

LECTURE 7

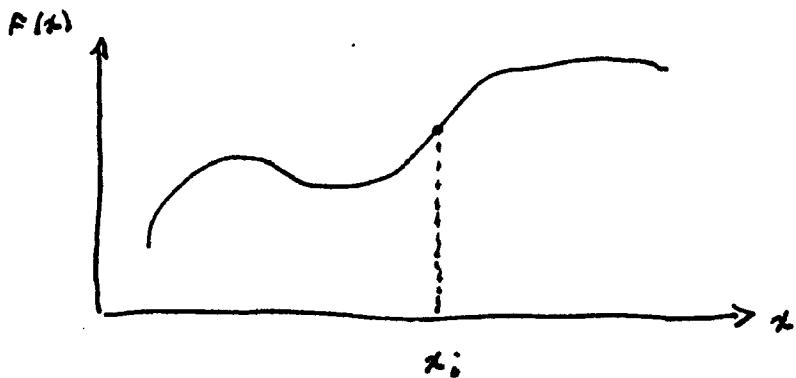
JULY 1, 2009

DIRAC DELTA FCN

FOURIER TRANSFORMS

THEIR RELATIONSHIP

DIRAC DELTA FUNCTION

(THING) THAT PICKS OUT $f(x_i)$

DIRAC:

$$\int (\text{THING}) f(x) dx = f(x_i)$$

\downarrow

$$\delta(x - x_i)$$

\nearrow NUMBER

KRONENCKER:

$$\sum (\text{THING}) a_i \delta_{ij} = a_j$$

\downarrow

$$\delta_{ij}$$

\nearrow NUMBER

DIRAC THING MAP: FUNCTION \rightarrow NUMBERKRONENCKER THING MAP: VECTOR \rightarrow NUMBEROLD WISDOM: DIRAC δ -FCNS ONLY MAKE SENSE INSIDE AN INTEGRAL

NEW WISDOM: FUNCTIONAL NOT A FUNCTION

TWO KINDS OF FUNCTIONS

GOOD OLD FUNCTIONS

D. PHYSICAL FCNS

P: PROPER FCNS ↗

M: SQUARE INTEGRABLE L^2

$$\langle f | f \rangle \text{ FINITE}$$

GENERALIZED FCNS

$$\langle x | x' \rangle \text{ INFINITE} \rightarrow \text{NOT SQUARE INTEGRABLE}$$

$$\langle x | f \rangle \text{ FINITE}$$

P: IMPROPER FCNS

M: DISTRIBUTIONS

DIRAC DELTA FCN

$$\langle x | g \rangle = \int (\text{THING}) g(x') dx' = g(x)$$

$$\delta(x-x')$$

well done!

$$\delta(x-x') = \langle x | x' \rangle = \langle x' | x \rangle$$

$$\delta(p-p') = \langle p | p' \rangle = \langle p' | p \rangle$$

PROPERTIES OF DIRAC DELTA FCN

$$(1) \quad \delta(x-x') = 0 \text{ WHEN } x \neq x'$$

$$(2) \quad \int_a^b \delta(x-x') dx' = 1 \quad \text{WHEN } a \leq x \leq b$$

$$(3) \quad \delta(x-x') = \delta(x'-x) \quad \text{EVEN FCN}$$

GAUSSIAN REPRESENTATION

$$g_\Delta(x-x') = \frac{1}{\sqrt{\pi} \Delta} e^{-{(x-x')}^2/\Delta^2}$$

$$\delta(x-x') = \lim_{\Delta \rightarrow 0} g_\Delta(x-x')$$

ACTION

$$\int \delta(x-x') f(x') dx' = f(x)$$

OLD WISDOM: DELTA FCNS ONLY MAKE SENSE

INSIDE INTEGRALS



NEW WISDOM: DIRAC'S δ

IS NOT A FCN, it is a FCNAL
FCN: number \rightarrow number
FCNAL: number \rightarrow FCN

