THE TWO POSTULATES

(1) The probability $P(b,a)$ of a particle moving from point a to point b is the square of a complex number $|U(b,a)|^2$

$$P(b,a) = |U(b,a)|^2$$

(2) The propagator $U(b,a)$ is given by the sum of the phase factor $e^{iS/\hbar}$, where S is the classical action, taken over all the possible paths from a to b:

$$U(b,a) = \sum_{\substack{\text{all} \\ \text{paths}}} A\, e^{iS/\hbar}$$

where the constant K can be determined by

$$U(c,a) = \sum_{\substack{\text{all} \\ \text{paths} \\ \text{all} \\ \text{points } b\text{'s}}} U(c,b)\, U(b,a)$$

RECIPE

(1) CONSTRUCT ALL PATHS

(2) CALCULATE THE ACTION $S[x(t)]$ FOR EACH PATH

$$S[x(t)] = \int_{t_1}^{t_2} L \, dt$$

LAGRANGIAN

(3) SUM (INTEGRATE) OVER ALL PATHS TO OBTAIN THE PROPAGATOR

$$U(x_2, t_2; x_1, t_1) = A \sum_{\text{all paths}} e^{i S[x(t)]/\hbar}$$

(4) USE THE PROPAGATOR

$$\psi(x_2, t_2) = \int U(x_2, t_2; x_1, t_1) \, \psi(x_1, t_1) \, dx_1$$

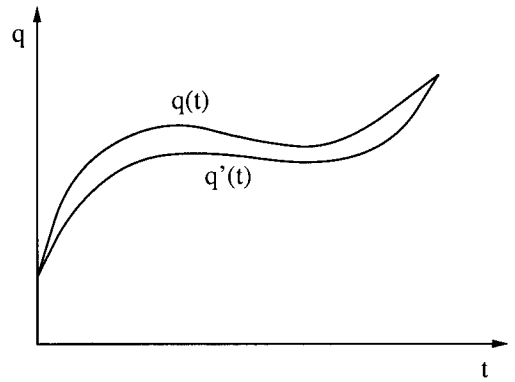


Figure 2: Two neighbouring paths.

However, this argument must be rethought for one exceptional path: that which extremizes the action, *i.e.*, the classical path, $q_c(t)$. For this path, $S[q_c + \eta] = S[q_c] + o(\eta^2)$. Thus the classical path and a very close neighbour will have actions which differ by much less than two randomly-chosen but equally close paths (Figure 3). This means that for fixed closeness

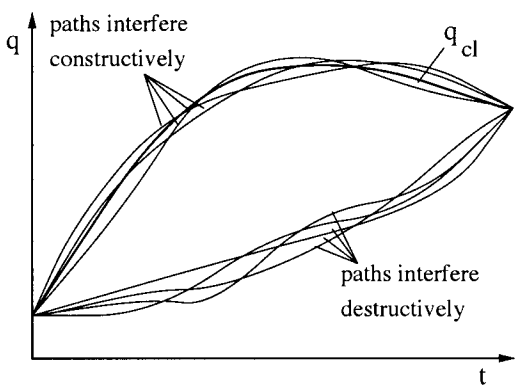
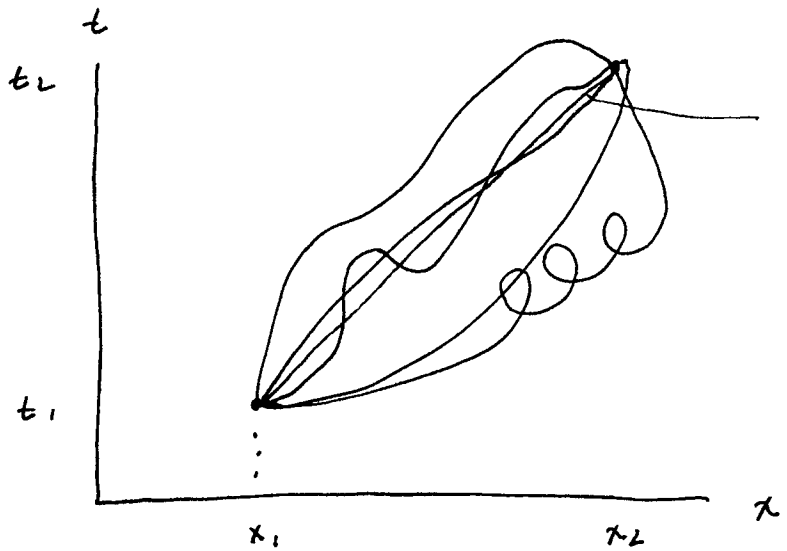


Figure 3: Paths near the classical path interfere constructively.

of two paths (I leave it as an exercise to make this precise!) and for fixed $\hbar$, paths near the classical path will on average interfere constructively (small phase difference) whereas for random paths the interference will be on average destructive.

Thus heuristically, we conclude that if the problem is classical (action $\gg \hbar$), the most important contribution to the PI comes from the region around the path which extremizes the PI. In other words, the particle's motion is governed by the principle that the action is stationary. This, of course, is none other than the Principle of Least Action from which the Euler-Lagrange equations of classical mechanics are derived.

