EXAMPLE

$$|\psi(t)\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$H = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Q: IF WE MEASURE energy (momentum) AT TIME t, WHAT RESULTS WILL WE FIND AND WITH WHAT PROBABILITES WILL WE FIND THEM?

STEP 1: CONSTRUCT OPERATORS ✓

STEP 2: FIND EV'S AND $e\vec{v}$'S OF OPERATORS

H

$$ev = 3 \qquad 4 \qquad 5$$

$$e\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

P

$$ev = 2 \qquad 3 \qquad 1$$

$$e\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

WHAT ENERGIES WILL WE FIND? 3, 4, 5

WHAT MOMENTA WILL WE FIND? 2, 3, 1

WITH WHAT PROBABILITIES WILL WE FIND THEM?

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SO, IF WE MEASURE THE ENERGY:

$$P(E=3) = \frac{|\langle \overset{E=3}{\cancel{E_3}} | \psi(t) \rangle|^2}{\langle \psi(t) | \psi(t) \rangle}$$

$$= \frac{\left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \frac{1}{14}$$

$$P(E=4) = \frac{\left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \frac{4}{14}$$

$$P(E=5) = \frac{\left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \frac{9}{14}$$

CHECK $\frac{1}{14} + \frac{4}{14} + \frac{9}{14} = 1$ ✓

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NOW IF WE MEASURE THE MOMENTUM

$$P(p=2) = \frac{\left| (1\ 0\ 0) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \frac{1}{14} = \frac{2}{28}$$

$$P(p=3) = \frac{\left| \frac{1}{\sqrt{2}} (0\ 1\ 1) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \frac{25}{28}$$

$$P(p=1) = \frac{\left| \frac{1}{\sqrt{2}} (0\ 1\ -1) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \frac{1}{28}$$

CHECK $\frac{2}{28} + \frac{25}{28} + \frac{1}{28} = 1$ ✓

H TREE

$$\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\xrightarrow[\frac{1}{14}]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\xrightarrow[4/14]{E=4} \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\xrightarrow[\frac{9}{14}]{E=6} \begin{pmatrix}0\\0\\1\end{pmatrix}$$

MEASURE E AGAIN

$$\begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix}$$

$$\begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix} \to \begin{pmatrix}\ \end{pmatrix}$$

ALWAYS GET SAME ENERGY e_v

STAY IN SAME ENERGY $e_{\vec{v}}$

P TREE

$$\begin{tikzcd}[column sep=large]
 & \begin{pmatrix}1\\0\\0\end{pmatrix} \arrow[r, "p=2", "100\%"'] & \begin{pmatrix}\ \\ \ \\ \ \end{pmatrix} \arrow[r] \\
\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} \arrow[ur, "p=2", "2/28"'] \arrow[r, "p=3", "25/28"'] & \begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[r, "p=3", "100\%"'] & \begin{pmatrix}0\\1\\1\end{pmatrix}
\end{tikzcd}$$

MEASURE IMMEDIATELY

$$\begin{tikzcd}[column sep=large]
 & & p=3 \quad \begin{pmatrix}0\\1\\1\end{pmatrix} \\
\begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[r, "\text{WAIT TIME } t", "\text{THEN MEASURE}"'] & \arrow[ur] \arrow[dr] & \\
 & & p=1 \quad \begin{pmatrix}0\\1\\-1\end{pmatrix}
\end{tikzcd}$$

REPEATED MEASUREMENTS WITH ZERO TIME DELAY

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{1}{14}]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow[100\%]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$|\psi(0^-)\rangle$ $\quad$ $|\psi(0^+)\rangle$ $\quad$ $|\psi(0^{++})\rangle$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{4}{14}]{E=4} \begin{pmatrix}0\\1\\0\end{pmatrix} \xrightarrow[100\%]{E=4} \begin{pmatrix}0\\1\\0\end{pmatrix}$$

IF IN AN $e\vec{v}_A$ OF H, THEN MEASURE ASSOCIATED ev OF H 100% OF THE TIME

IF IN AN $e\vec{v}$ OF P AND YOU MEASURE P_{op} THEN IN GENERAL YOU WILL NOT STAY IN $e\vec{v}$ OF P_{op} AT LATER TIMES

FOR ZERO TIME DELAY

SAME THING FOR P

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{25}{28}]{P=3} \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \xrightarrow[100\%]{P=3} \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$|\psi(0^-)\rangle$ $\qquad$ $|\psi(0^+)\rangle$

FOR ZERO TIME DELAY

WHAT IF YOU MEASURE E, P, E, P, ...

