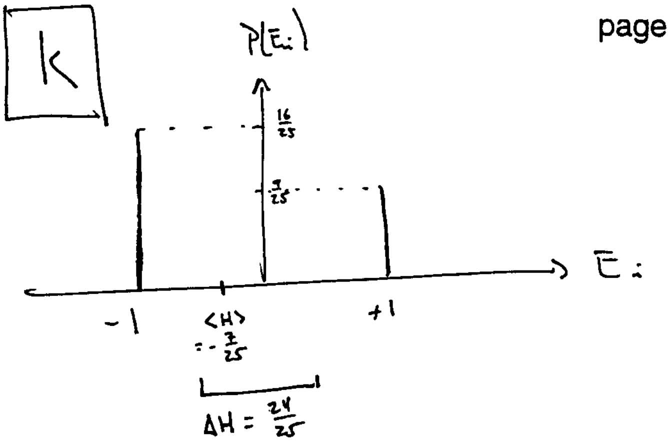
$= \frac{1}{50}\left[49\left(\frac{1}{625}\right) + 1\cdot\left(\frac{2401}{625}\right)\right]$

$= \frac{49}{625}$ $\quad \sqrt{\ } = \frac{7}{25}.$ ✓

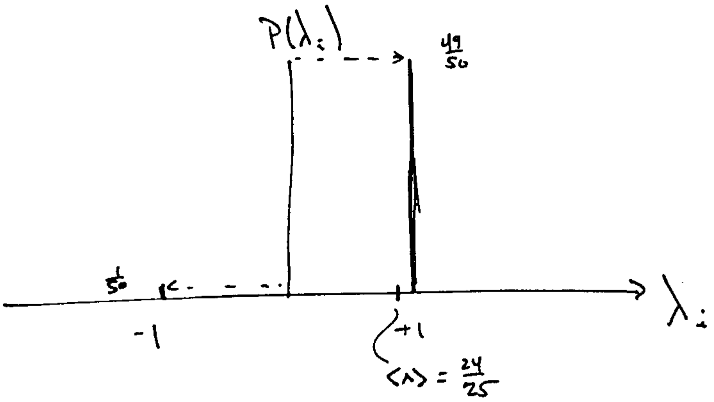
+10

SJB

k



+10



$\Delta\Lambda = \boxed{\frac{7}{25}}$ ~~depending on which method I believe. Algebra error somewhere...~~

l $\quad |\psi(0)\rangle = \frac{1}{5}\left[3|E=1\rangle + 4|E=-1\rangle\right]$ +10

$|\psi(t)\rangle = \frac{1}{5}\left[3|E=1\rangle e^{-\frac{it}{\hbar}} + 4|E=-1\rangle e^{\frac{it}{\hbar}}\right]$

m $\quad \langle E|\psi(t)\rangle = \frac{1}{5}\left[3\langle E|E=1\rangle e^{-\frac{it}{\hbar}} + 4\langle E|E=-1\rangle e^{\frac{it}{\hbar}}\right]$ <1>

$\langle E|$ will be $\langle E=1|$ or $\langle E=-1|$ so one of the terms will be unity and the other will be 0 since the energy eigenvectors are orthogonal. The remaining phase term → 1 w/ the $|\ |^2$, so there is no time dependence left to the energy measurements. You still find 1 w/ prob $\frac{9}{25}$ and -1 w/ P $\frac{16}{25}$.

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$\boxed{M}$ (cont.)

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for λ you don't get that orthogonality:

You can measure 1:

$$\langle\lambda=1|\psi\rangle = \frac{1}{5}\Big[3\underbrace{\langle\lambda=1|E=1\rangle}_{\frac{1}{\sqrt2}(1\ 1)\binom{1}{0}=\frac{1}{\sqrt2}}e^{-\frac{it}{\hbar}} + 4\underbrace{\langle\lambda=1|E=-1\rangle}_{\frac{1}{\sqrt2}(1\ 1)\binom{0}{1}=\frac{1}{\sqrt2}}e^{\frac{it}{\hbar}}\Big]$$

$$= \frac{1}{5\sqrt2}\left(3e^{-\frac{it}{\hbar}}+4e^{\frac{it}{\hbar}}\right)$$

$$|\langle\lambda=1|\psi\rangle|^2 = \frac{1}{50}\left(3e^{\frac{it}{\hbar}}+4e^{-\frac{it}{\hbar}}\right)\left(3e^{-\frac{it}{\hbar}}+4e^{\frac{it}{\hbar}}\right)$$