(1) CONSTRUCT ALL PATHS



(2) CALCULATE THE ACTION

$$L(x, \dot{x}, t) = T - V$$

CLASSICAL PATH HAS MINIMUM ACTION

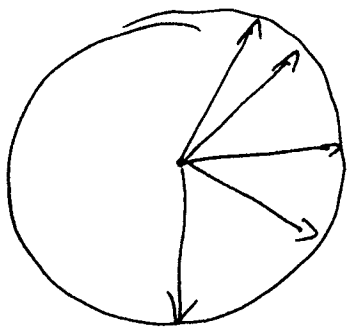
PHYSICS IS WHERE THE LEAST ACTION IS

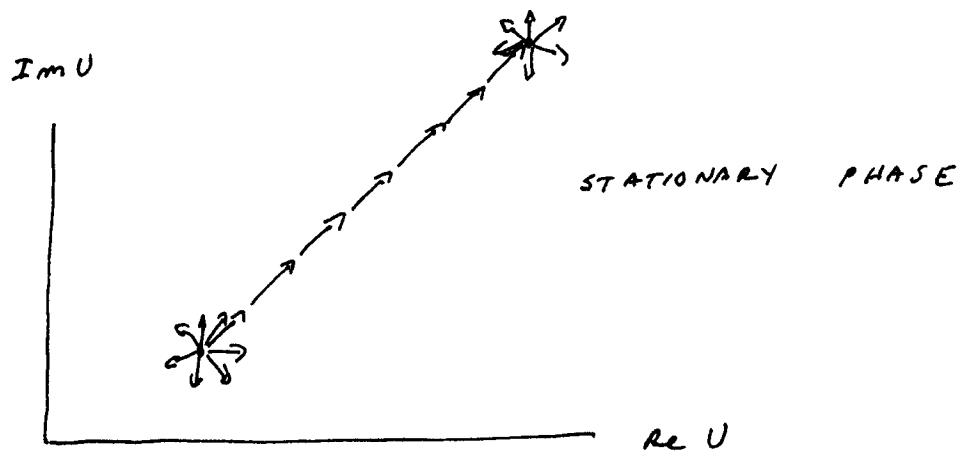
EXTREME ACTION IS

11

(3) CALCULATE THE PROPAGATOR

$$\sum_{\text{all paths}} e^{i S[x(t)]/\hbar}$$





PHASE COHERENCE

$$\frac{|S_{CLASSICAL} - S|}{\hbar} \leq \pi$$

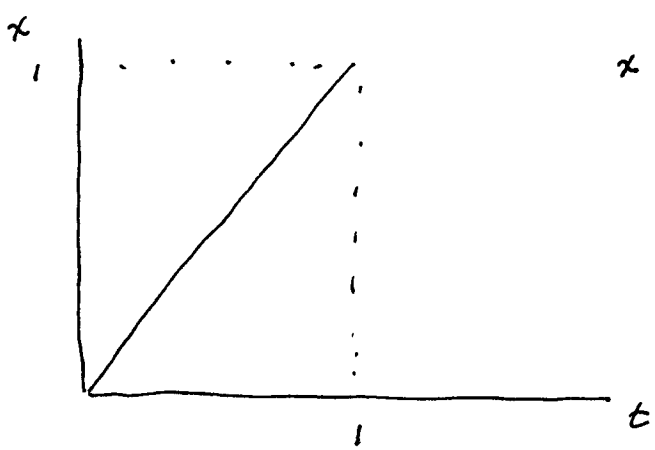
$$\Delta S \leq \pi \hbar$$

Q: Why do baseballs follow classical trajectories while electrons diffract, go thru both slits, etc.?

LAGRANGIAN for baseball and electron

$$L = T - V = \frac{1}{2} m v^2 - V(x) = \frac{1}{2} m v^2$$

THE CLASSICAL PATH



$x = v t \Rightarrow x \propto t$

CALCULATE THE ACTION FOR THE CLASSICAL PATH

$$S_{CL} = \int_0^1 L \, dt$$

$$= \int_0^1 \frac{1}{2} m v^2 \, dt$$

$$= \int_0^1 \frac{1}{2} m (1)^2 \, dt$$

$$S_{CL} = \frac{1}{2} m$$

NOW LET'S DO A NON CLASSICAL PATH

$$x = t^2$$

$$v = \frac{dx}{dt} = 2t$$

$$S_{NC} = \int_0^1 L \, dt = \int_0^1 \frac{1}{2} m (2t)^2 \, dt = \frac{4}{3} \left(\frac{1}{2} m\right)$$

$$S_{NC} = \frac{4}{3} S_{CL}$$

THE DIFFERENCE IS THE MASS!

BASEBALL $m \sim 200\,g$

$$S_{CL} = \frac{1}{2}(200g)(1\ cm/s)^2(1\ s)$$

$$= 100\ erg \cdot sec \approx 10^{29}\,\hbar$$

$$\Delta S = \frac{1}{3} S_{CL} = 3 \times 10^{28}\,\hbar \gg\gg \pi\hbar$$

SO, BASEBALL MUST STAY EXTREMELY CLOSE TO THE CLASSICAL PATH!

ELECTRON $m \sim 10^{-27}\,g$

$$S_{CL} = \frac{1}{2}(10^{-27})(1)^2(1)$$

$$= 5 \times 10^{-28}\ erg \cdot sec$$

$$\frac{1}{2}\hbar < \pi\hbar$$

$$\Delta S = \frac{1}{3} S_{CL} = \frac{1}{6}\hbar \ll \pi\hbar$$

SO, ELECTRON WILL FOLLOW THE NON CLASSICAL PATH!

