LECTURE 4 JUNE 26, 2007

YESTERDAY ev's AND $e\vec{v}$'s

$$\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$e\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad ev = 0$$

$$e\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad ev = 2$$

$$e\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad ev = 2$$

$\{ e\vec{v} \}$ ORTHONORMAL BASIS

MUTUALLY PERP

NORMALIZED

CAN MAKE ANY 3d VECTOR

TODAY : MACHINERY TO CHANGE BASIS

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

$$\Omega \longrightarrow U^{+}\Omega U$$

$U =$ UNITARY OP BUILT OUT OF ORTHONORMAL BASIS OF IT'S $\vec{e}\vec{v}$'S

$$\begin{array}{ccc} \omega = 2 & \omega = 2 & \omega = 0 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{array}$$

$$U = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

CONVENTION

HIGHEST . . . . . LOWEST

ev ev

WHY DOES THIS WORK?

THM: THE EV'S OF ANY HERMITEAN OPERATOR ARE REAL

$\Rightarrow$ ANY HERMITEAN OPERATOR HAS (AT LEAST ONE) SET OF ORTHONORMAL EV'S THAT ARE A BASIS

IN THIS BASIS, Ω IS DIAGONAL WITH ITS ω_i'S ON THE DIAGONAL

BUT HOW DOES IT WORK?

$\langle w | v \rangle$ inner product number $= |\langle w|| |v\rangle| \cos\theta$

CHANGE BASIS

$\langle w' | v' \rangle$ = SAME NUMBER

$$\langle w | v \rangle = \langle w | I | v \rangle$$
$$= \langle w | U U^{+} | v \rangle$$
$$= \langle w' | v' \rangle$$

$U U^{+} = I$ identity operator

CHANGE BASIS FOR OPERATORS

$$\langle w | \Omega | v \rangle = \langle w | I \, \Omega \, I | v \rangle$$

$$\langle w | \, U U^{+} \, \Omega \, U U^{+} | v \rangle$$

$$\left(\langle w | U \right) \left(U^{+} \Omega U \right) \left(U^{+} | v \rangle \right)$$

$$\langle w' | \quad \Omega' \quad | v' \rangle$$

$$U = \left(\begin{pmatrix} e^{\vec{v}_1} \end{pmatrix} \begin{pmatrix} e^{\vec{v}_2} \end{pmatrix} \begin{pmatrix} e^{\vec{v}_3} \end{pmatrix} \right)$$

$$U^{+} = (U^{T})^{*}$$

$$U^{+} = \begin{bmatrix} \left(e^{\vec{v}_1} \right)^{*} \\ \left(e^{\vec{v}_2} \right)^{*} \\ \left(e^{\vec{v}_3} \right)^{*} \end{bmatrix}$$

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APPLY U^+ TO THE $e\vec{v}$'s OF Ω

$$\begin{bmatrix} \overset{*}{\overset{\frown}{e\vec{v}_1}} \\ \overset{*}{\overset{\frown}{e\vec{v}_2}} \\ \overset{*}{\overset{\frown}{e\vec{v}_3}} \end{bmatrix} \begin{pmatrix} \\ e\vec{v}_1 \\ \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \\ e\vec{v}_2 \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \\ e\vec{v}_3 \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

SO U^+ CHANGES THE KET BASIS

$\Rightarrow$ U " " BRA BASIS

$$U^+ \Omega U = \Omega$$

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DOES IT WORK? CALCULATE $U^{+}\Lambda U$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 2\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

it works

SAY:

FIND ev's AND $\vec{ev}$'s OF H

DIAGONALIZE THE H

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THM 10: FOR EVERY HERMITEAN OPERATOR, THERE EXISTS A BASIS OF ITS ORTHONORMAL EIGENVECTORS

x, p, H are Hermitian

$\Rightarrow$ unique expansion in x
p
E

ANY COMMON EIGENVECTORS?

THM 13: IF Ω and Λ are commuting Hermitian operators then there exists a common basis of eigenvectors that diagonalizes them both.