CAN ALSO DEFINE DERIVATIVES OF DELTA

$$\delta'(x-x') \text{ def} = \frac{d}{dx} [\delta(x-x')]$$

$$= -\frac{d}{dx'} [\delta(x-x')]$$

↑
↓



ODD Fcn



ACTION OF δ'

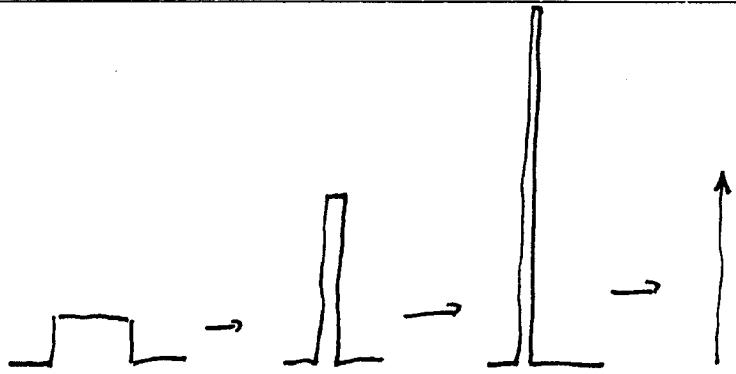
$$\int \delta'(x-x') f(x') dx' = \frac{df(x)}{dx}$$

GENERALIZE

$$\frac{d^n [\delta(x-x')]}{dx^m} = \delta(x-x') \frac{d^n}{dx'^n}$$

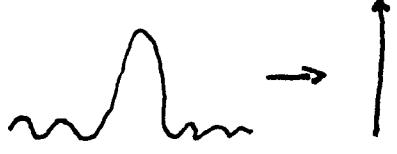
OTHER REPS

RECTANGLE FCN



SINC FCN

$$\text{SINC}(x) = \frac{\sin x}{x}$$



EXP FCN

$$\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(x-x')\xi} d\xi$$

Representations of the delta function

The delta function can be viewed as the limit of a sequence of functions

$$\delta(x) = \lim_{a \rightarrow 0} \delta_a(x),$$

where $\delta_a(x)$ is sometimes called a *nascent delta function*. This limit is in the sense that

$$\lim_{a \rightarrow 0} \int_{-\infty}^{\infty} \delta_a(x) f(x) dx = f(0)$$

for all continuous f .

The term *approximate identity* has a particular meaning in harmonic analysis, in relation to a limiting sequence to an identity element for the convolution operation (also on groups more general than the real numbers, e.g. the unit circle). There the condition is made that the limiting sequence should be of positive functions.

Some nascent delta functions are:

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} \quad \text{Limit of a Normal distribution}$$

$$\delta_a(x) = \frac{1}{\pi} \frac{a}{a^2 + x^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - |ak|} dk \quad \text{Limit of a Cauchy distribution}$$

$$\delta_a(x) = \frac{e^{-|x/a|}}{2a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + a^2 k^2} dk \quad \text{Cauchy } \varphi(\text{see note below})$$

$$\delta_a(x) = \frac{\text{rect}(x/a)}{a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{sinc}\left(\frac{ak}{2\pi}\right) e^{ikx} dk \quad \text{Limit of a rectangular function}$$

$$\delta_a(x) = \frac{1}{\pi x} \sin\left(\frac{x}{a}\right) = \frac{1}{2\pi} \int_{-1/a}^{1/a} \cos(kx) dk \quad \text{rectangular function } \varphi(\text{see note below})$$

$$\delta_a(x) = \partial_x \frac{1}{1 + e^{-x/a}} = -\partial_x \frac{1}{1 + e^{x/a}} \quad \text{Derivative of the sigmoid (or Fermi-Dirac) function}$$

$$\delta_a(x) = \frac{a}{\pi x^2} \sin^2\left(\frac{x}{a}\right)$$

$$\delta_a(x) = \frac{1}{a} A_i\left(\frac{x}{a}\right) \quad \text{Limit of the Airy function}$$