Likewise, the function $\chi^*(x, t)$ characterizes the experience, or, let us say, experiment to which the system is to be subjected. If a different region, r'' and different Lagrangian after t, were to give the same $\chi^*(x, t)$ *via* Eq. (16), as does region R'', then no matter what the preparation, ψ, Eq. (14) says that the chance of finding the system in R'' is always the same as finding it in r''. The two "experiments" R'' and r'' are equivalent, as they yield the same results. We shall say loosely that these experiments are to determine with what probability the system is in state χ. Actually, this terminology is poor. The system is really in state ψ. The reason we can associate a state with an experiment is, of course, that for an ideal experiment there turns out to be a unique state (whose wave function is $\chi(x, t)$) for which the experiment succeeds with certainty.

Thus, we can say: the probability that a system in state ψ will be found by an experiment whose characteristic state is χ (or, more loosely, the chance that a system in state ψ will appear to be in χ) is

$$\left|\int \chi^*(x, t)\psi(x, t)dx\right|^2. \tag{17}$$

These results agree, of course, with the principles of ordinary quantum mechanics. They are a consequence of the fact that the Lagrangian is a function of position, velocity, and time only.

6. THE WAVE EQUATION

To complete the proof of the equivalence with the ordinary formulation we shall have to show that the wave function defined in the previous section by Eq. (15) actually satisfies the Schroedinger wave equation. Actually, we shall only succeed in doing this when the Lagrangian L in (11) is a quadratic, but perhaps inhomogeneous, form in the velocities $\dot{x}(t)$. This is not a limitation, however, as it includes all the cases for which the Schroedinger equation has been verified by experiment.

The wave equation describes the development of the wave function with time. We may expect to approach it by noting that, for finite ϵ, Eq. (15) permits a simple recursive relation to be developed. Consider the appearance of Eq. (15) if we were to compute ψ at the next instant of time:

$$\psi(x_{k+1}, t+\epsilon) = \int_{R\text{-}} \exp\left[\frac{i}{\hbar}\sum_{i=-\infty}^{k} S(x_{i+1}, x_i)\right] \times \frac{dx_k}{A}\frac{dx_{k-1}}{A}\cdots. \tag{15'}$$

This is similar to (15) except for the integration over the additional variable x_k and the extra term in the sum in the exponent. This term means that the integral of (15') is the same as the integral of (15) except for the factor $(1/A)\exp(i/\hbar)S(x_{k+1}, x_k)$. Since this does not contain any of the variables x_i for i less than k, all of the integrations on dx_i up to dx_{k-1} can be performed with this factor left out. However, the result of these integrations is by (15) simply $\psi(x_k, t)$. Hence, we find from (15') the relation

$$\psi(x_{k+1}, t+\epsilon) = \int \exp\left[\frac{i}{\hbar}S(x_{k+1}, x_k)\right]\psi(x_k, t)dx_k/A. \tag{18}$$

This relation giving the development of ψ with time will be shown, for simple examples, with suitable choice of A, to be equivalent to Schroedinger's equation. Actually, Eq. (18) is not exact, but is only true in the limit $\epsilon \to 0$ and we shall derive the Schroedinger equation by assuming (18) is valid to first order in ϵ. The Eq. (18) *need* only be true for small ϵ to the first order in ϵ. For if we consider the factors in (15) which carry us over a finite interval of time, T, the number of factors is T/ϵ. If an error of order ϵ^2 is made in each, the resulting error will not accumulate beyond the order $\epsilon^2(T/\epsilon)$ or $T\epsilon$, which vanishes in the limit.

We shall illustrate the relation of (18) to Schroedinger's equation by applying it to the simple case of a particle moving in one dimension in a potential $V(x)$. Before we do this, however, we would like to discuss some approximations to the value $S(x_{i+1}, x_i)$ given in (11) which will be sufficient for expression (18).

The expression defined in (11) for $S(x_{i+1}, x_i)$ is difficult to calculate exactly for arbitrary ϵ from classical mechanics. Actually, it is only necessary that an approximate expression for $S(x_{i+1}, x_i)$ be

which may not be zero. The question is still more important in the coefficient of terms which are quadratic in the velocities. In these terms (19) and (20) are not sufficiently accurate representations of (11) in general. It is when the coefficients are constant that (19) or (20) can be substituted for (11). If an expression such as (19) is used, say for spherical coordinates, when it is not a valid approximation to (11), one obtains a Schroedinger equation in which the Hamiltonian operator has some of the momentum operators and coordinates in the wrong order. Equation (11) then resolves the ambiguity in the usual rule to replace p and q by the non-commuting quantities $(\hbar/i)(\partial/\partial q)$ and q in the classical Hamiltonian $H(p, q)$.

It is clear that the statement (11) is independent of the coordinate system. Therefore, to find the differential wave equation it gives in any coordinate system, the easiest procedure is first to find the equations in Cartesian coordinates and then to transform the coordinate system to the one desired. It suffices, therefore, to show the relation of the postulates and Schroedinger's equation in rectangular coordinates.