prob. of $\lambda=1$ $= \boxed{\frac{1}{50}\left(9+16+12e^{\frac{2it}{\hbar}}+12e^{-\frac{2it}{\hbar}}\right)}$

for $\lambda=-1$:

$$\langle\lambda=-1|\psi\rangle = \frac{1}{5}\Big[3\underbrace{\langle\lambda=-1|E=1\rangle}_{\frac{1}{\sqrt2}(1\ -1)\binom{1}{0}=\frac{1}{\sqrt2}}e^{-\frac{it}{\hbar}} + 4\underbrace{\langle\lambda=-1|E=-1\rangle}_{\frac{1}{\sqrt2}(1\ -1)\binom{0}{-1}=-\frac{1}{\sqrt2}}e^{\frac{it}{\hbar}}\Big]$$

$$|\langle\lambda=-1|\psi\rangle|^2 = \frac{1}{50}\left(3e^{\frac{it}{\hbar}}-4e^{-\frac{it}{\hbar}}\right)\left(3e^{-\frac{it}{\hbar}}-4e^{\frac{it}{\hbar}}\right)$$

$$= \frac{1}{50}\left(9+16-12e^{\frac{2it}{\hbar}}-12e^{-\frac{2it}{\hbar}}\right)$$

+10

Ah! I can simplify more.

$$\boxed{P(\lambda=-1,t) = \frac{1}{2} - \frac{6}{25}\cdot 2\cos\left(\frac{2t}{\hbar}\right)}$$ ✓

and

$$\boxed{P(\lambda=1,t) = \frac{1}{2} + \frac{6}{25}\cdot 2\cos\left(\frac{2t}{\hbar}\right)}$$ ✓

(a) Normalize $|\psi(0)\rangle = N\begin{bmatrix}3\\4\end{bmatrix}$

$\langle\psi(0)|\psi(0)\rangle = 1$

$N^2 \begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = 1$

$N^2 (9+16) = 1$

$N^2 = \frac{1}{25} \Rightarrow N = \frac{1}{\sqrt{25}} = \boxed{\frac{1}{5} = N}$ +10 $\Rightarrow \boxed{|\psi(0)\rangle = \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}}$

(b) Find ev and $e\vec{v}$ of H.

$H = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ An operator in its eigenbasis is diagonal with its eigenvalues on the diagonal.

$\therefore$ the ev's of H are +1, −1 eigenvalues

For a diagonal 2x2 matrix the eigenvectors are $\begin{bmatrix}1\\0\end{bmatrix}$ for the 1st one and $\begin{bmatrix}0\\1\end{bmatrix}$ for the 2nd one.

So by inspection the eigenvalues are: +1 −1

with corresponding eigenvectors: $|H=1\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ $|H=-1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ +10

show they work: ie. show $H|E\rangle = E|E\rangle$

for +1: $H|{+1}\rangle = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = (1)\begin{bmatrix}1\\0\end{bmatrix} = +1|{+1}\rangle$ ✓

for −1: $H|{-1}\rangle = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} = (-1)\begin{bmatrix}0\\1\end{bmatrix} = -1|{-1}\rangle$ ✓

(c) Find $\vec{ev}$ and ev of $\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

use $\det(\Lambda - \lambda I) = 0$

$\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0$

$(\lambda+1)(\lambda-1) = 0 \Rightarrow$ ev's are $+1, -1$

find $\vec{ev}$

$\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow \begin{matrix} -\lambda a + b = 0 \\ a - \lambda b = 0 \end{matrix}$

for $+1$, $-a + b = 0 \Rightarrow a = b \Rightarrow \vec{ev}: |\Lambda = +1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ normalized

for -1, $a + b = 0 \Rightarrow a = -b \Rightarrow \vec{ev}: |\Lambda = -1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ normalized

for Λ we have eigenvalues:
w/ corresponding eigenvectors:

$$\boxed{\begin{matrix} +1 & -1 \\ |\lambda=1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} & |\lambda=-1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{matrix}}$$

+10

Show that they work: $\Lambda|\lambda\rangle = \lambda|\lambda\rangle$

for 1, $\Lambda|+1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (+1)|+1\rangle$ ✓

for -1, $\Lambda|-1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)|-1\rangle$ ✓

(d) Calculate the $\mathbb{P}$ operators $\mathbb{P}_{E_i}$ and $\mathbb{P}_{\lambda_i}$ for all i $\qquad \mathbb{P}_\omega = |\omega\rangle\langle\omega|$