$$\delta_a(x) = \frac{1}{a} J_{1/a} \left(\frac{x+1}{a} \right)$$

Limit of a Bessel function

Note: If $\delta(a, x)$ is a nascent delta function which is a probability distribution over the whole real line (i.e. is always non-negative between $-\infty$ and $+\infty$) then another nascent delta function $\delta_\varphi(a, x)$ can be built from its characteristic function as follows:

$$\delta_\varphi(a, x) = \frac{1}{2\pi} \frac{\varphi(1/a, x)}{\delta(1/a, 0)}$$

where

$$\varphi(a, k) = \int_{-\infty}^{\infty} \delta(a, x) e^{-ikx} dx$$

is the characteristic function of the nascent delta function $\delta(a, x)$. This result is related to the localization property of the continuous Fourier transform.

The Dirac comb

Main article: Dirac comb

A so-called uniform "pulse train" of Dirac delta measures, which is known as a Dirac comb, or as the shah distribution, creates a sampling function, often used in digital signal processing (DSP) and discrete time signal analysis.

See also

- Kronecker delta
- Dirac comb
- Logarithmically-spaced Dirac comb
- Green's function
- Dirac measure

External links

- Delta Function (<http://mathworld.wolfram.com/DeltaFunction.html>) on MathWorld
- Dirac Delta Function (<http://planetmath.org/encyclopedia/DiracDeltaFunction.html>) on PlanetMath
- The Dirac delta measure is a hyperfunction (<http://www.osaka-kyoiku.ac.jp/~ashino/pdf/chinaproceedings.pdf>)
- We show the existence of a unique solution and analyze a finite element approximation when the source term is a Dirac delta measure (<http://pubs.siam.org/sam-bin/dbq/article/43178>)
- Non-Lebesgue measures on R. Lebesgue-Stieltjes measure, Dirac delta measure. (<http://www.mathematik.uni-muenchen.de/~lerdos/WS04/FA/content.html>)

~~DISCRETE~~ FOURIER TRANSFORMS

TIME

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

SYMMETRIC CONVENTION

ω TEMPORAL FREQUENCY

SPACE

κ

κ SPATIAL FREQUENCY

$$F(\kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\kappa x} dx$$

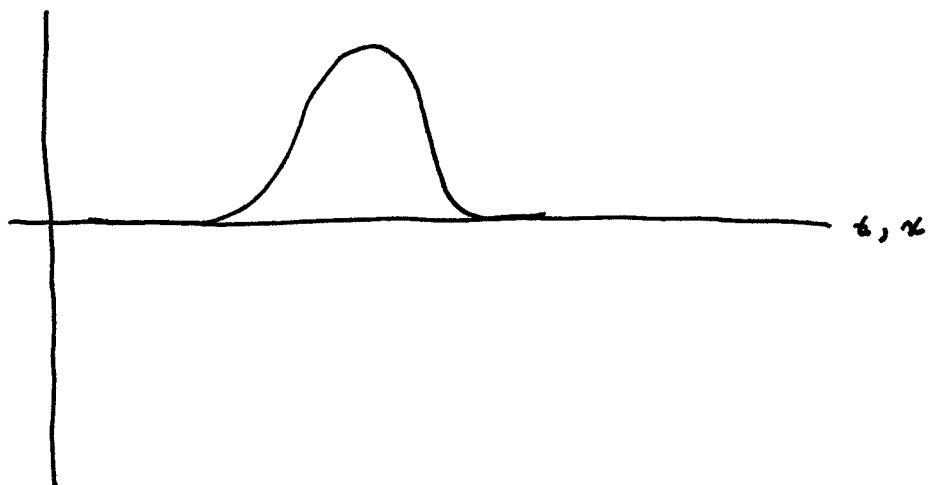
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\kappa) e^{i\kappa x} d\kappa$$