The derivation given here for one dimension can be extended directly to the case of three-dimensional Cartesian coordinates for any number, K, of particles interacting through potentials with one another, and in a magnetic field, described by a vector potential. The terms in the vector potential require completing the square in the exponent in the usual way for Gaussian integrals. The variable x must be replaced by the set $x^{(1)}$ to $x^{(3K)}$ where $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the coordinates of the first particle of mass m_1, $x^{(4)}$, $x^{(5)}$, $x^{(6)}$ of the second of mass m_2, etc. The symbol dx is replaced by $dx^{(1)}dx^{(2)}\cdots dx^{(3K)}$, and the integration over dx is replaced by a $3K$-fold integral. The constant A has, in this case, the value $A = (2\pi\hbar\epsilon i/m_1)^{\frac{1}{2}}(2\pi\hbar\epsilon i/m_2)^{\frac{1}{2}}\cdots(2\pi\hbar\epsilon i/m_K)^{\frac{1}{2}}$. The Lagrangian is the classical Lagrangian for the same problem, and the Schroedinger equation resulting will be that which corresponds to the classical Hamiltonian, derived from this Lagrangian. The equations in any other coordinate system may be obtained by transformation. Since this includes all cases for which Schroedinger's equation has been checked with experiment, we may say our postulates are able to describe what can be described by non-relativistic quantum mechanics, neglecting spin.

7. DISCUSSION OF THE WAVE EQUATION

The Classical Limit

This completes the demonstration of the equivalence of the new and old formulations. We should like to include in this section a few remarks about the important equation (18).

This equation gives the development of the wave function during a small time interval. It is easily interpreted physically as the expression of Huygens' principle for matter waves. In geometrical optics the rays in an inhomogeneous medium satisfy Fermat's principle of least *time*. We may state Huygens' principle in wave optics in this way: If the amplitude of the wave is known on a given surface, the amplitude at a near by point can be considered as a sum of contributions from all points of the surface. Each contribution is delayed in phase by an amount proportional to the *time* it would take the light to get from the surface to the point along the ray of least *time* of geometrical optics. We can consider (22) in an analogous manner starting with Hamilton's first principle of least *action* for classical or "geometrical" mechanics. If the amplitude of the wave ψ is known on a given "surface," in particular the "surface" consisting of all x at time t, its value at a particular nearby point at time $t+\epsilon$, is a sum of contributions from all points of the surface at t. Each contribution is delayed in phase by an amount proportional to the *action* it would require to get from the surface to the point along the path of least *action* of classical mechanics.[16]

Actually Huygens' principle is not correct in optics. It is replaced by Kirchoff's modification which requires that both the amplitude and its derivative must be known on the adjacent surface. This is a consequence of the fact that the wave equation in optics is second order in the time. The wave equation of quantum mechanics is first order in the time; therefore, Huygens' principle *is* correct for matter waves, action replacing time.

[16] See in this connection the very interesting remarks of Schroedinger, Ann. d. Physik 79, 489 (1926).

The equation can also be compared mathematically to quantities appearing in the usual formulations. In Schroedinger's method the development of the wave function with time is given by

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t}=\mathbf{H}\psi, \tag{31}$$

which has the solution (for any ϵ if $\mathbf{H}$ is time independent)

$$\psi(x, t+\epsilon)=\exp(-i\epsilon\mathbf{H}/\hbar)\psi(x, t). \tag{32}$$

Therefore, Eq. (18) expresses the operator $\exp(-i\epsilon\mathbf{H}/\hbar)$ by an approximate integral operator for small ϵ.

From the point of view of Heisenberg one considers the position at time t, for example, as an operator $\mathbf{x}$. The position $\mathbf{x}'$ at a later time $t+\epsilon$ can be expressed in terms of that at time t by the operator equation

$$\mathbf{x}'=\exp(i\epsilon\mathbf{H}/\hbar)\mathbf{x}\exp-(i\epsilon\mathbf{H}/\hbar). \tag{33}$$

The transformation theory of Dirac allows us to consider the wave function at time $t+\epsilon$, $\psi(x', t+\epsilon)$, as representing a state in a representation in which $\mathbf{x}'$ is diagonal, while $\psi(x, t)$ represents the same state in a representation in which $\mathbf{x}$ is diagonal. They are, therefore, related through the transformation function $(x'|x)_\epsilon$ which relates these representations:

$$\psi(x', t+\epsilon)=\int(x'|x)_\epsilon\psi(x,t)\,dx.$$

Therefore, the content of Eq. (18) is to show that for small ϵ we can set

$$(x'|x)_\epsilon=(1/A)\exp(iS(x', x)/\hbar) \tag{34}$$

with $S(x', x)$ defined as in (11).

The close analogy between $(x'|x)_\epsilon$ and the quantity $\exp(iS(x', x)/\hbar)$ has been pointed out on several occasions by Dirac.[1] In fact, we now see that to sufficient approximations the two quantities may be taken to be proportional to each other. Dirac's remarks were the starting point of the present development. The points he makes concerning the passage to the classical limit $\hbar\to 0$ are very beautiful, and I may perhaps be excused for briefly reviewing them here.