$\mathbb{P}_{E=1} = |+1\rangle\langle+1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{P}_{E=1}$ ✓

$\mathbb{P}_{E=-1} = |-1\rangle\langle-1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{P}_{E=-1}$ ✓

$\mathbb{P}_{\lambda=1} = |\lambda=1\rangle\langle\lambda=1| = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \mathbb{P}_{\lambda=1}$ ✓

$\mathbb{P}_{\lambda=-1} = |\lambda=-1\rangle\langle\lambda=-1| = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \mathbb{P}_{\lambda=-1}$ ✓

+10

(e) show that H and Λ form a basis for H,

$$\sum_i \mathbb{P}_{E_i} = \begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = \underline{I} \checkmark$$

+10

$$\sum_i \mathbb{P}_{\lambda_i} = \begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix} + \begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = \underline{I} \checkmark$$

(f) Do H and Λ commute, show

$$[H,\Lambda] = H\Lambda - \Lambda H = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} - \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}0&1\\-1&0\end{bmatrix} - \begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&2\\-2&0\end{bmatrix} \neq 0$$

$\boxed{\text{No, They don't commute}}$ +10

(g) At the t=0, measure E. What results and probabilities?

We would get H's eigenvalues: +1 or −1

with probabilities:

$$P(E) = |\langle E|\psi\rangle|^2$$

$$P(1) = \left|[1\ 0]\tfrac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right|^2 = \left|\tfrac{1}{5}(3+0)\right|^2 = \frac{9}{25}$$

$$P(-1) = \left|[0\ 1]\tfrac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right|^2 = \left|\tfrac{1}{5}(0+4)\right|^2 = \frac{16}{25}$$

$$\frac{9}{25} + \frac{16}{25} = 1 \checkmark$$

If you measure E you get values of:
with probability:

+1	−1
$\frac{9}{25}$	$\frac{16}{25}$

+10

(h) At t=0, measure Λ.

We'd get Λ's eigenvalues: +1 or −1 with prob:

$$P(\lambda) = |\langle\lambda|\psi\rangle|^2$$

$$P(1) = \left|\tfrac{1}{\sqrt{2}}[1\ 1]\tfrac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right|^2 = \left|\tfrac{1}{5\sqrt{2}}(\overset{7}{3+4})\right|^2 = \frac{49}{25(2)} = \frac{49}{50}$$

$$P(-1) = \left|\tfrac{1}{\sqrt{2}}[1\ -1]\tfrac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right|^2 = \left|\tfrac{1}{5\sqrt{2}}(\overset{-1}{3-4})\right|^2 = \frac{1}{25(2)} = \frac{1}{50}$$

$$\frac{49}{50} + \frac{1}{50} = 1 \checkmark$$

If you measure λ-ness you get valus of:
with probability

+1	−1
$\frac{49}{50}$	$\frac{1}{50}$

+10

(i) Calculate the expectation values of H and Λ at $t=0$.

$$\langle H(0)\rangle = \langle \psi(0)|H|\psi(0)\rangle = \frac{1}{5}[3\ 4]\begin{bmatrix}1&0\\0&-1\end{bmatrix}\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{25}[3\ 4]\begin{bmatrix}3\\-4\end{bmatrix} = \frac{1}{25}(\underbrace{9-16}_{-7}) = \boxed{\frac{-7}{25} = \langle H\rangle}$$

$$\langle \Lambda(0)\rangle = \langle \psi(0)|\Lambda|\psi(0)\rangle = \frac{1}{5}[3\ 4]\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{25}[3\ 4]\begin{bmatrix}4\\3\end{bmatrix} = \frac{1}{25}(\underbrace{12+12}_{24}) = \boxed{\frac{24}{25} = \langle \Lambda\rangle}$$

Show they agree with:

$$\langle H\rangle = \sum_i E_i P(E_i) \quad \text{and} \quad \langle \Lambda\rangle = \sum_i \lambda_i P(\lambda_i)$$

+10

$$\langle H\rangle = \sum_i E_i P(E_i) = (+1)\left(\frac{9}{25}\right) + (-1)\left(\frac{16}{25}\right) = \frac{9-16}{25} = \frac{-7}{25}$$ ✓ It agrees

$$\langle \Lambda\rangle = \sum_i \lambda_i P(\lambda_i) = (+1)\left(\frac{49}{50}\right) + (-1)\left(\frac{1}{50}\right) = \frac{49-1}{50} = \frac{48}{50} = \frac{24}{25}$$ ✓ It agrees