$$\epsilon = \kappa \omega$$

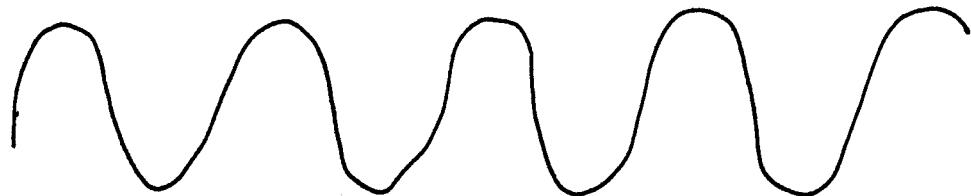
$$p = \kappa k$$

$$dp = \kappa dk$$

CARTOON VERSION



N.B.,
THESE HAVE
INFINITE
EXTENT

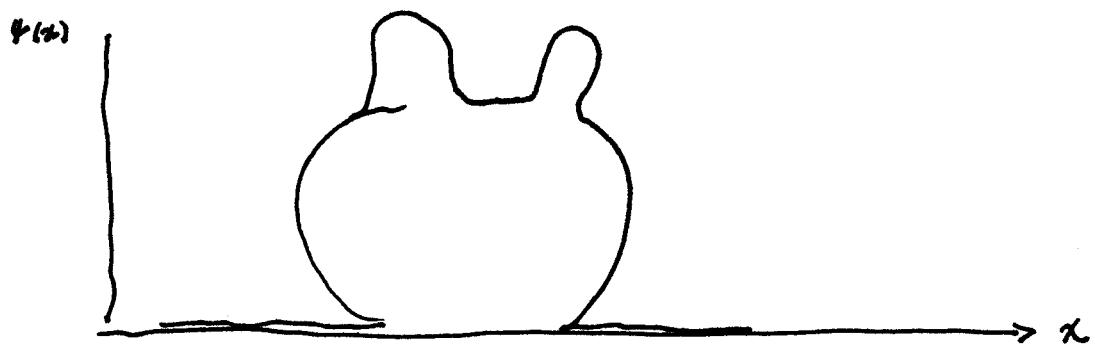


if complex \Rightarrow HELICES

ANY FUNCTION?

YES

HOW ABOUT THIS WAVEFCN?



NO MICKEY MOUSE WAVEFCNS!

$$\hat{q}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} q(x) e^{-ipx/\hbar} dx$$

$$q(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \hat{q}(p) e^{ipx/\hbar} dp$$

FT OF DELTA FUNCS

$$\hat{f}(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \delta(x-x') e^{-ikx} dx$$