First we note that the wave function at x'' at time t'' can be obtained from that at x' at time t' by

$$\psi(x'', t'')=\operatorname*{Lim}_{\epsilon\to 0}\int\cdots\int \times\exp\left[\frac{i}{\hbar}\sum_{i=0}^{j-1}S(x_{i+1}, x_i)\right] \times\psi(x', t')\frac{dx_0}{A}\frac{dx_1}{A}\cdots\frac{dx_{j-1}}{A}, \tag{35}$$

where we put $x_0\equiv x'$ and $x_j\equiv x''$ where $j\epsilon=t''-t'$ (between the times t' and t'' we assume no restriction is being put on the region of integration). This can be seen either by repeated applications of (18) or directly from Eq. (15). Now we ask, as $h\to 0$ what values of the intermediate coordinates x_i contribute most strongly to the integral? These will be the values most likely to be found by experiment and therefore will determine, in the limit, the classical path. If $\hbar$ is very small, the exponent will be a very rapidly varying function of any of its variables x_i. As x_i varies, the positive and negative contributions of the exponent nearly cancel. The region at which x_i contributes most strongly is that at which the phase of the exponent varies least rapidly with x_i (method of stationary phase). Call the sum in the exponent S;

$$S=\sum_{i=0}^{j-1}S(x_{i+1}, x_i). \tag{36}$$

Then the classical orbit passes, approximately, through those points x_i at which the rate of change of S with x_i is small, or in the limit of small $\hbar$, zero, i.e., the classical orbit passes through the points at which $\partial S/\partial x_i=0$ for all x_i. Taking the limit $\epsilon\to 0$, (36) becomes in view of (11)

$$S=\int_{t'}^{t''}L(\dot{x}(t), x(t))dt. \tag{37}$$

We see then that the classical path is that for which the integral (37) suffers no first-order change on varying the path. This is Hamilton's principle and leads directly to the Lagrangian equations of motion.

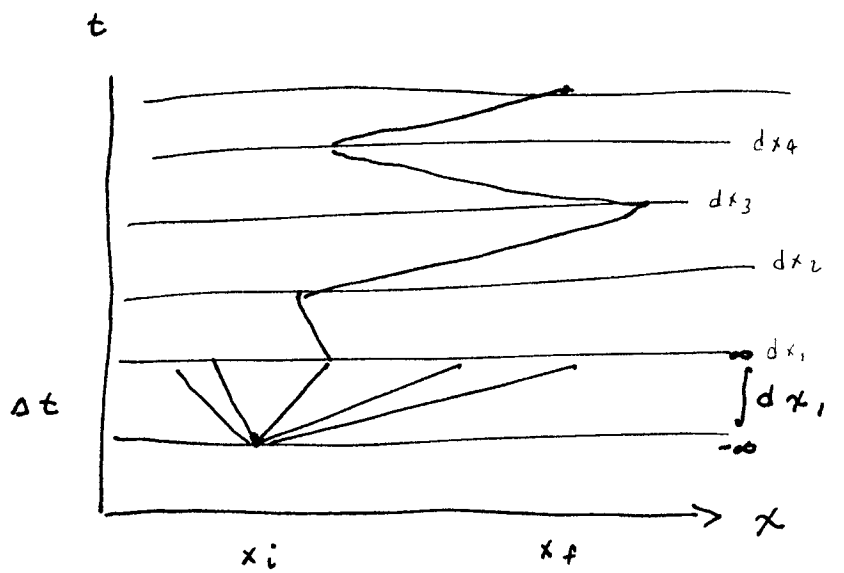
HOW TO DO THE PATH INTEGRAL?

$$A \sum_{\substack{\text{all} \\ \text{paths}}} e^{i S[x(t)]/\hbar}$$

$$A \int_{x_i}^{x_f} e^{i S[x(t)]/\hbar} \, D[x(t)]$$

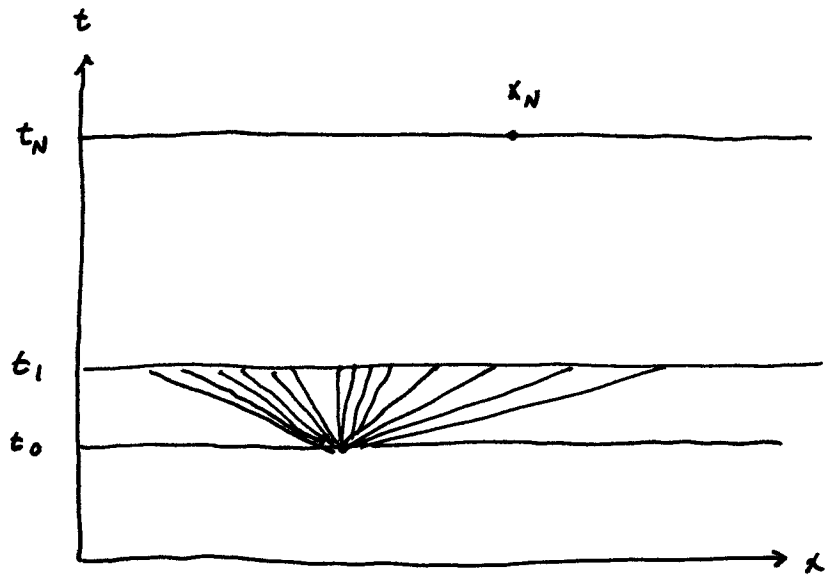
FUNCTIONAL INTEGRAL

$S[x(t)]$ IS A FUNCTIONAL



$\lim_{\Delta t \to 0}$

TIME SLICING



GO TO EVERY x_1

CONTINUOUS ACTION

$$S = \int_{t_0}^{t_N} L(t)\, dt = \int_{t_0}^{t_N} \frac{1}{2} m v^2\, dt$$

DISCRETE APPROXIMATION $\qquad x_i = x(t_i)$

$$S = \sum_{i=0}^{N-1} \frac{1}{2} m \left(\frac{x_{i+1} - x_i}{\epsilon} \right)^2 \epsilon$$