(j) calculate ΔH and $\Delta\Lambda$

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$$

$$\langle H^2\rangle = \langle \psi_0|H^2|\psi(0)\rangle = \frac{1}{5}[3\ 4]\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{5}[3\ 4]\overbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}^{I}\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{25}[3\ 4]\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{25}(\underbrace{9+16}_{25}) = 1$$

$$\langle \Lambda^2\rangle = \langle \psi_0|\Lambda^2|\psi(0)\rangle = \frac{1}{5}[3\ 4]\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{25}[3\ 4]\overbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}^{I}\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{25}[3\ 4]\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{25}(\underbrace{9+16}_{25}) = 1$$

So,

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \sqrt{1-\left(\frac{-7}{25}\right)^2} = \sqrt{1-\frac{49}{625}} = \sqrt{\frac{576}{625}} = \boxed{\frac{24}{25} = \Delta H}$$ ✓

$$\Delta \Lambda = \sqrt{\langle \Lambda^2\rangle - \langle \Lambda\rangle^2} = \sqrt{1-\left(\frac{24}{25}\right)^2} = \sqrt{1-\frac{576}{625}} = \sqrt{\frac{49}{625}} = \boxed{\frac{7}{25} = \Delta\Lambda}$$

+10

Show it agrees w/ $\Delta\Omega = \sqrt{\sum P(\omega_i)(\omega_i - \langle\Omega\rangle)^2}$

for H,

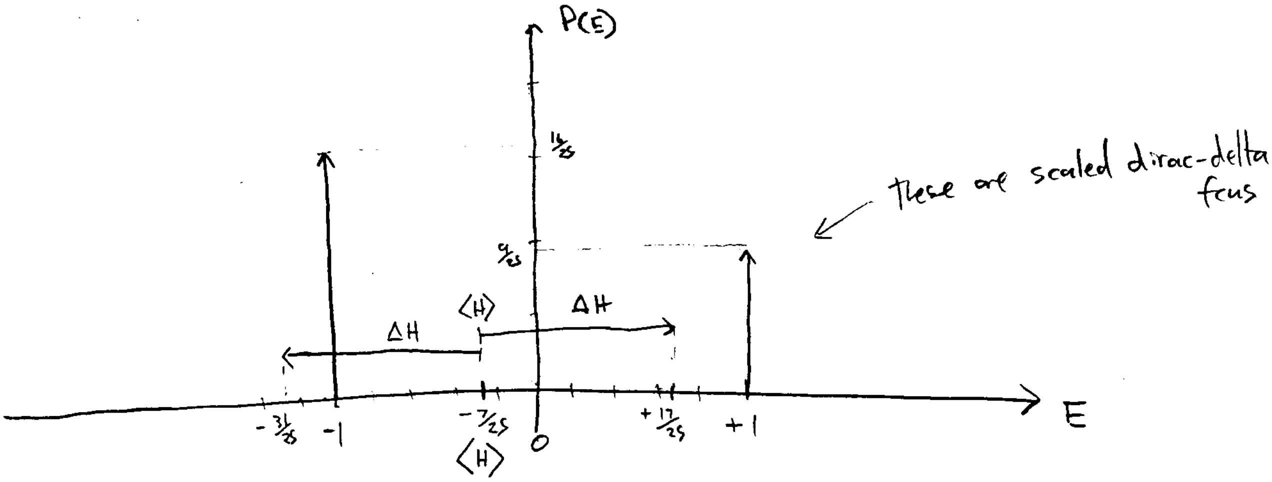
$$\Delta H = \sqrt{\left(\frac{9}{25}\right)\left(1+\left(\frac{+7}{25}\right)\right)^2 + \left(\frac{16}{25}\right)\left(-1+\left(\frac{+7}{25}\right)\right)^2} = \sqrt{\frac{9}{25}\left(\frac{32}{25}\right)^2 + \left(\frac{16}{25}\right)\left(\frac{-18}{25}\right)^2} = \sqrt{\frac{9216+5184}{15625}} = \frac{120}{125}$$