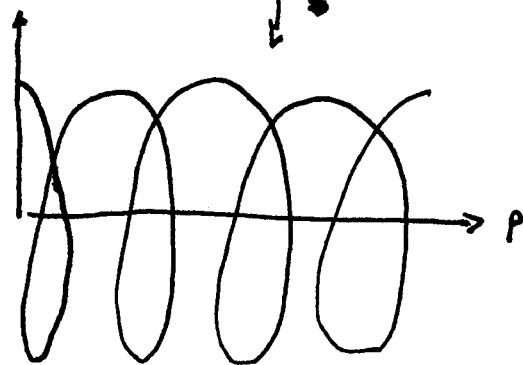
$$= (2\pi)^{-1/2} e^{-ikx'}$$

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \delta(k-k') e^{ikx} dk$$

$$= (2\pi)^{-1/2} e^{ik'x}$$

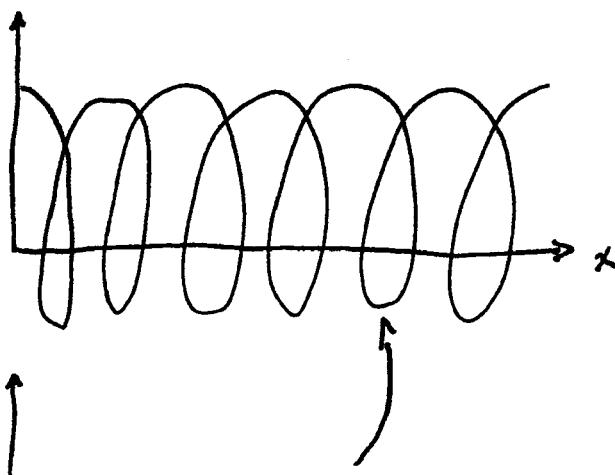


$$\delta(x - x')$$



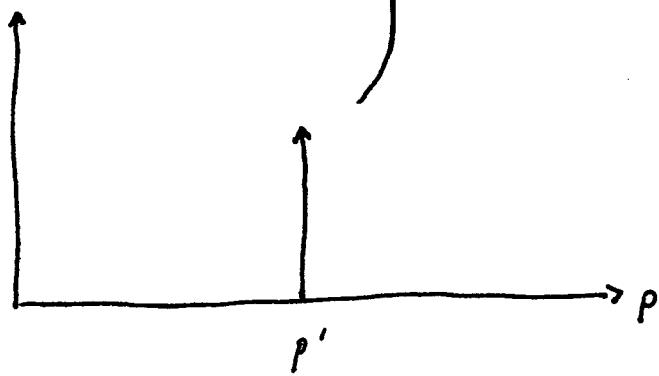
$$e^{-ipx'/\hbar}$$

$$e^{-ik'x'}$$



$$e^{ip'x/\hbar}$$

$$e^{ik'x}$$



$$\delta(p - p')$$

CONVOLUTION THM

CONVOLUTION MULTIPLICATION

IN IN

ONE SPACE THE OTHER
SPACE

OUT OUT

10

$$f \cdot g$$

۷۸

۲۷

$$FT \left\{ f(z) * g(z) \right\} = \hat{f}(k) \cdot \hat{g}(k)$$

$$\text{FT} \left(\begin{array}{c} \text{Graph} \\ * \end{array} \right) = \text{FT} \left(\begin{array}{c} \text{Graph} \\ \downarrow \end{array} \right)$$

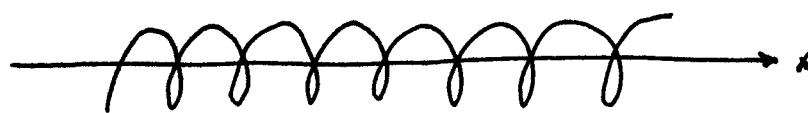
The diagram illustrates the conversion of 2,4-dihydroxyacetone (2,4-DHA) to 2-hydroxypropanoic acid (2-HPO). On the left, the structure of 2,4-DHA is shown as a three-carbon chain with hydroxyl groups at positions 2 and 4. An arrow points to the right, indicating the reaction. Below the arrow, two horizontal lines represent the equilibrium between 2,4-DHA and 2-HPO. The structure of 2-HPO is shown on the far right, featuring a three-carbon chain with a carboxylic acid group at position 1 and a hydroxyl group at position 2.

A horizontal line with a vertical arrow pointing upwards at its right end.

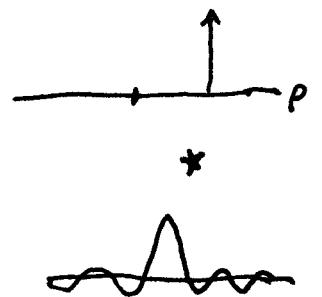
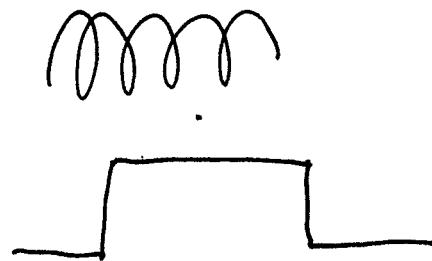
The image contains three separate hand-drawn line graphs. The first graph on the left shows a single broad, smooth peak. The second graph in the middle shows a series of four smaller, closely spaced, rounded peaks. The third graph on the right shows a single sharp, narrow peak with several shorter, jagged lines extending downwards from its main body, resembling a multi-lobed or fractal-like shape.