PATH INTEGRAL

$$U(x_N, t_N; x_0, t_0) = \int_{x_0}^{x_N} e^{i S[x(t)]/\hbar} \, D[x(t)]$$

$$= \lim_{\substack{N \to \infty \\ \epsilon \to 0}} A \int_{-\infty}^{\infty} \int \cdots \int e^{\frac{i}{\hbar} \frac{m}{2} \sum_{i=0}^{N} (x_{i+1} - x_i)^2 / \epsilon} \, dx_1 \ldots dx_{N-1}$$

BECAUSE THE INTEGRAND IS GAUSSIAN,

WE CAN DO THE INTEGRALS!

DO THE dx_1 INTEGRATION

$$\int_{-\infty}^{\infty} \exp\frac{i}{\epsilon}\left[(x_2 - x_1)^2 + (x_1 - x_0)^2 \right] dx_1 \sim e^{-(y_2 - y_0)^2/2i}$$

where $y_2 - y_1$ and $y_1 - y_2$ are over 2ϵ.

DO THE dx_2 INTEGRATION

$$\int dx_2 \sim e^{-(y_3 - y_0)^2/3i}$$

$$\vdots$$

$$\sim e^{-(y_N - y_0)^2/Ni}$$

$$\exp\left[\frac{im (y_N - x_0)^2}{2\hbar N\epsilon} \right]$$

$N \to \infty$

$\epsilon \to 0$

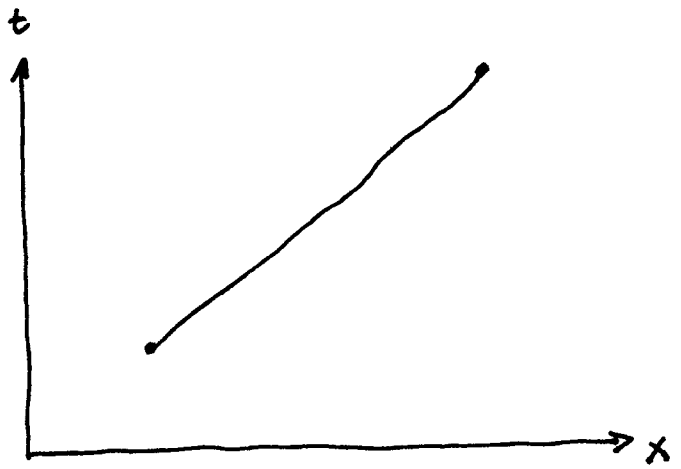
$N\epsilon \to t_N - t_0$

AUG 13, 2004

LAST TIME: PATH INTEGRAL

SUM OVER HISTORIES

LAGRANGEAN



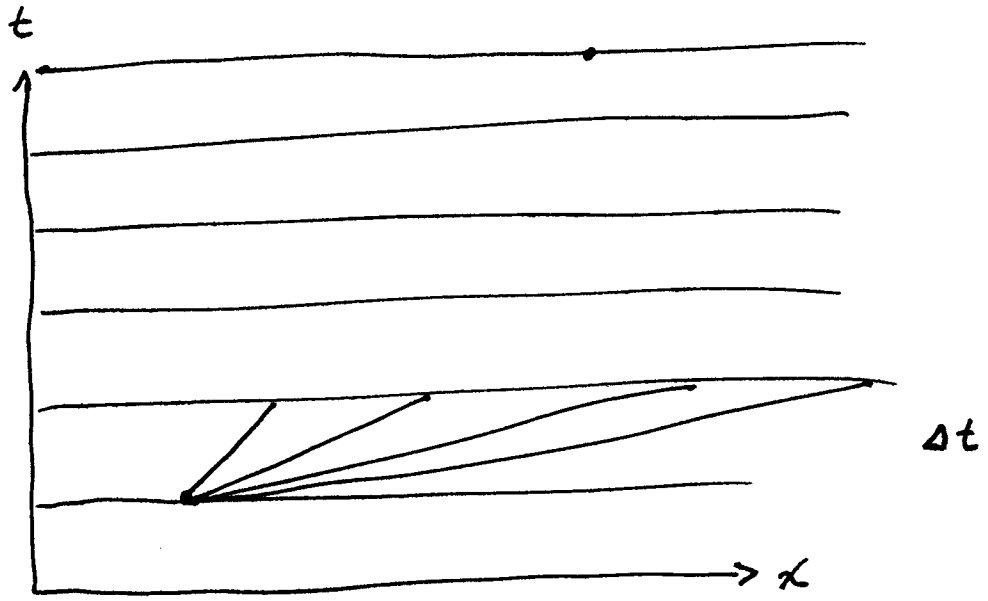
$$\sum_{\substack{\text{ALL}\\ \text{PATHS}}} e^{iS/\hbar} \Rightarrow \bullet \sum e^{i\varphi}$$