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{1}{14}]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow[100\%]{P=2} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow[100\%]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

H AND P SHARE one $\vec{ev} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{4}{14}]{E=4} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{cases} \xrightarrow[50\%]{P=3} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{cases} \xrightarrow[50\%]{E=4} \begin{pmatrix}0\\1\\0\end{pmatrix} \\ \xrightarrow[50\%]{E=5} \begin{pmatrix}0\\0\\1\end{pmatrix} \end{cases} \\ \xrightarrow[50\%]{P=1} \begin{pmatrix}0\\1\\-1\end{pmatrix} \begin{cases} \xrightarrow[50\%]{E=4} \begin{pmatrix}0\\1\\0\end{pmatrix} \\ \xrightarrow[50\%]{E=5} \begin{pmatrix}0\\0\\1\end{pmatrix} \end{cases} \end{cases}$$

H AND P DO NOT SHARE ~~A~~ E=4 $\vec{ev}$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{9}{14}]{E=5} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{cases} \xrightarrow[50\%]{P=3} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{cases} \xrightarrow{E=4} \\ \xrightarrow{E=5} \end{cases} \\ \xrightarrow[50\%]{P=1} \begin{pmatrix}0\\1\\-1\end{pmatrix} \end{cases}$$

(ZERO TIME DELAY)

IF MEASURE $P, E, P, E, \ldots$

SAME TREES

WHAT HAPPENS IF WE WAIT t BEFORE WE MAKE THE SECOND MEASUREMENT?

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow{E=3} \begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\text{MEASURE } E]{\text{WAIT } t} \begin{pmatrix}?\\?\\?\end{pmatrix}$$

$|\psi(0^-)\rangle$ $\qquad$ $|\psi(0^+)\rangle$ $\qquad$ $|\psi(t)\rangle$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow{P=3} \begin{pmatrix}0\\1\\1\end{pmatrix} \xrightarrow[\text{MEASURE } P]{\text{WAIT } t} \begin{pmatrix}?\\?\\?\end{pmatrix}$$

$\langle \vec{v} \rangle = ?$

$\langle v \rangle = ?$

$$\boxed{\text{TIME EVOLUTION POSTULATE 4}}$$

(1) EXPAND $|\psi(0)\rangle$ IN ENERGY EIGEN BASIS:

$$\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{2}{\sqrt{14}}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{3}{\sqrt{14}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\vec{e}$'s OF H OBEY TISE

$$H\,|E_i\rangle = E_i\,|E_i\rangle$$

$$H\,|E_i\rangle = i\hbar\frac{d}{dt}|E_i\rangle = E_i\,|E_i\rangle$$

$$|E_i(t)\rangle = e^{-iEt/\hbar}\,|E_i(0)\rangle$$

TIME EVOLUTION OF $\vec{e}$'S OF H IS VERY SIMPLE: ONLY THEIR PHASE CHANGES

(2) SIMPLY WRITE DOWN THE TIME DEPENDENCE:

$$|\psi(t)\rangle = \sum e^{-iE_it/\hbar}\,a_i\,|E_i\rangle$$

$$= \frac{1}{\sqrt{14}}e^{-i3t/\hbar}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{2}{\sqrt{14}}e^{-i4t/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$+ \frac{3}{\sqrt{14}}e^{-i5t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

eF's

e$\vec{v}$'s OF H ARE CALLED STATIONARY STATES

eS's

ONLY THEIR PHASE VARIES

$$\xrightarrow{E=3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} \xrightarrow[\text{TWO}]{\text{MEASURE IMMEDIATELY} \quad E=3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} \xrightarrow[\text{THEN MEASURE}]{\text{WAIT } t'} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} \right) e^{-i3t'/\hbar}$$