$$= \frac{24}{25}$$ ✓ It agrees

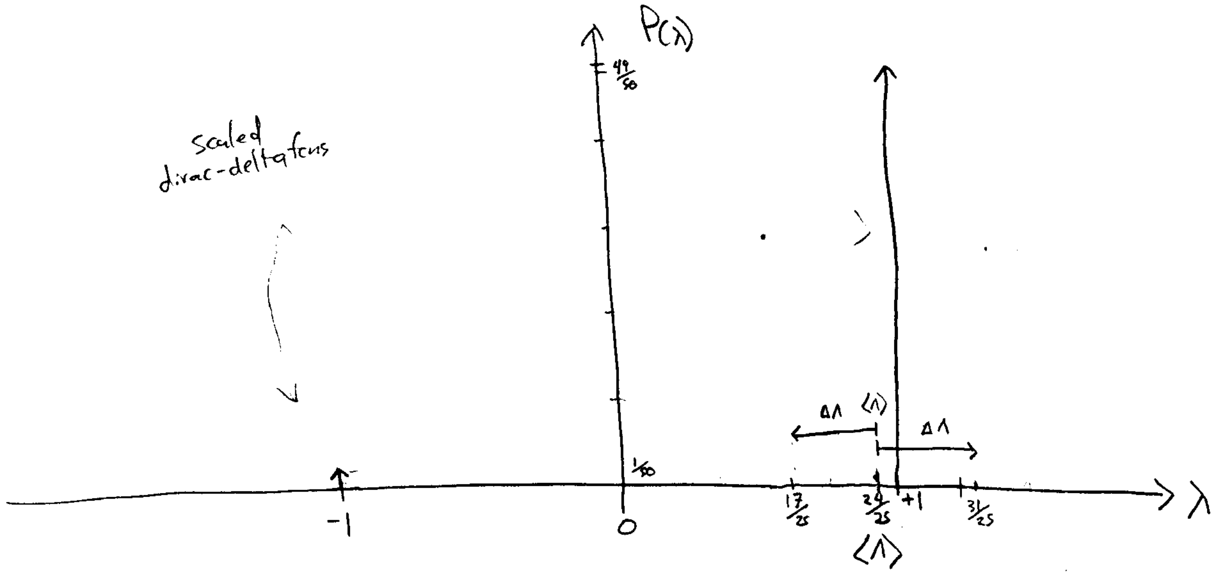
$$\Delta \Lambda = \sqrt{\left(\frac{49}{50}\right)\left(1-\left(\frac{24}{25}\right)\right)^2 + \left(\frac{1}{50}\right)\left(-1-\left(\frac{24}{25}\right)\right)^2} = \sqrt{\left(\frac{49}{50}\right)\left(\frac{1}{25}\right)^2 + \left(\frac{1}{50}\right)\left(-\frac{49}{25}\right)^2} = \sqrt{\frac{49+2401}{31250}} = \sqrt{\frac{1225}{15625}} = \frac{35}{125}$$

$$= \frac{7}{25}$$ ✓ It agrees

(k) Sketch $P(E_i)$ versum E and $P(\lambda_i)$ vs λ_i



+10



(l) Expand $|\psi(0)\rangle$ in the energy eigen basis and find $|\psi(t)\rangle$

$$|\psi(0)\rangle = \sum_{E_i} |E_i\rangle\langle E_i|\psi(0)\rangle = |E=1\rangle\left([1\ 0]\tfrac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right) + |E=-1\rangle\left([0\ 1]\tfrac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right)$$

$$= \tfrac{3}{5}|E=1\rangle + \tfrac{4}{5}|E=-1\rangle$$

+10

Energy eigenstate evolve in time according to $|E(t)\rangle = e^{-iEt/\hbar}|E(0)\rangle$

$\therefore\ |E=1(t)\rangle = e^{-it/\hbar}|E=1(0)\rangle$ and $|E=-1(t)\rangle = e^{it/\hbar}|E=-1(0)\rangle$

$\therefore\ \boxed{|\psi(t)\rangle = \tfrac{3}{5}e^{-it/\hbar}|E=1(0)\rangle + \tfrac{4}{5}e^{it/\hbar}|E=-1(0)\rangle}$ -or- $\tfrac{3}{5}|E=1(t)\rangle + \tfrac{4}{5}|E=-1(t)\rangle = |\psi(t)\rangle$

$|E=1\rangle = [1] \quad |E=-1\rangle\ [0]$

☆ See back of last sheet. ☆

The λ measurements are time dependent because Λ does not commute with the Hamiltonian, which is the generator of time translation. So, measuring Λ is measuring the probability of being in a state $|\lambda\rangle$ which is a combination of energy states and therefore has a time dependence as the energy states' phases make them add relatively (acording to their relative phase).

The E measurements are not time dependent because H commutes with itself. So measuring H is measuring the probability of being in a state $|E\rangle$ which has a simple time dependence with only one phase. So the probability can't change since $|e^{i\theta}| = 1$.