STATIONARY PHASE $\Rightarrow$ CORNU SPIRAL

$$\int f(x)\, dx$$ INTEGRAL

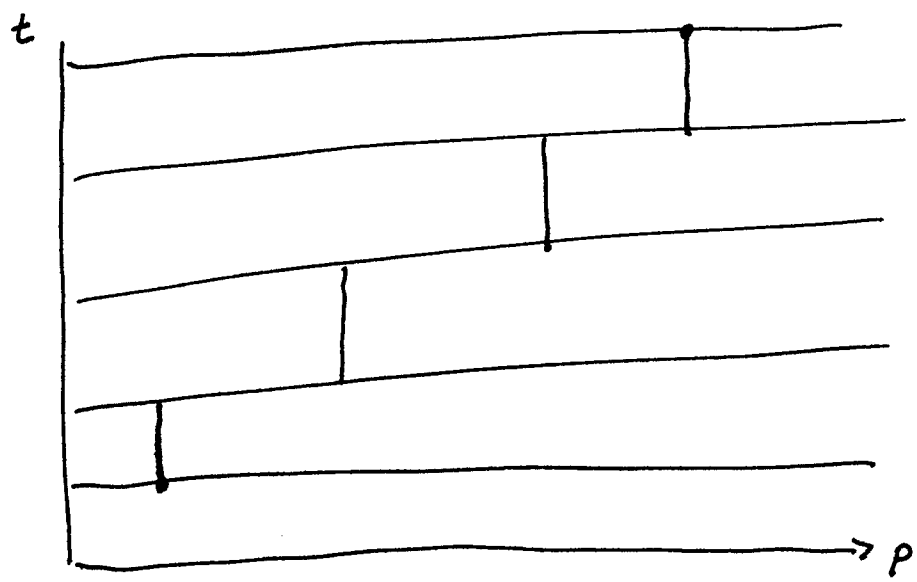
$$\int F[x(t)]\, d[x(t)]$$ FUNCTIONAL INTEGRAL

NRQM ⇒ INFINITE VELOCITIES

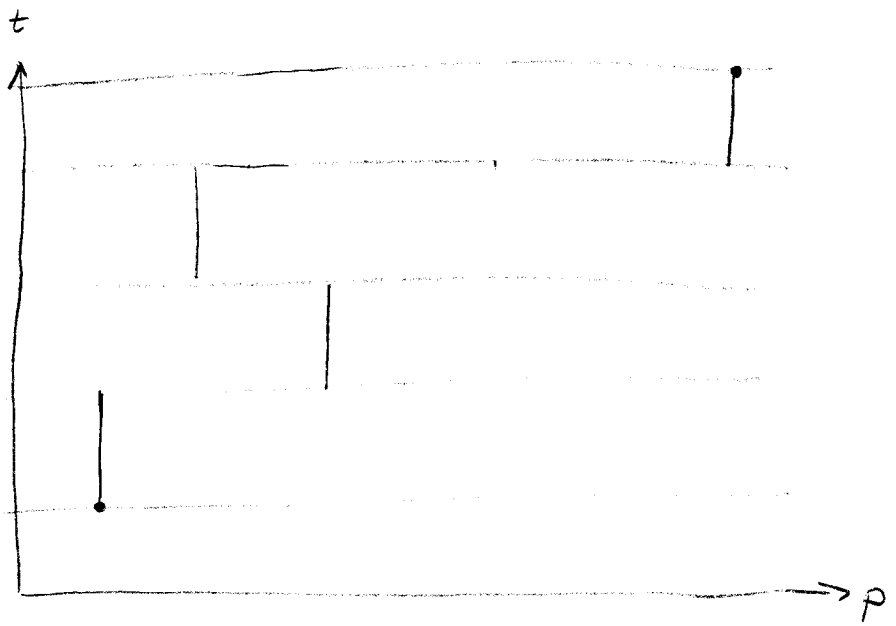


$$\text{PATH INTEGRAL} = \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots$$

MOMENTUM SPACE



TRAJECTORY IN MOMENTUM SPACE



PHASE SPACE PATH INTEGRAL

⇒ MORE GENERAL! WORKS EVEN WHEN p IS NOT QUADRATICALLY SEPARABLE IN $\mathcal{H}$

$$U(x_2, t_2; x_1, t_1) = \int \mathcal{D}[p(t)]\, \mathcal{D}[x(t)]$$

$$\exp\left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} [p\dot{x} - \mathcal{H}(x,p)]\, dt' \right\}$$

$2T \quad -(T+V)$

When the Hamiltonian is quadratic in the momentum, and the momentum is separable, we can do the $\mathcal{D}[p(t)]$ integration to obtain the COORDINATE SPACE PATH INTEGRAL!