ALWAYS GET $E=3$

ALWAYS IN STATE $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{i\theta_0}$

TIME EVOLUTION OF e$\vec{v}$ OF H:

$$e^{i\theta_0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{WAIT } t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{i\theta_0} \right] e^{-i3t/\hbar}$$

TIME EVOLUTION OF $\vec{e}\vec{v}$'S OF P

$$\text{E } p=2 \ \vec{e}\vec{v} \text{ OF } P_{op} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = E=3 \ \vec{e}\vec{v} \text{ OF } H_{op}$$

P=2 IS STATIONARY
BECAUSE $|P=2\rangle \equiv |E=3\rangle$
AND $|E=3\rangle$ IS STATIONARY

SO ITS TIME EVOLUTION IS EXACTLY THE SAME

AS THE $E=3$ $\vec{e}\vec{v}$ OF H

$$p=3 \ \vec{e}\vec{v} \text{ OF } P = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

NO EQUAL TO AN $\vec{e}\vec{v}$ OF H
SO $|p=3\rangle$ IS TIME
DEPENDENT

TO FIND
ITS TIME EVOLUTION

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

EXPAND IN $\vec{e}\vec{v}$'S OF H

BECAUSE WE KNOW THEIR
TIME EVOLUTION

PUT IN THEIR
TIME
DEPENDENCE

$$= \frac{1}{\sqrt{2}}\left[0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-i4t/\hbar} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-i5t/\hbar}\right]$$

IF $|\psi(0)\rangle = |p=3\rangle$ THEN $|\psi(t)\rangle \neq |p=3\rangle$

WHAT DOES THIS LOOK LIKE:

$$\frac{1}{\sqrt{2}}\left[e^{-i4t/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-i5t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

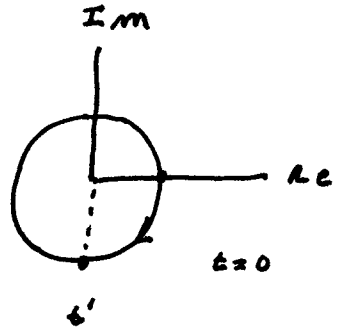
$$t=0 \quad \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

TIME-DEPENDENT PHASE FACTORS

$e^{-i4t/\hbar}$

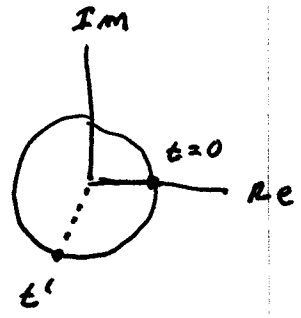
$e^{-i\omega t}$

$\omega = \frac{E}{\hbar}$



$e^{-i5t/\hbar}$

$e^{-i\omega' t}$



SO AT SOME TIME $e^{-i5t/\hbar}$ WILL BE 180°

AHEAD OF $e^{-i4t/\hbar}$

$$\frac{1}{\sqrt{2}}\left[e^{-i4t/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-i5t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$\frac{1}{\sqrt{2}}e^{-i4t/\hbar}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-i\cancel{5}t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}\right] \nearrow \begin{pmatrix}0\\1\\1\end{pmatrix}e^{i\theta}$$

$$\searrow \begin{pmatrix}0\\1\\-1\end{pmatrix}e^{i\theta'}$$

START IN $p = 3$ $e\vec{v}$ OF P_{op}

← IN BETWEEN 50 : 50

LATER IN $p = 1$ $e\vec{v}$ OF P_{op}

← AGAIN IN BETWEEN

BACK IN $p = 3$ $e\vec{v}$ OF P_{op}

JUST LIKE COUPLED PENDULA

IN BETWEEN, YOU ARE

IN A LINEAR COMBINATION

OF $|E=4\rangle$ AND $|E=5\rangle$