CONVOLUTION INTEGRAL

$$h(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du$$



$\downarrow FT$



$\text{sinc}(p)$



FTS AND DIRAC NOTATION

$$|q\rangle = |q\rangle$$

$$= I |q\rangle \quad \begin{matrix} \text{INSERT A COMPLETE SET} \\ \text{OF STATES} \end{matrix}$$

$$= \int dK |K\rangle \langle K| q\rangle$$

$$= \int dK |K\rangle \hat{q}(K)$$

$$\langle x | q\rangle = \langle x | \int dK |K\rangle \hat{q}(K)$$

$$q(x) = \int \langle x | K\rangle \hat{q}(K) dK$$

↓

$$(2\pi)^{-1/2} e^{\bullet i k x}$$

$$= (2\pi)^{-1/2} \int \hat{q}(K) e^{ikx} dK$$

⋮

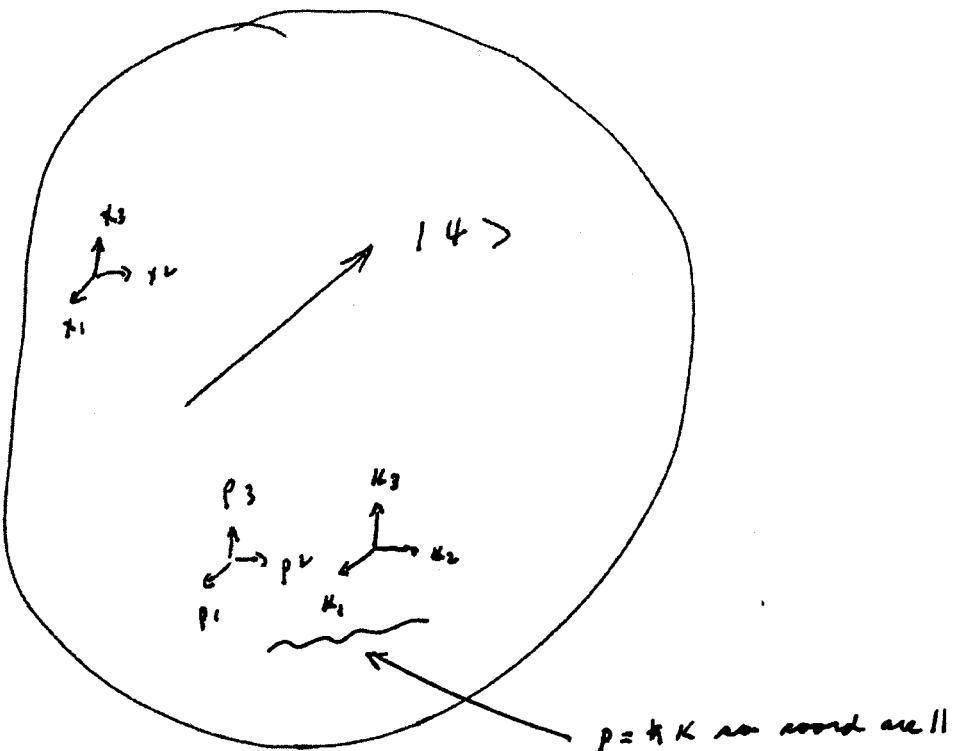
$$q(x)$$

EXERCISE:

$$I = \int dx |x\rangle \langle x|$$

$$\hat{g}(K) = (2\pi)^{-1/2} \int q(x) e^{-ikx} dx$$

FOURIER TRANSFORMS ARE JUST
A CHANGE OF BASIS



$$\begin{aligned} \langle x | \psi \rangle &= \psi(x) \\ \langle k | \psi \rangle &= \hat{\psi}(k) \\ \langle p | \psi \rangle &= \hat{\psi}(p) \\ \langle E | \psi \rangle &= \tilde{\psi}(E) \end{aligned} \quad \left. \begin{array}{l} \text{all have} \\ \text{all of the info} \\ \text{precisely the same} \\ \text{info} \end{array} \right\} \quad \begin{array}{l} \text{precisely the same} \\ \text{info} \end{array}$$

$|\psi\rangle$ is the real thing

FT's are just a change in basis

in QM!

ALL OBSERVABLES ...