COMMUTATOR

$$[\Omega, \Lambda] = 0 = \Omega\Lambda - \Lambda\Omega$$

THREE CASES:

(1) COMPATIBLE: SHARE ALL $\vec{ev}$

(2) INCOMPATIBLE: SHARE NO $\vec{ev}$

(3) MIXED: SHARE SOME $\vec{ev}$ BUT NOT ALL

$$\Omega \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

ev = 1 $\quad e\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad$ ev = 2 $\quad e\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

ev = 3 $\quad e\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Lambda \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

ev = 1 $\quad e\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

many different

ev = 2 $\quad e\vec{v} =$

$$\Omega\Lambda = \begin{pmatrix} 1\cdot 1 & 0 & 0 \\ 0 & 2\cdot 2 & 0 \\ 0 & 0 & 3\cdot 2 \end{pmatrix}$$

$$\Lambda\Omega = \begin{pmatrix} 1\cdot 1 & 0 & 0 \\ 0 & 2\cdot 2 & 0 \\ 0 & 0 & 2\cdot 3 \end{pmatrix}$$

$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = 0 \quad \Rightarrow$ THERE EXISTS A UNIQUE SET OF $e\vec{v}$'S THAT DIAGONALIZE BOTH

CSCO $\Rightarrow$ unique eigenbasis

FINITE DIMENSIONAL $\longrightarrow$ INFINITE DIMENSIONAL

MATRICES	OPERATORS
VECTORS	FUNCTIONS
(DISCRETE EV'S)	(CONTINUOUS OR DISCRETE)

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GENERALIZE ~~TH~~ INNER PRODUCT

DISCRETE:

$$\langle f | g \rangle = \sum_{i=1}^{n} f_i^* \, g_i$$

DEPEND ON BASIS

BUT SUM DOES NOT

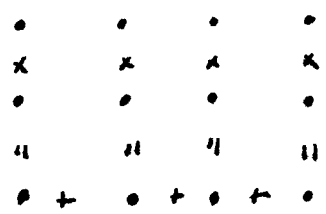
CONTINUOUS:

$$\langle f | g \rangle = \int_a^b f^*(x) \, g(x) \, dx$$

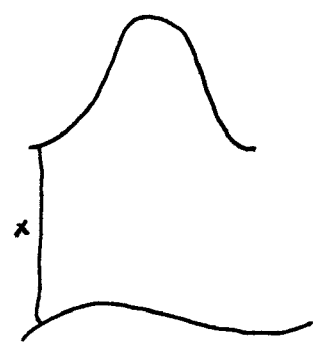
DEPEND ON BASIS

BUT INTEGRAL DOES NOT

DISCRETE



CONTINUOUS



MULTIPLY ~~POINT BY~~

COMPONENT BY COMPONENT

SUM

MULTIPLY

POINT BY POINT

INTEGRATE

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DISCRETE CASE

$$\Omega \, | \omega_i \rangle = \omega_i \, | \omega_i \rangle$$

CONTINUOUS CASE

$$\text{OP}\left(\text{FUNCTION}\right) = \left(\text{NUMBER}\right)\left(\text{FUNCTION}\right)$$

$$\frac{d}{dx}\left(\text{FUNCTION}\right) = \left(\text{NUMBER}\right) \text{FUNCTION}$$

$$\frac{d}{dx}\left(e^{mx}\right) = \underbrace{(m)}_{ev}\underbrace{\left(e^{mx}\right)}_{ef}$$

$m \in \mathbb{C}$

m can be any complex number

"SPECTRUM IS CONTINUOUS"

DELTA FUNCTIONS

DISCRETE:

KRONECKER DELTA

$$\delta_{ij} = 1 \quad \text{if } i = j$$

$$\delta_{ij} = 0 \quad \text{if } i \neq j$$

$$\langle e_i | e_j \rangle = \delta_{ij}$$

$$|v\rangle = \sum_i a_i \, |e_i\rangle$$

$$\langle e_j | v \rangle = \langle e_j | \sum_i a_i \, |e_i\rangle$$

$$= \sum_i a_i \underbrace{\langle e_j | e_i \rangle}_{\delta_{ji}}$$

$$= a_j$$

δ_{ij} PICKS OUT ONE OF THE EXPANSION COEFF'S

KRONECKER DELTA — DISCRETE BASIS

DIRAC DELTA — CONTINUOUS BASIS