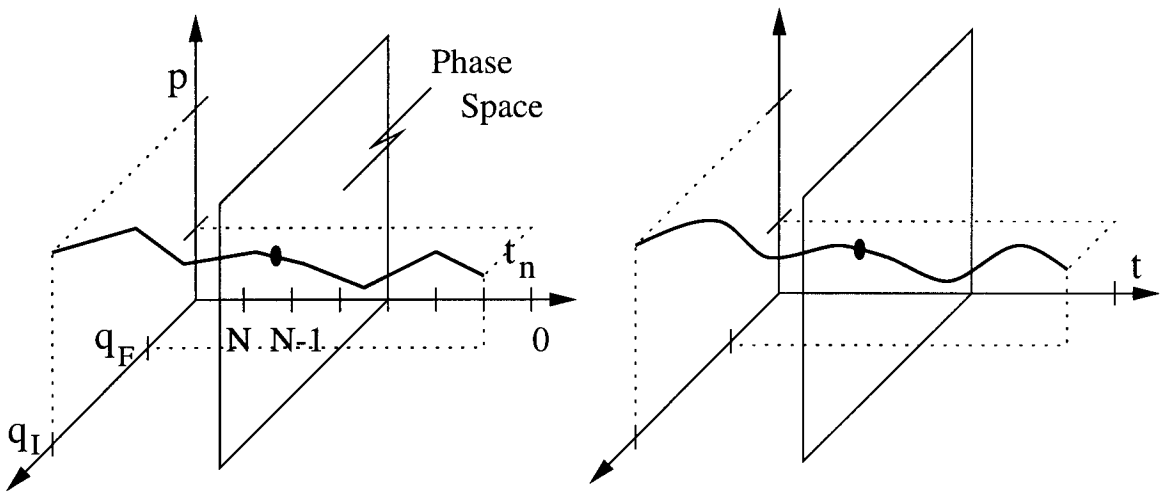
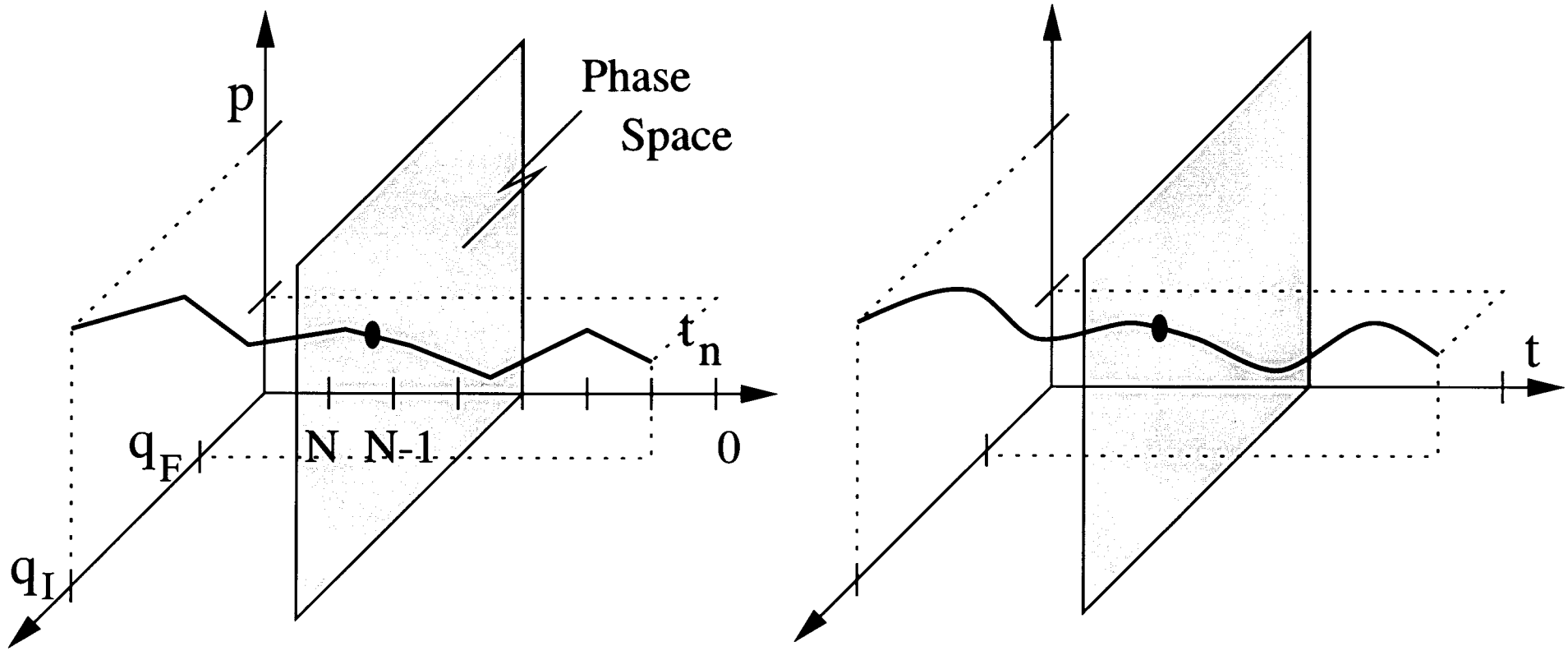


Figure 3.1: Left: visualization of a set of phase space points contributing to the discrete time configuration integral (3.5). Right: in the continuum limit, the set of points becomes a smooth curve.


p
Phase
Space
t_n
q_F
N N-1
0
q_I
t

PHASE SPACE PATH INTEGRAL

$$\int e^{i S/\hbar} \; D[p(t)] \; D[x(t)]$$

$$L = p\dot{x} - H(x, p)$$

$$\downarrow$$

$$2T - (T+V)$$

$$T - V$$

$$S = \int_{t_1}^{t_2} L \, dt$$

WHEN $H \propto p^2$ AND x AND p SEPARATE

DO $\int D[p(t)]$ => position space path integral

QUANTUM GRAVITY $\int D[g_{\mu\nu}] \, e^{i S_G/\hbar} \, D[\text{everything else}] \, e^{i S_M/\hbar}$